Stability of Neutral Functional Differential Equations with Infinite Delay

By

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§ 1. General theorems

Let $C$ be the space of bounded continuous functions mapping $(-\infty, 0]$ to $\mathbb{R}^n$, $C^1$ be a subspace of $C$, the elements of which have bounded continuous derivative. For $\varphi \in C$, we define the norm $\|\varphi\| = \sup_{s \in [0]} |\varphi(s)|$, where $\cdot$ is the Euclidean norm in $\mathbb{R}^n$. $C_H = \{\varphi \in C; \|\varphi\| < H\}$, $C_H^1 = \{\psi \in C_H \cap C^1; \varphi \in C_H\}$. For $A > 0$, $t_0 \in \mathbb{R}$, $x$: $(-\infty, t_0 + A) \rightarrow \mathbb{R}^n$, $t \in [t_0, t_0 + A]$ define $x_t: [-\infty, 0] \rightarrow \mathbb{R}^n$ as $x_t(\theta) = x(t + \theta)$ for $\theta \leq 0$.

Consider the neutral functional differential equation

$$\dot{x}(t) = f(t, x_t, \dot{x}_t) \quad t \geq t_0$$

where $f: \mathbb{R} \times C_H \times C_H \rightarrow \mathbb{R}^n$ is continuous, $f(t, 0, 0) \equiv 0$. Throughout this paper, we always suppose the solution $x(t; t_0, \varphi)$ of (1) through $(t_0, \varphi) \in \mathbb{R} \times C_H^1$ exists and is unique.

Definition 1. We say the zero solution of (1) is stable if for any $\varepsilon > 0$, $t_0 \in \mathbb{R}$, there exists a $\delta(\varepsilon, t_0) > 0$ such that $|x(t; t_0, \varphi)| < \varepsilon$ for any $\varphi \in C_1$ and $t \geq t_0$.

If $\delta$ is independent of $t_0$, then we say the zero solution of (1) is uniformly stable.

If the zero solution of (1) is stable and there is a $\delta(t_0) > 0$ such that for any $\varphi \in C_1^1(t_0)$, $\lim_{t \rightarrow \infty} x(t; t_0, \varphi) = 0$, then we say the zero solution of (1) is asymptotically stable.

Condition 1. There exist nonnegative continuous non-decreasing functions $k_1, k_2$ with $k_1(0) = k_2(0) = 0$ such that

$$|f(t, x_t, \dot{x}_t)| \leq k_1(\|x_t\|) + k_2(\|\dot{x}_t\|).$$

Condition 2. If $z \leq k_1(\sigma) + k_2(\tau)$, then $z \leq k(\sigma)$, where $k$ is a continuous, strictly increasing function with $k(s) > 0$ for $s > 0$.

Condition 3. There exists functions $p: \mathbb{R} \rightarrow \mathbb{R}$, $F: \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$p(t) \leq t, \quad p(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty$$

and for any $M > 0$, $\sup_{0 \leq x, y \leq M} F(t, x, y) \rightarrow 0$ as $t \rightarrow \infty$, 

\[\text{People's Republic of China, 1986} \]
\[ |f(t, x_t, \dot{x}_t)| \leq k_1( \sup_{p(t) \leq s \leq t} |x(s)|) + k_2( \sup_{p(t) \leq s \leq t} |\dot{x}(s)|) + F(t, \|x_t\|, \|\dot{x}_t\|) \]

where \( k_1, k_2 \) is defined as condition 1.

**Condition 4.**

i) There exists a continuous \( V: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^+ \) such that
\[ u(|x|) \leq V(t, x) \leq w(|x|) \]
where \( u, w \) are nonnegative and continuous, with \( u(0) = w(0) = 0 \), \( u \) is strictly increasing and \( w \) nondecreasing.

ii) When \( V(s, x(s)) \leq V(t, x(t)), |\dot{x}(s)| \leq k(u^{-1}(V(t, x(t)))) \) hold for \( s \leq t \),
\[ \dot{V}_{(1)}(t, x(t)) \leq W_1(t, V(t, x(t))) \]
where \( W_1: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is continuous and the zero solution of

\[ (2) \quad \dot{y}(t) = W_1(t, y(t)) \]

is uniformly stable.

**Condition 5.**

(i) The same as (i) of Condition 4.

(ii) The exists a function \( p: \mathbb{R}^+ \rightarrow \mathbb{R} \) such that \( p(t) \leq t \) and \( p(t) \rightarrow \infty \) as \( t \rightarrow \infty \) and when
\[ V(s, x(s)) \leq N, \quad |\dot{x}(s)| \leq k(u^{-1}(N)) \]
for an arbitrary \( N \) and all \( s \in [p(t), t] \),
\[ \dot{V}_{(1)}(t, x(t)) \leq F(t, V, N) \]
where
\[ F(t, V, V) \leq -W_2(V) + g_3(t)G(V) + g_4(t)Q(V) \]
\( W_2(r) \) is continuous and positive for \( r > 0; g_3(t) \geq 0, \int_0^{+\infty} g_3(s)ds < +\infty, G(V), Q(V) \geq 0 \) are continuous, \( g_4(t) \geq 0 \) and \( g_5(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

\[ |F(t, V, N_1) - F(t, V, N_2)| \leq [h_1(t) + h_2]|N_1 - N_2| \]
\( h_1(t) \geq 0, \int_0^{+\infty} h_1(t)dt < +\infty, h_2 \) is a nonnegative constant.

**Theorem 1.** If the Conditions 1, 2, 4 hold, then the zero solution of equation (1) is uniformly stable.

**Proof.** For any \( \varepsilon > 0 \), by the uniform stability of the zero solution of equation (2), there exists \( y_0(\varepsilon) > 0 \), such that for any \( t_0 \geq 0 \), the maximal solution \( y(t; t_0, y_0(\varepsilon)) \) of (2) through \( (t_0, y_0(\varepsilon)) \) satisfies
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\[ y(t; t_0, y_0(\epsilon)) < u(\epsilon) \quad \text{for } t \geq t_0. \]

Chosing \( \delta(\epsilon) > 0 \) such that

\[ w(\delta) < y_0(\epsilon) \quad \text{and} \quad \delta < k(u^{-1}(y_0(\epsilon))), \]

we shall prove that for any \( \varphi \in C_1^\delta \)

\[ V(t) \overset{\text{def}}{=}= V(t, x(t; t_0, \varphi)) \leq y(t; t_0, y_0(\epsilon)) \overset{\text{def}}{=} y(t). \]

Let \( y_m(t) \) be the solution of the following initial-value problem

\[
\begin{cases}
\dot{y}_m(t) = W_1(t, y_m(t)) + \frac{1}{m} & t \geq t_0 \\
y_m(t_0) = y_0
\end{cases}
\]

then

\[ y(t) = \lim_{m \to \infty} y_m(t). \]

Therefore, to prove \( V(t) \leq y(t) \), it is sufficient to prove \( V(t) \leq y_m(t) \) for \( m = 1, 2, \ldots, t \geq t_0. \)

Assume, on the contrary, there exist a positive integer \( m \) and \( t_1 > t_0 \) such that \( V(t_1) = y_m(t_1) \), \( V(t) \leq y_m(t) \) for \( t_0 \leq s < t_1 \) and there exists a sequence \( T_k > t_1 \) such that

\[ V(T_k) > y_m(T_k) \quad \text{for } k = 1, 2, \ldots \]

\( T_k \to t_1 \) as \( k \to \infty \). Thus,

(3) \[ \dot{V}(t_1) \geq \dot{y}_m(t_1) = W_1(t, y_m(t_1)) + \frac{1}{m}. \]

Obviously,

\[ V(s) \leq w(|x(s)|) \leq w(||\varphi||) \leq w(\delta) < y_0 < y_m(t_1) < V(t_1) \quad \text{for } s \leq t_0, \]

\[ V(s) \leq y_m(s) \leq y_m(t_1) = V(t_1) \quad \text{for } t_0 \leq s < t_1, \]

\[ |\dot{x}(s)| \leq \delta < k(u^{-1}(y_0)) \leq k(u^+(V(t_1))) \quad \text{for } s \leq t_0. \]

If \( |\dot{x}(t)| = \sup_{s \leq t} |\dot{x}(s)| \), \( t_0 \leq t \leq t_1 \), then

\[ |\dot{x}(t)| \leq k_1(||x_1||) + k_2(||\dot{x}_1||) \leq k_1(u^{-1}(V(t_1))) + k_2(||\dot{x}(t)||). \]

From Condition 2, it follows that

\[ |\dot{x}(t)| \leq k(u^{-1}(V(t_1))). \]

Then, from Condition 4, we get
\[ \dot{V}(t) \leq W_j(t, V(t)) = W_j(t, y_m(t)) , \]
which is contrary to (3).

Therefore, \( V(t) \leq y_m(t) \) for \( t \geq t_0 \), \( m = 1, 2, \ldots \), it follows that \( V(t) \leq y(t) < u(\varepsilon) \), \( |x(t)| < \varepsilon \). This completes the proof.

\textbf{Theorem 2.} If Conditions 1, 2, 3, 5 hold and the zero solution of equation (1) is stable, then the zero solution of (1) is asymptotically stable.

\textit{Proof.} For a given positive constant \( H_1 < H \), \( t_0 \geq 0 \), by the stability of equation (1), there exists \( \delta > 0 \) such that for \( \varphi \in C_1, t \geq t_0 \),
\[ |x(t; t_0, \varphi)| \leq H_1 < H. \]
From Condition 2, we get
\[ |\dot{x}(t; t_0, \varphi)| \leq \max (\delta, k(H_1)). \]

Let \( \lim_{t \to \infty} V(t, x(t)) = \sigma, \lim_{t \to \infty} V(t, x(t)) = q, \lim_{t \to \infty} |\dot{x}(t)| = q \) (where \( x(t) = x(t; t_0, \varphi) \)). For any \( \mu > 0 \), there exists \( \alpha \in (0, \mu) \) such that \( k(u^{-1}(\sigma + \mu)) \geq k(u^{-1}(\sigma)) + \alpha \).
From the definition of upper limit, there exists \( T \geq t_0 \) such that for \( t \geq T \),
\[ V(t, x(t)) \leq \sigma + \mu \]
\[ |\dot{x}(t)| \leq q + \alpha \leq q + \mu, \]
and hence \( |x(t)| \leq u^{-1}(\sigma + \mu) \).

Since \( p(t) \to \infty \) as \( t \to \infty \), there is \( T_1 \geq T \) such that \( p(t) \geq T \) for \( t \geq T_1 \), and so
\[ \sup_{p(t) \leq s \leq t} |x(s)| \leq u^{-1}(\sigma + \mu) \quad \text{for} \quad t \geq T_1 \]
\[ \sup_{p(t) \leq s \leq t} |\dot{x}(s)| \leq q + \alpha \quad \text{for} \quad t \geq T_1. \]

Since \( \lim_{t \to \infty} |\dot{x}(t)| = q \), we can find a sequence \( T_n \geq T_1, T_n \to \infty \) as \( n \to \infty \), such that
\[ |\dot{x}(T_n)| \geq q - \mu \quad n = 1, 2, \ldots. \]

By the Condition 3, we get
\[ q - \mu \leq |\dot{x}(T_n)| \leq k_1(u^{-1}(\sigma + \mu)) + k_2(q + \mu) + |F(T_n, ||x_{T_n}||, ||\dot{x}_{T_n}||)|. \]

Putting \( \mu \to 0, n \to \infty \), we get
\[ q \leq k_1(u^{-1}(\sigma)) + k_2(q). \]

From Condition 2, we have
\[ q \leq k(u^{-1}(\sigma)). \]
Thus, for \( t \geq T_1, s \geq p(t) \)
\[
V(s, x(s)) \leq \sigma + \mu \\
|\dot{x}(s)| \leq q + \alpha \leq k(u^{-1}(\sigma)) + \alpha \leq k(u^{-1}(\sigma + \mu)).
\]

Then, from Condition 5, we get
\[
\dot{V}(t) \leq F(t, V(t), \sigma + \mu) \\
\leq F(t, V(t), V(t)) + |F(t, V(t), \sigma + \mu) - F(t, V(t), V(t))| \\
\leq - W_2(V(t)) + g_1(t)G(V(t)) + g_2(t)Q(V(t)) + [h_1(t) + h_2]|\sigma + \mu - V|.
\]

If \( \sigma > \sigma \), then there exist two sequences \( k_n, V_n \) with \( k_n \rightarrow \infty \) (as \( n \rightarrow \infty \)) and \( V_n > 0 \), such that
\[
V(k_n) = \sigma - \mu, \quad V(k_n + n) = \sigma - \frac{n}{2}
\]
\[
\sigma - \mu \leq V(t) \leq \sigma - \frac{n}{2} \quad k_n \leq t \leq k_n + V_n.
\]

Choosing \( \mu \) small and \( n \) Large enough so that
\[
\inf_{\sigma - \mu < s < \sigma - \mu/2} W_2(s) - 2h_2(t) - g_3(t) \sup_{\sigma - \mu < s < \sigma - \mu/2} Q(s) \geq 0
\]
for \( t \in [k_n, k_n + V_n] \), we have
\[
\dot{V}(t) \leq g_1(t)Q(G(V(t))) + h_1(t)|\sigma + \mu - V(t)| \\
\leq g_1(t)M + 2\mu h_1(t) \quad t \in [k_n, k_n + V_n]
\]
where \( G(V(t)) \leq M \equiv \text{cons} t, \ t \geq t_0 \)

(4) \[
V(k_n + V_n) \leq V(k_n) + M \int_{k_n}^{k_n + V_n} g_1(s)ds + 2\mu \int_{k_n}^{k_n + V_n} h_1(s)ds
\]
\[
\frac{\mu}{2} \leq \int_{k_n}^{k_n + V_n} g_1(s)ds + 2\mu \int_{k_n}^{k_n + V_n} h_1(s)ds.
\]

Since \( g_1(t), h_1(t) \in L^1[0, \infty) \), we know
\[
\lim_{n \rightarrow \infty} \int_{k_n}^{k_n + V_n} g_1(s)ds = 0, \quad \lim_{n \rightarrow \infty} \int_{k_n}^{k_n + V_n} h_1(s)ds = 0.
\]

Putting \( n \rightarrow \infty \) in (4), we get a contradiction \( \mu \leq 0 \), which implies \( \sigma = \sigma \).

If \( \sigma = \sigma = \sigma \neq 0 \), then there exists \( T_2 > T_1, \mu > 0 \), such that
\[
0 < \sigma - \mu < V(t) \leq \sigma + \mu, \quad t \geq T_2
\]
\[
\inf_{\sigma - \mu < s < \sigma + \mu} W_2(s) - [2h_2(t) + g_3(t)] \sup_{\sigma - \mu < s < \sigma + \mu} Q(s) \geq \frac{1}{2}W_2(\sigma).
\]

Then, for \( t \geq T_2 \),
\[ V(t) \leq -W_2(V(t)) + g_1(t)Q(V(t)) + g_2(t)G(V(t)) + [h_1(t) + h_2] \sigma + \mu \]

Hence,
\[ V(t) \leq V(T_3) - \frac{1}{2} W_2(\sigma)(t-T_3) + M \int_{T_3}^{t} g_1(s)ds + 2\mu \int_{T_3}^{t} h_1(s)ds \]
where \( t \geq T_3 \geq T_2 \).

From the inequality above and Cauchy theorem, we know if \( T_3 \), \( t-T_3 \) are large enough, then \( V(t) < 0 \). This, obviously, is a contradiction and then \( \sigma = 0 \). Then \( \lim_{t \to \infty} x(t) = 0 \). The proof is completed.

**Corollary 1.** If condition 1-3, 5 hold and \( g_2(t) \equiv 0 \), \( \lim_{b \to \sigma} \int_{b}^{a} 1/G(u) \ du = \infty \), for any \( a > 0 \), then the zero solution of (1) is asymptotically stable.

This is an immediate consequence of Theorems 1 and 2.

\section*{§ 2. The stability of integrodifferential equations}

As the application of the previous theorems, we discuss the stability of the zero solution of the following integrodifferential equations

\[ \dot{x}(t) = Ax(t) + B_1x(p_1(t)) + B_2\dot{x}(p_2(t)) + \int_{-\infty}^{t} C_1(t-s)x(s)ds + \int_{-\infty}^{t} C_2(t-s)\dot{x}(s)ds, \]

where \( A, B, B_2 \) are \( n \times n \) constant matrices. \( p(t) \leq p_1(t) \leq t, \lim_{t \to \infty} p(t) = \infty, C_1, C_2 \) are continuous matrix functions defined on \([0, + \infty]\) and \( \int_{0}^{\infty} \|C_i(t)\|dt < + \infty \) \((i=1, 2)\).

Suppose \( A \) is stable, then there exist a positive definite matrix \( B \) and positive constants \( \lambda_{\min}, \lambda_{\max} \) such that

\[ \begin{cases} A^T B + BA = -E \\ \lambda_{\min} |x|^2 \leq x^T Bx \leq \lambda_{\max} |x|^2. \end{cases} \]

**Theorem 3.** Suppose

\( i) \ t - p(t) \to \infty, p(t) \to \infty \) as \( t \to \infty; \)

\( ii) \ \| B_2 \| + \int_{0}^{\infty} \| C_2(t) \|dt < 1; \)

\( iii) \ 1 - 2 \| B \| \frac{\lambda_{\max}}{\lambda_{\min}} \left[ \| B_1 \| + \| B_2 \| + \int_{0}^{\infty} \| C_1(t) \|dt + K_1 \int_{0}^{\infty} \| C_2(t) \|dt \right] \geq 0; \)

where
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\[
\begin{align*}
    k &= \frac{\|A\| + \|B_1\| + \int_0^{+\infty} \|C_1(t)\| \, dt}{1 - \left[\|B_2\| + \int_0^{+\infty} \|C_2(t)\| \, dt \right]} \\
\end{align*}
\]

then the zero solution of (5) is asymptotically stable.

**Proof.** Let

\[
\begin{align*}
    k_1(\sigma) &= \left[\|A\| + \int_0^{+\infty} \|C_1(u)\| \, du + \|B_1\| \right] \sigma \\
    k_2(\sigma) &= \left[\|B_2\| + \int_0^{+\infty} \|C_2(u)\| \, du \right] \sigma \\
    k(\sigma) &= K \sigma \\
    q_i(t) &= \int_{t-p(t)}^{+\infty} \|C_i(u)\| \, du \\
       & \quad (i=1, 2).
\end{align*}
\]

It is easy to prove Conditions 1 and 2 hold for the \(k_1, k_2, k\) defined above.

\[
\begin{align*}
    |Ax(t)+B_1x(p(t))+B_2\dot{x}(p_2(t))&+\int_{-\infty}^{t} C_1(t-s)x(s)ds+\int_{-\infty}^{t} C_2(t-s)\dot{x}(s)ds| \\
    &\leq k_1(\sup_{p(t)\leq s\leq t}|x(s)|) + k_2(\sup_{p(t)\leq s\leq t} |\dot{x}(s)|) + q_1(t)\|x_{t}\| + q_2(t)\|\dot{x}_{t}\|
\end{align*}
\]

From Condition (i), it follows that \(q_i(t)\to 0\) as \(t\to\infty\). So, the Condition 3 hold.

Let \(V(t, x) = x^T Bx\), \(U(|x|) = \lambda_{\min}|x|^\beta\), \(W(|x|) = \lambda_{\max}|x|^\beta\) then

\[
\begin{align*}
    \dot{V}_{(5)}(x(t)) &\leq -|x|^2 \left[1 - 2\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \|B\| \left(\|B_1\| + K\|B_2\| + \int_0^{+\infty} \|C_1(u)\| \, du + \int_0^{+\infty} \|C_2(u)\| \, du \right) \right] \\
    &\leq 0.
\end{align*}
\]

Thus, the zero solution of (5) is stable.
If for \( p(t) \leq s \leq t \), \( V(x(s)) \leq N(t) \), \( |\dot{x}(s)| \leq k(\sqrt{N(t)}/\lambda_{\text{min}}) \), then

\[
|\dot{x}(s)| \leq K \frac{\sqrt{N(t)}}{\lambda_{\text{min}}} \quad p(t) \leq s \leq t
\]

\[
|x(s)| \leq \sqrt{\frac{N(t)}{\lambda_{\text{min}}}} \quad p(t) \leq s \leq t
\]

\[
\dot{V}(x(t)) \leq F(t, V, N)
\]

\[
F(t, V, N) = -\frac{V}{\lambda_{\text{max}}} + \frac{2\|B\|N}{\lambda_{\text{min}}} \left( \|B_1\| + K\|B_2\| + \int_0^{+\infty} \|C_1(u)\| du + K\int_0^{+\infty} \|C_2(u)\| du \right)
\]

\[
+ 2g_2(t)M\|B\|
\]

where \( |x(t)|^2 \leq M; |\dot{x}(t)|^2 \leq M \)

\[
g_2(t) = \int_{t-p(t)}^{+\infty} \left( \|C_1(u)\| + \|C_2(u)\| \right) du \longrightarrow 0 \quad \text{as} \ t \rightarrow \infty
\]

\[
F(t, V, V)
\]

\[
= -\left[ \frac{1}{\lambda_{\text{max}}} - \frac{2\|B\|}{\lambda_{\text{min}}} \left( \|B_1\| + K\|B_2\| + \int_0^{+\infty} \|C_1(u)\| du + K\|C_2(u)\| du \right) \right] V
\]

\[
+ 2g_2(t)M\|B\|
\]

\[
|F(t, V, N_1) - F(t, V, N_2)|
\]

\[
\leq \frac{2\|B\|}{\lambda_{\text{min}}} \left[ \|B_1\| + K\|B_2\| + \int_0^{+\infty} \|C_1(u)\| du + K\|C_2(u)\| du \right] |N_1 - N_2|
\]

From Theorem 2, we get the asymptotical stability of the zero solution of (5).

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References


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