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Bistability and multistability in opinion dynamics models[☆]

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ABSTRACT

In this paper, we analyze a few opinion formation or election game dynamics models. For a simple model with two subpopulations with opposing opinions *A* and *B*, if the fraction of committed believers of opinion *A* ("*A* zealots") is less than a critical value, then the system has two bistable equilibrium points. When the fraction is equal to the critical value, the system undergoes a saddle-node bifurcation. When the fraction is larger than the critical value, a boundary equilibrium point is globally asymptotically stable, suggesting that the entire population reaches a consensus on *A*. We find a similar bistability property in the model in which both opinions have their own zealots. We also extend the model to include multiple competitors and show the dynamical behavior of multistability. The more competitors subpopulation *A* has, the fewer zealots opinion *A* needs to obtain consensus.

1. Introduction

Sociophysics studying the social and political behavior with modeling and simulation tools has been a research area with growing interest to physicists and mathematicians [5,6,19]. Opinion dynamics is one of the most active research subjects of sociophysics [4,7–9,16–18,20]. Motivated by the study of language competition dynamics, Castelló et al. used a modification of a voter model to study the ordering dynamics with two non-excluding options [1]. Colaiori and Castellano [3] studied the influence of both media and peer pressure on opinion dynamics. Built upon other work on opinion dynamics [21], Marvel et al. [12] modified a simple model of opinion spreading and found that only one of the modifications can significantly expand the moderate subpopulation. Nyczka et al. [14] studied a generalized version of the Sznajd model [11] and showed that opinion dynamics can be understood as a movement of a public opinion in a symmetric bistable effective potential. Inspired by the work in [14], we will analyze the simple model in the ref. [12] and its generalizations. There are other works studying opinion spreading models, such as [2,4,10,13,15,17]. The roles of informed agents and social power on opinion formation have been evaluated in [22–24]. Competition and cooperation in the study of social behavior have also been addressed by the evolutionary game theory [25,26].

The simple model in [12] studied the dynamics of three subpopulations holding three opinion states: an extreme opinion *A*, a conceptually opposing opinion *B*, and neither *A* nor *B* (the moderates). The model diagram is shown in Fig. 1(a) and the

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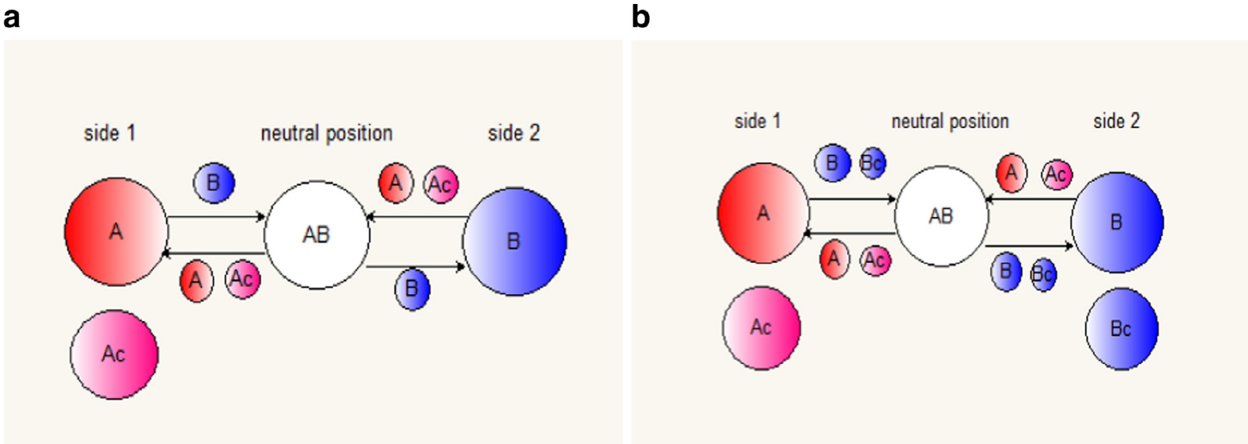

Fig. 1. (a) Diagram of model (1.1), adapted from the ref. [12]. (b) Diagram of model (3.1). *AB* represents the subpopulation that holds neither opinion *A* nor *B* (the moderates). The subscript *c* represents the subpopulation that holds the opinion indefinitely. A speaker can convert a listener from one subpopulation to another.

model is described by the following ordinary differential equations.

$$\begin{cases} \dot{n}_A = (p+n_A)n_{AB} - n_A n_B, \\ \dot{n}_B = n_B n_{AB} - (p+n_A)n_B, \\ \dot{n}_{AB} = n_A n_B + (p+n_A)n_B - (p+n_A)n_{AB} - n_B n_{AB}. \end{cases} \tag{1.1}$$

In the model, n_A, n_B and n_{AB} denote the proportions of those who currently hold opinion *A*, opinion *B*, and neither *A* nor *B*, respectively. A speaker can convert a listener from one subpopulation to another (for example, *B* can convert *A* to an undecided opinion state *AB* and *AB* can be converted by *A* to *A*, see the diagram in Fig. 1a). The model also considers people who hold opinion *A* indefinitely and cannot be influenced by others (the so-called "*A* zealots", denoted by A_c). Let p ($0 \le p \le 1$) be the proportion of *A* zealots in the total population.

Because $n_{AB} = 1 - p - n_A - n_B$, we can convert model (1.1) to the following system.

$$\begin{cases} \dot{n}_A = p(1-p) + (1-2p)n_A - pn_B - n_A^2 - 2n_A n_B, \\ \dot{n}_B = (1-2p)n_B - n_B^2 - 2n_A n_B. \end{cases} \tag{1.2}$$

In this paper, we will analyze model (1.2) and its generalizations. Bistability and multistability will be shown to appear in these opinion dynamics models.

2. Model analysis

2.1. Positivity and equilibria

Let $\mathscr{D}^* = \left\{(n_A, n_B, n_{AB}) \in \mathbb{R}_+^3 | n_A + n_B + n_{AB} = 1 - p\right\}$ and $\mathscr{D} = \left\{(n_A, n_B) \in \mathbb{R}_+^2 | n_A + n_B \le 1 - p\right\}$. We will show that system (1.1) or (1.2) is well-posed.

Theorem 2.1. *All solutions of system (1.1) are eventually confined in the compact subset $\mathscr{D}^*$. All solutions of system (1.2) are nonnegative. Moreover, $\mathscr{D}$ is a positive invariant for (1.2).*

Proof. For any positive initial conditions (n_A^0, n_B^0, n_{AB}^0), we assume that t_1 is the first time such that $n_A(t_1)n_B(t_1)n_{AB}(t_1) = 0$. Thus, there are three possible cases: (i) $n_A(t_1) = 0, n_B(t) \ge 0, n_{AB}(t) \ge 0$; (ii) $n_B(t_1) = 0, n_A(t) \ge 0, n_{AB}(t) \ge 0$; (iii) $n_{AB}(t_1) = 0, n_A(t) \ge 0, n_B(t) \ge 0$, for $t \in [0, t_1]$.

For case (i), it is obvious that $\dot{n}_A(t_1) < 0$. On the other hand, from the first equation of system (1.1), we have $\dot{n}_A(t_1) = pn_{AB}(t_1) \ge 0$, which is a contradiction.

For case (ii), from the second equation of system (1.1) we have $\dot{n}_B(t) \ge -(p+n_A)n_B$ for $t_2 \in [0, t_1]$. Thus $n_B(t_1) \ge n_B^0 e^{-(p+n_A)t} > 0$, which is contradiction with $n_B(t_1) = 0$.

For case (iii), it is clear that $\dot{n}_{AB}(t_1) < 0$. On the other hand, from the third equation of system (1.1) we have $\dot{n}_{AB}(t_1) = pn_B(t_1) + 2n_A(t_1)n_B(t_1) \ge 0$, which is a contradiction.

Thus we showed that all the solutions of system (1.1) with positive initial data are positive. Let (n_A, n_B, n_{AB}) be any nonnegative solution and $N = n_A + n_B + n_{AB}$. The time derivative along a solution of (1.1) is $\dot{N} = 0$. Thus, we know that $n_A + n_B + n_{AB}$ is a constant. Thus, the set $\mathscr{D}^*$ is a positive invariant with respect to (1.1). Since $n_{AB} = 1 - p - n_A - n_B$ is nonnegative, we have $n_A + n_B \le 1 - p$. Therefore, $\mathscr{D}$ is a positive invariant for (1.2).

To examine the existence of equilibrium points, we let $\Delta = 4p^2 - 8p + 1$. □

Theorem 2.2. *(i) There always exists a boundary equilibrium $E_0(1-p,0)$.*

(ii) If $0<p<1-\frac{\sqrt{3}}{2}$, system (1.2) has two positive equilibria: $E^{1*}(n_A^{1*},n_B^{1*})$ and $E^{2*}(n_A^{2*},n_B^{2*})$, where $n_A^{1*}=\frac{1-4p+\sqrt{\Delta}}{6}$, $n_B^{1*}=\frac{2(1-p)-\sqrt{\Delta}}{3}$, $n_A^{2*}=\frac{1-4p-\sqrt{\Delta}}{6}$ and $n_B^{2*}=\frac{2(1-p)+\sqrt{\Delta}}{3}$.

(iii) If $p=1-\frac{\sqrt{3}}{2}$, then there is a unique positive equilibrium $\bar{E}(\bar{n}_A,\bar{n}_B)$, where $\bar{n}_A=\frac{\sqrt{3}}{3}-\frac{1}{2}$ and $\bar{n}_B=\frac{\sqrt{3}}{3}$.

(iv) If $p>1-\frac{\sqrt{3}}{2}$, then there is no positive equilibrium but the boundary equilibrium E_0 still exists.

Proof. (i) It is clear that the boundary equilibrium $E_0(1-p,0)$ always exist. Next, we consider the other cases. A positive equilibrium of (1.2) must satisfy the following equations

$$\begin{cases} p(1-p)+(1-2p)n_A-pn_B-n_A^2-2n_An_B=0,\\ (1-2p)-n_B-2n_A=0. \end{cases} \tag{2.1}$$

From the second equation of (2.1), we have $n_B=1-2p-2n_A$. In view of the condition $0\le p\le 1$, we have

(ii) If $\Delta>0$, i.e., $0<p<1-\frac{\sqrt{3}}{2}$, then there are two equilibria $E^{1*}(n_A^{1*},n_B^{1*}), E^{2*}(n_A^{2*},n_B^{2*})$, where $n_A^{1*}=\frac{1-4p+\sqrt{\Delta}}{6}$, $n_B^{1*}=\frac{2(1-p)-\sqrt{\Delta}}{3}$, $n_A^{2*}=\frac{1-4p-\sqrt{\Delta}}{6}$ and $n_B^{2*}=\frac{2(1-p)+\sqrt{\Delta}}{3}$.

(iii) When $\Delta=0$, i.e., $p=1-\frac{\sqrt{3}}{2}$, there exists a unique positive equilibrium $\bar{E}(\bar{n}_A,\bar{n}_B)$.

(iv) When $\Delta<0$, i.e., $1-\frac{\sqrt{3}}{2}<p\le 1$, the equation $3n_A^2+(4p-1)n_A+p^2=0$ has no real solution. Thus, the system has no positive equilibrium. However, the boundary equilibrium E_0 still exists. □

2.2. Stabilities analysis

The following theorem provides stability results of the equilibria of model (1.2).

Theorem 2.3. *(i) The boundary equilibrium $E_0(1-p,0)$ is always a stable node.*

(ii) When $0<p<1-\frac{\sqrt{3}}{2}$, the equilibrium $E^{1*}(n_A^{1*},n_B^{1*})$ is a saddle and $E^{2*}(n_A^{2*},n_B^{2*})$ is a stable node.

(iii) When $p=1-\frac{\sqrt{3}}{2}$, the equilibria $E^{1*}(n_A^{1*},n_B^{1*})$ and $E^{2*}(n_A^{2*},n_B^{2*})$ become the same equilibrium $\bar{E}(\bar{n}_A,\bar{n}_B)=(\frac{\sqrt{3}}{3}-\frac{1}{2},\frac{\sqrt{3}}{3})$, which is a saddle-node point. System (1.2) undergoes a saddle-node bifurcation when p passes through $1-\frac{\sqrt{3}}{2}$.

(iv) When $1-\frac{\sqrt{3}}{2}<p\le 1$, the unique equilibrium $E_0(1-p,0)$ is stable.

Proof. Let $E^+(n_A^+,n_B^+)$ be an arbitrary equilibrium of (1.2). The Jacobian matrix of the linearized system of (1.2) at $E^+(n_A^+,n_B^+)$ is

$$\mathscr{J}=\begin{bmatrix} 1-2p-2n_A^+-2n_B^+ & -p-2n_A^+-2n_B^+ & 1-2p-2n_A^+-2n_B^+ \end{bmatrix}.$$

The characteristic equation is $|\lambda I-\mathscr{J}|=0$.

(i) For the boundary equilibrium $E_0(1-p,0)$, the double root of equation $|\lambda I-\mathscr{J}|=0$ is $\lambda=-1$. Thus, E_0 is a stable node.

(ii) When $0<p<1-\frac{\sqrt{3}}{2}$, we also let $E^+(n_A^+,n_B^+)$ be an arbitrary positive equilibrium of (1.2) for convenience. Using $n_B^+=1-2n_A^+-2p$, the characteristic equation can be written as $\lambda^2+a_1\lambda+a_2=0$, where $a_1=2n_B^+$ and $a_2=3n_B^{+2}+2pn_B^+-2n_B^+$. The root of the characteristic equation is $\lambda_{1,2}=-n_B^+\pm\sqrt{2n_B^+(1-n_B^+-p)}$. Because $n_B^{1*}=\frac{2(1-p)-\sqrt{\Delta}}{3}<\frac{2(1-p)}{3}$ and $2(1-n_B^{1*}-p)>2(1-\frac{2}{3}+\frac{2}{3}p-p)=\frac{2(1-p)}{3}$, we obtain $\lambda_2=-n_B^{1*}+\sqrt{2n_B^{1*}(1-n_B^{1*}-p)}>0$, which shows that the equilibrium $E^{1*}(n_A^{1*},n_B^{1*})$ is a saddle. Using a similar method, for the equilibrium $E^{2*}(n_A^{2*},n_B^{2*})$ we can show that $\lambda_2=-n_B^{2*}+\sqrt{2n_B^{2*}(1-n_B^{2*}-p)}<0$. Thus, E^{2*} is a stable node.

(iii) When $p=1-\frac{\sqrt{3}}{2}$, for $\bar{E}(\bar{n}_A,\bar{n}_B)$ the characteristic roots are $\lambda_1=0$ and $\lambda_2=-\frac{2\sqrt{3}}{3}$. Thus, $\bar{E}(\bar{n}_A,\bar{n}_B)$ is a saddle-node point and system (1.2) undergoes a saddle-node bifurcation when p passes through $1-\frac{\sqrt{3}}{2}$. The case (iv) is self evident from case (i). This finishes the proof of the theorem.

A special case is $p=0$, in which there are four equilibria: $E_0^0=(0,0), E_0^1=(1,0), E_0^2=(0,1), E_0^*=(\frac{1}{3},\frac{1}{3})$. In this case, if A and B have the same initial value, then the fractions of subpopulations A, B and AB are all equal to $\frac{1}{3}$. If the two initial fractions are different, then the entire population will reach a consensus on the opinion with a larger initial fraction.

In Fig. 2, we plotted the steady states of subpopulations A and B as p varies from 0 to 1. From the figure, we can see when $0\le p<1-\frac{\sqrt{3}}{2}$, system (1.2) has a bistable dynamical behavior (i.e. two stable states S_1 and S_2). When $p=1-\frac{\sqrt{3}}{2}$, the bistability disappears and system (1.2) experiences a saddle-node bifurcation. When $p>1-\frac{\sqrt{3}}{2}$, the entire population reaches a consensus on opinion A.

We determine if system (1.2) has a limit cycle. Let the right-hand sides of system (1.2) be $P(n_A,n_B)$ and $Q(n_A,n_B)$, respectively. We construct a Dulac function $D(n_A,n_B)=\frac{1}{n_An_B}$. It follows that

$$\frac{\partial(DP)}{\partial n_A}+\frac{\partial(DQ)}{\partial n_B}=-p-\frac{1}{n_B}-\frac{1}{n_A}-\frac{p(1-p-n_B)}{n_A^2n_B}<0,$$

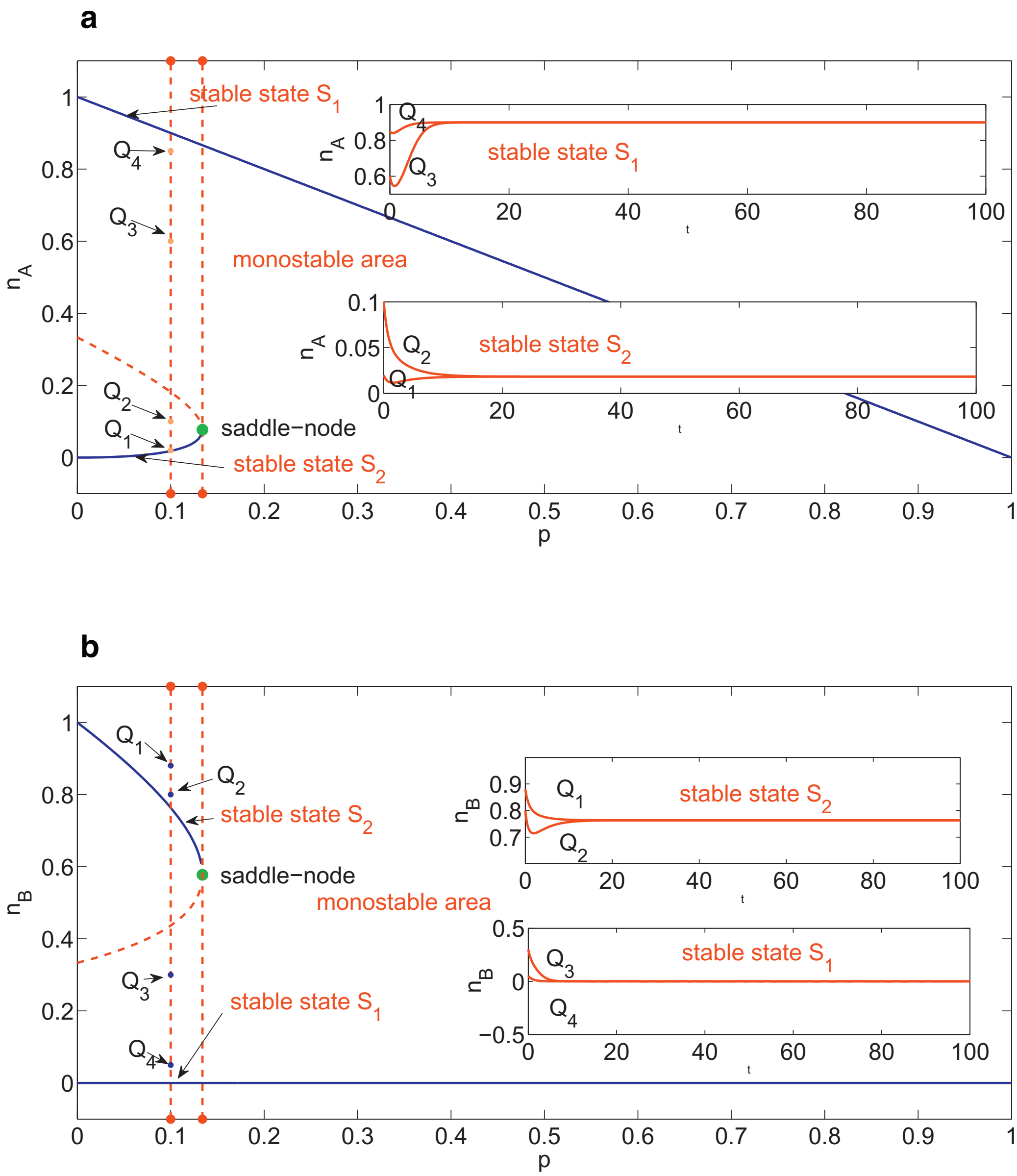

Fig. 2. Bistability and saddle-node bifurcation of model (1.2). In this simulation, p was chosen to be 0.1, which is in the bistable interval, $[0, 1-\frac{\sqrt{3}}{2})$. Initial conditions are $Q_1(0.02, 0.88)$, $Q_2(0.1, 0.8)$, $Q_3(0.6, 0.3)$ and $Q_4(0.85, 0.05)$. The solution with initial points Q_1 and Q_2 converges to a stable state S_1 and the solution with initial points Q_3 and Q_4 converges to another stable state S_2. When $0 \le p < 1-\frac{\sqrt{3}}{2}$, system (1.2) has a bistable dynamical behavior (i.e. two stable states S_1 and S_2). When $p = 1-\frac{\sqrt{3}}{2}$, the bistability disappears and system (1.2) experiences a saddle-node bifurcation. When $p > 1-\frac{\sqrt{3}}{2}$, the entire population reaches a consensus on opinion A.

for all $n_A > 0$, $n_B > 0$. Thus, by Dulac's criteria, system (1.2) does not has any limit cycle in int$\mathscr{D}$. Therefore, the boundary equilibrium $E_0(1-p, 0)$ is actually globally asymptotically stable when $1-\frac{\sqrt{3}}{2} < p \le 1$.

The above results show that when the proportion of committed believers of one opinion is small, the subpopulation holding this opinion will eventually be small or large, depending on the initial condition. However, when the proportion of committed believers exceeds a critical threshold, the entire population will eventually reach a consensus on the same opinion. □

3. The model with zealots of each opinion

In this section, we consider a model in which opinions A and B both have their own zealots. The model diagram is shown in Fig. 1(b) and the equations are given as follows. The parameters p_1 and p_2 are the fractions of A and B zealots in the total

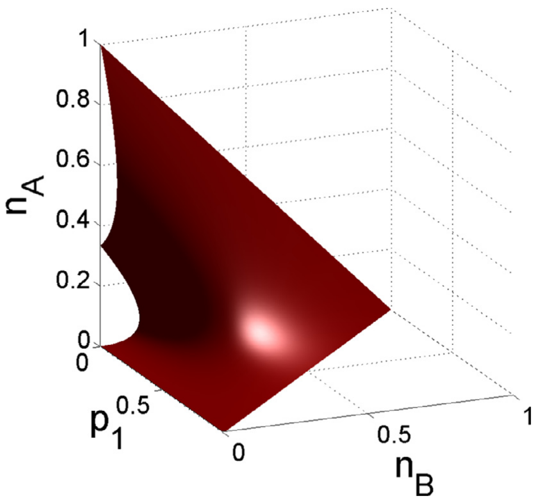

Fig. 3. The steady-state subpopulations of A and B as a function of the fraction of A zealots (model 3.1). There may exist one, two or three positive equilibria.

population, respectively.

$$\begin{cases} \dot{n}_A = (p_1 + n_A)n_{AB} - n_A(n_B + p_2), \\ \dot{n}_B = (p_2 + n_B)n_{AB} - (p_1 + n_A)n_B, \\ \dot{n}_{AB} = n_A(n_B + p_2) + (p_1 + n_A)n_B - (p_1 + n_A)n_{AB} - (p_2 + n_B)n_{AB}. \end{cases} \tag{3.1}$$

Because $n_{AB} = 1 - p_1 - p_2 - n_A - n_B$, we can write system (3.1) as the following system.

$$\begin{cases} \dot{n}_A = p_1(1 - p_1 - p_2) + (1 - 2p_1 - 2p_2)n_A - p_1 n_B - n_A^2 - 2n_A n_B, \\ \dot{n}_B = p_2(1 - p_1 - p_2) + (1 - 2p_1 - 2p_2)n_B - p_2 n_A - n_B^2 - 2n_A n_B. \end{cases} \tag{3.2}$$

Let the right side of system (3.2) be equal to zero, we have

$$n_A^2 + p_1 n_A + \frac{p_1}{2} - \left(\frac{n_A + p_1}{2}\right)\sqrt{(1 - 2p_1 - 2p_2 - 2n_A)^2 + 4p_2(1 - p_1 - p_2 - n_A)} = 0,$$

$$n_B^2 + p_2 n_B + \frac{p_2}{2} - \left(\frac{n_B + p_2}{2}\right)\sqrt{(1 - 2p_1 - 2p_2 - 2n_B)^2 + 4p_1(1 - p_1 - p_2 - n_B)} = 0.$$

In view of the symmetry of system (3.2), the positivity of solution can be obtained by the same method as Theorem 2.1. However, it is challenging to calculate the steady state. From the simulation in Fig. 3, we find there may exist one, two or three positive equilibria. Similar to the previous model, system (3.2) also has the property of bistability (see illustration in Fig. 4).

The system shows two saddle-node bifurcations, between which bistability appears. However, the bistability is sensitive to initial values. In the case of $p_1 = p_2$, when subpopulations A and B have the same initial value, each of subpopulations A, AB and B accounts for $\frac{1-p_1-p_2}{3}$ of the total population. When they have different initial values, the entire population reaches a consensus on the opinion that has a larger initial value.

4. The model with multiple competitors

In this section, we consider a model in which opinion A has multiple rival opinions $B_1, B_2, \ldots, B_n$. The model is given as follows.

$$\begin{cases} \dot{n}_A = (p + n_A)\left(1 - p - n_A - \sum\limits_{i=1}^{n} n_{B_i}\right) - n_A\left(\sum\limits_{i=1}^{n} n_{B_i}\right), \\ \dot{n}_{B_i} = n_{B_i}\left(1 - p - n_A - \sum\limits_{i=1}^{n} n_{B_i}\right) - n_{B_i}\left(p + n_A + \sum\limits_{j=1, j\neq i}^{n} n_{B_j}\right), i = 1, 2, \ldots, n. \end{cases} \tag{4.1}$$

To obtain the positive equilibrium, we need to find the roots of the equation $(2n+1)n_A^2 + (2p - 1 + 2np)n_A + p^2 + p(n - 1) = 0$. Let $\Delta_n = 4n^2p^2 - 8n^2p + 1$ and $p_{c_n} = 1 - \frac{\sqrt{4n^2-1}}{2n}$. It is clear that $\{p_{c_n}\}$ is a decreasing series. Using the same method as in Section 3, we have the following results:

(i) System (4.1) always has a boundary equilibrium $E_0(1 - p, 0, 0, \ldots, 0)$.

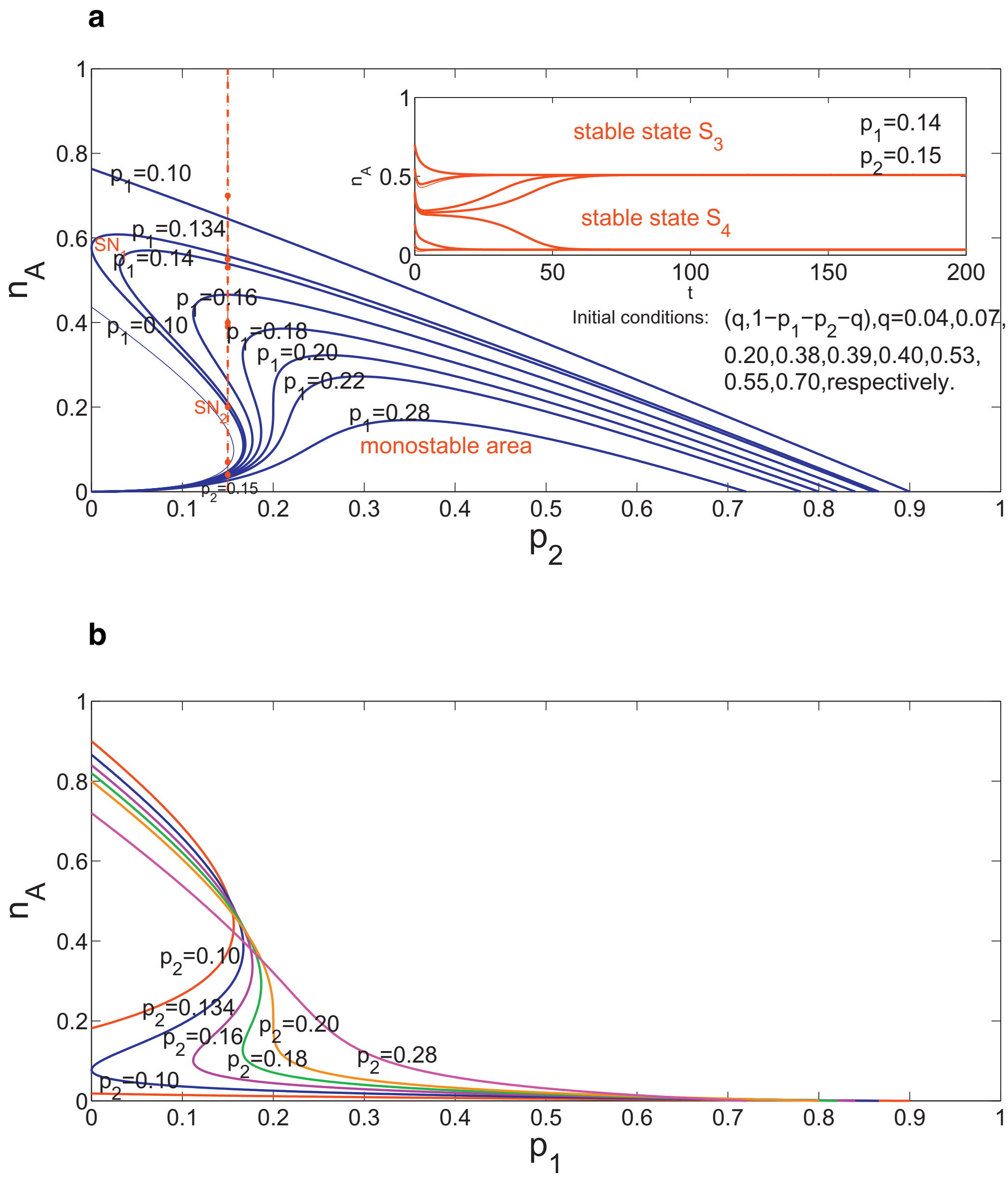

Fig. 4. Bistability and saddle-node bifurcation diagram of model (3.2). If opinion A (or B) has some zealots, system (3.1) or (3.2) may have two saddle-node points, SN_1 and SN_2, between which system (3.1) or (3.2) has a bistable dynamical behavior. (a) We chose $p_1 = 0.14$ and $p_2 = 0.15$. The solution with different initial point $(q, 1 - p_1 - p_2 - q)$ from different values of q converges to either a stable state S_3 or S_4. (b) If one opinion has more zealots or the rivals have fewer zealots, then it will be easier to reach consensus on the opinion than its rivals.

(ii) When $0 < p < p_{c_i}$, system (4.1) has $2i$ boundary equilibria: $E_i^+(n_A^{i+}, n_{B_1}^{i+}, n_{B_2}^{i+}, ..., n_{B_i}^{i+}, 0, ..., 0)$ and $E_i^-(n_A^{i-}, n_{B_1}^{i-}, n_{B_2}^{i-}, ..., n_{B_i}^{i-}, 0, ..., 0)$, ˜˜$i = 1, 2, ..., n-1$.

(iii) When $0 < p < p_{c_n}$, system (4.1) has two positive equilibria: $E_n^+(n_A^{n+}, n_{B_1}^{n+}, n_{B_2}^{n+}, ..., n_{B_n}^{n+})$ and $E_n^-(n_A^{n-}, n_{B_1}^{n-}, n_{B_2}^{n-}, ..., n_{B_n}^{n-})$.

(iv) When $p = p_{c_n}$, there exists a unique positive equilibrium $E_n(n_A^n, n_{B_1}^n, n_{B_2}^n, ..., n_{B_n}^n)$.

(v) When $p > p_{c_n}$, there is no positive equilibrium.

Using the same method as in Section 2, we can also obtain the multistability results for system (4.1) (see Table 4.1). If subpopulation A has n competitors, then system (4.1) will have $2n+1$ equilibria and $(n+1)$ stability. Table 4.2 gives the relationship between the number of competitors and the condition of achieving consensus on A. The more competitors subpopulation A has, the fewer zealots opinion A needs to obtain consensus.

Table 4.1
Multistablity of system (4.1).

Parameter p	The number of equilibria	System (4.1)
$p_{c_1} < p < 1$	1	Consensus on A
$p_{c_2} < p < p_{c_1}$	3	Bistable
$p_{c_3} < p < p_{c_2}$	5	Tristable
$p_{c_4} < p < p_{c_3}$	7	4-stable
...	...	...
$p_{c_n} < p < p_{c_{n-1}}$	$2n-1$	n-stable
$0 < p < p_{c_n}$	2n+1	(n+1)-stable

Table 4.2
Condition of achieving consensus on A.

The number of competitors of A	Condition of achieving consensus on A
1	$p > p_{c_1}$
2	$p > p_{c_2}$
3	$p > p_{c_3}$
...	...
n	$p > p_{c_n}$

5. Summary

In this paper, we studied the rich dynamics of a few ideological models. For a simple model of opinion dynamics, we showed there exists bistability when the fraction of committed believers of one opinion is less than a critical value. The system undergoes a saddle-node bifurcation when the fraction is equal to the critical value. To achieve consensus, the fraction needs to exceed the threshold value. We also extended the model to include multiple rival opinions and found that there exists multistability. We found that the more competitors subpopulation A has, the fewer zealots opinion A needs to obtain consensus. This is intuitive in opinion spreading. As opinion A has more zealots, the rivals have fewer zealots or split to form more parties holding different opinions, which makes it easier for opinion A to reach consensus. It would be interesting to extend the models studied in this paper to network systems and evaluate how the microscopic behavior affects the opinion dynamics. However, the analysis of the model will be much more complicated and remains to be investigated.

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