

§ 2.4 Exact Equations.

Recall: $F(x, y) = c$

gives an Implicit soln. of
the equation

$$\frac{\partial}{\partial x} F(x, y) dx + \frac{\partial}{\partial y} F(x, y) dy = 0$$

So if there exists F such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

then $F(x, y) = c$ solves

$$(*) \quad M dx + N dy = 0.$$

Eq (*) is called an Exact Equation

Question 1: When (*) is exact

Answer: (*) is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Ex: $(2xy^2+1)dx + 2x^2y dy$ is exact

(*) $dx + \frac{2x^2y}{2xy^2+1} dy = 0$ is NOT.

[Why: (*) \Leftrightarrow

$$(2xy^2+1)dx + 2x^2y dy = 0$$

$$\frac{\partial M}{\partial y} = 2x^2, \quad \frac{\partial N}{\partial x} = 4xy$$

(P1)

Question: How to solve an exact equation

$$M dx + N dy = 0 ?$$

Answer: Since $\frac{\partial F}{\partial x} = M$, we find

$$F(x, y) = \int M(x, y) dx + g(y)$$

↑
why a
function of y ?

Then use

$$N = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left[\int M(x, y) dx + g(y) \right]$$

to solve $g(y)$.

Similarly, you can solve

$$F(x, y) = \int N(x, y) dy + h(x)$$

and use $\frac{\partial F}{\partial x} = M$ to solve $h(x)$

Ex: $(1+e^x y + x e^x y) dx + (x e^x + 2) dy = 0$

Remark: Better to solve F by

$$F = \int N dy + h(x), \text{ first. Why???}$$

(P2)

2.4 EXERCISES

In Problems 1–8, classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

1. $(x^2y + x^4\cos x)dx - x^3dy = 0$
2. $(x^{10/3} - 2y)dx + xdy = 0$
3. $\sqrt{-2y - y^2}dx + (3 + 2x - x^2)dy = 0$
4. $(ye^{xy} + 2x)dx + (xe^{xy} - 2y)dy = 0$
5. $xydx + dy = 0$
6. $y^2dx + (2xy + \cos y)dy = 0$
7. $[2x + y\cos(xy)]dx + [x\cos(xy) - 2y]dy = 0$
8. $\theta dr + (3r - \theta - 1)d\theta = 0$

In Problems 9–20, determine whether the equation is exact. If it is, then solve it.

9. $(2xy + 3)dx + (x^2 - 1)dy = 0$
10. $(2x + y)dx + (x - 2y)dy = 0$
11. $(e^x\sin y - 3x^2)dx + (e^x\cos y + y^{-2/3}/3)dy = 0$
12. $(\cos x \cos y + 2x)dx - (\sin x \sin y + 2y)dy = 0$
13. $e^t(y - t)dt + (1 + e^t)dy = 0$
14. $(t/y)dy + (1 + \ln y)dt = 0$
15. $\cos \theta dr - (r \sin \theta - e^\theta)d\theta = 0$
16. $(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0$
17. $(1/y)dx - (3y - x/y^2)dy = 0$
18. $[2x + y^2 - \cos(x + y)]dx + [2xy - \cos(x + y) - e^y]dy = 0$
19. $\left(2x + \frac{y}{1 + x^2y^2}\right)dx + \left(\frac{x}{1 + x^2y^2} - 2y\right)dy = 0$
20. $\left[\frac{2}{\sqrt{1 - x^2}} + y\cos(xy)\right]dx + [x\cos(xy) - y^{-1/3}]dy = 0$

In Problems 21–26, solve the initial value problem.

21. $(1/x + 2y^2x)dx + (2yx^2 - \cos y)dy = 0, \quad y(1) = \pi$
22. $(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0, \quad y(1) = 1$
23. $(e^t y + te^t y)dt + (te^t + 2)dy = 0, \quad y(0) = -1$
24. $(e^x + 1)dt + (e^t - 1)dx = 0, \quad x(1) = 1$
25. $(y^2 \sin x)dx + (1/x - y/x)dy = 0, \quad y(\pi) = 1$
26. $(\tan y - 2)dx + (x \sec^2 y + 1/y)dy = 0, \quad y(0) = 1$

27. For each of the following equations, find the general function $M(x, y)$ so that the equation is exact.
 - $M(x, y)dx + (\sec^2 y - x/y)dy = 0$
 - $M(x, y)dx + (\sin x \cos y - xy - e^{-y})dy = 0$
28. For each of the following equations, find the general function $N(x, y)$ so that the equation is exact.
 - $[y \cos(xy) + e^x]dx + N(x, y)dy = 0$
 - $(ye^{xy} - 4x^3y + 2)dx + N(x, y)dy = 0$
29. Consider the equation

$$(y^2 + 2xy)dx - x^2dy = 0$$
 - Show that this equation is not exact.
 - Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact.
 - Use the solution of the resulting exact equation to solve the original equation.
 - Were any solutions lost in the process?

30. Consider the equation

$$(5x^2y + 6x^3y^2 + 4xy^2)dx + (2x^3 + 3x^4y + 3x^2y)dy = 0$$

- Show that the equation is not exact.
- Multiply the equation by $x^n y^m$ and determine values for n and m that make the resulting equation exact.
- Use the solution of the resulting exact equation to solve the original equation.

31. Argue that in the proof of Theorem 2 the function g can be taken as

$$g(y) = \int_{y_0}^y N(x, t)dt - \int_{y_0}^y \left[\frac{\partial}{\partial t} \int_{x_0}^x M(s, t)ds \right] dt$$

which can be expressed as

$$g(y) = \int_{y_0}^y N(x, t)dt - \int_{x_0}^x M(s, y_0)ds + \int_{x_0}^x M(s, y_0)ds$$

This leads ultimately to the representation

$$(18) \quad F(x, y) = \int_{y_0}^y N(x, t)dt + \int_{x_0}^x M(s, y_0)ds$$

Evaluate this formula directly with $x_0 = 0, y_0 = 0$ and rework

- Example 1.
- Example 2.
- Example 3.

32. Orthogonal family of curves. We may want to consider where equation is given