Although the majority of equations one is likely to encounter in practice fall into the *nonlinear* category, knowing how to deal with the simpler linear equations is an important first step (just as tangent lines help our understanding of complicated curves by providing local approximations).

## 1.1 EXERCISES

*In Problems 1–12, a differential equation is given along with the field or problem area in which it arises. Classify each as an ordinary differential equation (ODE) or a partial differential equation (PDE), give the order, and indicate the independent and dependent variables. If the equation is an ordinary differential equation, indicate whether the equation is linear or nonlinear.*

1. $5\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 2\cos 3t$

(mechanical vibrations, electrical circuits, seismology)

2. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

(Hermite's equation, quantum-mechanical harmonic oscillator)

3. $\frac{dy}{dx} = \frac{y(2-3x)}{x(1-3y)}$

(competition between two species, ecology)

4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(Laplace's equation, potential theory, electricity, heat, aerodynamics)

5. $y\left[1 + \left(\frac{dy}{dx}\right)^2\right] = C$, where $C$ is a constant

(brachistochrone problem,† calculus of variations)

6. $\frac{dx}{dt} = k(4-x)(1-x)$, where $k$ is a constant

(chemical reaction rates)

7. $\frac{dp}{dt} = kp(P-p)$, where $k$ and $P$ are constants

(logistic curve, epidemiology, economics)

8. $\sqrt{1-y}\,\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

(Kidder's equation, flow of gases through a porous medium)

9. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

(aerodynamics, stress analysis)

10. $8\frac{d^4y}{dx^4} = x(1-x)$

(deflection of beams)

11. $\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN$, where $k$ is a constant

(nuclear fission)

12. $\frac{d^2y}{dx^2} - 0.1(1-y^2)\frac{dy}{dx} + 9y = 0$

(van der Pol's equation, triode vacuum tube)

*In Problems 13–16, write a differential equation that fits the physical description.*

13. The rate of change of the population $p$ of bacteria at time $t$ is proportional to the population at time $t$.
14. The velocity at time $t$ of a particle moving along a straight line is proportional to the fourth power of its position $x$.
15. The rate of change in the temperature $T$ of coffee at time $t$ is proportional to the difference between the temperature $M$ of the air at time $t$ and the temperature of the coffee at time $t$.
16. The rate of change of the mass $A$ of salt at time $t$ is proportional to the square of the mass of salt present at time $t$.

17. **Drag Race.** Two drivers, Alison and Kevin, are participating in a drag race. Beginning from a standing start, they each proceed with a constant acceleration. Alison covers the last 1/4 of the distance in 3 seconds, whereas Kevin covers the last 1/3 of the distance in 4 seconds. Who wins and by how much time?

†*Historical Footnote:* In 1630 Galileo formulated the brachistochrone problem ($\beta\rho\acute{\alpha}\chi\iota\sigma\tau o\varsigma$ = shortest, $\chi\rho\acute{o}\nu o\varsigma$ = time), that is, to determine a path down which a particle will fall from one given point to another in the shortest time. It was reproposed by John Bernoulli in 1696 and solved by him the following year.

**Example 8** For the initial value problem

$$(11) \qquad 3\frac{dy}{dx} = x^2 - xy^3 , \qquad y(1) = 6 ,$$

does Theorem 1 imply the existence of a unique solution?

**Solution** Dividing by 3 to conform to the statement of the theorem, we identify $f(x, y)$ as $(x^2 - xy^3)/3$ and $\partial f/\partial y$ as $-xy^2$. Both of these functions are continuous in any rectangle containing the point $(1, 6)$, so the hypotheses of Theorem 1 are satisfied. It then follows from the theorem that the initial value problem (11) has a unique solution in an interval about $x = 1$ of the form $(1 - \delta, 1 + \delta)$, where $\delta$ is some positive number. ◆

**Example 9** For the initial value problem

$$(12) \qquad \frac{dy}{dx} = 3y^{2/3} , \qquad y(2) = 0 ,$$

does Theorem 1 imply the existence of a unique solution?

**Solution** Here $f(x, y) = 3y^{2/3}$ and $\partial f/\partial y = 2y^{-1/3}$. Unfortunately $\partial f/\partial y$ is not continuous or even defined when $y = 0$. Consequently, there is no rectangle containing $(2, 0)$ in which both $f$ and $\partial f/\partial y$ are continuous. Because the hypotheses of Theorem 1 do not hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution. We refer you to Problem 29 and Project G of Chapter 2 for the details. ◆

In Example 9 suppose the initial condition is changed to $y(2) = 1$. Then, since $f$ and $\partial f/\partial y$ are continuous in any rectangle that contains the point $(2, 1)$ but does not intersect the $x$-axis—say, $R = \{(x, y): 0 < x < 10, 0 < y < 5\}$—it follows from Theorem 1 that this *new* initial value problem has a unique solution in some interval about $x = 2$.

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## 1.2 EXERCISES

**1. (a)** Show that $\phi(x) = x^2$ is an explicit solution to

$$x\frac{dy}{dx} = 2y$$

on the interval $(-\infty, \infty)$.

**(b)** Show that $\phi(x) = e^x - x$ is an explicit solution to

$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$

on the interval $(-\infty, \infty)$.

**(c)** Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2 d^2y/dx^2 = 2y$ on the interval $(0, \infty)$.

**2. (a)** Show that $y^2 + x - 3 = 0$ is an implicit solution to $dy/dx = -1/(2y)$ on the interval $(-\infty, 3)$.

**(b)** Show that $xy^3 - xy^3 \sin x = 1$ is an implicit solution to

$$\frac{dy}{dx} = \frac{(x \cos x + \sin x - 1)y}{3(x - x \sin x)}$$

on the interval $(0, \pi/2)$.

*In Problems 3–8, determine whether the given function is a solution to the given differential equation.*

**3.** $y = \sin x + x^2 , \qquad \dfrac{d^2y}{dx^2} + y = x^2 + 2$

**4.** $x = 2 \cos t - 3 \sin t , \qquad x'' + x = 0$

**5.** $\theta = 2e^{3t} - e^{2t} , \qquad \dfrac{d^2\theta}{dt^2} - \theta\dfrac{d\theta}{dt} + 3\theta = -2e^{2t}$

6. $x = \cos 2t$, $\quad \dfrac{dx}{dt} + tx = \sin 2t$

7. $y = e^{2x} - 3e^{-x}$, $\quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

8. $y = 3\sin 2x + e^{-x}$, $\quad y'' + 4y = 5e^{-x}$

*In Problems 9–13, determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.*

9. $x^2 + y^2 = 4$, $\quad \dfrac{dy}{dx} = \dfrac{x}{y}$

10. $y - \ln y = x^2 + 1$, $\quad \dfrac{dy}{dx} = \dfrac{2xy}{y-1}$

11. $e^{xy} + y = x - 1$, $\quad \dfrac{dy}{dx} = \dfrac{e^{-xy} - y}{e^{-xy} + x}$

12. $x^2 - \sin(x + y) = 1$, $\quad \dfrac{dy}{dx} = 2x\sec(x + y) - 1$

13. $\sin y + xy - x^3 = 2$,
$$y'' = \frac{6xy' + (y')^3 \sin y - 2(y')^2}{3x^2 - y}$$

14. Show that $\phi(x) = c_1 \sin x + c_2 \cos x$ is a solution to $d^2y/dx^2 + y = 0$ for any choice of the constants $c_1$ and $c_2$. Thus, $c_1 \sin x + c_2 \cos x$ is a two-parameter family of solutions to the differential equation.

15. Verify that $\phi(x) = 2/(1 - ce^x)$, where $c$ is an arbitrary constant, is a one-parameter family of solutions to
$$\frac{dy}{dx} = \frac{y(y-2)}{2}.$$
Graph the solution curves corresponding to $c = 0$, $\pm 1$, $\pm 2$ using the same coordinate axes.

16. Verify that $x^2 + cy^2 = 1$, where $c$ is an arbitrary nonzero constant, is a one-parameter family of implicit solutions to
$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$
and graph several of the solution curves using the same coordinate axes.

17. Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $dy/dx - 3y = -3$ for any choice of the constant $C$. Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

18. Let $c > 0$. Show that the function $\phi(x) = (c^2 - x^2)^{-1}$ is a solution to the initial value problem $dy/dx = 2xy^2$, $y(0) = 1/c^2$, on the interval $-c < x < c$. Note that this solution becomes unbounded as $x$ approaches $\pm c$. Thus, the solution exists on the interval $(-\delta, \delta)$ with $\delta = c$, but not for larger $\delta$. This illustrates that in Theorem 1 the existence interval can be quite small (if $c$ is small) or quite large (if $c$ is large). Notice also that there is no clue from the equation $dy/dx = 2xy^2$ itself, or from the initial value, that the solution will "blow up" at $x = \pm c$.

19. Show that the equation $(dy/dx)^2 + y^2 + 4 = 0$ has no (real-valued) solution.

20. Determine for which values of $m$ the function $\phi(x) = e^{mx}$ is a solution to the given equation.

(a) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = 0$

(b) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$

21. Determine for which values of $m$ the function $\phi(x) = x^m$ is a solution to the given equation.

(a) $3x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} - 3y = 0$

(b) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 5y = 0$

22. Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear equation
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$
for any choice of the constants $c_1$ and $c_2$. Determine $c_1$ and $c_2$ so that each of the following initial conditions is satisfied.

(a) $y(0) = 2$, $\quad y'(0) = 1$

(b) $y(1) = 1$, $\quad y'(1) = 0$

*In Problems 23–28, determine whether Theorem 1 implies that the given initial value problem has a unique solution.*

23. $\dfrac{dy}{dx} = y^4 - x^4$, $\quad y(0) = 7$

24. $\dfrac{dy}{dt} - ty = \sin^2 t$, $\quad y(\pi) = 5$

25. $3x\dfrac{dx}{dt} + 4t = 0$, $\quad x(2) = -\pi$

26. $\dfrac{dx}{dt} + \cos x = \sin t$, $\quad x(\pi) = 0$

27. $y\dfrac{dy}{dx} = x$, $\quad y(1) = 0$

28. $\dfrac{dy}{dx} = 3x - \sqrt[3]{y-1}$, $\quad y(2) = 1$

29. (a) For the initial value problem (12) of Example 9, show that $\phi_1(x) \equiv 0$ and $\phi_2(x) = (x-2)^3$ are solutions. Hence, this initial value problem has multiple solutions. (See also Project G in Chapter 2.)

(b) Does the initial value problem $y' = 3y^{2/3}$, $y(0) = 10^{-7}$, have a unique solution in a neighborhood of $x = 0$?