Review of Linear Algebra

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2 \qquad (*)$$

Can be rewritten

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det A = a_{11} a_{22} - a_{12} a_{21}$$

$(*)$ has only only only one soln iff $\det A \neq 0$.

PO

§4.2 General Solutions of 2nd Linear Homogeneous Eqs

$$ay'' + by' + cy = 0 \qquad (*)$$

Recall: (i) If $a\lambda^2 + b\lambda + c = 0$ (C.E) has real solns λ_1 and λ_2, then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are solutions.

(ii) $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are linearly independent if $\lambda_1 \neq \lambda_2$.

(iii) If $\lambda_1 \neq \lambda_2$ then $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ is a general solution (CE)

(iv) If λ_1 is the only real soln. of (C.E.) then $y_1 = e^{\lambda_1 t}$ and $y_2 = te^{\lambda_1 t}$ are solutions, y_1 and y_2 are ~~$c_1 y_1 + c_2 y_2$~~ linearly independent. so $c_1 y_1 + c_2 y_2$ is a general soluti

Def: Two functions $y_1(t)$ and $y_2(t)$ are linearly dependent on an interval I if there exist c_1 and c_2 such that either $c_1 \neq 0$ or $c_2 \neq 0$ and $c_1 y_1(t) + c_2 y_2(t) = 0$ for all t in the interval.

Theorem: y_1 and y_2 are linearly dependent $\Leftrightarrow$ one is a constant multiple of another.

Special Criterion of linear independence of two solutions $y_1(t)$ and $y_2(t)$ of $(*)$:

Observation: If y_1 and y_2 solves $(*)$, then so is $c_1 y_1 + c_2 y_2$ for any constants c_1 and c_2.

Theorem: $(*)$ subject to an initial condition

(IV) $\quad y(t_0) = y_0; \quad y'(t_0) = y_1$

has one and only one soln.

Observation 2: The IVP [(*) + (IV)] with $y_0 = y_1 = 0$ has the soln. $y \equiv 0$

Th: Two solutions y_1 and y_2 are linearly dependent iff

$$\underbrace{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}_{\text{Wronskian.}} = 0$$

Pf:

y_1 and y_2 are linearly dependent

$\Rightarrow \exists\ c_1, c_2\ (c_1^2 + c_2^2 \neq 0)$ s.t.

$$c_1 y_1 + c_2 y_2 = 0$$

$\Rightarrow$

$$c_1 y_1'(t) + c_2 y_2'(t) = 0$$

$\Rightarrow$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = 0$$

$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = 0$$

$\Rightarrow$

$$A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \quad A = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$$

$$\det A = 0$$

Conversely, if $\det A = 0$, then $\exists$ c_1 and c_2 s.t $c_1^2 + c_2^2 \neq 0$ and

$$A\binom{c_1}{c_2} = 0$$

$\Rightarrow$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = 0$$

$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = 0$$

Let

$$y(t) = c_1 y_1 + c_2 y_2$$

y is a soln. + $y(t_0) = y'(t_0) = 0$

so $y = 0 \Rightarrow y_1$ and y_2 are linearly independent.

<u>Ex:</u> $\lambda_1 \neq \lambda_2$ are soln of (CE)

Ex $\lambda_1 = \lambda_2$ is the ~~soln~~ only soln. of (CE)

<u>Th:</u> If y_1 and y_2 are two linearly independent solutions, then $c_1 y_1 + c_2 y_2$ is a general soln.

HW: P164, #34.

4.3: Complex Roots

What happens if

$$a\lambda^2 + b\lambda + c$$

have complex roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = b^2 - 4ac < 0.$$

Write $\lambda_1 = \alpha + i\beta$

Then $\lambda_2 = \alpha - i\beta$.

Observation: $e^{\lambda_1 t}$ is a soln. of $(*)$ but $e^{\lambda_1 t}$ is not a real-valued function

$$e^{\lambda_1 t} = e^{\alpha t}\cos(\beta t) + i e^{\alpha t}\sin(\beta t).$$

Observation 2. If $y = y_1 + i y_2$ is a complex soln. of (*) then y_1 and y_2 are real solutions of (*)

Corollary. If $\lambda = \alpha + i\beta$ is a zero of the (CE)

$$a\lambda^2 + b\lambda + c = 0$$

then

$$e^{\alpha t}\cos(\beta t) \quad \text{and} \quad e^{\alpha t}\sin(\beta t)$$

are real solutions and

$$c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$$

is a general soln.

Ex. $y'' + y = 0.$

g. s. is $c_1 \sin t + c_2 \cos t.$

Ex. $\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = 0, \ y'(0) = 2 \end{cases}$

$y = 2e^{-t}\sin t$ is the soln.

Ex: $y'' + 2y' + 4y = 0$

G. S. $y = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)$

HW. #1, #21, #35

Next Lecture: §4.4:
Nonhomogeneous Equations

(P6)

4.2 EXERCISES

In Problems 1–12, find a general solution to the given differential equation.

1. $2y'' + 7y' - 4y = 0$
2. $y'' + 6y' + 9y = 0$
3. $y'' + 5y' + 6y = 0$
4. $y'' - y' - 2y = 0$
5. $y'' + 8y' + 16y = 0$
6. $y'' - 5y' + 6y = 0$
7. $6y'' + y' - 2y = 0$
8. $z'' + z' - z = 0$
9. $4y'' - 4y' + y = 0$
10. $y'' - y' - 11y = 0$
11. $4w'' + 20w' + 25w = 0$
12. $3y'' + 11y' - 7y = 0$

In Problems 13–20, solve the given initial value problem.

13. $y'' + 2y' - 8y = 0;\quad y(0) = 3,\quad y'(0) = -12$
14. $y'' + y' = 0;\quad y(0) = 2,\quad y'(0) = 1$
15. $y'' - 4y' + 3y = 0;\quad y(0) = 1,\quad y'(0) = 1/3$
16. $y'' - 4y' - 5y = 0;\quad y(-1) = 3,\quad y'(-1) = 9$
17. $y'' - 6y' + 9y = 0;\quad y(0) = 2,\quad y'(0) = 25/3$
18. $z'' - 2z' - 2z = 0;\quad z(0) = 0,\quad z'(0) = 3$
19. $y'' + 2y' + y = 0;\quad y(0) = 1,\quad y'(0) = -3$
20. $y'' - 4y' + 4y = 0;\quad y(1) = 1,\quad y'(1) = 1$

21. **First-Order Constant-Coefficient Equations.**
    (a) Substituting $y = e^{rt}$, find the auxiliary equation for the first-order linear equation
    $$ay' + by = 0,$$
    where a and b are constants with $a \neq 0$.
    (b) Use the result of part (a) to find the general solution.

In Problems 22–25, use the method described in Problem 21 to find a general solution to the given equation.

22. $3y' - 7y = 0$
23. $5y' + 4y = 0$
24. $3z' + 11z = 0$
25. $6w' - 13w = 0$

26. **Boundary Value Problems.** When the values of a solution to a differential equation are specified at two different points, these conditions are called **boundary conditions.** (In contrast, initial conditions specify the values of a function and its derivative at the same point.) The purpose of this exercise is to show that for boundary value problems there is no existence–uniqueness theorem that is analogous to Theorem 1. Given that every solution to
    $$y'' + y = 0 \tag{17}$$
    is of the form
    $$y(t) = c_1 \cos t + c_2 \sin t,$$
    where c_1 and c_2 are arbitrary constants, show that
    (a) There is a unique solution to (17) that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.
    (b) There is no solution to (17) that satisfies $y(0)$ and $y(\pi) = 0$.
    (c) There are infinitely many solutions to (17) satisfy $y(0) = 2$ and $y(\pi) = -2$.

In Problems 27–32, use Definition 1 to determine whether functions y_1 and y_2 are linearly dependent on the interval

27. $y_1(t) = \cos t \sin t,\quad y_2(t) = \sin 2t$
28. $y_1(t) = e^{3t},\quad y_2(t) = e^{-4t}$
29. $y_1(t) = te^{2t},\quad y_2(t) = e^{2t}$
30. $y_1(t) = t^2 \cos(\ln t),\quad y_2(t) = t^2 \sin(\ln t)$
31. $y_1(t) = \tan^2 t - \sec^2 t,\quad y_2(t) = 3$
32. $y_1(t) = 0,\quad y_2(t) = e^t$

33. Explain why two functions are linearly dependent interval I if and only if there exist constants c_1 and both zero, such that
    $$c_1 y_1(t) + c_2 y_2(t) = 0 \quad \text{for all } t \text{ in } I.$$

34. **Wronskian.** For any two differentiable functions and y_2, the function
    $$W[y_1, y_2](t) = y_1(t) y_2'(t) - y_1'(t) y_2(t) \tag{18}$$
    is called the *Wronskian*[†] of y_1 and y_2. This function a crucial role in the proof of Theorem 2.
    (a) Show that $W[y_1, y_2]$ can be conveniently expre as the 2×2 determinant
    $$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$
    (b) Let $y_1(t), y_2(t)$ be a pair of solutions to the mogeneous equation $ay'' + by' + cy = 0$ ($a \neq 0$) on an open interval I. Prove that $y_1(t)$ $y_2(t)$ are linearly independent on I if and only their Wronskian is never zero on I. [*Hint:* This is a reformulation of Lemma 1.]
    (c) Show that if $y_1(t)$ and $y_2(t)$ are any two differentiable functions that are linearly dependent on I, their Wronskian is identically zero on I.

35. **Linear Dependence of Three Functions.** Three functions $y_1(t), y_2(t)$, and $y_3(t)$ are said to be linearly dependent on an interval I if, on I, at least one of these functions is a linear combination of the remaining two [e.g., if $y_1(t) = c_1 y_2(t) + c_2 y_3(t)$]. Equivalently (compare Problem 33), y_1, y_2, and y_3 are linearly dependent on I if there exist constants C_1, C_2, and C_3, *not all zero*, such that
    $$C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t) = 0 \quad \text{for all } t \text{ in } I.$$
    Otherwise, we say that these functions are linearly independent on I.

[†]*Historical Footnote:* The Wronskian was named after the Polish mathematician H. Wronski (1778–1863).

For each
given three f
independent
(a) $y_1(t) =$
(b) $y_1(t) =$
(c) $y_1(t) =$
(d) $y_1(t) =$

36. Using the d
and r_3 are
$e^{r_1 t}, e^{r_2 t}$, and
[*Hint:* Assum
for all t. Di
and then di
are linearly

In Problems 37
(see Problem 35
and write a gen
tion of these.

37. $y''' + y'' -$
38. $y''' - 6y'' -$
39. $z''' + 2z'' -$
40. $y''' - 7y' +$
41. $y''' + 3y'' -$

42. (True or F
on $(-\infty,$
$(-\infty, \infty)$
on $(-\infty,$

43. Solve the
$y''' -$
$y'(0$

4.3 A

For each of the following, determine whether the given three functions are linearly dependent or linearly independent on $(-\infty, \infty)$:

(a) $y_1(t) = 1, \quad y_2(t) = t, \quad y_3(t) = t^2.$

(b) $y_1(t) = -3, \quad y_2(t) = 5\sin^2 t, \quad y_3(t) = \cos^2 t.$

(c) $y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = t^2e^t.$

(d) $y_1(t) = e^t, \quad y_2(t) = e^{-t}, \quad y_3(t) = \cosh t.$

36. Using the definition in Problem 35, prove that if $r_1, r_2,$ and r_3 are distinct real numbers, then the functions $e^{r_1 t}, e^{r_2 t}$, and $e^{r_3 t}$ are linearly independent on $(-\infty, \infty)$. [*Hint:* Assume to the contrary that, say, $e^{r_1 t} = c_1 e^{r_2 t} + c_2 e^{r_3 t}$ for all t. Divide by $e^{r_2 t}$ to get $e^{(r_1 - r_2)t} = c_1 + c_2 e^{(r_3 - r_2)t}$ and then differentiate to deduce that $e^{(r_1 - r_2)t}$ and $e^{(r_3 - r_2)t}$ are linearly dependent, which is a contradiction. (Why?)]

In Problems 37–41, find three linearly independent solutions (see Problem 35) of the given third-order differential equation and write a general solution as an arbitrary linear combination of these.

37. $y''' + y'' - 6y' + 4y = 0$

38. $y''' - 6y'' - y' + 6y = 0$

39. $z''' + 2z'' - 4z' - 8z = 0$

40. $y''' - 7y'' + 7y' + 15y = 0$

41. $y''' + 3y'' - 4y' - 12y = 0$

42. (True or False): If f_1, f_2, f_3 are three functions defined on $(-\infty, \infty)$ that are *pairwise* linearly independent on $(-\infty, \infty)$, then f_1, f_2, f_3 form a linearly independent set on $(-\infty, \infty)$. Justify your answer.

43. Solve the initial value problem:

$$y''' - y' = 0; \quad y(0) = 2,$$
$$y'(0) = 3, \quad y''(0) = -1.$$

44. Solve the initial value problem:

$$y''' - 2y'' - y' + 2y = 0;$$
$$y(0) = 2, \quad y'(0) = 3, \quad y''(0) = 5.$$

45. By using Newton's method or some other numerical procedure to approximate the roots of the auxiliary equation, find general solutions to the following equations:

(a) $3y''' + 18y'' + 13y' - 19y = 0.$

(b) $y^{\text{iv}} - 5y'' + 5y = 0.$

(c) $y^{\text{v}} - 3y^{\text{iv}} - 5y''' + 15y'' + 4y' - 12y = 0.$

46. One way to define hyperbolic functions is by means of differential equations. Consider the equation $y'' - y = 0$. The *hyperbolic cosine*, $\cosh t$, is defined as the solution of this equation subject to the initial values: $y(0) = 1$ and $y'(0) = 0$. The *hyperbolic sine*, $\sinh t$, is defined as the solution of this equation subject to the initial values: $y(0) = 0$ and $y'(0) = 1$.

(a) Solve these initial value problems to derive explicit formulas for $\cosh t$ and $\sinh t$. Also show that $\frac{d}{dt}\cosh t = \sinh t$ and $\frac{d}{dt}\sinh t = \cosh t$.

(b) Prove that a general solution of the equation $y'' - y = 0$ is given by $y = c_1 \cosh t + c_2 \sinh t$.

(c) Suppose a, b, and c are given constants for which $ar^2 + br + c = 0$ has two distinct real roots. If the two roots are expressed in the form $\alpha - \beta$ and $\alpha + \beta$, show that a general solution of the equation $ay'' + by' + cy = 0$ is $y = c_1 e^{\alpha t}\cosh(\beta t) + c_2 e^{\alpha t}\sinh(\beta t)$.

(d) Use the result of part (c) to solve the initial value problem: $y'' + y' - 6y = 0, y(0) = 2, y'(0) = -17/2$.

4.3 Auxiliary Equations with Complex Roots

The *simple harmonic equation* $y'' + y = 0$, so called because of its relation to the fundamental vibration of a musical tone, has as solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$. Notice, however, that the auxiliary equation associated with the harmonic equation is $r^2 + 1 = 0$, which has imaginary roots $r = \pm i$, where i denotes $\sqrt{-1}$.[†] In the previous section, we expressed the solutions to a linear second-order equation with constant coefficients in terms of exponential functions. It would appear, then, that one might be able to attribute a meaning to the forms e^{it} and e^{-it} and that these "functions" should be related to $\cos t$ and $\sin t$. This matchup is accomplished by Euler's formula, which is discussed in this section.

When $b^2 - 4ac < 0$, the roots of the auxiliary equation

$$\text{(1)} \qquad ar^2 + br + c = 0$$

[†]Electrical engineers frequently use the symbol j to denote $\sqrt{-1}$.

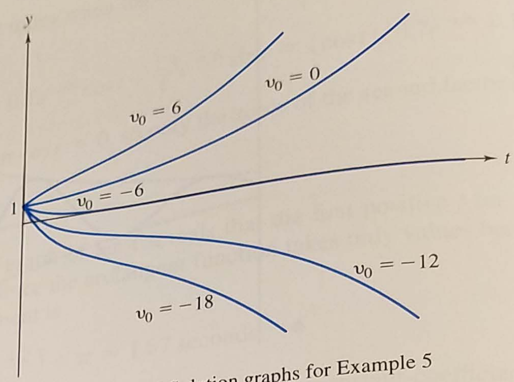

Figure 4.8 Solution graphs for Example 5

r_1, r_2 of the auxiliary equation (1) are, in general, also complex but not necessarily conju of each other. When $r_1 \neq r_2$, a general solution to equation (2) still has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} ,$$

but c_1 and c_2 are now arbitrary complex-valued constants, and we have to resort to the cl calculations of Example 1.

We also remark that a complex differential equation can be regarded as a system of real differential equations since we can always work separately with its real and imag parts. Systems are discussed in Chapters 5 and 9.

4.3 EXERCISES

In Problems 1–8, the auxiliary equation for the given differential equation has complex roots. Find a general solution.

1. $y'' + 9y = 0$
2. $y'' + y = 0$
3. $z'' - 6z' + 10z = 0$
4. $y'' - 10y' + 26y = 0$
5. $w'' + 4w' + 6w = 0$
6. $y'' - 4y' + 7y = 0$
7. $4y'' + 4y' + 6y = 0$
8. $4y'' - 4y' + 26y = 0$

In Problems 9–20, find a general solution.

9. $y'' - 8y' + 7y = 0$
10. $y'' + 4y' + 8y = 0$
11. $z'' + 10z' + 25z = 0$
12. $u'' + 7u = 0$
13. $y'' - 2y' + 26y = 0$
14. $y'' + 2y' + 5y = 0$
15. $y'' - 3y' - 11y = 0$
16. $y'' + 10y' + 41y = 0$
17. $y'' - y' + 7y = 0$
18. $2y'' + 13y' - 7y = 0$
19. $y''' + y'' + 3y' - 5y = 0$
20. $y''' - y'' + 2y = 0$

In Problems 21–27, solve the given initial value problem.

21. $y'' + 2y' + 2y = 0 ; \quad y(0) = 2 , \quad y'(0) = 1$
22. $y'' + 2y' + 17y = 0 ; \quad y(0) = 1 , \quad y'(0) = -1$
23. $w'' - 4w' + 2w = 0 ; \quad w(0) = 0 , \quad w'(0) = 1$
24. $y'' + 9y = 0 ; \quad y(0) = 1 , \quad y'(0) = 1$
25. $y'' - 2y' + 2y = 0 ; \quad y(\pi) = e^{\pi} , \quad y'(\pi) = 0$
26. $y'' - 2y' + y = 0 ; \quad y(0) = 1 , \quad y'(0) = -2$
27. $y''' - 4y'' + 7y' - 6y = 0 ; \quad y(0) = 1 , \quad y'(0) =$ $y''(0) = 0$

28. To see the effect of changing the parameter b in the value problem

$$y'' + by' + 4y = 0 ; \quad y(0) = 1 , \quad y'(0) =$$

solve the problem for $b = 5$, 4, and 2 and sketch solutions.

29. Find a general solution to the following higher-equations.
 (a) $y''' - y'' + y' + 3y = 0$
 (b) $y''' + 2y'' + 5y' - 26y = 0$
 (c) $y^{\text{iv}} + 13y'' + 36y = 0$

30. Using the representation for $e^{(\alpha + i\beta)t}$ in (6), verify the ferentiation formula (7).

31. Using the mass–spring analogy, predict the behavior as $t \to +\infty$ of the solution to the given initial value problem. Then confirm your prediction by actually solving the problem.
(a) $y'' + 16y = 0; \quad y(0) = 2, \quad y'(0) = 0$
(b) $y'' + 100y' + y = 0; \quad y(0) = 1, \quad y'(0) = 0$
(c) $y'' - 6y' + 8y = 0; \quad y(0) = 1, \quad y'(0) = 0$
(d) $y'' + 2y' - 3y = 0; \quad y(0) = -2, \quad y'(0) = 0$
(e) $y'' - y' - 6y = 0; \quad y(0) = 1, \quad y'(0) = 1$

32. **Vibrating Spring without Damping.** A vibrating spring without damping can be modeled by the initial value problem (11) in Example 3 by taking $b = 0$.
(a) If $m = 10$ kg, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec, find the equation of motion for this undamped vibrating spring.
(b) After how many seconds will the mass in part (a) first cross the equilibrium point?
(c) When the equation of motion is of the form displayed in (9), the motion is said to be **oscillatory** with **frequency** $\beta/2\pi$. Find the frequency of oscillation for the spring system of part (a).

33. **Vibrating Spring with Damping.** Using the model for a vibrating spring with damping discussed in Example 3:
(a) Find the equation of motion for the vibrating spring with damping if $m = 10$ kg, $b = 60$ kg/sec, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec.
(b) After how many seconds will the mass in part (a) first cross the equilibrium point?
(c) Find the frequency of oscillation for the spring system of part (a). [*Hint:* See the definition of frequency given in Problem 32(c).]
(d) Compare the results of Problems 32 and 33 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution?

34. ***RLC* Series Circuit.** In the study of an electrical circuit consisting of a resistor, capacitor, inductor, and an electromotive force (see Figure 4.9), we are led to an initial value problem of the form

$$L\frac{dI}{dt} + RI + \frac{q}{C} = E(t)\,; \tag{20}$$
$$q(0) = q_0\,,$$
$$I(0) = I_0\,,$$

where L is the inductance in henrys, R is the resistance in ohms, C is the capacitance in farads, $E(t)$ is the electromotive force in volts, $q(t)$ is the charge in coulombs on the capacitor at time t, and $I = dq/dt$ is the current in amperes. Find the current at time t if the charge on the capacitor is initially zero, the initial current is zero, $L = 10$ H, $R = 20\ \Omega$, $C = (6260)^{-1}$ F, and $E(t) = 100$ V. [*Hint:* Differentiate both sides of the differential equation in (20) to obtain a homogeneous linear second-order equation for $I(t)$. Then use (20) to determine dI/dt at $t = 0$.]

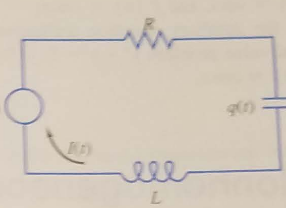

Figure 4.9 *RLC* series circuit

35. **Swinging Door.** The motion of a swinging door with an adjustment screw that controls the amount of friction on the hinges is governed by the initial value problem

$$I\theta'' + b\theta' + k\theta = 0\,; \quad \theta(0) = \theta_0\,, \quad \theta'(0) = v_0\,,$$

where θ is the angle that the door is open, I is the moment of inertia of the door about its hinges, $b > 0$ is a damping constant that varies with the amount of friction on the door, $k > 0$ is the spring constant associated with the swinging door, θ_0 is the initial angle that the door is opened, and v_0 is the initial angular velocity imparted to the door (see Figure 4.10). If I and k are fixed, determine for which values of b the door will *not* continually swing back and forth when closing.

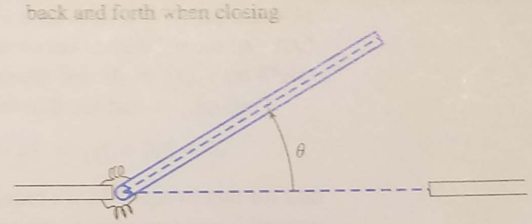

Figure 4.10 Top view of swinging door

36. Although the real general solution form (9) is convenient, it is also possible to use the form

$$d_1 e^{(\alpha + i\beta)t} + d_2 e^{(\alpha - i\beta)t} \tag{21}$$

to solve initial value problems, as illustrated in Example 1. The coefficients d_1 and d_2 are complex constants.
(a) Use the form (21) to solve Problem 21. Verify that your form is equivalent to the one derived using (9).
(b) Show that, in general, d_1 and d_2 in (21) must be complex conjugates in order that the solution be real.

37. The auxiliary equations for the following differential equations have repeated complex roots. Adapt the "repeated root" procedure of Section 4.2 to find their general solutions:
(a) $y^{iv} + 2y'' + y = 0$.
(b) $y^{iv} + 4y''' + 12y'' + 16y' + 16y = 0$. [*Hint:* The auxiliary equation is $(r^2 + 2r + 4)^2 = 0$.]