

Math 2271: The third Test, November 10, 2017

Use only papers provided.

Time: 50 Minutes

Your name:

Solution

Your student ID:

Your signature:

- Problem 1: Your Mark 20
- Problem 2: Your Mark 20
- Problem 3: Your Mark 20
- Problem 4: Your Mark 20
- Problem 5: Your Mark 20

Problem 1(20 points): Consider the equation

$$t^2y'' - 3ty' + 3y = 0, \quad t > 0.$$

- (a) Check if $y = t$ is a solution;
- (b) Using the method of reduction to find another linearly independent solution.

(a) $y = t'$, $y'' = 0$

$$t^2y'' - 3ty' + 3y = 0 \cdot y'' - 3t \cdot 1 + 3t = 0$$

$y = t$ is a solution 4pts

(b) Let $y = tv$ 2pts

$$y' = v + tv' \quad \text{2pts}$$

$$y'' = 2v' + tv'' \quad \text{2pts}$$

$$t^2y'' - 3ty' + 3y$$

$$= t^2(2v' + tv'') - 3t(v + tv') + 3tv$$

$$= t^2[2v' + tv'' - 3v']$$

$$= t^2[tv'' - v'] \quad \text{2pts}$$

$$= 0$$

$$\Rightarrow v'' = \frac{1}{t}v'$$

Let $z = v'$ 2pts

$$z' = \frac{1}{t}v \quad \text{2pts}$$

$$z = e^{\int \frac{1}{t} dt} = t \quad \text{2pts}$$

$$v' = z \Rightarrow v = \frac{1}{2}t^2 \quad \text{2pts}$$

Another solution is $y_2 = t^3$ 2pts

Problem 2 (20 points) Find the inverse Laplace transform of the function

$$\frac{s^2 + 5s}{(s^2 - 2s + 5)(s+1)}.$$

$$F(s) = \frac{s^2 + 5s}{(s^2 - 2s + 5)(s+1)} = \frac{s^2 + 5s}{[(s-1)^2 + 4](s+1)} \quad \text{2pts}$$

$$= \frac{A(s-1) + B}{(s-1)^2 + 4} + \frac{C}{s+1} \quad \text{2pts}$$

$$\Rightarrow [A(s-1) + B][s+1] + C[(s-1)^2 + 4] = s^2 + 5s$$

$$s = -1 \Rightarrow C[(-1-1)^2 + 4] = (-1)^2 + 5(-1) = -4$$

$$\stackrel{\text{''}}{\partial} C \Rightarrow C = -\frac{1}{2} \quad \text{2pts}$$

$$s = 1 \Rightarrow B(1+1) + C(4) = 6$$

$$2B + 2 = 6, \quad B = 4 \quad \text{2pts}$$

$$s = 0 \Rightarrow (-A+B)(1) + C(1+4) = 0$$

$$-A + B - \frac{5}{2} = 0$$

$$A = \frac{3}{2} \quad \text{2pts}$$

$$\mathcal{L}^{-1}(F(s)) = \frac{3}{2} \mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2 + 2^2}\right) + 2 \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{(s-1)^2 + 2^2}\right)$$

$$- \frac{1}{2} \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \quad \text{5pts}$$

$$= \frac{3}{2} e^{st} \cos(2t) + 2e^{st} \sin(2t) - \frac{1}{2} e^{-st}$$

5pts

Problem 3 (20 points) Using the definition of the Laplace transforms to find the Laplace transform of the following function

$$f(t) = \begin{cases} 1, & 0 < t < 2; \\ 0, & 2 < t < 4; \\ e^{-3t}, & t > 4. \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

10 pts

$$= \int_0^2 e^{-st} (1) dt + \int_2^4 e^{-st} (0) dt + \int_4^\infty e^{-st} e^{-3t} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^2 + 0 - \frac{1}{s+3} e^{-(s+3)t} \Big|_4^\infty$$

$$= -\frac{1}{s} e^{-2s} + \frac{1}{s} + \frac{1}{s+3} e^{-(s+3)4}$$

5 pts

5 pts

Problem 4 (20 points): Use the Laplace transform to solve the following initial value problem

$$y''(t) + 4y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0$$

where

$$g(t) = \begin{cases} 1, & 0 < t < 1; \\ 0, & 1 < t < 2; \\ 0, & t > 2. \end{cases}$$

Let $\mathcal{L}(y) = Y(s)$

$$\mathcal{L}(y'') = s^2 Y - s y(0) - y'(0) = s^2 Y \quad \underline{\underline{2pt}}$$

~~Let~~ $g(t) = u(t) - u(t-1) + u(t-2)$ ~~4pt~~

$$\mathcal{L}(g(t)) = \mathcal{L}(u(t)) - \mathcal{L}(u(t-1)) + \mathcal{L}(u(t-2))$$

$$= \frac{1}{s} - \frac{1}{s} e^{-s} + \frac{1}{s} e^{-2s} \quad \underline{\underline{4pt}}$$

$$\Rightarrow s^2 Y + 4Y = \frac{1}{s} - \frac{1}{s} e^{-s} + \frac{1}{s} e^{-2s} \quad \underline{\underline{2pt}}$$

$$Y = \frac{1}{s^2+4} \left[\frac{1}{s} - \frac{1}{s} e^{-s} + \frac{1}{s} e^{-2s} \right]$$

$$\frac{1}{s^2+4} \cdot \frac{1}{s} = \frac{As+B}{s^2+4} + \frac{C}{s}$$

$$(As+B)s + C(s^2+4) = 1 \quad \underline{\underline{2pt}}$$

$$A + C = 0$$

$$B = 0 \quad \Rightarrow \quad C = \frac{1}{4}$$

$$4C = 1$$

$$A = -\frac{1}{4}$$

$$\begin{aligned} Y &= \frac{1}{4} \left(\frac{-s}{s^2+2^2} + \frac{1}{s} \right) \left(1 - e^{-s} + e^{-2s} \right) \\ &= \frac{1}{4} \left[-\cos(2t) + t \right] + \frac{1}{4} \left[-\cos 2(t-1) + (t-1) \right] u(t-1) \\ &\quad + \frac{1}{4} \left[-\cos 2(t-2) + t-2 \right] u(t-2) \end{aligned} \quad \underline{\underline{6pt}}$$

Problem 5 (20 points): Using the method of Laplace transforms to solve the initial value problem

$$y'' - y = t - 2; \quad y(2) = 3, y'(2) = 0.$$

Let

$$w(t) = y(t+2) \quad 2pts$$

$$w(0) = y(2) = 3 \quad 2pts$$

$$w'(0) = y'(2) = 0 \quad 2pts$$

$$w''(t) - w(t) = t + 2 - 2 = t$$

Let

$$W = \mathcal{L}(w), \quad 2pts$$

$$s^2 W - s w(0) - w'(0) - w = \mathcal{L}(t) = \frac{1}{s} \quad 1pt$$

$$s^2 W - 3s - w = \frac{1}{s} \quad 2pts$$

$$(s^2 - 1)w = 3s + \frac{1}{s} \quad 2pts$$

$$w = \frac{1}{(s-1)(s+1)} \left[3s + \frac{1}{s} \right]$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s} \quad 2pts$$

$$A(s+1)s + B(s-1)s + C(s-1)(s+1)$$

$$= 3s^2 + 1$$

$$s=0 \Rightarrow C(-1) = 1 \Rightarrow C = -1$$

$$s=1 \Rightarrow 2A = 4 \Rightarrow A = 2$$

$$s=-1 \Rightarrow 2B = 4 \Rightarrow B = 2$$

$$w = \frac{2}{s-1} + \frac{2}{s+1} - \frac{1}{s} \quad \text{Handwritten notes: } \cancel{\frac{2}{s-1}}, \cancel{\frac{2}{s+1}}, \cancel{-\frac{1}{s}}$$

$$w(t) = \mathcal{L}^{-1}(w) = 2e^{-t} + 2e^{+t} - 1 \quad 1pt$$

$$\Rightarrow y(t) = w(t-2) = 2e^{-(t-2)} + 2e^{+(t-2)} - 1 \quad 2pts$$