P1

7.5: Solving IVPs

Ex: $y'' - 2y' + 5y = -8e^{-t}$ Partial Fraction

$y(0) = 2, \quad y'(0) = 12$

$y(t) = 3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t}$

Ex: $y'' + 4y' - 5y = \cancel{t}e^{t}$

$y(0) = 1, \quad y'(0) = 0$

$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}$$

$$Y(s) = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

$A = \frac{35}{216}, \quad B = \frac{181}{216}, \quad C = -\frac{1}{36}, \quad D = \frac{1}{6}$

$$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{31}te^{t} + \frac{1}{12}t^2e^{t}$$

Non-zero Initial Time.

Ex: $w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}$

$$w(\pi) = 2, \quad w'(\pi) = 12$$

Initial time $t_0 = \pi$

Transform to $t_0 = 0$ by letting

$$x(t) = w(t_0 + t) = w(\pi + t)$$

$$x(0) = 2,$$

$$x'(0) = 12$$

$$x'(t) = w'(\pi + t), \quad x'' = w''(\pi + t)$$

$$x'' - 2x' + 5y = -8e^{\pi - (\pi + t)}$$

$$= -8e^{-t}$$

$$x(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}$$

Returning to w:

$$w(t) = x(t - \pi)$$

$$w(t) = 3e^{t-\pi}\cos(2t) + 4e^{t-\pi}\sin 2t - e^{-(\pi - t)}$$

§ 7-5 P3

Ex: $y'' + 2ty' - 4y = 1,$

$y(0) = y'(0) = 0$

$Y = \mathcal{L}(y)$

$\mathcal{L}(y'') = s^2 \cancel{Y''} - sy(0) - y'(0) = s^2 Y(s)$

$\mathcal{L}(ty') = (-1)\frac{d}{ds}\mathcal{L}(y') = (-1)[sY(s) - y(0))]$

$= -sY'(s) - Y(s)$

$\mathcal{L}(1) = \frac{1}{s}$

$\Longrightarrow$ $\cancel{-sY'(s) + s^2 Y(s)}$

$s^2 Y(s) - 2sY'(s) - 2Y(s) - 4Y = \frac{1}{s}$

$s^2 Y(s) - 6Y(s) - 2sY'(s) = \frac{1}{s}$

$\cancel{Y'(s) + 6}$

$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right) Y = \frac{-1}{2s^2}$

Linear 1st-order eq. : Integrating factor

$\mu = e^{\int\left(\frac{3}{s} - \frac{s}{2}\right)ds}$

$\cancel{3\ln s\, e} = s^3 e^{-\frac{s^2}{4}}$

$s^3 e^{-\frac{s^2}{4}}\left[Y' + \left(\frac{3}{s} - \frac{s}{2}\right)\right] = -\frac{1}{2} s e^{-\frac{s^2}{4}}$

P4

$$\left[s^3 e^{-\frac{s^2}{4}} Y(s) \right]' = -\frac{1}{2} s e^{-\frac{s^2}{4}}$$

$$s^3 e^{-\frac{s^2}{4}} Y(s) = -\int \frac{1}{2} s e^{-\frac{s^2}{4}} ds$$

$$= e^{-\frac{s^2}{4}} + C$$

$$Y(s) = \frac{1}{s^3} + C \frac{e^{\frac{s^2}{4}}}{s^3}$$

Using the fact : $Y(s) \to 0$ as $s \to \infty$

to rule out $C = 0$

$$Y(s) = \frac{1}{s^3}$$

$$y(t) = \frac{t^2}{2}$$

HW §7.5

1, # 9, # 13, # 35, # 37

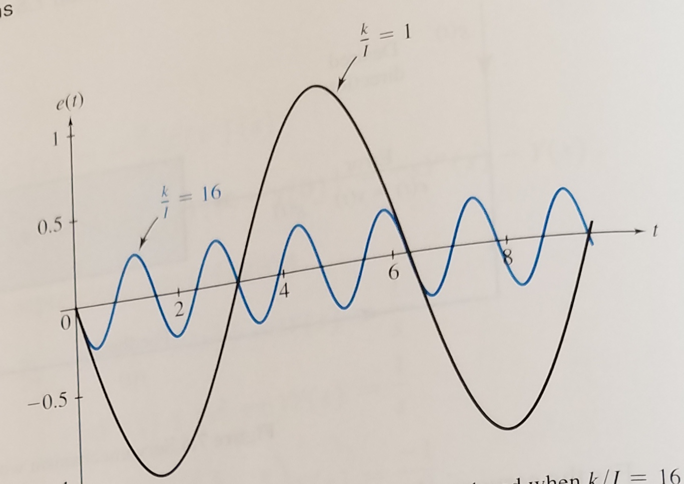

Figure 7.7 Error for automatic pilot when $k/I = 1$ and when $k/I = 16$

error small by making k large relative to I, but then the term $\sqrt{k/I}$ becomes large, causing the error to oscillate more rapidly. (See Figure 7.7.) As with vibrations, the oscillations or oversteering can be controlled by introducing a damping torque proportional to $e'(t)$ but opposite in sign (see Problem 40).

7.5 EXERCISES

In Problems 1–14, solve the given initial value problem using the method of Laplace transforms.

1. $y'' - 2y' + 5y = 0$; $y(0) = 2$, $y'(0) = 4$
2. $y'' - y' - 2y = 0$; $y(0) = -2$, $y'(0) = 5$
3. $y'' + 6y' + 9y = 0$; $y(0) = -1$, $y'(0) = 6$
4. $y'' + 6y' + 5y = 12e^{t}$; $y(0) = -1$, $y'(0) = 7$
5. $w'' + w = t^2 + 2$; $w(0) = 1$, $w'(0) = -1$
6. $y'' - 4y' + 5y = 4e^{3t}$; $y(0) = 2$, $y'(0) = 7$
7. $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5$, $y'(0) = -4$
8. $y'' + 4y = 4t^2 - 4t + 10$; $y(0) = 0$, $y'(0) = 3$
9. $z'' + 5z' - 6z = 21e^{t-1}$; $z(1) = -1$, $z'(1) = 9$
10. $y'' - 4y = 4t - 8e^{-2t}$; $y(0) = 0$, $y'(0) = 5$
11. $y'' - y = t - 2$; $y(2) = 3$, $y'(2) = 0$
12. $w'' - 2w' + w = 6t - 2$; $w(-1) = 3$; $w'(-1) = 7$
13. $y'' - y' - 2y = -8\cos t - 2\sin t$; $y(\pi/2) = 1$, $y'(\pi/2) = 0$
14. $y'' + y = t$; $y(\pi) = 0$, $y'(\pi) = 0$

In Problems 15–24, solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.

15. $y'' - 3y' + 2y = \cos t$; $y(0) = 0$, $y'(0) = -1$
16. $y'' + 6y = t^2 - 1$; $y(0) = 0$, $y'(0) = -1$
17. $y'' + y' - y = t^3$; $y(0) = 1$, $y'(0) = 0$
18. $y'' - 2y' - y = e^{2t} - e^{t}$; $y(0) = 1$, $y'(0) = 3$
19. $y'' + 5y' - y = e^{t} - 1$; $y(0) = 1$, $y'(0) = 1$
20. $y'' + 3y = t^3$; $y(0) = 0$, $y'(0) = 0$
21. $y'' - 2y' + y = \cos t - \sin t$; $y(0) = 1$, $y'(0) = 3$

22. $y'' - 6y' + 5y = te^t$; $y(0) = 2$, $y'(0) = -1$

23. $y'' + 4y = g(t)$; $y(0) = -1$, $y'(0) = 0$,
where
$$g(t) = \begin{cases} t, & t < 2, \\ 5, & t > 2 \end{cases}$$

24. $y'' - y = g(t)$; $y(0) = 1$, $y'(0) = 2$,
where
$$g(t) = \begin{cases} 1, & t < 3, \\ t, & t > 3 \end{cases}$$

In Problems 25–28, solve the given third-order initial value problem for $y(t)$ using the method of Laplace transforms.

25. $y''' - y'' + y' - y = 0$;
$y(0) = 1$, $y'(0) = 1$, $y''(0) = 3$

26. $y''' + 4y'' + y' - 6y = -12$;
$y(0) = 1$, $y'(0) = 4$, $y''(0) = -2$

27. $y''' + 3y'' + 3y' + y = 0$;
$y(0) = -4$, $y'(0) = 4$, $y''(0) = -2$

28. $y''' + y'' + 3y' - 5y = 16e^{-t}$;
$y(0) = 0$, $y'(0) = 2$, $y''(0) = -4$

In Problems 29–32, use the method of Laplace transforms to find a general solution to the given differential equation by assuming $y(0) = a$ and $y'(0) = b$, where a and b are arbitrary constants.

29. $y'' - 4y' + 3y = 0$

30. $y'' + 6y' + 5y = t$

31. $y'' + 2y' + 2y = 5$

32. $y'' - 5y' + 6y = -6te^{2t}$

33. Use Theorem 6 in Section 7.3, page 364, to show that
$$\mathcal{L}\{t^2y'(t)\}(s) = sY''(s) + 2Y'(s),$$
where $Y(s) = \mathcal{L}\{y\}(s)$.

34. Use Theorem 6 in Section 7.3, page 364, to show that
$$\mathcal{L}\{t^2y''(t)\}(s) = s^2Y''(s) + 4sY'(s) + 2Y(s),$$
where $Y(s) = \mathcal{L}\{y\}(s)$.

In Problems 35–38, find solutions to the given initial value problem.

35. $y'' + 3ty' - 6y = 1$; $y(0) = 0$, $y'(0) = 0$

36. $ty'' - ty' + y = 2$; $y(0) = 2$, $y'(0) = -1$

37. $ty'' - 2y' + ty = 0$; $y(0) = 1$, $y'(0) = 0$
[*Hint:* $\mathcal{L}^{-1}\{1/(s^2+1)^2\}(t) = (\sin t - t\cos t)/2$.]

38. $y'' + ty' - y = 0$;
$y(0) = 0$, $y'(0) = 3$

39. Determine the error $e(t)$ for the automatic pilot in Example 5, page 381, if the shaft is initially at rest in the zero direction and the desired direction is $g(t) = a$, where a is a constant.

40. In Example 5 assume that in order to control oscillations, a component of torque proportional to $e'(t)$, but opposite in sign, is also fed back to the steering shaft. Show that equation (17) is now replaced by
$$Iy''(t) = -ke(t) - \mu e'(t),$$
where μ is a positive constant. Determine the error $e(t)$ for the automatic pilot with mild damping (i.e., $\mu < 2\sqrt{Ik}$) if the steering shaft is initially at rest in the zero direction and the desired direction is given by $g(t) = a$, where a is a constant.

41. In Problem 40 determine the error $e(t)$ when the desired direction is given by $g(t) = at$, where a is a constant.

7.6 Transforms of Discontinuous Functions

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities. As we saw in the mixing problem of Section 7.1, jump discontinuities occur naturally in any physical situation that involves switching. Finding the [illegible] of such functions is straightforward; however, we need some theory for [illegible] Oliver Heaviside introduced the following step