

§ 2.5: Special Integrating Factors

Recall,

$$M(x, y)dx + N(x, y)dy = 0 \quad (*)$$

is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Integrating Factor:

If $(*)$ is not exact, ~~but~~ but multiplying $\mu(x, y)$ to $(*)$ yields an exact equation

$$\mu(x, y)M(x, y)dx + \cancel{\mu(x, y)}N(x, y)dy = 0$$

then $\mu(x, y)$ is called an integrating factor.

Question: How to determine $\mu(x, y)$ is an integrating factor?

Check if $\frac{\partial}{\partial y}[\mu(x, y)M(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)N(x, y)]$.

Ex: $\mu(x, y) = xy^2$ is an integrating factor for

$$(2y - 6x)dx + (3x - 4x^2y^{-1})dy = 0$$

Question: How to find an integrating factor?

Note: Note every equation has an integrating factor!!!

(P1)

Special Cases:

(i) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only then

$$M(x) = \exp \left[\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) dx \right]$$

is an integrating factor.

(ii) If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only then

$$M(y) = \exp \left[\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \right) dy \right]$$

is an integrating factor.

Ex. Solve

$$(2x^2 + y)dx + (x^2y - x)dy = 0$$

Solu.:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -\frac{2}{x}$$

is an integrating factor. Now solve

$$x^{-2}(2x^2 + y)dx + x^{-3}(x^2y - x)dy = 0$$

Hw: §2.5, # 1, 3, 5, 7, 9, 11.

(P2)

Because (12) is not exact, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{2(1 - xy)}{-x(1 - xy)} = \frac{-2}{x}.$$

We obtain a function of only x , so an integrating factor for (12) is given by formula (8). That is,

$$\mu(x) = \exp\left(\int \frac{-2}{x} dx\right) = x^{-2}.$$

When we multiply (12) by $\mu = x^{-2}$, we get the exact equation

$$(2 + yx^{-2}) dx + (y - x^{-1}) dy = 0.$$

Solving this equation, we ultimately derive the implicit solution

$$(13) \quad 2x - yx^{-1} + \frac{y^2}{2} = C.$$

Notice that the solution $x \equiv 0$ was lost in multiplying by $\mu = x^{-2}$. Hence, (13) and $x \equiv 0$ are solutions to equation (12). ♦

There are many differential equations that are not covered by Theorem 3 but for which an integrating factor nevertheless exists. The major difficulty, however, is in finding an explicit formula for these integrating factors, which in general will depend on both x and y .

2.5 EXERCISES

In Problems 1–6, identify the equation as separable, linear, exact, or having an integrating factor that is a function of either x alone or y alone.

1. $(2x + yx^{-1}) dx + (xy - 1) dy = 0$
2. $(2y^3 + 2y^2) dx + (3y^2x + 2xy) dy = 0$
3. $(2x + y) dx + (x - 2y) dy = 0$
4. $(y^2 + 2xy) dx - x^2 dy = 0$
5. $(x^2 \sin x + 4y) dx + x dy = 0$
6. $(2y^2x - y) dx + x dy = 0$

In Problems 7–12, solve the equation.

7. $(2xy) dx + (y^2 - 3x^2) dy = 0$
8. $(3x^2 + y) dx + (x^2y - x) dy = 0$
9. $(x^4 - x + y) dx - x dy = 0$
10. $(2y^2 + 2y + 4x^2) dx + (2xy + x) dy = 0$
11. $(y^2 + 2xy) dx - x^2 dy = 0$
12. $(2xy^3 + 1) dx + (3x^2y^2 - y^{-1}) dy = 0$

In Problems 13 and 14, find an integrating factor of the form

$x^n y^m$ and solve the equation.

13. $(2y^2 - 6xy) dx + (3xy - 4x^2) dy = 0$
14. $(12 + 5xy) dx + (6xy^{-1} + 3x^2) dy = 0$

15. (a) Show that if $(\partial N/\partial x - \partial M/\partial y)/(xM - yN)$ depends only on the product xy , that is,

$$\frac{\partial N/\partial x - \partial M/\partial y}{xM - yN} = H(xy),$$

then the equation $M(x, y) dx + N(x, y) dy = 0$ has an integrating factor of the form $\mu(xy)$. Give the general formula for $\mu(xy)$.

- (b) Use your answer to part (a) to find an implicit solution to

$$(3y + 2xy^2) dx + (x + 2x^2y) dy = 0,$$

satisfying the initial condition $y(1) = 1$.

16. (a) Prove that $M dx + N dy = 0$ has an integrating factor that depends only on the sum $x + y$ if and only if the expression

$$\frac{\partial N/\partial x - \partial M/\partial y}{M - N}$$

depends only on $x + y$.

- (b) Use part (a) to solve the equation

$$(3 + y + xy) dx + (3 + x + xy) dy = 0.$$