

7.3. Properties of L-transform.

Let $\bar{F}(s) = \int_0^\infty e^{-st} f(t) dt$

$$\frac{d}{ds} \bar{F}(s) = \int_0^\infty (-1) e^{-st} f(t) dt$$

$$= - \int_0^\infty e^{-st} (t f(t)) dt$$

$$\Rightarrow L(t f(t))(s) = -1 \frac{d}{ds} \bar{F}(s)$$

$$\frac{d^2}{ds^2} \bar{F}(s) = \int_0^\infty e^{-st} (t^2 \cancel{\frac{d}{ds} f(t)}) f(t) dt$$

$$\Rightarrow \cancel{\int_0^\infty t^2} L(t^2 f(t)) = (-1)^2 \frac{d^2 \bar{F}(s)}{ds^2}$$

$$L(t^n f(t)) = (-1)^n \frac{d^n \bar{F}(s)}{ds^n}$$

Table 7.2:

Ex: $L(t \sin t)$

Table 7.1:

check the solution

$$i \neq = \text{if } f$$

$$f(s) = \frac{1}{s} \Leftrightarrow s < 0$$

$$\frac{s}{1-s} = \frac{s}{1} - 1 = (s)(1-s)$$

$$\begin{matrix} 1 \\ + \\ \end{matrix} \quad \begin{matrix} 0 \\ \downarrow \\ \end{matrix}$$

$$\frac{s}{1} = (s)f - (s)h = f(s)$$

$$\frac{s}{1} = (h)f - (h)h$$

$$\frac{s}{1} = (h)f - = (h-h)f$$

$$(h)f = (h) \quad \begin{matrix} \text{if } h \neq 0 \\ \text{if } h = 0 \end{matrix}$$

$$\begin{cases} h \neq 0 \\ h = 0 \end{cases}$$

$$f = h - h$$

$$\text{Ex. } S_{11} \text{ II. II}$$

T.4. Inverse Laplace Transforms

(1)

$$= \left(\frac{L + s - S}{1 - S} \right) f$$

$$= \left(\frac{b + S}{S} \right) f$$

$$= \left(\frac{S}{S^2} \right) f$$

$$n^2 = \left(\frac{1 + S}{S} \right) f$$

$$e^{as} = \left(\frac{S + a - S}{S - a} \right) f$$

$$(1 + g)^{-1} = \left(\frac{S + a}{S} \right) f$$

$$e^{-as} = \left(\frac{1}{S - a} \right) f$$

$$1 = \left(\frac{1}{S} \right) f$$

$$1 = f^{-1}(F)$$

Def: A cut of f such that $f(f^{-1}(F)) = F$
 i.e. called the inverse of F , denoted

(2c)

7.4

1, 3, 7, 9, 21, 23, 25, 31

$$f(t) = 3e^{-3t} \cos(2t) + 4e^{-5t} \sin(2t) - e^{-t}$$

$$A=3, B=4, C=-1$$

$$= \frac{(s+1)^2 + 2^2}{A(s-1) + B(2)} + \frac{C}{s+1}$$

$$= \frac{(s^2 - 2s + 5)(s+1)}{2s^2 + 10s} = \frac{[(s-1)^2 + 4][s+1]}{2s^2 + 10s}$$

$$f(t) = \frac{(s^2 - 2s + 5)(s+1)}{2s^2 + 10s}$$

General Particular Function

$$\text{Ans: } f_1 = \frac{3s+2}{s^2 + 2s + 10} = 3e^{-s} \cos(3t) - \frac{1}{3} e^{-s} \sin(3t)$$

$$f_2 = \frac{s-6}{s^2 + 4s + 6} = -\frac{6s}{s^2 + 4s + 6} + \frac{s^2 + 9}{s^2 + 4s + 6}$$

$$f_2 = -C_1 F_1 + C_2 F_2$$

$$\text{Ans: } f_2 = C_1 (F_1 + F_2) = f_1(F_1) + f_1(F_2)$$

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and, hence, $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4),$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$0 = [A(-1) + 2B](1) + C(5),$$

$$0 = -A + 8 - 5,$$

$$A = 3.$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}.$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}\right\}(t)$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 2^2}\right\}(t)$$

$$+ 4\mathcal{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 2^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t)$$

$$= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}. \quad \blacklozenge$$

In Section 7.8, we discuss a different method (involving convolutions) for computing inverse transforms that does not require partial fraction decompositions. Moreover, the convolution method is convenient in the case of a rational function with a repeated quadratic factor in the denominator. Other helpful tools are described in Problems 33–36 and 38–43.

7.4 EXERCISES

In Problems 1–10, determine the inverse Laplace transform of the given function.

1. $\frac{6}{(s - 1)^4}$

2. $\frac{2}{s^2 + 4}$

3. $\frac{s + 1}{s^2 + 2s + 10}$

4. $\frac{4}{s^2 + 9}$

5. $\frac{1}{s^2 + 4s + 8}$

6. $\frac{3}{(2s + 5)^3}$

7. $\frac{2s + 16}{s^2 + 4s + 13}$

8. $\frac{1}{s^5}$

9. $\frac{3s - 15}{2s^2 - 4s + 10}$

10. $\frac{s - 1}{2s^2 + s + 6}$

In Problems 11–20, determine the partial fraction expansion for the given rational function.

11. $\frac{s^2 - 26s - 47}{(s - 1)(s + 2)(s + 5)}$

12. $\frac{-s - 7}{(s + 1)(s - 2)}$

13. $\frac{-2s^2 - 3s - 2}{s(s + 1)^2}$

14. $\frac{-8s^2 - 5s + 9}{(s + 1)(s^2 - 3s - 2)}$

15. $\frac{8s - 2s^2 - 14}{(s + 1)(s^2 - 2s + 5)}$

16. $\frac{-5s - 36}{(s + 2)(s^2 + 9)}$

17. $\frac{3s + 5}{s(s^2 + s - 6)}$

18. $\frac{3s^2 + 5s + 3}{s^4 + s^3}$

19. $\frac{1}{(s-3)(s^2+2s+2)}$ 20. $\frac{s}{(s-1)(s^2-1)}$

In Problems 21–30, determine $\mathcal{L}^{-1}\{F\}$.

21. $F(s) = \frac{6s^2 - 13s + 2}{s(s-1)(s-6)}$

22. $F(s) = \frac{s+11}{(s-1)(s+3)}$

23. $F(s) = \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)}$

24. $F(s) = \frac{7s^2 - 41s + 84}{(s-1)(s^2 - 4s + 13)}$

25. $F(s) = \frac{7s^2 + 23s + 30}{(s-2)(s^2 + 2s + 5)}$

26. $F(s) = \frac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}$

27. $s^2F(s) - 4F(s) = \frac{5}{s+1}$

28. $s^2F(s) + sF(s) - 6F(s) = \frac{s^2 + 4}{s^2 + s}$

29. $sF(s) + 2F(s) = \frac{10s^2 + 12s + 14}{s^2 - 2s + 2}$

30. $sF(s) - F(s) = \frac{2s + 5}{s^2 + 2s + 1}$

31. Determine the Laplace transform of each of the following functions:

(a) $f_1(t) = \begin{cases} 0, & t = 2, \\ t, & t \neq 2. \end{cases}$

(b) $f_2(t) = \begin{cases} 5, & t = 1, \\ 2, & t = 6, \\ t, & t \neq 1, 6. \end{cases}$

(c) $f_3(t) = t.$

Which of the preceding functions is the inverse Laplace transform of $1/s^2$?

32. Determine the Laplace transform of each of the following functions:

(a) $f_1(t) = \begin{cases} t, & t = 1, 2, 3, \dots, \\ e^t, & t \neq 1, 2, 3, \dots \end{cases}$

(b) $f_2(t) = \begin{cases} e^t, & t \neq 5, 8, \\ 6, & t = 5, \\ 0, & t = 8. \end{cases}$

(c) $f_3(t) = e^t.$

Which of the preceding functions is the inverse Laplace transform of $1/(s-1)$?

Theorem 6 in Section 7.3 on page 364 can be expressed in terms of the inverse Laplace transform as

$$\mathcal{L}^{-1}\left\{\frac{d^n F}{ds^n}\right\}(t) = (-t)^n f(t),$$

where $f = \mathcal{L}^{-1}\{F\}$. Use this equation in Problems 33–36 to compute $\mathcal{L}^{-1}\{F\}$.

33. $F(s) = \ln\left(\frac{s+2}{s-5}\right)$

34. $F(s) = \ln\left(\frac{s-4}{s-3}\right)$

35. $F(s) = \ln\left(\frac{s^2+9}{s^2+1}\right)$

36. $F(s) = \arctan(1/s)$

37. Prove Theorem 7, page 368, on the linearity of the inverse transform. [Hint: Show that the right-hand side of equation (3) is a continuous function on $[0, \infty)$ whose Laplace transform is $F_1(s) + F_2(s)$.]

38. Residue Computation. Let $P(s)/Q(s)$ be a rational function with $\deg P < \deg Q$ and suppose $s - r$ is a non-repeated linear factor of $Q(s)$. Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $s - r$ is

$$\frac{A}{s-r},$$

where A (called the **residue**) is given by the formula

$$A = \lim_{s \rightarrow r} \frac{(s-r)P(s)}{Q(s)} = \frac{P(r)}{Q'(r)}.$$

39. Use the residue computation formula derived in Problem 38 to determine quickly the partial fraction expansion for

$$F(s) = \frac{2s+1}{s(s-1)(s+2)}.$$

40. Heaviside's Expansion Formula.[†] Let $P(s)$ and $Q(s)$ be polynomials with the degree of $P(s)$ less than the degree of $Q(s)$. Let

$$Q(s) = (s-r_1)(s-r_2) \cdots (s-r_n),$$

where the r_i 's are distinct real numbers. Show that

$$\mathcal{L}^{-1}\left\{\frac{P}{Q}\right\}(t) = \sum_{i=1}^n \frac{P(r_i)}{Q'(r_i)} e^{r_i t}.$$

[†]Historical Footnote: This formula played an important role in the "operational solution" to ordinary differential equations developed by Oliver Heaviside in the 1890s.