On the boundedness of solutions of nonautonomous differential equations

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Dedicated to L. Pintér on his 60th birthday

1. Introduction

In the study of existence of periodic solutions and almost periodic solutions as well as behavior of limiting sets of solutions of ordinary differential equations, the uniform boundedness and uniform ultimate boundedness of solutions are frequently needed [1—4, 9]. These properties of solutions can be regarded as either the instability of infinity or a special case of some kind of stability of a set. Therefore, there exists a close relation between Lyapunov's direct method and the boundedness of solutions. A typical result showing this relation is Theorem 10.4 in [3]. In this theorem the uniform ultimate boundedness is guaranteed by the existence of an appropriate Lyapunov function having a negative definite derivative along the solutions. However, in practice it is very difficult to construct such a Lyapunov function. For example, for mechanical systems the total mechanical energy, which is a typical Lyapunov function, never has a negative definite derivative along the motions with respect to the generalized coordinates.

The purpose of this paper is to study the boundedness and ultimate boundedness of solutions of nonautonomous differential equations by Lyapunov's direct method when the derivative of the Lyapunov function along the solutions is only semidefinite. The results generalize V. M. Matrosov's theorem [5] on the asymptotic stability to the boundedness of solutions. An application is given to the boundedness of the motions of a holonomic scleronomic mechanical system of \( n \) degrees of freedom being under the action of potential, dissipative and gyroscopic forces.

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2. Notations and definitions

Consider the system

\[ \dot{x} = X(t, x), \]

where \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+ = [0, \infty)\) and \(X: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous. Throughout this paper, for simplicity, we assume that for any \((t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n\), there exists a unique solution \(x(t; t_0, x_0)\) of (2.1) through \((t_0, x_0)\) defined for all \(t \geq t_0\).

Definition 2.1 [3]. A solution \(x(t; t_0, x_0)\) of (2.1) is bounded, if

\[ \sup_{t \geq t_0} |x(t; t_0, x_0)| < \infty. \]

The solutions of (2.1) are uniformly bounded (U.B.) if for every \(\alpha > 0\) there exists a \(\beta(\alpha) > 0\) such that \([t_0 \equiv 0, |x_0| < \alpha, t \equiv t_0]\) imply \(|x(t; t_0, x_0)| < \beta(\alpha)\).

The solutions of (2.1) are equiultimately bounded (E.U.B.) for some bound \(B\) if for every \(\alpha > 0\) and \(t_0 \equiv 0\) there exists a \(T(t_0, \alpha) > 0\) such that \([|x_0| < \alpha, t \equiv t_0 + T(t_0, \alpha)]\) imply \(|x(t; t_0, x_0)| < B\).

The solutions of (2.1) are uniformly ultimately bounded (U.U.B.) for some bound \(B\) if for every \(\alpha > 0\) there exists a \(T(\alpha) > 0\) such that \([t_0 \equiv 0, |x_0| < \alpha, t \equiv t_0 + T(\alpha)]\) imply \(|x(t; t_0, x_0)| < B\).

By a pseudo wedge \(W\) we mean a continuous and strictly increasing function \(W: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) with \(W(r) > 0\) if \(r > 0\). A pseudo wedge \(W\) is called unbounded if \(\lim_{r \to \infty} W(r) = +\infty\).

Denote by \([a]_+\) and \([a]_-\) the positive and negative part of the real number \(a\), respectively, that is, \([a]_+ = \max\{a, 0\}\), \([a]_- = \max\{-a, 0\}\).

Definition 2.2 [5]. A measurable function \(\lambda: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is said to be integrally positive if

\[ \int J \lambda(t) \, dt = \infty \]

holds on every set \(J = \bigcup_{m=1}^{\infty} [a_m, b_m]\) such that \(a_m < b_m \equiv a_{m+1}\) and \(b_m - a_m \equiv \delta > 0\) \((m = 1, 2, \ldots)\) for a constant \(\delta > 0\).

Definition 2.3 [7]. A measurable function \(\lambda: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is said to be weakly integrally positive if for every \(\delta > 0, \Delta > 0\) and for every set \(J = \bigcup_{m=1}^{\infty} [a_m, b_m]\) with \(a_m + \delta \equiv b_m \equiv a_{m+1} < b_m + \Delta\) \((m = 1, 2, \ldots)\) the relation \(\int J \lambda(t) \, dt = \infty\) holds.

Lemma 2.1. If a measurable function \(\lambda: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is integrally positive, then for every \(\alpha > 0\) and \(\delta > 0\) there exists a positive integer \(K(\alpha, \delta)\) such that for every set \(J = \bigcup_{m=1}^{K} [a_m, b_m]\) with \(a_m < a_m + \delta \equiv b_m \equiv a_{m+1}\) for \(1 \equiv m \equiv K - 1\), we have

\[ \int_J \lambda(t) \, dt \equiv \alpha. \]
Proof. It is easy to see that $\lambda$ is integrally positive if and only if for every $\delta > 0$ the inequality

$$\liminf_{t \to \infty} \int_{t}^{t+\delta} \lambda(s) \, ds > 0$$

holds. Consequently, for any given $\delta > 0$ there are $T = T(\delta) > 0$ and $\mu(\delta) > 0$ such that $t \equiv T(\delta)$ implies

$$\int_{t}^{t+\delta} \lambda(s) \, ds \equiv \mu(\delta).$$

Let $\alpha > 0$ and $\delta > 0$ be given, and define $K(\alpha, \delta) = [T(\delta)/\delta]+1+\lfloor z/\mu(\delta) \rfloor + 1$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$, that is, $[a] = \max \{z : z$ is an integer with $z \equiv a\}$. Then the number $K(\alpha, \delta)$ has the property mentioned in the assertion.

The following assertion can be easily proved by making use of (2.2).

Lemma 2.2. If a measurable function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is integrally positive, then

$$\lim_{T \to \infty} \int_{t_0}^{t_0+T} \lambda = \infty$$

uniformly with respect to $t_0 \in \mathbb{R}^+$. 

Remark 2.1. The property of weak integral positivity and property (2.3) are independent of one another. E.g., $\lambda(t) = 1/(1+t)$ is weakly integrally positive, but it does not satisfy (2.3) and so it is not integrally positive. On the other hand, weak integral positivity and (2.3) together do not imply integral positivity. E.g., the function

$$\lambda(t) = \begin{cases} 1/(1+t) & n \equiv t \equiv n+1/2 \\ 1 & n+1/2 < t < n+1 \end{cases}$$

is weakly integrally positive and satisfy (2.3) but it is not integrally positive.

With a continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ we associate the function

$$\dot{V}(t, x) = \limsup_{h \to 0^+} (1/h) \{V(t+h, x+hX(t, x)) - V(t, x)\},$$

which called the derivative of $V$ with respect to (2.1).

It can be proved (see [3], p. 3) that if $V$ is locally Lipschitz, then for an arbitrary solution $x(t)$ of (2.1) we have

$$V(t_2, x(t_2)) - V(t_1, x(t_1)) = \int_{t_1}^{t_2} \dot{V}(t, x(t)) \, dt, \quad (t_1, t_2 \in \mathbb{R}^+).$$
3. The theorems and their proofs

Theorem 3.1. Suppose that there exist nonnegative constants $B$ and $D$, nonnegative locally Lipschitz functions $V(t, x)$, $P(t, x)$ and continuous $K(t, x)$ defined for \( t \leq 0 \), $|x| \leq B$ satisfying the following conditions:

(i) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$, where $W_1$ and $W_2$ are unbounded pseudo wedges;

(ii) the derivative of $V$ with respect to (2.1) satisfies the inequality

\[
\dot{V}_{(2.1)}(t, x) \equiv -K(t, x) \quad \text{for } t \leq 0, \quad |x| \leq B;
\]

(iii) for each $M \geq B$ there are $k=k(M) > 0$ and $H=H(M) \leq 0$ such that

\[
[t \leq 0, \quad B \leq |x| \leq M, \quad P(t, x) \leq H] \implies K(t, x) \leq k;
\]

(iv) for each $M > B$ there exists an $L(M) > 0$ such that

\[
[t \leq 0, \quad B \leq |x| \leq M, \quad H(M) \leq P(t, x) \leq 2H(M)] \implies \dot{P}_{(2.1)}(t, x) \leq L(M);
\]

(v) for each $M > B$ there is a $T(M) > 0$ such that for any solution $x(t)$ of (2.1) with $B \leq |x(t)| \leq M$ and $P(t, x(t)) \leq 2H(M)$ for \( t_0 \leq t \leq t_0 + T(M) \) there exists \( \delta \in [t_0, t_0 + T(M)] \) with $|x(\delta)| < D$.

Then the solutions of (2.1) are U.B. and U.U.B.

Proof. For any $\alpha > 0$, define $\beta(\alpha) = W_1^{-1}(W_2(\max \{B, \alpha\}))$. It is easy to prove that \( [t_0 \leq 0, \quad x_0 \leq \alpha] \implies |x(t; t_0, x_0)| \leq \beta(\alpha) \) for \( t \geq t_0 \). Therefore, the solutions of (2.1) are U.B. Throughout the remainder of this proof we use the notations $x(t) = x(t; t_0, x_0)$, $V(t) = V(t, x(t))$ and $\dot{V}(t) = \dot{V}_{(2.1)}(t, x(t))$.

To prove the uniform ultimate boundedness, we consider the following two cases:

(a) there exists a $t_9 \geq t_0$ with $|x(t_9)| \leq B$;

(b) $|x(t)| \equiv B$ for all $t \geq t_0$.

In case (a) $|x(t)| \leq \beta(t)$ for $t \geq t_9$.

In case (b) we have \( \dot{V}(t) \equiv -K(t, x(t)) \) for all $t \geq t_0$. By (iii) there exist $k=k(\beta(\alpha)) > 0$ and $H=H(\beta(\alpha)) > 0$ such that $P(t, x(t)) \leq H$ implies $K(t, x(t)) \equiv k$.

Let $t \equiv t_9$ be fixed, and choose a constant $S=S(\alpha) > W_2(\beta(\alpha))/k$. Then by (3.1) the nonnegativeness of $V$ implies the existence of a $t_9 \in \mathbb{I} \cup \{S(\alpha)\}$ such that $P(t_9, x(t_9)) \leq H$. By (v), there exists $T=T(\beta(\alpha)) > 0$ such that if $P(t, x(t)) \leq 2H$ for $t \in [t_9, t_9 + T]$, then there is an $s \in [t_9, t_9 + T]$ with $|x(s)| < D$, which implies $|x(t)| < \beta(D)$ for $t \equiv t_9 + T$, especially, for $t \equiv t_9 + S + T$.

Therefore, only two cases may occur:

\[ \text{(b)}_1 \quad P(t, x(t)) \leq 2H \quad \text{for all } t \in [t_9, t_9 + T]. \]

In this case, $|x(t)| \equiv \beta(D)$ for $t \equiv t_9 + T + S$.

\[ \text{(b)}_2 \quad \text{there exists } t_4 \in [t_9, t_9 + T] \text{ with } P(t_4, x(t_4)) \equiv 2H. \]

In this case, there are $t_6, t_6$ such that $t_6 \leq t \leq t_6 \equiv t_6$, $P(t_6, x(t_6)) = H$, $P(t_6, x(t_6)) =$
\[ = 2H \quad \text{and} \quad H \leq P(t, x(t)) \leq 2H \quad \text{for} \quad t_{0} \leq t \leq t_{0}. \] 
By (iv), we get 
\[ t_{0} - t_{0} \leq H/L(\beta(\alpha)). \]
On the other hand, by \[ V(t) \leq -K(t, x(t)) \leq -k \quad \text{for} \quad t \in [t_{0}, t_{0}] \]
we obtain

\[ (3.2) \quad V(t_{0}) \leq V(t_{0}) - kH/L(\beta(\alpha)). \]

Since in case (b) \[ V(t) \leq -K(t, x(t)) \leq 0 \quad \text{for all} \quad t \geq t_{0}, \]
we get \[ V(t + S + T) \leq V(t) - kH/L(\beta(\alpha)). \]
Let \( t = t_{0} + m(S + T) \), where \( m \) is a nonnegative integer. 
Then from the argument above we get either

\[ (c_{m}) \quad |x(t)| \leq \max \{ \beta(B), \beta(D) \} \quad \text{for} \quad t \geq t_{0} + (m + 1)(S + T), \]
or

\[ (d_{m}) \quad V(t_{0} + m(S + T)) \leq V(t_{0} + m(S + T)) - kH/L(\beta(\alpha)). \]

Choose a positive integer \( N = N(\alpha) \) such that

\[ (3.3) \quad N(\alpha)kH/L(\beta(\alpha)) > W_{0}(\beta(\alpha)). \]

Then by the nonnegativeness of \( V \), \( (d_{m}) \) holds for at most \( m = 0, 1, \ldots, N - 1 \), and thus \( |x(t)| \leq \max \{ \beta(B), \beta(D) \} \) \quad \text{for} \quad t \geq t_{0} + N(S + T). \] 
This completes the proof.

Remark 3.1. Using the same argument as one above, the comparison method and Lemma 2.1, we can prove the following assertion:

If conditions (i), (iii)—(v) of Theorem 3.1 are satisfied and if for each \( M > B \) there exists a weakly integrally positive function \( \lambda_{M} : \mathbb{R}^{+} \to \mathbb{R}^{+} \) such that

\[ \mathcal{V}(x, t, x) \equiv -\lambda(t)K(t, x, t) = F(t, V(t, x)) \quad \text{for} \quad t \geq 0 \]
and \( B \leq |x| \leq M \), where \( F : \mathbb{R}^{+} \times \mathbb{R}^{+} \to \mathbb{R}^{+} \) is continuous, the solutions of \( \dot{z} = F(t, z) \)
are uniformly bounded, and \( \int_{0}^{\infty} \sup_{0 \leq z \leq M} F(t, z) dt < \infty \) for \( r \equiv 0 \), then the solutions of (2.1) are U.B. and E.U.B. If, in addition, \( \lambda_{M} \) is integrally positive, then the solutions (2.1) are U.B. and U.U.B.

Remark 3.2. If conditions (i), (iii) and (v) of Theorem 3.1 are satisfied and if

(a) \( \mathcal{V}(x, t, x) \equiv -\lambda(t)K(t, x, t) + F(t, V(t, x)) \) \quad \text{for} \quad t \geq 0 \) and \( |x| \equiv B \), where \( \lambda : \mathbb{R}^{+} \to \mathbb{R}^{+} \) is measurable and satisfies condition (2.3), and \( F \) is of the same kind as in Remark 3.1;

(b) for any \( M > 0 \) there exists a \( \mu = \mu(M) > 0 \) such that \( B \leq |x| \leq M \), \( H(M) \equiv \leq P(t, x) \equiv 2H[M] \) imply

\[ \mathcal{V}(x, t, x) \equiv -\mu^{\hat{\lambda}}(t, x) + F(t, V(t, x)), \]

then the solutions of (2.1) are U.B. and U.U.B.
To prove this remark it is sufficient to replace (3.2) and (3.3) in the proof of Theorem 3.1 by

$$V(t_0) \leq V(t_0) - \mu(\beta(\alpha))H(\beta(\alpha)) + \int_{t_0}^{t_i} \max \left\{ F(t, z) : 0 \leq z \leq \omega(z, \beta(\alpha)) \right\} dt$$

and

$$N\mu(\beta(\alpha))H(\beta(\alpha)) \geq \omega(z, \beta(\alpha)) + \int_{0}^{\infty} \max \left\{ F(t, z) : 0 \leq z \leq \omega(z, \beta(\alpha)) \right\} dt,$$

respectively.

**Remark 3.3.** Condition (iv) in Theorem 3.1 can be weakened as follows: for any \( M > B \) there exists a continuous function \( L_M : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \int_{0}^{t} L_M \) is uniformly continuous on \([0, \infty)\) and either

\[
[\hat{P}(t, x)]_+ \leq L_M(t) \quad \text{for} \quad t \geq 0, \quad B \leq |x| < M \quad \text{and} \quad H(M) \leq P(t, x) \leq 2H(M),
\]

or

\[
[\hat{P}(t, x)]_- \leq L_M(t) \quad \text{for} \quad t \geq 0, \quad B \leq |x| < M \quad \text{and} \quad H(M) \leq P(t, x) \leq 2H(M).
\]

**Remark 3.4.** Condition (i) in Theorem 3.1 can be replaced by \( 0 \leq V(t, x) \leq \omega_{\omega}(\|x\|) \) if the solutions of (2.1) are U.B.

**Example 3.1.** Consider a Liénard equation with forcing term

$$\ddot{x} + f(x) \dot{x} + g(t, x) = e(t),$$

where \( f(x), g(t, x), \partial g(t, x)/\partial t \) and \( e(t) \) are continuous for \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\) and \( \int_{0}^{\infty} |e(t)| dt < \infty \). Besides, we assume that there exist unbounded pseudo wedges \( W_1, W_2 \), a continuous \( W_3 : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( W_3(r) > 0 \) for \( r > 0 \) and an integrally positive function \( \lambda : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

$$W_1(|x|) \equiv \int_{0}^{x} g(t, x) dx \leq W_2(|x|),$$

$$g(t, x)F(x) - \int_{0}^{x} (\partial g(t, r)/\partial t) dr \equiv \lambda(t)W_3(|x|),$$

where \( F(x) = \int_{0}^{x} f(s) ds \). Obviously, (3.4) is equivalent to

$$\dot{x} = y - F(x), \quad \dot{y} = -g(t, x) + e(t).$$
Let $V(t, x, y) = \left[ y^2 + \int_0^t g(t, r) \, dr \right]^{1/2} + \int_t^\infty |e(s)| \, ds$, then

$$[y^2 + 2W_1(|x|)]^{1/2} \equiv V(t, x, y) \equiv [y^2 + 2W_5(|x|)]^{1/2} + \int_0^\infty |e(s)| \, ds$$

$$\dot{V}(t, x, y) \equiv -\lambda(t) W_5(|x|) [y^2 + 2W_6(|x|)]^{-1/2},$$

Let $K(t, x, y) = W_3(|x|) [y^2 + 2W_5(|x|)]^{-1/2}, \quad P(t, x, y) = |x|, \quad B = 1$ and $H = 1$. Then for each $M > 1$ and for $t \geq 0, \quad 1 \leq |x| + |y| \leq M$ and $|x| \geq 1$, we have $K(t, x, y) \geq \min \{ W_3(r) : 1 \leq r \leq M \} (M^2 + 2W_6(M))^{-1/2}$. Therefore, conditions (i)—(iv) of Theorem 3.1 hold (see also Remark 3.1). Now we check condition (v).

Let $E = \max \{ |F(x)| + 1 : |x| \leq 2 \}$. Define $T(M) = 2M + 1$. Suppose that $(x(t), y(t))$ is a solution of (3.5) with $1 \leq |x(t)| + |y(t)| \leq M$ and $|x(t)| \leq 2$ for $t \in [t_0, t_0 + T(M)]$. If $|x(t)| + |y(t)| \geq E + 2$ for all $t \in [t_0, t_0 + T(M)]$, then $|y(t)| \equiv E$, e.g. $y(t) \equiv 0$, and consequently $y(t) = y(t) - F(x(t)) \equiv E - \max_{|x| \leq 2} F(x) \leq 1$. Hence we obtain the inequality $2M \geq |x(t_0 + T(M)) - x(t_0)| \geq T(M) = 2M + 1$, which is a contradiction. Therefore, there is a $\varepsilon \in [t_0, t_0 + T(M)]$ with $|x(\varepsilon)| + |y(\varepsilon)| = D = E + 2$, i.e. condition (v) in Theorem 3.1 holds.

Consequently, under our conditions the solutions of (3.5) are U.B. and U.U.B.

Notice that if $P(t, x) = |x|$, then condition (iv) in Theorem 3.1 can be dropped. (Indeed, if condition (i)—(iii), (v) are satisfied for $P(t, x) = |x|$, then all the conditions of the theorem are satisfied for the new auxiliary function $\bar{P}(t, x) = V(t, x)$. If, in addition, $H$ in (iii) is constant, then (v) obviously holds. This special case initiates the following generalization of T. YOSHIZAWA’s theorem (3), Theorem 10.4):

**Theorem 3.2.** Suppose that there exist a constant $B \geq 0$, a locally Lipschitz function $V(t, x)$ and a continuous function $K(t, x)$ defined for $t \geq 0$ and $|x| \leq B$ satisfying the following conditions:

(i) $W_1(|x|) \equiv V(t, x) \equiv W_5(|x|)$, where $W_1$ and $W_5$ are unbounded pseudo wedges;

(ii) $\dot{V}(t, x) \equiv -\lambda(t) K(t, x)$ for $t \geq 0$ and $|x| \leq B$, where $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is measurable with $\lim_{t \to \infty} \int_{t_0}^t \lambda(s) \, ds = \infty$ for any $t_0 \geq 0$;

(iii) for each $M > B$ there exists $k(M) > 0$ such that $B \leq |x| \leq M$ implies $K(t, x) \equiv k(M)$.

Then the solutions of (2.1) are U.B. and E.U.B. If, in addition, $\lambda$ satisfies condition (2.3), then the solutions of (2.1) are U.B. and U.U.B.

**Proof.** For any $\alpha > 0$, define $\beta(\alpha) = W_1^{-1}(W_5(\max \{ B, \alpha \})$. Let $x(t; t_0, x_0)$ be a solution of (2.1) with $|x_0| < \alpha$. Then $|x(t; t_0, x_0)| < \beta(\alpha)$ for all $t \geq t_0$, i.e. the solutions are U.B.
For a given \( t_0 \geq 0 \) choose \( T(t_0, \alpha) > 0 \) such that
\[
\int_{t_0}^{t_0 + T(t_0, \alpha)} \lambda(s) \, ds > W_2(\beta(\alpha))/k(\beta(\alpha)).
\]
It is easy to prove that \( |x(t; t_0, x_0)| \leq \beta(B) \) for all \( t \geq t_0 + T(t_0, \alpha) \).
The second conclusion can be proved similarly.

The following theorem is a generalization of V. M. Matrosov's stability theorem [5] to the boundedness of solutions.

**Theorem 3.3.** Suppose that there exist a constant \( B \geq 0 \) and nonnegative locally Lipschitz functions \( V(t, x), W(t, x), P(t, x) \), a continuous function \( F(t, u) \) defined for \( t \geq 0, \, |x| \leq B, \, u \geq 0 \) and such that
(i) \( W_1(|x|) \leq V(t, x) \leq W_2(|x|) \), where \( W_1 \) and \( W_2 \) are unbounded pseudo wedges;
(ii) for every \( M > B \) there is a measurable function \( \lambda_M : \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[
\dot{V}_{(a,1)}(t, x) \leq -\lambda_M(t) P(t, x) + F(t, V(t, x)) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad B \leq |x| \leq M,
\]
where
(a) \( \lambda_M \) is weakly integrally positive;
(b) the solutions of the equation \( \dot{z} = F(t, z) \) are U.B., and \( \int_0^\infty \left[ \sup_{0 \leq z \leq r} F(t, z) \right] dt < \infty \)
for every \( r > 0 \);
(iii) for every \( M > B \) there exists a continuous function \( L_M : \mathbb{R}^+ \to \mathbb{R}^+ \)
such that \( \int_0^r L_M(t) \) is uniformly continuous on \( \mathbb{R}^+ \) and either \( [\dot{P}_{(a, 1)}(t, x)]_{+} \leq L_M(t) \)
or \( [\dot{P}_{(a, 1)}(t, x)]_{-} \leq L_M(t) \) for \( t \geq 0, \, B \leq |x| \leq M \);
(iv) for every \( M > B \) there exists a constant \( A(M) > 0 \) such that \( |W(t, x)| \leq A(M) \) for \( t \geq 0 \) and \( B \leq |x| \leq M \);
(v) there exists a constant \( D \geq B \) and for any \( M > B \) there exists a continuous function \( W_3 : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( W_3(r) \geq 0 \) for \( r \geq D \) such that
\[
\max \{ P(t, x), |\dot{W}_{(a, 1)}(t, x)| \} \leq W_3(|x|) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad D \leq |x| \leq M.
\]

Then the solutions of (2.1) are U.B. and E.U.B. If, in addition, \( \lambda_M(t) \) is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

**Proof.** First we show that under the assumptions of the theorem condition (v) in Theorem 3.1 is satisfied.

For any \( M > D \), choose \( H(M) > 0 \) such that \( 2H < \alpha(M) = \min_{D \leq r \leq M} W_3(r) \) and define \( T(M) = [2A(M) + 1]/\alpha \). Let \( x(t) \) be a solution of (2.1), with \( B \leq |x(t)| \leq M \) and \( P(t, x(t)) \leq 2H(M) \) for \( t \in [t_0, t_0 + T(M)] \). If \( |x(t)| \leq D \) for all \( t \in [t_0, t_0 + T(M)] \) then according to condition (v) we get \( |\dot{W}_{(a, 1)}(t, x(t))| \leq \alpha \), hence, \( 2A(M) \geq 

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\[ W(t_0 + T(M), x(t_0 + T(M))) - W(t_0, x(t_0)) \geq \alpha T(M) = 2A(M) + 1, \text{ which is a contradiction. Therefore, condition (v) of Theorem 3.1 holds.} \]

An application of Theorem 3.1, Remark 3.1 and Remark 3.3 completes the proof.

Remark 3.5. Condition (v) of Theorem 3.3 can be weakened by asking there is a constant \( D \geq B \) such that for every \( M > D \), there are \( B_\alpha(M) > 0 \) and a continuous function \( \mu_M : \mathbb{R}^+ \to \mathbb{R}^+ \) with property (2.3) and such that \( \{ t \geq 0, \quad D \geq |x| \leq M, \quad P(t, x) \leq B_\alpha \} \) imply \( |W_{(2,1)}(t, x)| \leq \mu_M(t) \).

An application of this theorem to a holonomic scleronomic mechanical system will be given in Section 4.

As we have seen so far, the key step in the application of Theorem 3.1 is to check condition (v). Now we establish a sufficient condition for this property by Lyapunov's direct method.

Lemma 3.1. Suppose that there exist \( H_\alpha > 0, \quad D > B \) and a locally Lipschitz function \( Q(t, x) \) defined on the set \( \{ (t, x) : \ t \geq 0, \quad |x| \leq D, \quad P(t, x) \leq 2H_\alpha \} \) such that

(i) for each \( M > D \) there are continuous functions \( \gamma, g : \mathbb{R}^+ \to \mathbb{R}^+ \) and a number \( H \in (0, H_\alpha) \) such that \( \gamma \) has property (2.3), the function \( \int_0^t [g(s)]_+ \, ds \) is bounded on \( \mathbb{R}^+ \), and \( \{ t \geq 0, \quad D \geq |x| \leq M, \quad P(t, x) \leq 2H \} \) imply \( \dot{Q}_{(2,1)}(t, x) \leq - \gamma(t) + g(t) \);

(ii) for each \( M > D \) there exists \( L(M) > 0 \) with \( |Q(t, x)| \leq L(M) \) for \( t \geq 0 \) and \( D \leq |x| \leq M \).

Then condition (v) of Theorem 3.1 holds with these numbers \( H \) and \( D \).

Proof. Let \( M > D \) be given and let a solution \( x(t) \) of (2.1) satisfy \( B \geq |x(t)| \leq M \) and \( P(t, x(t)) \leq 2H(M) \) for \( t \in [t_0, \ t_0 + T(M)] \), where \( T(M) > 0 \) is a constant such that

\[ \int_{t_0}^{t_0 + T(M)} \gamma(s) \, ds > 2L(M) + \int_0^\infty [g(s)]_+ \, ds \quad \text{for all } \ t_0 \geq 0. \]

If \( |x(t)| \geq D \) for \( t \in [t_0, \ t_0 + T(M)] \), then we get

\[-L(M) \leq Q(t_0 + T(M), x(t_0 + T(M))) \leq L(M) - \int_{t_0}^{t_0 + T(M)} \gamma(t) \, dt + \int_0^\infty [g(s)]_+ \, ds \]

which yields a contradiction to the choice of \( T(M) \). Consequently, there is \( s \in [t_0, \ t_0 + T(M)] \) with \( |x(s)| < D \), and the proof is complete.

Example 3.2. Consider the equation

\[ \ddot{x} + a(t) \dot{x} + f(x) = e(t) \]

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and suppose that the continuous functions \( a, e : \mathbb{R}^+ \to \mathbb{R}, \ f : \mathbb{R} \to \mathbb{R} \) satisfy the following conditions:

(i) \( a(t) \equiv 0 \) for \( t \in \mathbb{R}^+ \), \( a \) is weakly integrally positive, and there exist constant \( \varepsilon > 0, T > 0 \) such that \( [t_0 \equiv 0, t \equiv T] \) imply \((1/t) \int_{t_0}^{t} a(s) ds \equiv \varepsilon \).

(ii) \( e \in L^1[0, \infty) \);

(iii) there is an \( r_0 > 0 \) such that \( x f(x) > 0, \ |f(x)| > 0 \) provided \( |x| > r_0 \), and \( F(x) = \int_{0}^{x} f(s) ds \to \infty \), as \( |x| \to \infty \).

Then the solutions of equation (3.6) and their derivatives are U.B. and E.U.B. If, in addition, the function \( a(t) \) is integrally positive, then the solutions and their derivatives are U.B. and U.U.B.

Equation (3.6) is equivalent to the system

\[
\dot{x} = y, \quad \dot{y} = -f(x) - a(t)y + e(t).
\]

Define \( V(t, x, y) = [y^2 + 2F(x)]^{1/2} + \int_{t}^{\infty} |e(s)| ds \). Then

\[
\dot{V}_{(a, \eta)}(t, x, y) \equiv -a(t)y^2[y^2 + 2F(x)]^{-1/2}.
\]

Choose \( K(t, x, y) = y^2[y^2 + 2F(x)]^{-1/2}, \ P(t, x, y) = y^2 \). Then

\[
[\dot{P}_{(a, \eta)}(t, x, y)]_+ = [-f(x)y - a(t)y^2 + e(t)y]_+ \equiv |f(x)||y| + |e(t)||y|.
\]

Let \( B > 0 \) be fixed arbitrarily. For \( M > B \) let \( K_M = \max \{|f(x)|: 0 \leq |x| \leq M\} \) and suppose \( B \equiv |x| + |y| \equiv M \). Then \( \dot{P}_{(a, \eta)}(t, x, y)]_+ \equiv [K_M + |e(t)|]M \) and \( \int_{0}^{t} (K_M + |e(s)|) M ds \) is uniformly continuous in \( \mathbb{R}^+ \). Consequently, conditions (i)–(iv) of Theorem 3.1 (see also Remark 3.3) are met with arbitrary \( H > 0 \), and the solutions are U.B.

Now define \( D = r_0 + 1, \ H_0 = 1/2 \), and

\[
Q(t, x, y) = \begin{cases} y & \text{if } x \equiv r_0, \\ -y & \text{if } x \equiv -r_0, \end{cases}
\]

whose derivative is

\[
\dot{Q}_{(a, \eta)}(t, x, y) = \begin{cases} -f(x) - a(t)y + e(t) & \text{if } x \equiv r_0, \\ f(x) + a(t)y - e(t) & \text{if } x \equiv -r_0. \end{cases}
\]

For a given \( M > D \) introduce the notation \( m(M) = \min \{|f(x)|: r_0 \leq |x| \leq M\} \). By the conditions, \( m(M) > 0 \), and \( [t \equiv 0, D \equiv |x| + |y| \equiv M, y^2 \equiv 2H] \) imply the inequality

\[
\dot{Q}_{(a, \eta)}(t, x, y) \equiv -m(M) + a(t)[2H]^{1/2} + e(t).
\]
Let \( H = \min \left\{ \frac{1}{2} [m(M)(d+1)]^2, \frac{1}{2} \right\} \), \( \gamma(t) = m(M) - (2H)^{1/2}a(t) \) and \( g(t) = |e(t)| \).

Then \( \hat{Q}_{(a, \eta)}(t, x, y) \leq -\gamma(t) + g(t) \) and for sufficiently large \( T > 0 \),

\[
\int_{t_0}^{t_0+T} \gamma(t) \, dt = m(M)T - (2H)^{1/2} \int_{t_0}^{t_0+T} a(t) \, dt \geq m(M)d/(d+1)T \rightarrow \infty
\]

as \( T \rightarrow \infty \) uniformly with respect to \( t_0 \geq 0 \), and so all the conditions of Lemma 3.1 are satisfied.

This completes the proof.

Consider now the system

\[
(3.8) \quad \dot{x} = X(t, x, y), \quad \dot{y} = Y(t, x, y)
\]

where \( x \in \mathbb{R}^n, y \in \mathbb{R}^k \); \( X: \mathbb{R}^+ \times \mathbb{R}^{m+k} \rightarrow \mathbb{R}^m \) and \( Y: \mathbb{R}^+ \times \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k \) are continuous. The following theorem shows that the function \( Q \) in Lemma 3.1 can be constructed from the reduced subsystem

\[
(3.9) \quad \dot{y} = Y(t, 0, y).
\]

**Theorem 3.4.** Suppose that

(i) There exist constants \( B, H \geq 0 \) and a locally Lipschitz function \( V(t, x, y) \) defined for \( t \geq 0 \) and \(|x| + |y| \leq B \) such that

(a) \( \dot{W}_4(|x| + |y|) \equiv V(t, x, y) \equiv W_5(|x| + |y|) \), where \( W_4 \) and \( W_5 \) are unbounded pseudo wedges;

(b) \( \dot{V}_{(3.8)}(t, x, y) \equiv -\lambda(t)K(x, y) \) for \( t \geq 0 \) and \(|x| + |y| \leq B \), where \( \lambda(t) \) is weakly integrally positive, \( K(x, y) \equiv 0 \) for \(|x| + |y| \equiv B \), and for any \( M > B \) there exists \( k(M) > 0 \) such that \( K(x, y) \equiv k(M) \) for \( H \equiv |x|, B \equiv |x| + |y| \equiv M \);

(ii) there exist a constant \( B_1 > 0 \), a continuous \( N: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( N(s) > 0 \) for \( s \equiv B_1 \) and a locally Lipschitz function \( Q(t, y) \) defined for \( t \geq 0 \) and \(|y| \equiv B \) such that

(c) \( 0 \equiv Q(t, y) \equiv W_5(|y|) \), \( W_5 \) is a pseudo wedge;

(d) \( \dot{Q}_{(3.9)}(t, y) \equiv -W_4(|y|) \) for \(|y| \equiv B_1 \), where \( W_4 \) is a pseudo wedge;

(e) \( |Q(t, y) - Q(t, \tilde{y})| \equiv N(\max\{|y|, |\tilde{y}|\})|y - \tilde{y}| \);

(iii) for any \( M \geq 0 \) there exists \( L(M) = 0 \) such that \(|X(t, x, y)| \equiv L(M) \) if \(|x| + |y| \equiv M \);

(iv) there exist continuous \( P_1, P_2: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( P_1(s) > 0 \) for \( s \equiv B_1 \) such that \(|Y(t, x, y) - Y(t, 0, y)| \equiv P_1(|y|)P_2(|x|)\);  

(v) \( \lim_{r \rightarrow \infty} W_4(r)/P_1(r)N(r) = \infty \).

Then the solutions of (3.8) are U.B. and E.U.B. If, in addition, \( \lambda \) is integrally positive, then the solutions of (3.8) are U.B. and U.U.B.

**Proof.** Obviously, (i)–(iv) of Theorem 3.1 hold with \( P(t, x, y) = |x| \).

Choose \( D > 0 \) such that \( D - 2H \equiv B_1 \), \( W_4(r)/N(r)P_1(r) \equiv \max\{P_1(s): |s| \equiv 2H\} + 1 \)
for \( s \equiv D - 2H \). Then if \( D \equiv |x| + |y| \equiv M \), \( |x| \equiv 2H \), then \( |y| \equiv D - 2H \equiv B_1 \), and thus

\[
\mathcal{Q}_{(a, b)}(t, y) \equiv \mathcal{Q}_{(a, 0)}(t, y) + N(|y|)|Y(t, x, y) - Y(t, 0, y)| \equiv -W_4(|y|) + \\
+ N(|y|)P_1(|y|)P_2(|x|) \equiv -N(|y|)P_1(|y|) \left[ \frac{W_4(|y|)}{N(|y|)P_1(|y|)} - P_3(|x|) \right] - \\
- N(|y|)P_1(|y|) \equiv - \inf \{ N(r)P_1(r): B_1 \equiv r \equiv M \}.
\]

Therefore, condition (v) of Theorem 3.1 holds by Lemma 3.1, and so the proof is complete.

**Example 3.3.** Consider now the system

\[
(3.10) \quad \dot{x} = f_1(t, x) + by, \quad \dot{y} = f_2(t, x) + dy + e(t),
\]

where \( f_1, f_2 \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}) \) with \( f_1(t, 0) = 0 \), \( f_2(t, 0) = 0 \), \( e(t) \) is a bounded continuous function on \( \mathbb{R}^+ \) with \( e \in L^1[0, \infty) \), \( b, d \) are constants with \( db \neq 0 \). Besides, we assume

(i) \( \sup \{|f_1(t, x)| + |f_2(t, x)|: t \equiv 0, |x| \equiv M| < \infty \) for any \( M > 0 \);

(ii) \( [df_1(t, x) - bf_2'(t, x)]/x \equiv \alpha(x) > 0 \) for \( t \equiv 0 \) and \( x \equiv 0 \), where \( \alpha \) is continuous and \( \lim_{|x| \rightarrow \infty} \int_0^x \alpha(r) r \, dr = \infty \);

(iii) \( \int f_1(t, x) + dx)[bf_2(t, x) - df_1(t, x)] - \int_0^x \left[ (dbf_1(t, r)/\partial t) - (dbf_2(t, r)/\partial t) \right] \, dr \equiv \lambda(t)\beta(x) \), where \( \lambda(t) \) is integrally positive, \( \beta \) is continuous with \( \beta(x) > 0 \) if \( x \neq 0 \).

Under these conditions the solutions of (3.10) are U.B. and U.U.B.

Indeed, let

\[
V(t, x, y) = \left[ (dx - by)^2 + 2 \int_0^t [df_1(t, r) - bf_2(t, r)] \, dr \right]^{1/2} + b \int_t^\infty |e(s)| \, ds.
\]

Then

\[
\mathcal{V}_{(a, 10)}(t, x, y) \equiv \\
[bf_2(t, x) - df_1(t, x)][f_1(t, x) + dx] + \int_0^x \left[ \frac{d}{\partial t} f_1(t, r) - b \frac{\partial}{\partial t} f_2(t, r) \right] \, dr \\
\equiv \lambda(t)K(x, y),
\]

where

\[
K(x, y) = \beta(x) \left[ (dx - by)^2 + 2 \sup_{t \equiv 0} \int_0^t [df_1(t, r) - bf_2(t, r)] \, dr \right]^{-1/2}.
\]
It is easy to prove that for any $M>0$ there exists $k=k(M)>0$ such that $|x|+|y|\leq M$, $|x|\leq H$ imply $K(x,y)>k(M)$. Therefore, (i) of Theorem 3.4 holds.

On the other hand, for the subsystem

$$
(3.11)
\dot{y} = dy + e(t)
$$

and for $Q(t,y)=y^2/2$, $N(r)=r$, we have

$$
\hat{Q}_{(3.11)}(t,y) \leq d|y|\left[|y|+(1/d)\sup_{t\leq 0}|e(t)|\right] \leq (1/2) dy^2 \quad \text{for} \quad |y| \leq -(2/d)\sup_{t\geq 0}|e(t)|.
$$

Therefore, after making the choice $P_1(r)=1$, $P_3(r)=\sup\{|f_3(t,x)|: t\geq 0, |x|\leq r\}$ all the conditions of Theorem 3.4 are met, and our assertion is true.

**Theorem 3.5.** For system (3.8), suppose that

(i) there exist continuous functions $P_1, P_2: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $P_1(s)>0$ for $s>0$ such that $|Y(t,x,y)-Y(t,0,y)| \leq P_1(|y|) P_2(|x|)$;

(ii) there exist a constant $B_2>0$ and a locally Lipschitz function $V_1(t,x,y)$ defined for $t\geq 0$, $|x| \leq B_1$ and $y \in \mathbb{R}^k$ such that

$$
W_1(|x|) \leq V_1(t,x,y) \leq W_2(|x|),
$$

where $W_1$ and $W_2$ are unbounded pseudo wedges and $W_2: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous with $W_2(r)>0$ for $r \geq B_1$;

(iii) there exist a constant $B_2>0$, a locally Lipschitz function $V_2(t,y)$ defined for $t \geq 0$ and $|y| \leq B_2$, and a positive continuous function $N: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $N(r)>0$ for $r \geq B_2$ and such that

$$
W_3(|y|) \leq V_2(t,y) \leq W_6(|y|),
$$

where $W_3, W_6$ are unbounded pseudo wedges, $W_6$ is nonnegative and continuous with $\lim_{r \rightarrow \infty} W_6(r)/(N(r) P_1(r)) = \infty$.

Then the solutions of (3.8) are U.B. and U.U.B.

**Proof.** First, we shall prove the uniform boundedness. For any $\alpha=\max\{B_2, B_3\}$, there exist $\beta_1(\alpha), \beta_2(\alpha)$ and $\beta_3(\alpha)>0$ such that $W_1(\beta(\alpha))>W_5(\alpha)$, $\beta_2(\alpha)\geq \beta_1(\alpha) \geq \alpha$, $W_6(s)/N(s) P_1(s) - \max_{\beta(\alpha)} P_2(r) \equiv 1$ for $s \geq \beta_1(\alpha)$, and $W_3(\beta(\alpha)) \geq W_6(\beta(\alpha))$. Then for any solution $(x(t), y(t))$ with $|x(t_0)|<\alpha$, and $|y(t_0)|<\alpha$, we have $x(t)<\beta(\alpha)$ and $|y(t)|<\beta_2(\alpha)$ for $t \geq t_0$.

If this is not true, then only two cases may occur:
Case 1. There exist \( t_2 \geq t_1 \geq t_0 \) with \( |y(t_2)| = \beta_4(\alpha) \), \( |y(t_0)| = \beta_2(\alpha) \), \( \beta_4(\alpha) < \beta_3(\alpha) \), \( \beta_2(\alpha) < \beta_3(\alpha) \) for \( t \in (t_1, t_2) \) and \( |x(t)| < \beta(\alpha) \) for \( t \in [t_0, t_2] \).

Case 2. There exist \( t_3 \geq t_2 \geq t_0 \) such that \( |x(t_3)| = \alpha \), \( |x(t_2)| = \beta(\alpha) \), \( \alpha < |x(t)| < \beta(\alpha) \) for \( t \in (t_2, t_3) \) and \( |y(t)| \leq \beta_2(\alpha) \) for \( t \in [t_2, t_3] \).

In Case 1, for \( t \in [t_1, t_2] \), we have

\[
\dot{V}_{2(a, a)}(t, y(t)) \leq -W_6(|y(t)|) + N(|y(t)|)P_1(|y(t)|)P_3(|x(t)|) \leq -N(|y(t)|)P_1(|y(t)|)[W_6(|y(t)|)/N(|y(t)|)]P_1(|y(t)|)P_3(|x(t)|) \leq -N(|y(t)|)P_1(|y(t)|) \leq 0.
\]

Therefore, \( W_4(\beta_2(\alpha)) \leq V_2(t_2, y(t_2)) \leq V_2(t_1, y(t_1)) = W_4(\beta_1(\alpha)) \). This contradicts \( W_4(\beta_2(\alpha)) > W_4(\beta_1(\alpha)) \).

In Case 2, for \( t \in [t_2, t_3] \), we have \( \dot{V}_1(t, x(t), y(t)) \leq 0 \), thus

\[
W_1(\beta(\alpha)) \leq V_1(t_2, x(t_2), y(t_2)) \leq V_1(t_3, x(t_3), y(t_3)) = W_6(\alpha),
\]

which contradicts \( W_7(\beta(\alpha)) > W_6(\alpha) \).

Therefore, \( |x(t; t_0, x_0, y_0)| < \beta(\alpha) \) and \( |y(t; t_0, x_0, y_0)| < \beta_2(\alpha) \) for \( t \geq t_0 \) if \( |x_0| < \alpha \) and \( |y_0| < \alpha \). This completes the proof of uniform boundedness.

Let \( v_1(\alpha) = \min \{ W_6(r); B_1 + 1 \geq r \geq \beta(\alpha) \} \) and \( T_1(\alpha) = W_6(\alpha)/v_1(\alpha) \). If \( |x(t)| \geq \alpha \) for \( t \in [t_0, t] \) holds for \( t \geq t_0 + T_1(\alpha) \) then

\[
W_4(B_2 + 1) \leq V_1(t, x(t), y(t)) \leq V_4(t_0, x(t_0), y(t_0)) = v_1(\alpha)(t - t_0) < W_4(\alpha) - v_1(\alpha)W_6(\alpha)/v_1(\alpha) = 0,
\]

which yields a contradiction. Therefore, there exists \( t_5 \in [t_0, t_0 + T_1(\alpha)] \) with \( |x(t_5)| \geq \alpha \). Following the same argument as in the proof of uniform boundedness, we get \( |x(t)| < \beta(B_1 + 1) \) for \( t \geq t_5 \), especially for \( t \geq t_0 + T_1(\alpha) \).

Choose \( B_3 = B_2 \) with \( W_6(s)/N(s)R_1(s) - \max \{ P_2(r); |r| < \beta(B_1 + 1) \} \geq 1 \) for \( s \geq B_1 \). If \( |y(t)| \leq B_3 \) for \( t \geq t_0 + T_1(\alpha) \), then there exists \( v_2(\alpha) > 0 \) such that

\[
R_1(|y(t)|)N(|y(t)|) \leq v_2(\alpha),
\]

and so

\[
\dot{V}_{2(a, a)}(t, y(t)) \leq -R_1(|y(t)|)N(|y(t)|)[W_6(|y(t)|)/N(|y(t)|)]R_1(|y(t)|)P_3(|x(t)|) \leq -N(|y(t)|)P_1(|y(t)|) \leq -v_2(\alpha).
\]

Therefore, if \( |y(t)| \geq B_3 \) for \( t \geq t_0 + T_1(\alpha) \), then

\[
V_2(t_0 + T_1(\alpha) + \tau, y(t_0 + T_1(\alpha) + \tau)) \leq V_2(t_0 + T_1(\alpha), y(t_0 + T_1(\alpha))) = W_6(\beta(\alpha) - v_2(\alpha)\tau).
\]

If \( \tau \geq T_2(\alpha) \), where \( T_2(\alpha) = (W_6(\beta(\alpha)) - W_6(B_2))/v_2(\alpha) \), then

\[
W_4(B_3) \leq V_2(t_0 + T_1(\alpha) + \tau, y(t_0 + T_1(\alpha) + \tau)) \leq W_6(\beta(\alpha) - v_2(\alpha)T_2(\alpha) = W_4(B_3),
\]

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which yields a contradiction. Therefore, there exists \( t_0 \in [t_0 + T_1(\alpha), t_0 + T_2(\alpha) + T_3(\alpha)] \) with \( |y(t_0)| < B_3 \), and thus \( |x(t_0)| < B_4 \) and \( |y(t_0)| < B_4 \), where \( B_4 = \max \{ B_3, \beta(B_4 + 1) \} \). This implies \( |x(t)| < \beta(B_4) \) and \( |y(t)| < \beta(B_4) \) for \( t \geq t_0 + T_1(\alpha) + T_2(\alpha) \). This completes the proof.

Sometimes in practice it is very difficult to find a Lyapunov function satisfying the condition \( V_1(t, x, y) = V_2(|x|) \) (see Example 3.4). Now we give a modification of Theorem 3.5 asking the much milder property \( V_1(t, x, y) = V_2(|x| + |y|) \).

**Theorem 3.6.** Suppose that

(i) conditions (i), (iii) of Theorem 3.5 hold;

(ii) there exist a constant \( B_1 > 0 \) and a continuous function \( V_1(t, x, y) \) defined for \( t \geq 0 \), \( (x, y) \in \mathbb{R}^{n+k} \) and such that

\[
V_1(|x|) \leq V_1(t, x, y) \leq W_2(|x| + |y|),
\]

\[
\dot{V}_1(t, x, y) \leq -W_2(x, y),
\]

where \( W_1 \) and \( W_2 \) are unbounded pseudo wedges, and \( W_2: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^+ \) is continuous and \( |x| \leq B_1 \) implies \( W_2(x, y) = 0 \);

(iii) for any \( M > 0 \) there exists \( L(M) = 0 \) such that \( [t \geq 0, |x| + |y| \leq M] \) imply \( |X(t, x, y)| \leq L(M) \);

Then the solutions of (3.8) are U.B. and U.U.B.

**Proof.** Obviously, by (ii) for any \( \alpha > 0 \), if \( |x_0| + |y_0| < \alpha \), then \( |x(t; t_0, x_0, y_0)| = W_1^{-1}(W_2(\alpha) = \beta(\alpha)) \) provided that \( (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0)) \) exists. Following the same argument as in the proof of Theorem 3.5, there exists \( \beta(\alpha) > 0 \) such that \( |y(t; t_0, x_0, y_0)| = \beta(\alpha) \) provided that \( |x_0| + |y_0| < \alpha \) and \( (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0)) \) exists. Then the solutions of (3.8) are U.B. Throughout the remainder of the proof denote \( x(t) = x(t; t_0, x_0, y_0), y(t) = y(t; t_0, x_0, y_0) \).

Let \( T_1(\alpha) = W_2(\beta(\alpha) + \beta(\alpha)) \), \( B_1 + 1 \leq |x| \leq \beta(\alpha), |y| \leq \beta(\alpha) \). Then by (ii), for any \( \tilde{t} \geq t_0 \) there is a \( \tilde{t} \in [\tilde{t}, \tilde{t} + T_1(\alpha)] \) with \( |x(\tilde{t})| = \beta(\alpha) \).

Suppose that for all \( t \in [t_1, \tilde{t} + T_1(\alpha) + \tilde{t}] \) we have \( |x(t)| < B_2 + 2 \) and \( |y(t)| < B_3 \), where \( B_3 = B_2 \) is a fixed constant such that

\[
W_2(r)/N(r)P_1(r) - \max \{ P_2(s); 0 \leq s \leq B_1 + 2 \} \geq 1 \quad \text{for} \quad r \geq B_3.
\]

Then from

\[
\dot{V}_2(x, y) \geq -N(|y(t)|)P_1(|y(t)|) \left[ \frac{W_2(|y(t)|)}{N(|y(t)|)P_1(|y(t)|)} - P_2(|x(t)|) \right] \geq -\min \{ N(r)P_1(r); B_3 \leq r \leq \beta(\alpha) \} = -m
\]

we get

\[
0 \geq V_2(t + T_1(\alpha) + \tilde{t}, y(t + T_1(\alpha) + \tilde{t})) \geq V_2(t_1, y(t_1)) - m[t^* + T_1(\alpha) + \tilde{t} - t_1] \leq W_2(\beta(\alpha)) - m[t^* + T_1(\alpha) + \tilde{t} - t_1].
\]
Therefore, \( t^* = T_3(\alpha) = \lfloor W_0(\beta_2(\alpha)) + 1 \rfloor / m \). This shows only two cases may occur:

Case 1. \( |x(t)| = B_1 + 2 \) for all \( t \in [t_4, t + T_1(\alpha) + T_2(\alpha)] \) and there exists \( t_6 \in [t_4, t + T_1(\alpha) + T_2(\alpha)] \) with \( |y(t_6)| = B_3 \). In this case, \( |x(t)| < \beta(B_1 + B_3 + 2) \) and \( |y(t)| < \beta_2(B_1 + B_3 + 2) \) for \( t \equiv t + T_1(\alpha) + T_2(\alpha) \).

Case 2. There exists \( t_6 \in [t_4, t + T_1(\alpha) + T_2(\alpha)] \) such that \( |x(t_6)| = B_1 + 2 \). In this case, there exist \( t_4, t_6 \in [t_4, t_6] \) with \( |x(t_4)| = B_1 + 1 \) and \( |x(t_6)| = B_1 + 2 \) and \( B_1 + 1 \leq |x(t)| \leq B_1 + 2 \) for \( t \in (t_4, t_6) \). By condition (iii) \( t_6 = t_4 \equiv 1/L(\beta(\alpha) + \beta_2(\alpha)) \), and (ii) implies \( V_1(t + T_1(\alpha) + T_2(\alpha)) \equiv V_1(t) \equiv V_1(t_4) - (t_6 - t_4) m(\alpha) \equiv V_1(t) - v(\alpha) \), where \( V_1(t) = V_1(t, x(t), y(t)) \), \( v(\alpha) = [L(\beta(\alpha) + \beta_2(\alpha))]^{-1} m(\alpha) \), and \( m(\alpha) = \min \{W_0(x, y) : B_1 + 1 \leq |x| \leq \beta(\alpha), |y| \leq \beta_2(\alpha) \} \). Making the choice \( t = t_6 = t_0 + m \left( T_1(\alpha) + T_2(\alpha) \right) \) \((m = 0, 1, 2, \ldots)\) we get that either \( |x(t)| < \beta(B_1 + B_3 + 2) \) and \( |y(t)| < \beta_2(B_1 + B_3 + 2) \) for \( t \equiv t_6 = t_0 + m \left( T_1(\alpha) + T_2(\alpha) \right) \), or

(Note 12)

On the other hand, \( 0 \equiv V_1(t) \equiv W_0(\beta(\alpha) + \beta_2(\alpha)) \) for \( t \equiv t_6 \), and so (Note 12) can not be true for \( m = 0, 1, \ldots, N \), where \( N = N(\alpha) \) is a positive integer such that \( N(\alpha) v(\alpha) \geq W_0(\beta(\alpha) + \beta_2(\alpha)) \). Therefore, \( |x(t)| < \beta(B_1 + B_3 + 2) \) and \( |y(t)| < \beta_2(B_1 + B_3 + 2) \) for \( t \equiv t_0 + N(\alpha) \left( T_1(\alpha) + T_2(\alpha) \right) \). This completes the proof.

Example 3.4. Consider the Liénard equation with forcing term

(3.13)

\[ x^2 + f(x) \dot{x} + g(x) = p(t), \]

where \( f(x) \) and \( g(x) \) are continuous for \( x \in \mathbb{R} \) and \( p(t) \) is continuous for \( t \equiv 0 \). Besides, we assume that

(i) \( f(\alpha) \geq 0 \);

(ii) \( x \{ g(x) - x[f(x) - 1] \} \geq 0 \);

(iii) \( \int_0^\infty |p(s)| \, ds < \infty \).

Then the solutions of (3.13) are U.B. and U.U.B.

Proof. System (3.13) is equivalent to

(3.14)

\[ \dot{x} = -x + y, \quad \dot{y} = -\{g(x) - x[f(x) - 1] - [f(x) - 1] y\} + p(t). \]

Let \( V(t, x, y) = \left[ y^2 + 2 \int_0^t \{ g(r) - r[f(r) - 1] \} \, dr \right]^{1/2} + \int_t^\infty |p(s)| \, ds \).

Then

\[ \dot{V}(3.14)(t, x, y) \leq \frac{\left[ y^2 + 2 \int_0^t \{ g(r) - r[f(r) - 1] \} \, dr \right]^{1/2}}{x} = -W(x, y). \]
Then $|y|>0$ implies $W(x, y)>0$. On the other hand, for the subsystem $\dot{x}=-x$ the auxiliary function $V_2(t, x)=x^2$, $N(r)=2r$ and $W_6(r)=2r^2$ satisfy condition (iii) of Theorem 3.5 and so the solutions of (3.13) are U.B. and U.U.B. by Theorem 3.6.

4. An application to a holonomic scleronomic mechanical system

Consider a holonomic scleronomic mechanical system of $n$ degrees of freedom being under the action of potential, dissipative and gyroscopic forces. The motions such a system can be described by the Lagrangian equation

$$
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial \pi}{\partial q} - B\dot{q} + G\dot{q},
$$

where $q, \dot{q} \in \mathbb{R}^n$ are the vectors of the generalized coordinates and velocities, respectively, $\pi=\pi(t, q)$ is the potential energy, $T=T(q, \dot{q})=(1/2)\dot{q}^TA(q)\dot{q}$ is the kinetic energy where $A(q)$ is a symmetric $n \times n$ matrix function ($v^T$ denotes the transposed of $v \in \mathbb{R}^n$); $B=B(t, q)$ is the symmetric positive semi-definite $n \times n$ matrix function of dissipation, and $G=G(t, \dot{q})$ is the antisymmetric $n \times n$ matrix of the gyroscopic coefficients.

By the Hamiltonian variables $q, p=A(q)\dot{q}$ system (4.1) can be rewritten into the form

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + (G-B) \frac{\partial H}{\partial p},
$$

where $H=H(t, p, q)$ is the total mechanical energy:

$$
H = H(t, q, p) = T+\pi = (1/2)p^TA^{-1}(q)p + \pi(t, q).
$$

Choose the auxiliary functions $V=H(t, p, q)$, $W=p^Tq$. Their derivatives with respect to (4.2) read as follows:

$$
\dot{H} = \left(\frac{\partial H}{\partial p}\right)^T (G-B) \frac{\partial H}{\partial p} + \frac{\partial \pi}{\partial t} = -p^TA^{-1}(q)B(t, q)A^{-1}(q)p + \frac{\partial \pi(t, q)}{\partial t} \equiv -\beta(t, q)A^{-1}(q)p^TA^{-1}(q)p + \left[\frac{\partial \pi(t, q)}{\partial t}\right],
$$

where $\beta(t, q)$ denotes the smallest eigenvalue of the matrix $B(t, q)$; $A(q)$ denotes the largest eigenvalue of $A(q)$. It is known from the mechanics that the kinetic energy is a positive definite quadratic form of the velocities, consequently $A(q)>0$ for all $q \in \mathbb{R}^n$. 
Let
\[ A^{-1}(q) = (a_{ij}^{-1}(q))_{n \times n}, \]
\[ d_{ij} = \left( \frac{\partial a_{ij}^{-1}(q)}{\partial q_1}, \ldots, \frac{\partial a_{ij}^{-1}(q)}{\partial q_n} \right) A^{-1}(q)p, \quad D = (d_{ij})_{n \times n}, \]
\[ e_k = \sum_{i,j=1}^{n} \frac{\partial a_{ij}^{-1}(q)}{\partial q_k} p_i p_j, \quad e = (e_1, \ldots, e_n)^T. \]

Then for \( P = p^T A^{-1}(q)p \), its derivative with respect to (4.2) is
\[
\dot{p} = \left[ -\frac{\partial \pi}{\partial q} - \frac{1}{2} \frac{\partial}{\partial q} p^T A^{-1}(q)p + (G - B) A^{-1}(q)p \right]^T A^{-1}(q)p + \\
+ p^T A^{-1}(q) \left[ -\frac{\partial \pi}{\partial q} - \frac{1}{2} \frac{\partial}{\partial q} p^T A^{-1}(q)p + (G - B) A^{-1}(q)p \right] p^T D p = \\
= -2 \left[ \frac{\partial \pi(t, q)}{\partial q} \right]^T A^{-1}(q)p + p^T A^{-1}(q)[(G - B)^T + (G - B)] A^{-1}(q)p - \\
- p^T A^{-1}(q) \frac{\partial}{\partial q} [p^T A^{-1}(q)p] + p^T D p = -2 \left[ \frac{\partial \pi(t, q)}{\partial q} \right]^T A^{-1}(q)p - \\
- 2p^T A^{-1}(q)BA^{-1}(q)p - p^T A^{-1}(q)e + p^T D p; \\
[\dot{p}]_+ \equiv \left| \frac{\partial}{\partial q} \pi(t, q) \right| F_5(q, p) + F_5(q, p),
\]
where
\[ F_5(q, p) = 2A^{-1}(q)p, \quad F_5(q, p) = |p||A^{-1}(q)||e| + |D|p^2. \]

Similarly,
\[ \dot{\theta} = \dot{p}^T q + p^T \dot{q} = -\left[ \frac{\partial \pi(t, q)}{\partial q} \right]^T q + \frac{1}{2} e^T q + p^T A^{-1}(q)(G - B)^T q + p^T A^{-1}(q)p, \]
\[ |\dot{\theta}| \equiv q^T \left| \frac{\partial \pi(t, q)}{\partial q} \right| - |G(t, q) - B(t, q)||F_6(q, p) - F_4(q, p)|,
\]
where
\[ F_4(q, p) = \frac{1}{2} |e||q| + |A^{-1}(q)||p^2, \quad F_6(q, p) = |A^{-1}(q)||q||p|. \]

It is easy to prove that \( F_i(q, p) \) are continuous for \( p, q \in \mathbb{R}^n \), and for every \( M > 0 \), \( \lim_{p \to 0} F_i(q, p) = 0 \) for \( i = 2, \ldots, 5 \). Therefore, from Theorem 3.3 and Remark 3.5, we get the following

**Corollary 4.1.** Suppose that there are \( B \geq 0 \) and unbounded pseudo wedges \( W_1, W_2 \) such that
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(i) \( W_{\ell}(\|q\|) \leq \pi(t, q) \leq W_{0}(\|q\|) \) for \( t > 0 \) and \( q \in \mathbb{R}^n \);
(ii) for every \( M > 0 \) the function \( \beta_M(t) = \min \{ \beta(t, q) : 0 \leq \|q\| \leq M \} \) is weakly integrally positive;
(iii) there is a continuous function \( r : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( r(t, u) \) is increasing with respect to \( u \) for every \( t \in \mathbb{R}^+ \) and \( [\partial \pi(t, q)/\partial t]+ \equiv r(t, \pi(t, q)) \) for \( t \in \mathbb{R}^+ \) and \( q \in \mathbb{R}^n \);
(iv) for every \( u_0 > 0 \) there is a \( u(t) > u_0 \) with \( \int_0^\infty r(s, u(t)) \, ds < u(t) - u_0 \);
(v) for every \( M > 0 \) the function \( |\partial \pi(t, q)/\partial q| \) is bounded for \( t \equiv 0 \) and \( |q| \leq M \);
(vi) for every \( M > 0 \) there are \( \mu_M > 0 \) and \( K_M > 0 \) such that \( |q^T \partial \pi(t, q)/\partial q| \equiv \mu_M \) \( G(t, q) - B(t, q) \equiv K_M \) for \( t \equiv 0 \) and \( B \equiv |q| \leq M \).

Then the motions are U.B. and E.U.B.

If, in addition, \( \beta_M(t) \) is integrally positive, then the motions are U.B. and U.U.B.

References