8.6 Method of Frobenius

In the previous section we showed that a homogeneous Cauchy–Euler equation has a solution of the form $y(x) = x^r$, $x > 0$, where r is a certain constant. Cauchy–Euler equations have, of course, a very special form with only one singular point (at $x = 0$). In this section we show how the theory for Cauchy–Euler equations generalizes to other equations that have a special type of singularity.

To motivate the procedure, let's rewrite the Cauchy–Euler equation,

$$ax^2y''(x) + bxy'(x) + cy(x) = 0\,, \qquad x > 0\,, \tag{1}$$

in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \qquad x > 0\,, \tag{2}$$

where

$$p(x) = \frac{p_0}{x}\,, \qquad q(x) = \frac{q_0}{x^2}\,,$$

and p_0, q_0 are the constants b/a and c/a, respectively. When we substitute $w(r, x) = x^r$ for $y(x)$ into equation (2), we get

$$[r(r-1) + p_0r + q_0]x^{r-2} = 0\,,$$

which yields the indicial equation

$$r(r-1) + p_0r + q_0 = 0\,. \tag{3}$$

Thus, if r_1 is a root of (3), then $w(r_1, x) = x^{r_1}$ is a solution to equations (1) and (2).

Let's now assume, more generally, that (2) is an equation for which $xp(x)$ and $x^2q(x)$, instead of being constants, are *analytic functions*. That is, in some open interval about $x = 0$,

$$xp(x) = p_0 + p_1x + p_2x^2 + \cdots = \sum_{n=0}^{\infty} p_nx^n\,, \tag{4}$$

$$x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots = \sum_{n=0}^{\infty} q_nx^n\,. \tag{5}$$

It follows from (4) and (5) that

$$\lim_{x\to 0} xp(x) = p_0 \qquad \text{and} \qquad \lim_{x\to 0} x^2q(x) = q_0\,, \tag{6}$$

and hence, for x near 0 we have $xp(x) \approx p_0$ and $x^2q(x) \approx q_0$. Therefore, it is reasonable to expect that the solutions to (2) will behave (for x near 0) like the solutions to the Cauchy-Euler equation

$$x^2y'' + p_0xy' + q_0y = 0\,.$$

When $p(x)$ and $q(x)$ satisfy (4) and (5), we say that the singular point at $x = 0$ is regular. More generally, we state the following.

Regular Singular Point

Definition 3. A singular point x_0 of

(7) $$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

is said to be a **regular singular point** if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.[†] Otherwise x_0 is called an **irregular singular point.**

Example 1 Classify the singular points of the equation

(8) $$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0 .$$

Solution Here

$$p(x) = \frac{x+1}{(x^2-1)^2} = \frac{1}{(x-1)^2(x+1)} ,$$

$$q(x) = \frac{-1}{(x^2-1)^2} = \frac{-1}{(x-1)^2(x+1)^2} ,$$

from which we see that ± 1 are the singular points of (8). For the singularity at 1, we have

$$(x-1)p(x) = \frac{1}{(x-1)(x+1)} ,$$

which is not analytic at $x = 1$. Therefore, $x = 1$ is an irregular singular point.

For the singularity at -1, we have

$$(x+1)p(x) = \frac{1}{(x-1)^2} , \qquad (x+1)^2 q(x) = \frac{-1}{(x-1)^2} ,$$

both of which are analytic at $x = -1$. Hence, $x = -1$ is a regular singular point. ◆

Let's assume that $x = 0$ is a regular singular point for equation (7) so that $p(x)$ and $q(x)$ satisfy (4) and (5); that is,

(9) $$p(x) = \sum_{n=0}^{\infty} p_n x^{n-1} , \qquad q(x) = \sum_{n=0}^{\infty} q_n x^{n-2} .$$

The idea of the mathematician Frobenius[‡] was that since Cauchy–Euler equations have solutions of the form x^r, then for the regular singular point $x = 0$, there should be solutions to (7) of the form x^r *times an analytic function.* Hence we seek solutions to (7) of the form

(10) $$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} , \qquad x > 0 .$$

In writing (10), we have assumed a_0 is the first nonzero coefficient, so we are left with determining r and the coefficients a_n, $n \geq 1$. Differentiating $w(r, x)$ with respect to x, we have

(11) $$w'(r, x) = \sum_{n=0}^{\infty} (n + r)a_n x^{n+r-1} ,$$

(12) $$w''(r, x) = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-2} .$$

[†] In the terminology of complex variables, p has a pole of order at most 1, and q has a pole of order at most 2, at x_0.

[‡] *Historical Footnote:* George Frobenius (1848–1917) developed this method in 1873. He is also known for his research on group theory.

If we substitute the above expansions for $w(r, x)$, $w'(r, x)$, $w''(r, x)$, $p(x)$, and $q(x)$ into (4), we obtain

$$\text{(13)} \quad \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \left(\sum_{n=0}^{\infty} p_n x^{n-1}\right)\left(\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}\right)$$
$$+ \left(\sum_{n=0}^{\infty} q_n x^{n-2}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 .$$

Now we use the Cauchy product to perform the series multiplications and then group like powers of x, starting with the lowest power, x^{r-2}. This gives

$$\text{(14)} \quad [r(r-1) + p_0 r + q_0]a_0 x^{r-2}$$
$$+ [(r+1)ra_1 + (r+1)p_0 a_1 + p_1 r a_0 + q_0 a_1 + q_1 a_0]x^{r-1} + \cdots = 0 .$$

For the expansion on the left-hand side of equation (14) to sum to zero, each coefficient must be zero. Considering the first term, x^{r-2}, we find

$$\text{(15)} \quad [r(r-1) + p_0 r + q_0]a_0 = 0 .$$

We have assumed that $a_0 \neq 0$, so the quantity in brackets must be zero. This gives the indicial equation; it is the same as the one we derived for Cauchy–Euler equations.

Indicial Equation

Definition 4. If x_0 is a regular singular point of $y'' + py' + qy = 0$, then the **indicial equation** for this point is

$$\text{(16)} \quad r(r-1) + p_0 r + q_0 = 0 ,$$

where

$$p_0 := \lim_{x \to x_0} (x - x_0)p(x) , \qquad q_0 := \lim_{x \to x_0} (x - x_0)^2 q(x) .$$

The roots of the indicial equation are called the **exponents (indices)** of the singularity x_0.

Example 2 Find the indicial equation and the exponents at the singularity $x = -1$ of

$$\text{(17)} \quad (x^2 - 1)^2 y''(x) + (x+1)y'(x) - y(x) = 0 .$$

Solution In Example 1 we showed that $x = -1$ is a regular singular point. Since $p(x) = (x+1)^{-1}(x-1)^{-2}$ and $q(x) = -(x+1)^{-2}(x-1)^{-2}$, we find

$$p_0 = \lim_{x \to -1} (x+1)p(x) = \lim_{x \to -1} (x-1)^{-2} = \frac{1}{4} ,$$

$$q_0 = \lim_{x \to -1} (x+1)^2 q(x) = \lim_{x \to -1} [-(x-1)^{-2}] = -\frac{1}{4} .$$

Substituting these values for p_0 and q_0 into (16), we obtain the indicial equation

$$\text{(18)} \quad r(r-1) + \frac{1}{4}r - \frac{1}{4} = 0 .$$

Multiplying by 4 and factoring gives $(4r+1)(r-1) = 0$. Hence, $r = 1, -1/4$ are the exponents. ◆

As we have seen, we can use the indicial equation to determine those values of r for which the coefficient of x^{r-1} in (14) is zero. If we set the coefficient of x^r in (14) equal to zero, we have

(19) $$[(r+1)r + (r+1)p_0 + q_0]a_1 + (p_1 r + q_1)a_0 = 0 .$$

Since a_0 is arbitrary and we know the p_i's, q_i's, and r, we can solve equation (19) for a_1 provided the coefficient of a_1 in (19) is not zero. This will be the case if we take r to be the *larger*† of the two roots of the indicial equation (see Problem 43, page 464). Similarly, when we set the coefficient of x^{r+1} equal to zero, we can solve for a_2 in terms of the p_i's, q_i's, r, a_0, and a_1. Continuing in this manner, we can recursively solve for the a_n's. The procedure is illustrated in the following example.

Example 3 Find a series expansion about the regular singular point $x = 0$ for a solution to

(20) $$(x+2)x^2y''(x) - xy'(x) + (1+x)y(x) = 0 , \quad x > 0 .$$

Solution Here $p(x) = -x^{-1}(x+2)^{-1}$ and $q(x) = x^{-2}(x+2)^{-1}(1+x)$, so

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0}[-(x+2)^{-1}] = -\frac{1}{2} ,$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0}(x+2)^{-1}(1+x) = \frac{1}{2} .$$

Since $x = 0$ is a regular singular point, we seek a solution to (20) of the form

(21) $$w(r,x) = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} .$$

By the previous discussion, r must satisfy the indicial equation (16). Substituting for p_0 and q_0 in (16), we obtain

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0 ,$$

which simplifies to $2r^2 - 3r + 1 = (2r-1)(r-1) = 0$. Thus, $r = 1$ and $r = 1/2$ are the roots of the indicial equation associated with $x = 0$.

Let's use the larger root $r = 1$ and solve for a_1, a_2, etc., to obtain the solution $w(1, x)$. We can simplify the computations by substituting $w(r, x)$ directly into equation (20), where the coefficients are polynomials in x, rather than dividing by $(x+2)x^2$ and having to work with the rational functions $p(x)$ and $q(x)$. Inserting $w(r, x)$ in (20) and recalling the formulas for $w'(r, x)$ and $w''(r, x)$ in (11) and (12) gives (with $r = 1$)

(22) $$(x+2)x^2\sum_{n=0}^{\infty}(n+1)na_nx^{n-1} - x\sum_{n=0}^{\infty}(n+1)a_nx^n + (1+x)\sum_{n=0}^{\infty}a_nx^{n+1} = 0 ,$$

which we can write as

(23) $$\sum_{n=0}^{\infty}(n+1)na_nx^{n+2} + \sum_{n=0}^{\infty}2(n+1)na_nx^{n+1} - \sum_{n=0}^{\infty}(n+1)a_nx^{n+1} + \sum_{n=0}^{\infty}a_nx^{n+1} + \sum_{n=0}^{\infty}a_nx^{n+2} = 0 .$$

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†"Larger" in the sense of Problem 43.

Figure 8.[illegible]

Method of Frobenius

To derive a series solution [illegible]

(29) [illegible]

(a) Set [illegible]

[illegible] remaining steps [illegible]

(b) Let

(30) [illegible]

and, using termwise differentiation, substitute [illegible] equation (29) to obtain an equation of the form

$A_0(x - x_0)$ [illegible]

(c) Set the coefficients [illegible] equal to zero. Notice that the equation [illegible] is just a constant multiple of the indicial equation [illegible]

(d) Use the system of equations

$$A_0 = 0, \quad A_1 = 0, \quad \ldots, \quad A_k = 0$$

to find a recurrence relation involving a_k and $a_0, a_1, \ldots, a_{k-1}$.

(e) Take $r = r_1$, the larger root of the indicial equation, and use the relation obtained in step (d) to determine $a_1, a_2, \ldots$ recursively in terms of a_0 and r_1.

(f) A series expansion of a solution to (29) is

$$(31) \qquad w(r_1, x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x > x_0,$$

where a_0 is arbitrary and the a_n's are defined in terms of a_0 and r_1.

Next we shift the indices so that each summation in (23) is in powers x^k. With $k = n + 2$ in the first and last summations and $k = n + 1$ in the rest, (23) becomes

(24) $$\sum_{k=2}^{\infty}[(k-1)(k-2)+1]a_{k-2}x^k + \sum_{k=1}^{\infty}[2k(k-1)-k+1]a_{k-1}x^k = 0.$$

Separating off the $k = 1$ term and combining the rest under one summation yields

(25) $$[2(1)(0)-1+1]a_0x + \sum_{k=2}^{\infty}[(k^2-3k+3)a_{k-2}+(2k-1)(k-1)a_{k-1}]x^k = 0.$$

Notice that the coefficient of x in (25) is zero. This is because $r = 1$ is a root of the indicial equation, which is the equation we obtained by setting the coefficient of the lowest power of x equal to zero.

We can now determine the a_k's in terms of a_0 by setting the coefficients of x^k in equation (25) equal to zero for $k = 2, 3$, etc. This gives the recurrence relation

(26) $$(k^2-3k+3)a_{k-2}+(2k-1)(k-1)a_{k-1} = 0,$$

or, equivalently,

(27) $$a_{k-1} = -\frac{k^2-3k+3}{(2k-1)(k-1)}a_{k-2}, \qquad k \geq 2.$$

Setting $k = 2, 3$, and 4 in (27), we find

$$a_1 = -\frac{1}{3}a_0 \qquad (k=2),$$

$$a_2 = -\frac{3}{10}a_1 = \frac{1}{10}a_0 \qquad (k=3),$$

$$a_3 = -\frac{1}{3}a_2 = -\frac{1}{30}a_0 \qquad (k=4).$$

Substituting these values for r, a_1, a_2, and a_3 into (21) gives

(28) $$w(1,x) = a_0x^1\left(1-\frac{1}{3}x+\frac{1}{10}x^2-\frac{1}{30}x^3+\cdots\right),$$

where a_0 is arbitrary. In particular, for $a_0 = 1$, we get the solution

$$y_1(x) = x-\frac{1}{3}x^2+\frac{1}{10}x^3-\frac{1}{30}x^4+\cdots \qquad (x>0).$$

See Figure 8.8 on page 459. ◆

To find a second linearly independent solution to equation (20), we could try setting $r = 1/2$ and solving for $a_1, a_2, \ldots$ to obtain a solution $w(1/2, x)$ (see Problem 44, page 464). In this particular case, the approach would work. However, if we encounter an indicial equation that has a repeated root, then the method of Frobenius would yield just one solution (apart from constant multiples). To find the desired second solution, we must use another technique, such as the reduction of order procedure discussed in Section 4.7 or Exercises 6.1, Problem 31, page 327. We tackle the problem of finding a second linearly independent solution in the next section.

One important question that remains concerns the radius of [illegible] series that appears in [illegible]. The following theorem contains an answer [illegible]

Frobenius's Theorem

Theorem 6. If x_0 is a regular singular point of equation (29), then there exists at least one series solution of the form [illegible] is the larger root of the associated indicial equation. Moreover, this series converges for all x such that [illegible] R, where R is the distance from x_0 to the nearest other singular point (real or complex) of (29).

For simplicity, in the examples [illegible] consider only series expansions about regular singular points [illegible] for which the associated indicial equation has real roots.

The following three examples not only illustrate the method of Frobenius but also [illegible] important models [illegible]

Example 4 Find a series solution about the [illegible]

(32) [illegible]

Solution Here [illegible] and [illegible] that [illegible] regular singular point of (32), so we compute

$$p_0 = \lim [illegible]$$

$$q_0 = \lim [illegible]$$

Then the indicial equation is

$$r(r - 1) - r + 1 = r^2 - 2r + 1 = (r - 1)^2 = 0 ,$$

which has the roots $r_1 = r_2 = 1$.

Next we substitute

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{33}$$

into (32) and obtain

$$x^2 \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} + (1 - x) \sum_{n=0}^{\infty} a_n x^{n+r} = 0 , \tag{34}$$

which we write as

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n + r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0 . \tag{35}$$

For a proof of this theorem, see *Ordinary Differential Equations*, by E. L. Ince (Dover Publications, New York, 1956), Chapter 16.

[illegible] the indices so that [illegible] summation in (35) is in powers [illegible] we take [illegible] in the [illegible] summation and [illegible] in the second. This gives

[illegible]

[illegible] out the term corresponding to [illegible] and combining the rest under one summation [illegible]

[illegible]

When we set the coefficients [illegible] zero, we recover the indicial equation

(38) [illegible]

[illegible] obtain, for [illegible] the recurrence relation

(39) [illegible]

which reduces to

(40) [illegible]

Relation (40) can be used to [illegible]

(41) [illegible]

Setting [illegible] in (39) [illegible] gives

(42) $$a_k = \frac{1}{k^2} a_{k-1}, \quad k \geq 1$$

For $k = 1, 2,$ and 3, we now find

$$a_1 = \frac{1}{1^2} a_0 = a_0$$

$$a_2 = \frac{1}{2^2} a_1 = \frac{1}{(2 \cdot 1)^2} a_0 = \frac{1}{4} a_0$$

$$a_3 = \frac{1}{3^2} a_2 = \frac{1}{(3 \cdot 2 \cdot 1)^2} a_0 = \frac{1}{36} a_0$$

In general, we have

(43) $$a_k = \frac{1}{(k!)^2} a_0$$

Hence, equation (32) has a series solution given by

(44) $$\begin{aligned} w(1, x) &= a_0 x \left\{ 1 + x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \cdots \right\} \\ &= a_0 x \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k, \quad x > 0. \ \blacklozenge \end{aligned}$$

Since $x = 0$ is the only singular point for equation (32), it follows from Frobenius's theorem or directly by the ratio test that the series solution (44) converges for all $x > 0$.

In the next two examples, we only outline the method; we leave it to you to supply the intermediate steps.

Example 5 Find a series solution about the regular singular point $x = 0$ of

$$xy'' + 4y' - xy = 0, \quad x > 0. \tag{45}$$

Solution Since $p(x) = 4/x$ and $q(x) = -1$, we see that $x = 0$ is indeed a regular singular point and

$$p_0 = \lim_{x \to 0} xp(x) = 4, \quad q_0 = \lim_{x \to 0} x^2 q(x) = 0.$$

The indicial equation is

$$r(r - 1) + 4r = r^2 + 3r = 0,$$

with roots $r_1 = 0$ and $r_2 = -3$.

Now we substitute

$$w(r, x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{46}$$

into (45). After a little algebra and a shift of indices, we get

$$[r(r - 1) + 4r] a_0 x^{r-1} + [(r + 1)r + 4(r + 1)] a_1 x^r + \sum_{k=1}^{\infty} [(k + r + 1)(k + r)a_{k+1} + 4(k + r + 1)a_{k+1} - a_{k-1}] x^{k+r} = 0. \tag{47}$$

Next we set the coefficients equal to zero and find

$$[r(r - 1) + 4r] a_0 = 0, \tag{48}$$

$$[(r + 1)r + 4(r + 1)] a_1 = 0, \tag{49}$$

and, for $k \geq 1$, the recurrence relation

$$(k + r + 1)(k + r + 4)a_{k+1} - a_{k-1} = 0. \tag{50}$$

For $r = r_1 = 0$, equation (48) becomes $0 \cdot a_0 = 0$ and (49) becomes $4 \cdot a_1 = 0$. Thus, although a_0 is arbitrary, a_1 must be zero. Setting $r = r_1 = 0$ in (50), we find

$$a_{k+1} = \frac{1}{(k + 1)(k + 4)} a_{k-1}, \quad k \geq 1, \tag{51}$$

from which it follows (after a few experimental computations) that $a_{2k+1} = 0$ for $k \geq 0$ and

$$\begin{aligned} a_{2k} &= \frac{1}{[2 \cdot 4 \cdots (2k)][5 \cdot 7 \cdots (2k + 3)]} a_0 \\ &= \frac{1}{2^k k! [5 \cdot 7 \cdots (2k + 3)]} a_0, \quad k \geq 1. \end{aligned} \tag{52}$$

Hence equation (45) has the power series solution

$$w(0, x) = a_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{2^k k! [5 \cdot 7 \cdots (2k + 3)]} x^{2k} \right\}, \quad x > 0. \tag{53}$$

If in Example 5 we had worked with the root $r = r_2 =$ [illegible], then we would actually have obtained two linearly independent solutions (see Problem 45).

Example 6 Find a series solution about the regular singular point of $x = 0$ of

(54) [illegible]

Solution Since $p(x) =$ [illegible] and $q(x) =$ [illegible], we see that $x = 0$ is a regular singular point. Moreover

$$p_0 = \lim_{x \to 0} xp(x) = \text{[illegible]}, \quad q_0 = \lim_{x \to 0} x^2 q(x) = 0.$$

So the indicial equation is

(55) [illegible] $r(r + 2) = 0$,

with roots $r_1 = 0$ and $r_2 = -2$.

Substituting

(56) [illegible]

into (54) ultimately gives

(57) [illegible]

Setting the coefficients equal to zero, we have

(58) [illegible]

(59) [illegible]

and, for $k \geq 1$, the recurrence relation

(60) $(k + r + 1)(k + r + 3)a_{k+1} - a_{k-1} = 0$.

With $r = r_1 = 0$, these equations lead to the following formulas: [illegible]

and

(61) $$a_{2k} = \frac{1}{[2 \cdot 4 \cdots (2k)][4 \cdot 6 \cdots (2k+2)]} a_0 = \frac{1}{2^{2k}k!(k+1)!} a_0, \quad k \geq 0.$$

Hence equation (54) has the power series solution

(62) $$w(0, x) = a_0 \sum_{k=0}^{\infty} \frac{1}{2^{2k}k!(k+1)!} x^{2k}, \quad x > 0. \; \blacklozenge$$

Unlike in Example 5, if we work with the second root $r = r_2 = -2$ in Example 6, then we do *not* obtain a second linearly independent solution (see Problem 46).

In the preceding examples we were able to use the method of Frobenius to find a series solution valid to the right ($x > 0$) of the regular singular point $x = 0$. For $x < 0$, we can use the change of variables $x = -t$ and then solve the resulting equation for $t > 0$.

The method of Frobenius also applies to higher-order linear equations (see Problems 35–38).

8.6 EXERCISES

[illegible]

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]
5. [illegible]
6. [illegible]
7. [illegible]
8. [illegible]
9. [illegible]
10. [illegible]

In Problems 11–18, [illegible]

11. [illegible]
12. [illegible]
13. [illegible]
14. [illegible]
15. [illegible]
16. [illegible]
17. [illegible]
18. $4x \sin x\, y'' - 3y = 0$, at $x = 0$

In Problems 19–24, use the method of Frobenius to find at least the first four nonzero terms in the series expansion about $x = 0$ for a solution to the given equation for $x > 0$.

19. $9x^2y'' + 9x^2y' + 2y = 0$
20. $2x(x-1)y'' + 3(x-1)y' - y = 0$
21. $x^2y'' + xy' + x^2y = 0$
22. $xy'' + y' - 4y = 0$
23. $x^2z'' + (x^2 + x)z' - z = 0$
24. $3xy'' + (2 - x)y' - y = 0$

In Problems 25–30, use the method of Frobenius to find a general formula for the coefficient a_n in a series expansion about $x = 0$ for a solution to the given equation for $x > 0$.

25. $4x^2y'' + 2x^2y' - (x + 3)y = 0$
26. $x^2y'' + (x^2 - x)y' + y = 0$
27. $xw'' - w' - xw = 0$
28. $3x^2y'' + 8xy' + (x - 2)y = 0$
29. $xy'' + (x - 1)y' - 2y = 0$
30. $x(x + 1)y'' + (x + 5)y' - 4y = 0$

[illegible]

31. [illegible]
32. [illegible]
33. [illegible]
34. [illegible]

[illegible]

35. [illegible]
36. [illegible]
37. [illegible]
38. [illegible]

[illegible]

39. [illegible]
40. [illegible]

[illegible] an expansion about the point at infinity. [illegible] use the change of variables [illegible] In Problems 41 and 42, show that [illegible] point of the given differential equation [illegible] a regular singular point for the [illegible] find at least the first four nonzero terms [illegible] about infinity of a solution to the [illegible]

41. [illegible]
42. [illegible]

43. Show that if r_1 and r_2 are roots of the indicial [illegible] (16) on page 456, with r_1 the larger root [illegible] then the coefficient of a_k in equation (19) on page [illegible] not zero when $r = r_1$.

44. To obtain a second linearly independent solution [illegible] tion (20):
    (a) Substitute $w(r, x)$ given in (21) into [illegible] clude that the coefficients a_k [illegible] must satisfy the recurrence relation

$$(k + r - 1)(2k + 2r - 1)a_k$$
$$+ \{(k + r - 1)(k + r - 2) [illegible]$$