

CASCADE FLOCKING WITH FREE-WILL

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This work is dedicated to the memory of Professor Paul Waltman

ABSTRACT. We consider a self-organized system with a hierarchy structure to allow multiple leaders in the highest rank, and with free-will. In the model, we use both Cucker-Smale and Motsch-Tadmor functions for the pair influence of agents, and we derive sufficient conditions for such a system to converge to a flock, where agents ultimately move in the same velocity. We provide examples to show our sufficient conditions are sharp, and we numerically observe that such a self-organized system may have agents moving in different (final) velocities but maintain finite distance from each other due to the free-will.

1. Introduction. Flocking, a dynamic outcome of a self-organized system with multiple agents, is achieved via a process of adjusting the individual velocity of each agent according to the agent's relative locations with others in order to reach a certain consensus (to move in the same speed). Key factors leading to flocking is the hierarchy structure of the system under consideration, the degree to which each agent can adjust its velocity according to the external environment (free will), and the pairwise interaction (influence).

Pairwise influence. We start with the well recognized Cucker-Smale model [9, 10] and its extension, the Motsch-Tadmor model [27]. In these models, a self-organized system consists of agents $i = 1, \dots, N$ where agent i is characterized by x_i (the location) and $v_i \in R^d$ (the velocity). The system adjusts its relative locations of all agents simultaneously, following the rule:

$$\frac{d}{dt}x_i(t) = v_i(t), \quad (1)$$

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$$\frac{d}{dt}v_i(t) = \alpha \sum_{j=1, j \neq i}^N b_{ij}(\|x_j - x_i\|)(v_j(t) - v_i(t)),$$

for $i = 1, 2, \dots, N$. In what follows, we let $\mathbb{N} = \{1, \dots, N\}$.

In the model above, α measures the strength of self-adaptation from the pairwise influence, b_{ij} is the pairwise function which is given by $b_{ij} = a_{ij}^{CS}$ (for the Cucker-Smale influence function) and by a_{ij}^{MT} (for the Motsch-Tadmor function), where

$$a_{ij}^{CS}(\|x_j - x_i\|) = \frac{\chi(\|x_j - x_i\|)}{N},$$

$$a_{ij}^{MT}(\|x_j - x_i\|) = \frac{\chi(\|x_j - x_i\|)}{\sum_{1 \leq k \leq N} \chi(\|x_k - x_i\|)},$$

and $\chi(r) = \frac{1}{(1+r^2)^\beta}$ with a given constant $\beta \geq 0$.

Flocking phenomena has been observed in a wide range of fields in biology, ecology, robotics and control theory, sensor networks, sociology and economics where interacting agents use their internal relationships to achieve a consensus eventually. For example, a price system may emerge from a complex market environment, and a common language may emerge from the evolution and interaction of multiple languages. See [30, 11, 21] and references for more examples.

Vicsek developed a kinematic model to examine the motor behaviour in systems of biological particles with interaction, and interesting simulations based on his model illustrated the dynamic processes how a group reaches consensus[29]. The model proposed by Cucker and Smale in 2007 [9, 10] represents some significant simplification of the Vicsek model, and this study has since inspired a lot of modelling research (see [25, 21, 32] and references therein for more details) including the modification by Motsch and Tadmor that was used to reflect the observation that influence between agents should perhaps be measured according to their relative distances. This extension led to the aforementioned pairwise influence function which is non-symmetric. In what follows, we will refer to CS-model (influence function) and MT-model (influence function) whenever the influence function is given by a_{ij}^{CS} and a_{ij}^{MT} respectively.

Hierarchical organization. In both CS-model and MT-model, agents are assumed to be equal in terms of their influence in self-regulating the entire system. Substantial research activities and evidence, including those in [16, 21, 18, 15], however show that there exists leader-follower relationship in aggregation. The experiments conducted by the Vicsek's group using 10 homing pigeons tracked by high-resolution lightweight GPS devices clearly supported the existence of leader-follower relationship and its role in flocking, and indicated that the hierarchical organization among the flight group makes migration more efficient than egalitarian one [24]. There are some mathematical models developed to reflect this phenomenon including the work of Shen [21] that considers hierarchical leadership where each agent is influenced only by its superiors in a specified hierarchy. This model was later extended in Li and Xue [32] (using a discrete-in-time model) in which rooted leadership, where there is an overall leader who influences all agents either directly or indirectly, is incorporated. On the other hand, Ballerini and his collaborators discovered that each agent only impacts on an average fixed number of neighbours (about six or seven in the discovery)[23]. Therefore, in many applications, not every pair of agents in the group can have direct influence. Inspired by these studies, we

propose in this paper to consider flocking for a self-organized (continuous- in-time) system with rank hierarchy. A precise definition of such a system will be given in the next section, but vaguely speaking, a system with rank hierarchy is a system with different ranks, where agents in a given rank can only be influenced by the agents in the same rank i (within rank influence) and/or by agents from the rank directly above. In comparison with [15], we use continuous-in-time model since many consensus dynamics such as bird flocking and fish schooling take place in continuous-in-time processes. Note also that we will consider the case where the highest rank may have multiple agents, so the system can have multiple leaders with coordinated leadership.

Free will. Our model will also incorporate free-will, with an aim to addressing whether this impact on the system’s flocking. It was noted in that free-will exists in every life from cellular organisms to humans [8]. It is natural to ask what kind of free-will will not destroy the flocking of interacting agents. Cucker and Huepe added this free-will term in the CS model, where the free-will is considered as depending on the relative velocity [12]. Shen also incorporated the free-will mechanism in his work, and Shen’s results on flocking were later improved in [20] using discrete-in-time models. See also the work of Li and Xue [33] for discrete dynamical systems with free-will. Here, we consider the CS or MT model and a more general model with rank hierarchy with free-will using continuous-in-time dynamical systems, and we obtain sharp sufficient conditions for the systems to possess the flocking behaviour.

2. Model description: Hierarchy of social ranks and free-will. We start with the definition of a hierarchy of social ranks.

Definition 2.1. (HR model) A HR model is defined as a self-organized system with N agents, where we assume that a). there exists K ranks (where $K > 1$ is an integer), and the m -th Rank R_m has N_m agents; b). for agent i is in R_m with $m > 1$ we have

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha \sum_{j \in R_{m-1} \cup R_m, j \neq i} b_{ij}(|x_j - x_i|)(v_j - v_i); \end{aligned} \tag{2}$$

and c). for agent $i \in R_1$ we have

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha \sum_{j \in R_1, j \neq i} b_{ij}(|x_j - x_i|)(v_j - v_i), \end{aligned} \tag{3}$$

with α and b_{ij} as defined earlier.

Note that if there is only one agent in R_1 , the HR model is just the natural generalization of the hierarchical fellowship (HL) model from discrete- to continuous-in time dynamical systems. We refer to Figure 1 for an illustration.

Similarly, we can define a HR model with free-will, characterized by functions f_i :

Definition 2.2. (HR model with free-will) A HR model is defined as a self-organized system with N agents, where we assume that a). there exists K ranks

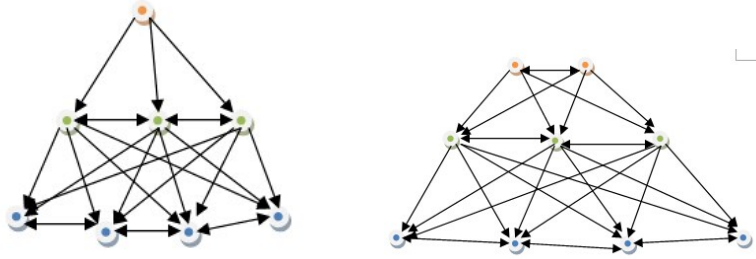


FIGURE 1. RH models with a single leader (left) and multiple leaders (right).

(where $K > 1$ is an integer), and the m -th Rank R_m has N_m agents; b). for agent i is in R_m with $m > 1$ we have

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha \sum_{j \in R_{m-1} \cup R_m, j \neq i} b_{ij}(\|x_j - x_i\|)(v_j - v_i) + f_i(t); \end{aligned} \quad (4)$$

and c). for agent $i \in R_1$ we have

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha \sum_{j \in R_1, j \neq i} b_{ij}(\|x_j - x_i\|)(v_j - v_i) + f_i(t), \end{aligned} \quad (5)$$

with α and b_{ij} as defined earlier. $f_i(t)$ represents the free-will of the agent “ i ”.

Let $\{(x_i, v_i)\}_{i=1}^N$ be a solution of a HR-model. As usual, we use d_X and d_V denote the corresponding diameters in position and velocity phase spaces. Namely, we define

$$\begin{aligned} d_X &= \max_{i,j \in N} \|x_i - x_j\|, \\ d_V &= \max_{i,j \in N} \|v_i - v_j\|. \end{aligned} \quad (6)$$

Here $\|\cdot\|$ denotes European norm.

We say the model system converges to a flock if

$$\sup_{t>0} d_X(t) < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} d_V(t) = 0 \quad (7)$$

for every solution $\{(x_i, v_i)\}_{i=1}^N$ of the model.

Before stating our main results, we make a remark about the influence function b_{ij} for both CS-function and MT-function. For $i \in R_1$, the influence function is

$$b_{ij} = \frac{\chi(\|x_j - x_i\|)}{N_1}, \quad j \in R_1 \quad \text{and} \quad j \neq i$$

or

$$b_{ij} = \frac{\chi(\|x_j - x_i\|)}{\sum_{k \in R_1} \chi(\|x_k - x_i\|)}, \quad j \in R_1 \quad \text{and} \quad j \neq i.$$

From the definition of χ and by rescaling α if necessary, we have

$$\chi(r) < 1, \quad \sum_{j \in R_1, i \neq j} b_{ij}(\|x_j - x_i\|) < 1.$$

Therefore, let $b_{ii} = 1 - \sum_{i \neq j} b_{ij}$, we have

$$\sum_{j \in R_1} b_{ij} = 1.$$

For $i \in R_m$ with $m > 1$, we have

$$b_{ij} = \frac{\chi(\|x_j - x_i\|)}{N_{m-1} + N_m}, \quad j \in R_{m-1} \cup R_m \quad \text{and} \quad j \neq i$$

or

$$b_{ij} = \frac{\chi(\|x_j - x_i\|)}{\sum_{k \in R_{m-1} \cup R_m} \chi(\|x_k - x_i\|)} \quad j \in R_{m-1} \cup R_m \quad \text{and} \quad j \neq i.$$

From the definition of χ and by rescaling α if necessary, we have

$$\chi(r) < 1, \quad \sum_{j \in R_{m-1} \cup R_m, i \neq j} b_{ij}(\|x_j - x_i\|) < 1.$$

Therefore, let $b_{ii} = 1 - \sum_{i \neq j} b_{ij}$, we have

$$\sum_{j \in R_{m-1} \cup R_m} b_{ij} = 1.$$

3. RH flocking. In this section, we first consider the flocking for a system with a single leader and then use an induction argument to extend the results for the HR models with a general cascade influence structure.

3.1. Flocking with a leader. We start with a special HR model with an overall leader who can directly lead all followers, where all followers can influence each other.

Definition 3.1. (HR with a leader) A HR with a leader is a self-organized system with $N + 1$ agents, with the agent “p” being the leader with a constant velocity denoted by $v_p \in R^n$ and position denoted by $x_p(t) \in R^n$. All other agents, agents ($i \in \mathbb{N}$), are called followers and they have mutual influence on each other.

Following the definition,

$$\begin{aligned} \frac{dx_p}{dt} &= v_p, \\ \frac{dv_p}{dt} &= 0; \end{aligned} \tag{8}$$

and for the followers, $\{(x_i, v_i)\}_{i \in \mathbb{N}}$ satisfy

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha \sum_{j \in \mathbb{N} \cup \{p\}, j \neq i} b_{ij}(\|x_j - x_i\|)(v_j - v_i). \end{aligned} \tag{9}$$

In what follows, we use D^+ to denote the upper-Dini derivative. We also define $d_{X_1} = \max_{i \in \mathbb{N}} \|x_i - x_p\|$ and $d_{V_1} = \max_{i \in \mathbb{N}} \|v_i - v_p\|$.

Lemma 3.2.

$$\begin{aligned} D^+ d_{X_1} &\leq d_{V_1}, \\ D^+ d_{V_1} &\leq -\alpha \frac{1}{N+1} \chi(d_{X_1}) d_{V_1}. \end{aligned}$$

Proof. First, we can easily show that $D^+ d_{X_1} \leq d_{V_1}$ using the same argument for Theorem 3.5 in [27].

Secondly, we consider $d_{V_1}(t)$. v_i with $i \in \mathbb{N}$ satisfies

$$\begin{aligned} \langle \dot{v}_i - \dot{v}_p, v_i - v_p \rangle &= \langle \alpha \sum_{j \in \mathbb{N} \cup \{p\}, j \neq i} b_{ij} (\|x_j - x_i\|) (v_j - v_i) - v_p + v_p, v_i - v_p \rangle \\ &= \alpha \sum_{j \in \mathbb{N} \cup \{p\}, j \neq i} b_{ij} \langle v_j - v_p, v_i - v_p \rangle - \alpha \langle v_i - v_p, v_i - v_p \rangle \\ &\leq \alpha \sum_{j \in \mathbb{N}} b_{ij} \langle v_j - v_p, v_i - v_p \rangle + \alpha b_{ip} d_{V_1}^2 - \alpha a_{ip} d_{V_1}^2 \\ &\quad - \alpha \langle v_i - v_p, v_i - v_p \rangle \\ &\leq \alpha \sum_{j \in \mathbb{N}} b_{ij} d_{V_1}^2 + \alpha b_{ip} d_{V_1}^2 - \alpha b_{ip} d_{V_1}^2 - \alpha \langle v_i - v_p, v_i - v_p \rangle \\ &= \alpha \sum_{j \in \mathbb{N} \cup \{p\}} b_{ij} d_{V_1}^2 - \alpha b_{ip} d_{V_1}^2 - \alpha \langle v_i - v_p, v_i - v_p \rangle. \end{aligned}$$

As $\sum_{j \in \mathbb{N} \cup \{p\}} b_{ij} = 1$, we have

$$\langle \dot{v}_i - \dot{v}_p, v_i - v_p \rangle \leq \alpha d_{V_1}^2 - \alpha b_{ip} d_{V_1}^2 - \alpha \langle v_i - v_p, v_i - v_p \rangle.$$

For a given time t , we can choose agent “ i ”, such that $d_{V_1}(t) = \|v_i(t) - v_p(t)\|$. If we choose $b_{ip}(t) = a_{ip}^{CS}(t)$, then,

$$D^+ d_{V_1} \leq -\alpha b_{ip} d_{V_1} \leq -\alpha \frac{1}{N+1} \chi(d_{X_1}) d_{V_1},$$

Therefore, if we choose $b_{ip}(t) = a_{ip}^{MT}(t)$, then,

$$D^+ d_{V_1} \leq -\alpha b_{ip} d_{V_1}^2 \leq -\alpha \frac{\chi(\|v_i(t) - v_p(t)\|)}{\sum_{k \in \mathbb{N} \cup \{p\}} \chi(\|v_i(t) - v_k(t)\|)} d_{V_1}.$$

For $0 < \chi(\|v_i(t) - v_k(t)\|) < 1$, we have $0 < \sum_{k \in \mathbb{N} \cup \{p\}} \chi(\|v_i(t) - v_k(t)\|) < N + 1$,

then,

$$D^+ d_{V_1} \leq -\alpha \frac{1}{N+1} \chi(d_{X_1}) d_{V_1}.$$

Therefore, for both $b_{ip}(t) = a_{ip}^{CS}(t)$ and $b_{ip}(t) = a_{ip}^{MT}(t)$, we have

$$D^+ d_{V_1} \leq -\alpha \frac{1}{N+1} \chi(d_{X_1}) d_{V_1},$$

completing the proof. \square

Theorem 3.3. *If the function $\chi(r)$ satisfies $\int_0^\infty \chi(r) = \infty$, the subsystem of all followers converges to a flocking, namely if (x_p, v_p) and (x_i, v_i) is an arbitrarily given solution of (8)-(9), then $\sup_{t>0} d_{X1}(t) < \infty$ and $\lim_{t \rightarrow \infty} d_{V1}(t) = 0$.*

Proof. We used the energy function introduced by Ha and Liu as follows:

$$E_1(t) = d_{V1}(t) + \alpha \frac{1}{N+1} \int_0^{d_{X1}(t)} \chi(s) ds.$$

We apply Lemma 3.2 to $D^+ E_1(t)$ to obtain

$$\begin{aligned} D^+ E_1(t) &\leq D^+ d_{V1}(t) + \alpha \frac{1}{N+1} \chi(d_{X1}(t)) D^+ d_{X1}(t) \\ &\leq -\alpha \frac{1}{N+1} \chi(d_{X1}) d_{V1}(t) + \alpha \frac{1}{N+1} \chi(d_{X1}) d_{V1}(t) = 0. \end{aligned}$$

So the energy function $E_1(t)$ is decreasing. For $\int_0^\infty \chi(r) dr = \infty$, we can choose a constant $C_1 > d_{X1}(0)$ such that $d_{V1}(0) = \alpha \frac{1}{N+1} \int_{d_{X1}(0)}^{C_1} \chi(s) ds$, and hence

$$d_{V1}(t) + \alpha \frac{1}{N+1} \int_0^{d_{X1}(t)} \chi(s) ds \leq d_{V1}(0) + \alpha \frac{1}{N+1} \int_0^{d_{X1}(0)} \chi(s) ds,$$

from which it follows that

$$d_{V1}(t) \leq \alpha \frac{1}{N+1} \int_{d_{X1}(0)}^{C_1} \chi(s) ds + \alpha \frac{1}{N+1} \int_{d_{X1}(t)}^{d_{X1}(0)} \chi(s) ds,$$

and hence

$$d_{V1}(t) \leq \alpha \frac{1}{N+1} \int_{d_{X1}(t)}^{C_1} \chi(s) ds.$$

As $d_{V1}(t)$ and $\chi(s)$ are both positive, we deduce that $d_{X1}(t) < C_1$ for $t \in (0, \infty)$. Using the decreasing property of $\chi(t)$, we obtain $\chi(d_{X1}(t)) \geq \chi(C_1)$ and

$$D^+ d_{V1} \leq -\alpha \frac{1}{N+1} \chi(C_1) d_{V1}(t).$$

By Gronwall's inequality, we have $d_{V1}(t) \leq d_{V1}(0) e^{-C_1^* t}$, where $C_1^* = \alpha \frac{1}{N+1} \chi(C_1)$. Obviously, $\lim_{t \rightarrow \infty} d_{V1}(t) = 0$. □

Remark 1. The model considered in the above theorem is a natural extension of the analogous RL model from discrete-time systems to continuous-time systems. We used the idea of Motsch and Tadmor [27] to allow the influence between a pair of leader and follower or a pair of followers rely on relative positions and velocities, however we cannot apply Lemma 3.1 in paper [27] due to the lack of symmetry between the leader and followers.

3.2. Flocking with multiple leaders. We now extend the results to HR systems with multiple leaders. We consider

Definition 3.4. (HR with multiple leaders) A HR system with multiple leaders is a self-organized system which contains N agents, with N_1 leaders forming the rank R_1 and N_2 followers forming the rank R_2 as follows: for an agent “ i ” (one of the

leaders), its position x_i and velocity v_i satisfy

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha \sum_{k \in R_1, k \neq i} b_{ij}(\|x_k - x_i\|)(v_k - v_i); \end{aligned} \tag{10}$$

for an agent “j” (one of the followers), its position x_j and velocity v_j satisfy

$$\begin{aligned} \frac{dx_j}{dt} &= v_j \\ \frac{dv_j}{dt} &= \alpha \sum_{l \in R_1 \cup R_2, l \neq j} b_{jl}(\|x_l - x_j\|)(v_l - v_j). \end{aligned} \tag{11}$$

In what follows, the diameters of position and velocity are given by

$$d_{X_{ii}} = \max_{k, l \in R_i} \|x_k - x_l\|, d_{X_{ij}} = \max_{k \in R_i, l \in R_j} \|x_k - x_l\|$$

and

$$d_{V_{ii}} = \max_{k, l \in R_i} \|v_k - v_l\|, d_{V_{ij}} = \max_{k \in R_i, l \in R_j} \|v_i - v_j\|.$$

Theorem 3.5. *If the influence function $b_{ij} = a_{ij}^{CS}$ and $\chi(r)$ satisfies $\int_0^\infty \chi(r) = \infty$, then the system (10)-(11) converges to a flock. If the influence function $b_{ij} = a_{ij}^{MT}$ and $\chi(r)$ satisfies $\int_0^\infty \chi^2(r) = \infty$, then the system (10)-(11) converges to a flock.*

Proof. Using the same proof of Theorem 3.3, we have $D^+d_{X_{12}} \leq d_{V_{12}}$.

For the system (10) of leaders, if the influence function $b_{ij} = a_{ij}^{CS}$ and $\chi(r)$ satisfies $\int_0^\infty \chi(r) = \infty$, we use the argument of theorem 3.2 in [25], or if the influence function $b_{ij} = a_{ij}^{MT}$ and $\chi(r)$ satisfies $\int_0^\infty \chi^2(r) = \infty$, then we use the argument of theorem 3.4 in [27], we can find positive constants $\beta_{11} > 0$ and $d_{11} > 0$ so that $d_{V_{11}}(t) \leq d_{V_{11}}(0)e^{-\beta_{11}t}$ and $d_{X_{11}}(t) < d_{11}$ for all $t \geq 0$.

We now consider $d_{V_{12}}(t)$, when agent “i” is one of the leaders and agent “j” is one of the followers. Then

$$\begin{aligned} \frac{dv_i}{dt} &= \alpha \sum_{k \in R_1, k \neq i} b_{ik}(\|x_k - x_i\|)(v_k - v_i), \\ \frac{dv_j}{dt} &= \alpha \sum_{l \in R_1 \cup R_2, l \neq j} b_{jl}(\|x_l - x_j\|)(v_l - v_j). \end{aligned}$$

From the remark at the end of Section 2, we have $\sum_{k \in R_1} b_{ik} = 1$ and $\sum_{l \in R_1 \cup R_2} b_{jl} = 1$.

Then,

$$\begin{aligned} \frac{d}{dt} \|v_i - v_j\|^2 &= 2 \langle \dot{v}_i - \dot{v}_j, v_i - v_j \rangle \\ &= 2 \langle \alpha \sum_{k \in R_1, k \neq i} b_{ik}(v_k - v_i) - \alpha \sum_{l \in R_1 \cup R_2, l \neq j} b_{jl}(v_l - v_j), v_i - v_j \rangle \\ &= 2 \langle \alpha \sum_{k \in R_1} b_{ik}(v_k - v_i) - \alpha \sum_{l \in R_1 \cup R_2} b_{jl}(v_l - v_j), v_i - v_j \rangle \end{aligned}$$

$$\begin{aligned}
 &= 2\alpha \sum_{k \in R_1} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\
 &= 2\alpha \sum_{k \in R_1, k \neq l} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\
 &\quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle
 \end{aligned}$$

For a given time t , we choose agents “i” and “j” such that $d_{V_{12}} = \|v_i - v_j\|$. We distinguish the following two cases:

Case 1. $d_{V_{12}} \leq d_{V_{11}}$. In this case, we have

$$\begin{aligned}
 &D^+ d_{V_{12}}^2 \\
 &\leq D^+ \|v_i - v_j\|^2 \\
 &= 2\alpha \sum_{k \in R_1} \sum_{l \in R_1 \cup R_2, l \neq k} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\
 &\quad + 2\alpha \sum_{k \in R_1} \sum_{l \in R_1 \cup R_2, l=k} b_{ik} b_{jl} d_{V_{12}} d_{V_{11}} \\
 &\quad - 2\alpha \sum_{k \in R_1} \sum_{l \in R_1 \cup R_2, l=k} b_{ik} b_{jl} d_{V_{12}} d_{V_{11}} \\
 &\quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\
 &\leq 2\alpha d_{V_{12}} d_{V_{11}} - 2\alpha \sum_{k \in R_1} \sum_{l \in R_1 \cup R_2, l=k} b_{ik} b_{jl} d_{V_{12}} d_{V_{11}} \\
 &\quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle .
 \end{aligned}$$

If $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$, then

$$b_{ik}(t)b_{jl}(t) = \frac{1}{N_1} \frac{1}{N_1 + N_2} \chi(\|x_i(t) - x_k(t)\|) \chi(\|x_j(t) - x_l(t)\|);$$

If $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, then

$$\begin{aligned}
 b_{ik}(t)b_{jl}(t) &= \frac{\chi(\|x_i(t) - x_k(t)\|)}{\sum_{m \in R_1} \chi(\|x_i(t) - x_m(t)\|)} \frac{\chi(\|x_j(t) - x_l(t)\|)}{\sum_{n \in R_1 + R_2} \chi(\|x_j(t) - x_n(t)\|)} \\
 &\geq \frac{1}{N_1} \frac{1}{N_1 + N_2} \chi(\|x_i(t) - x_k(t)\|) \chi(\|x_j(t) - x_l(t)\|).
 \end{aligned}$$

Therefore, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, we have

$$\begin{aligned}
 D^+ d_{V_{12}} &\leq \alpha d_{V_{11}} - \alpha \left(1 + \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}})\right) d_{V_{12}} \\
 &\leq \alpha d_{V_{11}}(0) e^{-\beta_{11} t} - \alpha \left(1 + \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}})\right) d_{V_{12}}.
 \end{aligned}$$

Case 2. $d_{V_{12}} > d_{V_{11}}$. Using a similar analysis in **Case 1**, we conclude for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, that

$$D^+ d_{V_{12}}^2 \leq 2\alpha d_{V_{12}}^2 - 2\alpha \sum_{k \in R_1} \sum_{l \in R_1, l=k} b_{ik} b_{jl} d_{V_{12}}^2 - 2\alpha d_{V_{12}}^2$$

$$\begin{aligned}
&= -2\alpha \sum_{k \in R_1} \sum_{l \in R_1, l=k} b_{ik} b_{jl} d_{V_{12}}^2 \\
&\leq -2\alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}}) d_{V_{12}}^2.
\end{aligned}$$

Then $D^+ d_{V_{12}} \leq -\alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}}) d_{V_{12}}$ since $-\alpha(1 + \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}})) < -\alpha \frac{1}{N_j} \chi(d_{11}) \chi(d_{X_{12}})$.

From the analysis above, regardless which case occurs, we have

$$D^+ d_{V_{12}} \leq \alpha d_{V_{11}}(0) e^{-\beta_{11}t} - \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}}) d_{V_{12}}.$$

We now introduce an energy function

$$E(d_{X_{12}}, d_{V_{12}})(t) = d_{V_{12}}(t) + \frac{\alpha}{\beta_{11}} d_{V_{11}}(0) e^{-\beta_{11}t} + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_0^{d_{X_{12}}(t)} \chi(s) ds.$$

Then,

$$\begin{aligned}
&D^+ E(d_{X_{12}}, d_{V_{12}})(t) \\
&\leq D^+ d_{V_{12}}(t) - \alpha d_{V_{11}}(0) e^{-\beta_{11}t} + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{X_{12}}(t)) d_{V_{12}}(t) \leq 0.
\end{aligned}$$

So $E(d_{X_{12}}, d_{V_{12}})(t)$ is decreasing on $[0, +\infty)$, and we obtain

$$\begin{aligned}
&d_{V_{12}}(t) + \frac{\alpha}{\beta_{11}} d_{V_{11}}(0) e^{-\beta_{11}t} + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_0^{d_{X_{12}}(t)} \chi(s) ds \\
&\leq d_{V_{12}}(0) + \frac{\alpha}{\beta_{11}} d_{V_{11}}(0) + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_0^{d_{X_{12}}(0)} \chi(s) ds.
\end{aligned}$$

Then,

$$d_{V_{12}}(t) \leq d_{V_{12}}(0) + \frac{\alpha}{\beta_{11}} d_{V_{11}}(0) + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_{d_{X_{12}}(t)}^{d_{X_{12}}(0)} \chi(s) ds.$$

For $\int_0^\infty \chi(s) ds = \infty$, we can find a constant $d_{12} \geq d_{X_{12}}(0)$ such that $d_{V_{12}}(0) + \frac{\alpha}{\beta_{11}} d_{V_{11}}(0) = \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_{d_{X_{12}}(0)}^{d_{12}} \chi(s) ds$. Then,

$$\begin{aligned}
d_{V_{12}}(t) &\leq -\alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_0^{d_{X_{12}}(t)} \chi(s) ds \\
&\quad + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_{d_{X_{12}}(0)}^{d_{12}} \chi(s) ds + \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_0^{d_{X_{12}}(0)} \chi(s) ds \\
&\leq \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \int_{d_{X_{12}}(t)}^{d_{12}} \chi(s) ds.
\end{aligned}$$

For $t \geq 0$, we can easily deduce that $d_{X_{12}}(t) \leq d_{12}$. Furthermore,

$$\frac{d}{dt} d_{V_{12}} \leq \alpha d_{V_{11}}(0) e^{-\beta_{11}t} - \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{12}) d_{V_{12}}(t). \quad (12)$$

Let $C_{12} = \alpha \frac{1}{N_1 + N_2} \chi(d_{11}) \chi(d_{12})$.

(i). If $C_{12} \neq \beta_{11}$, then (12) implies the following:

$$\begin{aligned} e^{C_{12}t} \frac{d}{dt} d_{V_{12}} &\leq \alpha d_{V_{11}}(0) e^{-\beta_{11}t} e^{C_{12}t} - C_{12} e^{C_{12}t} d_{V_{12}}(t); \\ \frac{d}{dt} e^{C_{12}t} d_{V_{12}} &\leq \alpha d_{V_{11}}(0) e^{-\beta_{11}t} e^{C_{12}t}; \\ d_{V_{12}}(t) &\leq (d_{V_{12}}(0) - \frac{\alpha}{C_{12} - \beta_{11}} d_{V_{11}}(0)) e^{-C_{12}t} + \frac{\alpha}{C_{12} - \beta_{11}} d_{V_{11}}(0) e^{-\beta_{11}t}. \end{aligned}$$

We can easily deduce that

$$0 < (d_{V_{12}}(0) - \frac{\alpha}{C_{12} - \beta_{11}} d_{V_{11}}(0)) e^{-C_{12}t} + \frac{\alpha}{C_{12} - \beta_{11}} d_{V_{11}}(0) e^{-\beta_{11}t}$$

for all $t > 0$.

We can choose constants $\beta_{12} = \min\{C_{12}, \beta_{11}\}$ and

$$0 < A_{12} = 2 \max\{|d_{V_{12}}(0) - \frac{\alpha}{C_{12} - \beta_{11}} d_{V_{11}}(0)|, |\frac{\alpha}{C_{12} - \beta_{11}} d_{V_{11}}(0)|\}$$

such that $d_{V_{12}}(t) \leq A_{12} e^{-\beta_{12}t}$.

(ii). If $C_{12} = \beta_{11}$, similar argument as above, we can find a constant $0 < A_{12}^{**}$ and $0 < \beta_{12}^{**} < \beta_{12}$, such that $d_{V_{12}}(t) \leq A_{12}^{**} e^{-\beta_{12}^{**}t}$.

For $d_{X_{22}} \leq d_{X_{11}} + d_{X_{12}}$ and $d_{V_{22}} \leq d_{V_{11}} + d_{V_{12}}$, we have $d_{X_{22}}(t) < \infty$ for $t > 0$, and $\lim_{t \rightarrow \infty} d_{V_{22}}(t) = 0$. From the above argument, we have $d_X(t) < \infty$ for $t > 0$ and $\lim_{t \rightarrow \infty} d_V(t) = 0$ □

3.3. Cascade flocking. We now extend our results to general hierarchy with ranks. We first make the following observation:

Lemma 3.6. *Let (x_i, v_i) be an arbitrarily given solution of (2) and (3). Define $d_{X_{i,j}} = \max_{k \in R_i, l \in R_j} \|x_k - x_l\|$ and $d_{V_{i,j}} = \max_{k \in R_i, l \in R_j} \|v_k - v_l\|$. If $d_{X_{i,i+1}}(t) < \infty$ and $\lim_{t \rightarrow \infty} d_{V_{i,i+1}}(t) = 0$ for all i , then the system converges to a flock.*

Theorem 3.7. *If the influence function $b_{ij} = a_{ij}^{CS}$ and $\chi(r)$ satisfies $\int_0^\infty \chi(r) = \infty$, then the system (2)-(3) converges to a flock. If the influence function $b_{ij} = a_{ij}^{MT}$ and $\chi(r)$ satisfies $\int_0^\infty \chi^2(r) = \infty$, then the system (2)-(3) converges to a flock.*

Proof. We prove the theorem by induction.

When $K = 2$, from Theorem 3.5, we conclude that the system converges to a flock and $d_{V_{12}} \leq A_{12} e^{-\beta_{12}t}$.

Assume $1 < k \leq m$, and that there exist positive constants $A_{k-1,k}$, $\beta_{k-1,k}$ and d_{kk} such that $d_{V_{k-1,k}} < A_{k-1,k} e^{-\beta_{k-1,k}t}$ and $d_{X_{k-1,k}} < d_{k-1,k}$. We want to show that there exists a constant $A_{m,m+1}$ such that $d_{V_{R_{m+1}R_m}} < A_{m,m+1} e^{-\beta_{m,m+1}t}$.

Choose $i \in R_m$ and $j \in R_{m+1}$. We note that

$$\begin{aligned} \frac{d}{dt} \|v_i - v_j\|^2 &= 2 \langle \dot{v}_i - \dot{v}_j, v_i - v_j \rangle \\ &= 2 \langle \alpha \sum_{k \in R_{m-1} \cup R_m, k \neq i} b_{ik}(\|x_k - x_i\|)(v_k - v_i) \\ &\quad - \alpha \sum_{l \in R_m \cup R_{m+1}, l \neq j} b_{jl}(\|x_l - x_j\|)(v_l - v_j), v_i - v_j \rangle \\ &= 2 \langle \alpha \sum_{k \in R_{m-1} \cup R_m} b_{ik}(\|x_k - x_i\|)(v_k - v_i) \end{aligned}$$

$$\begin{aligned}
& -\alpha \sum_{l \in R_m \cup R_{m+1}} b_{ji}(\|x_l - x_j\|)(v_l - v_j), v_i - v_j \rangle \\
& = 2\alpha \sum_{k \in R_{m-1} \cup R_m} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\
& \quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\
& = 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\
& \quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle.
\end{aligned}$$

For a given time t , we choose an agent “ i ” in rank “ m ” and an agent “ j ” in rank “ $m+1$ ” such that $d_{V_m} := d_{V_{m,m+1}} = \|v_i - v_j\|$. We estimate d_{V_m} in different cases.

Case 1. $\max\{d_{V_{m,m+1}}, d_{V_{m-1,m}}, d_{V_{m-1,m+1}}, d_{V_{m,m}}\} = d_{V_{m-1,m}}$. In this case, we have

$$\begin{aligned}
& D^+ d_{V_{m,m+1}}^2 \\
& = \frac{d}{dt} \|v_i - v_j\|^2 \\
& \leq 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\
& \quad + 2\alpha \sum_{k \in R_m \cup R_{m-1}, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m-1,m}} d_{V_{m,m+1}} \\
& \quad - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m-1,m}} d_{V_{m,m+1}} - 2\alpha d_{V_{m,m+1}}^2 \\
& \leq -2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m,m+1}} d_{V_{m,m+1}} - 2\alpha d_{V_{m,m+1}}^2 \\
& \quad + 2\alpha d_{V_{m-1,m}} d_{V_{m,m+1}}.
\end{aligned}$$

If $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$, then

$$b_{ik}(t)b_{jl}(t) = \frac{1}{N_{m-1} + N_m} \frac{1}{N_m + N_{m+1}} \chi(\|x_i(t) - x_k(t)\|) \chi(\|x_j(t) - x_l(t)\|);$$

If $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, then

$$\begin{aligned}
b_{ik}(t)b_{jl}(t) & = \frac{\chi(\|x_i(t) - x_k(t)\|)}{\sum_{m \in R_{m-1} + R_m} \chi(\|x_i(t) - x_m(t)\|)} \frac{\chi(\|x_j(t) - x_l(t)\|)}{\sum_{n \in R_m + R_{m+1}} \chi(\|x_j(t) - x_n(t)\|)} \\
& \geq \frac{1}{N_{m-1} + N_m} \frac{1}{N_m + N_{m+1}} \chi(\|x_i(t) - x_k(t)\|) \chi(\|x_j(t) - x_l(t)\|)
\end{aligned}$$

Therefore, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$,

$$\begin{aligned}
D^+ d_{V_{m,m+1}} & \leq \alpha d_{V_{m-1,m}} - \alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m,m+1}} - \alpha d_{V_{m,m+1}} \\
& \leq \alpha A_{m-1,m} e^{-\beta_{m-1,m} t} - \alpha \left(1 + \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{mm}) \chi(d_{X_{m,m+1}})}{N_m + N_{m+1}}\right) d_{V_{m,m+1}}(t).
\end{aligned}$$

Case 2. $\max\{d_{V_{m,m+1}}, d_{V_{m-1,m}}, d_{V_{m-1,m+1}}, d_{V_{m,m}}\} = d_{V_{m,m}}$. In this case, we can easily deduce that $d_{V_{m,m}} \leq 2d_{V_{m-1,m}}$. So using a similar argument to that for Case

1, for both $b_{ij}(t) = a_{ij}^{CS}(t)$ and $b_{ij}(t) = a_{ij}^{MT}(t)$, we have

$$D^+ d_{V_{m,m+1}} \leq 2\alpha A_{m-1,m} e^{-\beta_{m-1,m}t} - \alpha \left(1 + \frac{N_m}{N_{m-1} + N_m} \frac{2\chi(d_{m,m})\chi(d_{X_{m,m+1}})}{N_m + N_{m+1}}\right) d_{V_{m,m+1}}(t).$$

Case 3. $\max\{d_{V_{m,m+1}}, d_{V_{m-1,m}}, d_{V_{m-1,m+1}}, d_{V_{m,m}}\} = d_{V_{m,m+1}}$. In this case, we have

$$\begin{aligned} & D^+ d_{V_{m,m+1}}^2 \\ &= \frac{d}{dt} \|v_i - v_j\|^2 \\ &= 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\ &\quad + 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m,m+1}}^2 \\ &\quad - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m,m+1}}^2 - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\ &\leq 2\alpha d_{V_{m,m+1}}^2 - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m,m+1}}^2 - 2\alpha d_{V_{m,m+1}}^2. \end{aligned}$$

Using a similar argument to that for Case 1, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, we have

$$D^+ d_{V_{m,m+1}}^2 \leq -2\alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m-1,m})\chi(d_{X_{m,m+1}})}{N_m + N_{m+1}} d_{V_{m,m+1}}^2.$$

Therefore,

$$D^+ d_{V_{m,m+1}} \leq -\alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m,m})\chi(d_{X_{m,m+1}})}{N_m + N_{m+1}} d_{V_{m,m+1}}.$$

Case 4. $\max\{d_{V_{m,m+1}}, d_{V_{m-1,m+1}}, d_{V_{m-1,m}}, d_{V_{m,m}}\} = d_{V_{m-1,m+1}}$. Using the triangle inequality, we first obtain $d_{V_{m-1,m+1}} \leq d_{V_{m,m+1}} + d_{V_{m-1,m}}$ and then

$$\begin{aligned} & D^+ d_{V_{m,m+1}}^2 \\ &= \frac{d}{dt} \|v_i - v_j\|^2 \\ &= 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle \\ &\quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\ &\leq 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} (d_{V_{R_{m+1}R_m}} + d_{V_{R_m R_{m-1}}}) d_{V_{R_{m+1}R_m}} \\ &\quad + 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} (d_{V_{R_{m+1}R_m}} + d_{V_{R_m R_{m-1}}}) d_{V_{R_{m+1}R_m}} \\ &\quad - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} (d_{V_{R_{m+1}R_m}} + d_{V_{R_m R_{m-1}}}) d_{V_{R_{m+1}R_m}} \\ &\quad - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \end{aligned}$$

$$\begin{aligned} &\leq 2\alpha(d_{V_{R_{m+1}R_m}} + d_{V_{R_m R_{m-1}}})d_{V_{R_{m+1}R_m}} - 2\alpha d_{V_{R_{m+1}R_m}}^2 \\ &\quad - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{R_{m+1}R_m}}^2. \end{aligned}$$

Therefore, using a similar argument to that for Case 1, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, we have

$$\begin{aligned} &D^+ d_{V_{m,m+1}} \\ &\leq \alpha d_{V_{m-1,m}} - \alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{V_{m,m+1}}^2 \\ &\leq \alpha A_{m-1,m} e^{-\beta_{m-1,m} t} - \alpha \left(1 + \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m,m})\chi(d_{X_{m,m+1}})}{N_m + N_{m+1}}\right) d_{V_{m,m+1}}(t). \end{aligned}$$

Let $\bar{A}_{m,m+1} = 2\alpha A_{m-1,m}$ and $B_{m,m+1} = \alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m,m})}{N_m + N_{m+1}}$. From the argument above, regardless which one of the above 4 cases occurs, we have

$$D^+ d_{V_{m,m+1}} \leq \bar{A}_{m,m+1} e^{-\beta_{m-1,m} t} - B_{m,m+1} \chi(d_{X_{m,m+1}}) d_{V_{m,m+1}}(t).$$

We now consider the energy function

$$E_c = d_{V_{m,m+1}}(t) + \frac{\bar{A}_{m,m+1}}{\beta_{m-1,m}} e^{-\beta_{m-1,m} t} + B_{m,m+1} \int_0^{d_{X_{m,m+1}}(t)} \chi(s) ds.$$

There exist positive constants $d_{m,m+1}$ and $C_{m,m+1}$ such that $d_{X_{m,m+1}}(t) \leq d_{m,m+1}$ for all $t \geq 0$. Let $C_{m,m+1} = B_{m,m+1} \chi(d_{m,m+1})$, then

$$D^+ d_{V_{m,m+1}} \leq \bar{A}_{m,m+1} e^{-\beta_{m-1,m} t} - C_{m,m+1} d_{V_{m,m+1}}(t).$$

This implies that

(i) If $C_{m,m+1} \neq \beta_{m-1,m}$, we have

$$\begin{aligned} d_{V_{m,m+1}}(t) &\leq (d_{V_{m,m+1}}(0) - \frac{\bar{A}_{m,m+1}}{C_{m,m+1} - \beta_{m-1,m}}) e^{-C_{m,m+1} t} \\ &\quad + \frac{\bar{A}_{m,m+1}}{C_{m,m+1} - \beta_{m-1,m}} e^{-\beta_{m-1,m} t}. \end{aligned}$$

We choose $0 < A_{m,m+1} = 2 \max\{d_{V_{m,m+1}}(0) - \frac{\bar{A}_{m,m+1}}{C_{m,m+1} - \beta_{m-1,m}}, \frac{\bar{A}_{m,m+1}}{C_{m,m+1} - \beta_{m-1,m}}\}$ and $\beta_{m,m+1} = \min\{C_{m,m+1}, \beta_{m-1,m}\}$, then $d_{V_{m,m+1}} \leq A_{m,m+1} e^{-\beta_{m,m+1} t}$.

(ii) If $C_{m,m+1} = \beta_{m-1,m}$, we can find a constant $0 < A_{m,m+1}^{**}$ and $0 < \beta_{m,m+1}^{**} < \beta_{m,m+1}$, such that $d_{V_{m,m+1}} \leq A_{m,m+1}^{**} e^{-\beta_{m,m+1}^{**} t}$.

Thus we have concluded that $d_{X_{m,m+1}} < \infty$ for all t , and that $\lim_{t \rightarrow \infty} d_{V_{m,m+1}}(t) = 0$. This completes the proof. \square

4. Flocking with free-will. We start with a general model with free-will

$$\begin{aligned} \frac{d}{dt} x_i(t) &= v_i(t), \\ \frac{d}{dt} v_i(t) &= \alpha \sum_{j=1, j \neq i}^N b_{ij} (\|x_j - x_i\|) (v_j(t) - v_i(t)) + f_i(t), \end{aligned} \tag{13}$$

for $i = 1, 2, \dots, N$.

Theorem 4.1. *Assume the function $\chi(t)$ satisfies $\int_0^\infty \chi^2(s)ds = \infty$, and the free-will $f_i(t)$ satisfies $\int_0^\infty \|f_i(t) - f_j(t)\|dt = \delta_{ij} < \infty$ and $\lim_{t \rightarrow \infty} \|f_i(t) - f_j(t)\| = 0$, where $i, j = 1, 2, \dots, N$, δ_{ij} is a positive constant. Then system (13) converges to a flock.*

Proof. Let (x_i, v_i) be an arbitrarily given solution of system (13). For any pair of agents “i” and “j”, their velocities v_i and v_j satisfy

$$\begin{aligned}
 & 2 < \dot{v}_i - \dot{v}_j, v_i - v_j > \\
 & = 2 < \alpha \sum_{k=1, k \neq i}^N b_{ik}(|x_k - x_i|)(v_k - v_i) - \alpha \sum_{l=1, l \neq j}^N b_{jl}(|x_l - x_j|)(v_l - v_j), v_i - v_j > \\
 & \quad + 2 < f_i - f_j, v_i - v_j > \\
 & = 2 < \alpha \sum_{k=1}^N b_{ik}(|x_k - x_i|)(v_k - v_i) - \alpha \sum_{l=1}^N b_{ji}(|x_l - x_j|)(v_l - v_j), v_i - v_j > \\
 & \quad + 2 < f_i - f_j, v_i - v_j > \\
 & = 2\alpha \sum_{k=1}^N \sum_{l=1}^N b_{ik}b_{jl} < v_k - v_l, v_i - v_j > - 2\alpha < v_i - v_j, v_i - v_j > \\
 & \quad + 2 < f_i - f_j, v_i - v_j > \\
 & = 2\alpha \sum_{k=1, k \neq l}^N \sum_{l=1}^N b_{ik}b_{jl} < v_k - v_l, v_i - v_j > - 2\alpha < v_i - v_j, v_i - v_j > \\
 & \quad + 2 < f_i - f_j, v_i - v_j > \\
 & = 2\alpha \sum_{k=1, k \neq l}^N \sum_{l=1}^N b_{ik}b_{jl} < v_k - v_l, v_i - v_j > + 2\alpha \sum_{k=1, k=l}^N \sum_{l=1}^N b_{ik}b_{jl}d_V^2 \\
 & \quad - 2\alpha \sum_{k=1, k=l}^N \sum_{l=1}^N b_{ik}b_{jl}d_V^2 - 2\alpha < v_i - v_j, v_i - v_j > + 2 < f_i - f_j, v_i - v_j > \\
 & \leq 2\alpha d_V^2 - 2\alpha \sum_{k=1, k=l}^N \sum_{l=1}^N b_{ik}b_{jl}d_V^2 - 2\alpha < v_i - v_j, v_i - v_j > + 2 < f_i - f_j, v_i - v_j > .
 \end{aligned}$$

For a given time t, we can choose two agents i and j, such that $d_V = \|v_i - v_j\|$. Then,

$$\begin{aligned}
 D^+ d_V^2 & \leq 2\alpha d_V^2 - 2\alpha \sum_{k=1, k=l}^N \sum_{l=1}^N b_{ik}b_{jl}d_V^2 - 2\alpha d_V^2 + 2 \sum_{i,j \in N} \|f_i - f_j\|d_V \\
 & = - 2\alpha \sum_{k=1, k=l}^N \sum_{l=1}^N b_{ik}b_{jl}d_V^2 + 2 \sum_{i,j \in N} \|f_i - f_j\|d_V .
 \end{aligned}$$

Using a similar argument to that for Lemma 3.2, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, we have

$$D^+ d_V^2 \leq -2\alpha \frac{1}{N} \chi^2(d_X) d_V^2 + 2 \sum_{i,j \in N} \|f_i - f_j\|d_V$$

Therefore,

$$\frac{d}{dt}d_V \leq -\alpha \frac{1}{N} \chi^2(d_X) d_V + \sum_{i,j \in N} \|f_i - f_j\|.$$

Using the function

$$E_2(t) = d_V(t) - \sum_{i,j \in N} \int_0^t \|f_i(s) - f_j(s)\| ds + \alpha \frac{1}{N} \int_0^{d_X} \chi^2(s) ds,$$

we can find a positive constant d_{11}^* such that $d_X \leq d_{11}^*$. For $\chi(r)$ is non-increasing and positive, we have $\chi^2(d_X) \geq \chi^2(d_{11}^*)$. Let $C_{11}^* = \alpha \frac{1}{N} \chi^2(d_{11}^*)$. Then we have

$$\begin{aligned} \frac{d}{dt}d_V &\leq -C_{11}^* d_V + \sum_{i,j \in N} \|f_i - f_j\|; \\ e^{C_{11}^* t} \frac{d}{dt}d_V &\leq -C_{11}^* e^{C_{11}^* t} d_V + e^{C_{11}^* t} \sum_{i,j \in N} \|f_i - f_j\|; \\ \frac{d}{dt}e^{C_{11}^* t} d_V &\leq e^{C_{11}^* t} \sum_{i,j \in N} \|f_i - f_j\|; \\ \int_0^t \frac{d}{ds} e^{C_{11}^* s} d_V &\leq \sum_{i,j \in N} \int_0^t e^{C_{11}^* s} \|f_i(s) - f_j(s)\| ds; \\ d_V(t) &\leq e^{-C_{11}^* t} d_V(0) + \sum_{i,j \in N} e^{-C_{11}^* t} \int_0^t e^{C_{11}^* s} \|f_i(s) - f_j(s)\| ds. \end{aligned}$$

For $\lim_{t \rightarrow \infty} e^{-C_{11}^* t} d_V(0) = 0$ and $\lim_{t \rightarrow \infty} e^{-C_{11}^* t} \int_0^t e^{C_{11}^* s} \|f_i(s) - f_j(s)\| ds = 0$, we have $\lim_{t \rightarrow \infty} d_V(t) = 0$. \square

We finally consider the HR model with free-will (4)-(5). For agent ‘‘i’’ in rank i and agent ‘‘j’’ in rank j , let (x_i, v_i) be a solution of system (4)-(5), we define the diameters of positions and velocities as $d_{\bar{X}_{ij}} = \lim_{k \in R_i, l \in R_j} \|x_k - x_l\|$ and $d_{\bar{V}_{ij}} = \lim_{k \in R_i, l \in R_j} \|v_k - v_l\|$.

Theorem 4.2. *Assume that $\int_0^\infty \chi^2(s) ds = \infty$, and the free-will $f_i(t)$ satisfies $\int_0^\infty \|f_i(t) - f_j(t)\| dt = \delta_{ij} < \infty$ and $\lim_{t \rightarrow \infty} \|f_i(t) - f_j(t)\| = 0$ (where $i, j = 1, 2, \dots, N$, δ_{ij} is a positive constant). Then system (4)-(5) converges to a flock.*

Proof. We want to prove this theorem by induction. Let (x_i, v_i) be a given solution of system (4)-(5).

The case where $k = 1$ is covered by Theorem 4.1, we have $d_{V_{11}}(t) \leq e^{-C_{11}^* t} d_V(0) + \sum_{i,j \in N} e^{-C_{11}^* t} \int_0^t e^{C_{11}^* s} \|f_i(s) - f_j(s)\| ds$, where C_{11}^* is a positive constant.

For $k = 2$, and for any fixed agent ‘‘i’’ ($i \in R_1$) and agent ‘‘j’’ ($j \in R_2$), we have

$$\begin{aligned} &2 < \dot{v}_i - \dot{v}_j, v_i - v_j > \\ &= 2 < \alpha \sum_{k \in R_1, k \neq i} b_{ik}(|x_k - x_i|)(v_k - v_i) \\ &\quad - \alpha \sum_{l \in R_1 \cup R_2, l \neq j} b_{jl}(|x_l - x_j|)(v_l - v_j), v_i - v_j > + 2 < f_i - f_j, v_i - v_j > \end{aligned}$$

$$\begin{aligned}
 &= 2 < \alpha \sum_{k \in R_1} b_{ik}(|x_k - x_i|)(v_k - v_i) - \alpha \sum_{l \in R_1 \cup R_2} b_{jl}(|x_l - x_j|)(v_l - v_j), v_i - v_j > \\
 &\quad + 2 < f_i - f_j, v_i - v_j > \\
 &= 2\alpha \sum_{k \in R_1} \sum_{l \in R_1 \cup R_2} b_{ik}b_{jl} < v_k - v_l, v_i - v_j > - 2\alpha < v_i - v_j, v_i - v_j > \\
 &\quad + 2 < f_i - f_j, v_i - v_j > \\
 &= 2\alpha \sum_{k \in R_1, k \neq l} \sum_{l \in R_1 \cup R_2} b_{ik}b_{jl} < v_k - v_l, v_i - v_j > - 2\alpha < v_i - v_j, v_i - v_j > \\
 &\quad + 2 < f_i - f_j, v_i - v_j >.
 \end{aligned}$$

For a given time t , we can choose a pair of agent i ($i \in R_i$) and agent j ($j \in R_2$) such that $d_{\bar{V}_{12}} = \|v_i - v_j\|$. There are two cases.

Case 1. $d_{\bar{V}_{12}} \leq d_{\bar{V}_{11}}$. In this case, we have

$$\begin{aligned}
 &D^+ d_{\bar{V}_{12}}^2 \\
 &= 2 < \dot{v}_i - \dot{v}_j, v_i - v_j > \\
 &= 2\alpha \sum_{k \in R_1, k \neq l} \sum_{l \in R_1 \cup R_2} b_{ik}b_{jl} < v_k - v_l, v_i - v_j > \\
 &\quad + 2\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik}b_{jl}d_{\bar{V}_{12}}d_{\bar{V}_{11}} - 2\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik}b_{jl}d_{\bar{V}_{12}}d_{\bar{V}_{11}} \\
 &\quad - 2\alpha < v_i - v_j, v_i - v_j > + 2 < f_i - f_j, v_i - v_j > \\
 &\leq 2\alpha d_{\bar{V}_{11}}d_{\bar{V}_{12}} - 2\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik}b_{jl}d_{\bar{V}_{12}}^2 - 2\alpha d_{\bar{V}_{12}}^2 \\
 &\quad + 2 \sum_{i \in R_1, j \in R_2} \|f_i - f_j\|d_{\bar{V}_{12}}.
 \end{aligned}$$

Therefore, Using a similar argument to that for theorem 3.5, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, we have

$$\begin{aligned}
 D^+ d_{\bar{V}_{12}} &\leq \alpha d_{\bar{V}_{11}} - \alpha \left(\frac{\chi(d_{11}^*)\chi(d_{\bar{X}_{12}})}{N_1 + N_2} + 1 \right) d_{\bar{V}_{12}} + \sum_{i \in R_1, j \in R_2} \|f_i - f_j\| \\
 &\leq \alpha d_{\bar{V}_{11}} - \alpha \frac{\chi(d_{11}^*)\chi(d_{\bar{X}_{12}})}{N_1 + N_2} d_{\bar{V}_{12}} + \sum_{i \in R_1, j \in R_2} \|f_i - f_j\|.
 \end{aligned}$$

Let $g_{11}(t) = e^{-C_{11}^*t}d_{\bar{V}_{11}}(0) + \sum_{i,j \in N} e^{-C_{11}^*t} \int_0^t e^{C_{11}^*s} \|f_i(s) - f_j(s)\| ds$. Then, we have $\lim_{t \rightarrow \infty} g_{11}(t) = 0$ and $\int_0^\infty g_{11}(t) dt < \infty$. Define $h_{12}(t) = \alpha g_{11}(t) + \sum_{i \in R_1, j \in R_2} \|f_i(t) - f_j(t)\|$ and $A_{12}^* = \alpha \frac{\chi(d_{11}^*)}{N_1 + N_2}$. As

$$d_{\bar{V}_{11}}(t) \leq e^{-C_{11}^*t}d_{\bar{V}_{11}}(0) + \sum_{i,j \in N} e^{-C_{11}^*t} \int_0^t e^{C_{11}^*s} \|f_i(s) - f_j(s)\| ds,$$

we have

$$\frac{d}{dt} d_{\bar{V}_{12}} \leq h_{12}(t) - \alpha A_{12}^* \chi(d_{\bar{X}_{12}}) d_{\bar{V}_{12}}$$

Case 2. $d_{\bar{V}_{12}} \geq d_{\bar{V}_{11}}$. In this case, we have

$$\begin{aligned} D^+ d_{\bar{V}_{12}}^2 &= 2 \langle \dot{v}_i - \dot{v}_j, v_i - v_j \rangle \\ &= 2\alpha \sum_{k \in R_1, k \neq l} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle + 2\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} d_{\bar{V}_{12}}^2 \\ &\quad - 2\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} d_{\bar{V}_{12}}^2 - 2\alpha \langle v_i - v_j, v_i - v_j \rangle + 2 \langle f_i - f_j, v_i - v_j \rangle \\ &\leq 2\alpha d_{\bar{V}_{12}}^2 - 2\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} d_{\bar{V}_{12}}^2 - 2\alpha d_{\bar{V}_{12}}^2 + 2 \sum_{i \in R_1, j \in R_2} \|f_i - f_j\| d_{\bar{V}_{12}}. \end{aligned}$$

Therefore,

$$D^+ d_{\bar{V}_{12}} \leq -\alpha \sum_{k \in R_1, k=l} \sum_{l \in R_1 \cup R_2} b_{ik} b_{jl} d_{\bar{V}_{12}} + \sum_{i \in R_1, j \in R_2} \|f_i - f_j\|.$$

As $h_{12}(t) > \sum_{i \in R_1, j \in R_2} \|f_i - f_j\|$, we conclude that regardless of Case 1 or Case 2, we have

$$\frac{d}{dt} d_{\bar{V}_{12}} \leq -A_{12}^* \chi(d_{\bar{X}_{12}}) d_{\bar{V}_{12}} + h_{12}(t).$$

From the definition of $h_{12}(t)$, we know that there exists a constant $\gamma_{12} > 0$ such that $\int_0^\infty h_{12}(t) dt \leq \gamma_{12}$. Now we can find a function $E_3(t) = d_{\bar{V}_{12}}(t) - \int_0^t h_{12}(s) ds + A_{12}^* \int_0^{d_{\bar{X}_{12}}(t)} \chi(s) ds$. Using a similar argument to that for Theorem 3.3, we can show that $E_3'(t) \leq 0$ and there exists a constant d_{12}^* such that $d_{\bar{X}_{12}}(t) \leq d_{12}^*$ for $t \in [0, \infty)$. Let $C_{12}^* = A_{12}^* \chi(d_{12}^*) > 0$. Then,

$$\frac{d}{dt} d_{\bar{V}_{12}} \leq -C_{12}^* d_{\bar{V}_{12}} + h_{12}(t). \tag{14}$$

From this, we deduce $\lim_{t \rightarrow \infty} d_{\bar{V}_{12}}(t) = 0$.

Assume for all k with $1 < k \leq m$, we have shown that there exist constants $d_{m-1,m}^*, C_{m-1,m}^*, \gamma_{m-1,m}$ and a function $h_{m-1,m}(t) \geq 0$ such that $\int_0^\infty h_{m-1,m}(t) dt \leq \gamma_{m-1,m}$ and $\lim_{t \rightarrow \infty} h_{m-1,m}(t) = 0, d_{\bar{X}_{m-1,m}}(t) \leq d_{m-1,m}^*$ and

$$D^+ d_{\bar{V}_{m-1,m}} \leq -C_{m-1,m}^* d_{\bar{V}_{m-1,m}} + h_{m-1,m}(t),$$

and

$$d_{\bar{V}_{m-1,m}}(t) \leq e^{-C_{m-1,m}^* t} d_{\bar{V}_{m-1,m}}(0) + e^{-C_{m-1,m}^* t} \int_0^t e^{C_{m-1,m}^* s} h_{m-1,m}(s) ds.$$

Now we consider $k = m + 1$, we want to prove that there exists a constant $d_{m,m+1}^*$ such that $d_{\bar{X}_{m,m+1}}(t) \leq d_{m,m+1}^*$ for all $t > 0$ and $\lim_{t \rightarrow \infty} d_{\bar{V}_{m-1,m}}(t) = 0$. For an agent “i” in rank m and a agent “j” in rank $m+1$, we have

$$\begin{aligned} &2 \langle \dot{v}_i - \dot{v}_j, v_i - v_j \rangle \\ &= 2 \langle \alpha \sum_{k \in R_{m-1} \cup R_m, k \neq i} b_{ik} (|x_k - x_i|) (v_k - v_i) \end{aligned}$$

$$\begin{aligned}
 & -\alpha \sum_{l \in R_m \cup R_{m+1}, l \neq j} b_{jl}(|x_l - x_j|)(v_l - v_j), v_i - v_j \rangle + 2 \langle f_i - f_j, v_i - v_j \rangle \\
 = & 2 \langle \alpha \sum_{k \in R_{m-1} \cup R_m} b_{ik}(|x_k - x_i|)(v_k - v_i) \\
 & -\alpha \sum_{l \in R_m \cup R_{m+1}} b_{ji}(|x_l - x_j|)(v_l - v_j), v_i - v_j \rangle + 2 \langle f_i - f_j, v_i - v_j \rangle \\
 = & 2\alpha \sum_{k \in R_{m-1} \cup R_m, l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\
 & + 2 \langle f_i - f_j, v_i - v_j \rangle \\
 = & 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\
 & + 2 \langle f_i - f_j, v_i - v_j \rangle .
 \end{aligned}$$

For a given time t , we can choose agents “ i ” and “ j ”, such that $d_{\bar{V}_{m,m+1}} = \|v_i - v_j\|$. We consider $\max\{d_{\bar{V}_{m,m+1}}, d_{\bar{V}_{m-1,m+1}}, d_{\bar{V}_{m-1,m}}, d_{\bar{V}_{m,m}}\}$, one of the following situations will occur. Using a similar argument to that for theorem 3.5, for both $b_{ik}(t)b_{jl}(t) = a_{ik}^{CS}(t)a_{jl}^{CS}(t)$ and $b_{ik}(t)b_{jl}(t) = a_{ik}^{MT}(t)a_{jl}^{MT}(t)$, we have

Caes 1. $\max\{d_{\bar{V}_{m,m+1}}, d_{\bar{V}_{m-1,m+1}}, d_{\bar{V}_{m-1,m}}, d_{\bar{V}_{m,m}}\} = d_{\bar{V}_{m-1,m}}$. In this case, we have

$$\begin{aligned}
 & D^+ d_{\bar{V}_{m,m+1}}^2 \\
 = & 2 \langle \dot{v}_i - \dot{v}_j, v_i - v_j \rangle \\
 = & 2\alpha \sum_{k \in R_{m-1} \cup R_m, k \neq l \in R_m \cup R_{m+1}} b_{ik} b_{jl} \langle v_k - v_l, v_i - v_j \rangle - 2\alpha \langle v_i - v_j, v_i - v_j \rangle \\
 & + 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{\bar{V}_{m-1,m}} d_{\bar{V}_{m,m+1}} \\
 & - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{\bar{V}_{m-1,m}} d_{\bar{V}_{m,m+1}} + 2 \langle f_i - f_j, v_i - v_j \rangle \\
 \leq & 2\alpha d_{\bar{V}_{m-1,m}} d_{\bar{V}_{m,m+1}} - 2\alpha d_{\bar{V}_{m,m+1}}^2 + 2 \sum_{i,j} \|f_i - f_j\| d_{\bar{V}_{m,m+1}} \\
 & - 2\alpha \sum_{k \in R_{m-1} \cup R_m, k=l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{\bar{V}_{m,m+1}}^2 .
 \end{aligned}$$

Then,

$$\begin{aligned}
 & D^+ d_{\bar{V}_{m,m+1}} \\
 \leq & \alpha d_{\bar{V}_{m-1,m}} + \sum_{i,j} \|f_i - f_j\| - \alpha \left(\sum_{k \in R_{m-1} \cup R_m, k=l \in R_m \cup R_{m+1}} b_{ik} b_{jl} + 1 \right) d_{\bar{V}_{m,m+1}} \\
 \leq & \alpha d_{\bar{V}_{m-1,m}} + \sum_{i,j} \|f_i - f_j\| - \alpha \sum_{k \in R_{m-1} \cup R_m, k=l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{\bar{V}_{m,m+1}} \\
 \leq & \alpha d_{\bar{V}_{m-1,m}} + \sum_{i,j} \|f_i - f_j\| - \alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m-1,m}^*) \chi(d_{\bar{X}_{m,m+1}})}{N_m + N_{m+1}} d_{\bar{V}_{m,m+1}} .
 \end{aligned}$$

Caes 2. $\max\{d_{\bar{V}_{m,m+1}}, d_{\bar{V}_{m-1,m+1}}, d_{\bar{V}_{m-1,m}}, d_{\bar{V}_{m,m}}\} = d_{\bar{V}_{m,m}}$. Clearly, $d_{\bar{V}_{m,m}} \leq 2d_{\bar{V}_{m-1,m}}$. Then from Caes 1, we can obtain

$$\begin{aligned} & D^+ d_{\bar{V}_{m,m+1}} \\ & \leq 2\alpha d_{\bar{V}_{m-1,m}} + \sum_{i,j} \|f_i - f_j\| - 2\alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m-1,m}^*)\chi(d_{\bar{X}_{m,m+1}})}{N_m + N_{m+1}} d_{\bar{V}_{m,m+1}} \end{aligned}$$

Caes 3. $\max\{d_{\bar{V}_{m,m+1}}, d_{\bar{V}_{m-1,m+1}}, d_{\bar{V}_{m-1,m}}, d_{\bar{V}_{m,m}}\} = d_{\bar{V}_{m,m+1}}$. We have

$$\begin{aligned} & D^+ d_{\bar{V}_{m,m+1}} \\ & \leq \sum_{i,j} \|f_i - f_j\| - \alpha \sum_{k \in R_{m-1} \cup R_m, k=l} \sum_{l \in R_m \cup R_{m+1}} b_{ik} b_{jl} d_{\bar{V}_{m,m+1}} \\ & \leq \sum_{i,j} \|f_i - f_j\| - \alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m-1,m}^*)\chi(d_{\bar{X}_{m,m+1}})}{N_m + N_{m+1}} d_{\bar{V}_{m,m+1}} \end{aligned}$$

Caes 4. $\max\{d_{\bar{V}_{m,m+1}}, d_{\bar{V}_{m-1,m+1}}, d_{\bar{V}_{m-1,m}}, d_{\bar{V}_{m,m}}\} = d_{\bar{V}_{m-1,m+1}}$. For $d_{\bar{V}_{m-1,m+1}} \leq d_{\bar{V}_{m,m+1}} + d_{\bar{V}_{m-1,m}}$. Using a similar argument from that for Case 1, we can easily deduce

$$\begin{aligned} & D^+ d_{\bar{V}_{m,m+1}} \\ & \leq \alpha d_{\bar{V}_{m-1,m}} + \sum_{i,j} \|f_i - f_j\| - \alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m-1,m}^*)\chi(d_{\bar{X}_{m,m+1}})}{N_m + N_{m+1}} d_{\bar{V}_{m,m+1}} \end{aligned}$$

Let $h_{m,m+1}(t) = 2\alpha d_{\bar{V}_{m-1,m}} + \sum_{i,j} \|f_i - f_j\|$ and $A_{m,m+1}^* = \alpha \frac{N_m}{N_{m-1} + N_m} \frac{\chi(d_{m-1,m}^*)}{N_m + N_{m+1}}$.

From the analysis above, regardless which one of the above cases above occurs, $d_{\bar{V}_{m,m+1}}(t)$ satisfies

$$D^+ d_{\bar{V}_{m,m+1}} \leq h_{m,m+1}(t) - A_{m,m+1}^* \chi(d_{\bar{X}_{m,m+1}}) d_{\bar{V}_{m,m+1}}.$$

Then we can use the function

$$E_4(t) = d_{\bar{V}_{m,m+1}}(t) - \int_0^t h_{m,m+1}(s) ds + A_{m,m+1} \int_0^{d_{\bar{X}_{m,m+1}}(t)} \chi(s) ds$$

Easily we can prove that $E_4'(t) \leq 0$. Also we can find a constant $\gamma_{m,m+1} > 0$ such that

$$\int_0^\infty (e^{-C_{m-1,m}^* t} d_{\bar{V}_{m-1,m}}(0) + e^{-C_{m-1,m}^* t} \int_0^t e^{C_{m-1,m}^* s} h_{m-1,m}(s) ds) dt \leq \gamma_{m,m+1}.$$

Then we can deduce that $\int_0^\infty h_{m,m+1}(t) dt < \infty$. Using a similar argument to that for theorem 3.3, we can find a constant $d_{m,m+1}^* > 0$, such that $d_{\bar{X}_{m,m+1}}(t) < d_{m,m+1}^*$ for all $t > 0$. and $\lim_{t \rightarrow \infty} d_{\bar{V}_{m,m+1}}(t) = 0$. By Lemma 3.6, we know that the system converges to a flock. \square

5. Conclusion and remark. We considered here a general self-organized system with leadership and free-will, and we show that the system converges to a flock under very generate conditions on the hierarchy structure of the system, the free-will responding to the external environment, and mutual influence with the hierarchy.

We emphasize that our sufficient condition in Theorem 3.7 for flocking is very sharp. Consider an example of system (2)- (3), where we assume 11 agents moving

in a 2-dimensional space, where individual influence is governed by the influence function $b_{ij}(t) = a_{ij}^{CS}(t)$. The group is divided into 4 ranks: the first and second rank each has 2 agents, the third rank has 4 agents, and the 4th has 3 agents. Let $\beta = \frac{1}{3}$, so the condition $\int_0^\infty \chi(r) = \infty$ is met, and let $\alpha = 0.5$. Figure 2 illustrates that the system converges to a flock, however, Figure 3 shows that flocking is no longer true when $\beta = 0.52, \alpha = 0.5$.

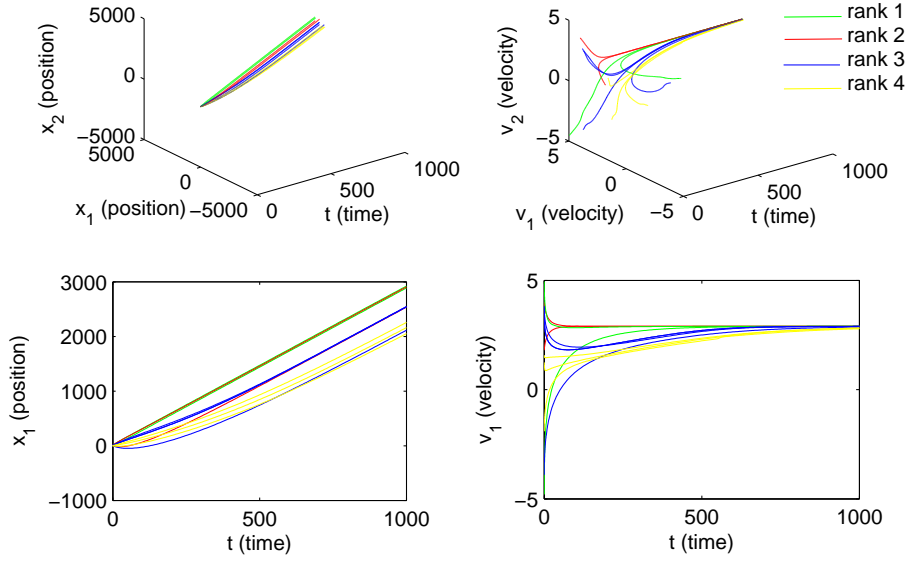


FIGURE 2. Flocking in a HR model with 11 agents: the parameter values are $\alpha = 0.5, \beta = 1/3$ and the condition $\int_0^\infty \chi(r) = \infty$ is met.

Similar remarks apply to Theorem 4.1, as illustrated in Figure 4 and Figure 5. In system (13), we consider the case where the system has 7 agents moving in 2 plane. Again, we use the influence function $b_{ij} = a_{ij}^{CS}$. When $\beta = \frac{1}{3}$, the condition $\int_0^\infty \chi^2(r)dr = \infty$ in Theorem 4.1 holds. We first consider the case where the free-will functions $\{f_i(t)\}_{i=1}^7$ are given by $f_1(t) = \langle e^{-t}, 0 \rangle$, $f_3(t) = \langle 0, \frac{1}{t^2} \rangle$ and others being zero. These free-will functions satisfy the condition in Theorem 4.1, and Figure 4 shows that such a system converges to a flock. However, when $f_1(t) = \langle \cos t, 0 \rangle$, $f_2(t) = \langle 0, \sin t \rangle$, $f_7(t) = \langle \frac{1}{t^2}, 0 \rangle$ and other free-will function zero, we have $\int_0^\infty \cos t dt < \infty$, while $\int_0^\infty \|\cos t\| dt = \infty$. These given free-will functions do not meet the condition in Theorem 4.1, and agents' velocities are not convergent to the same, as shown in Figure 5.

We conclude with a final example to demonstrate Theorem 4.2, Figures 6 and 7. In system (4)-(5), we consider the case where the system has 11 agents, with movement in the plane. The group is divided into 4 ranks, the first and second rank each has 2 agents, the third rank has 4 agents and the fourth rank has 3 agents. Set the influence function as $b_{ij} = a_{ij}^{CS}$. We choose the parameters $\beta = \frac{1}{3}$ so the condition $\int_0^\infty \chi^2(r) = \infty$ in Theorem 4.2 is met. In the Figure 6, we set

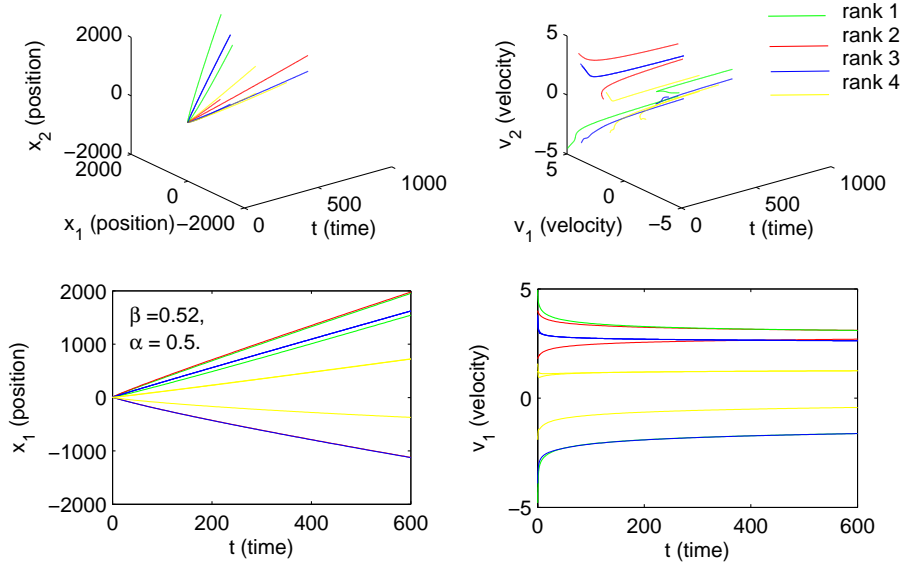


FIGURE 3. Flocking in a HR model with 11 agents is no longer true: the parameter values are $\alpha = 0.5, \beta = 0.52$ so the condition $\int_0^\infty \chi(r) = \infty$ is not satisfied.

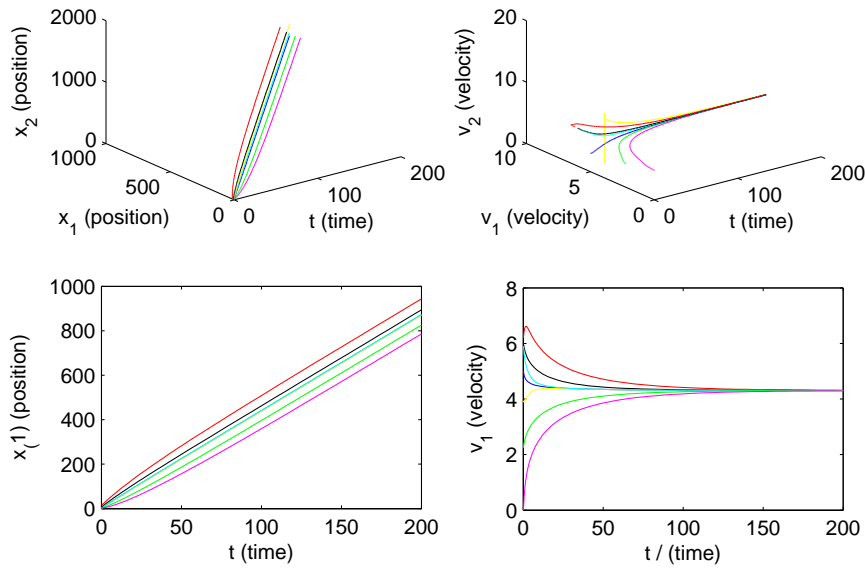


FIGURE 4. An illustrative example of flocking under free-will. The parameter values are $\alpha = 0.5, \beta = 1/3$.

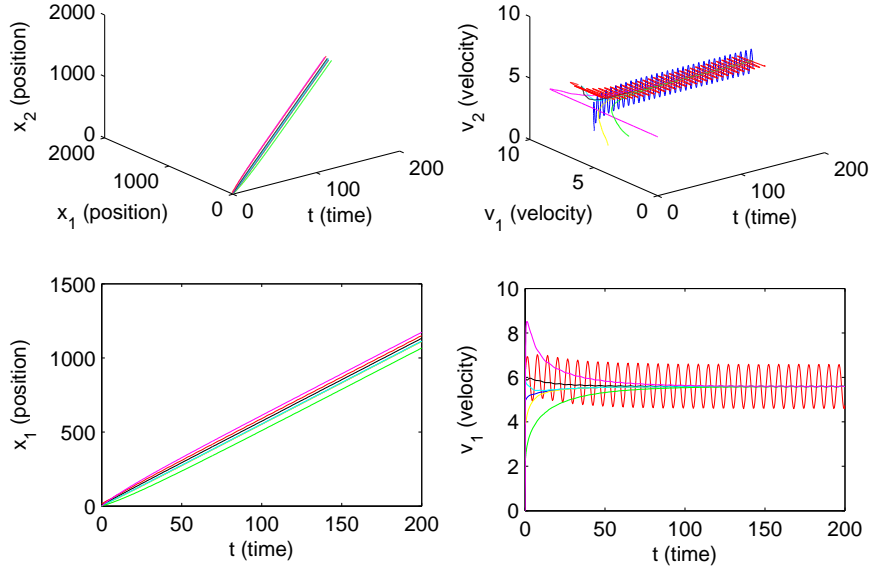


FIGURE 5. An illustrative example of CS-model with free will, where $\int_0^\infty \cos t dt < \infty$ while $\int_0^\infty \|\cos t\| dt = \infty$.

$f_1(t) = \langle e^{-t}, 0 \rangle$, $f_3(t) = \langle e^{-t} \cos t, \frac{1}{t^2} \rangle$, $f_6 = \langle 0, \sin t \frac{1}{t^2} \rangle$, $f_{10} = \langle e^{-t}, 0 \rangle$ and set other free-will functions as zero. These free-will functions satisfy the conditions in Theorem 4.1, so this system converges to a flock. However, when we set $f_1(t) = \langle e^{-t}, -\sin t \rangle$, $f_3(t) = \langle \cos t, \frac{1}{t^2} \rangle$, $f_6 = \langle e^{-t}, \sin t \rangle$, $f_{10} = \langle \cos t, e^{-t} \rangle$ and other free-will functions as zero, we note that $\int_0^\infty (e^{-t} - \cos t) dt < \infty$ while $\int_0^\infty \|\cos t\| dt = \infty$. Thus the free-will functions do not satisfy the conditions in Theorem 4.1, and Figure 7 shows some interesting behaviours: the velocities can not converge to a same function but the distance between any two agents remains bounded.

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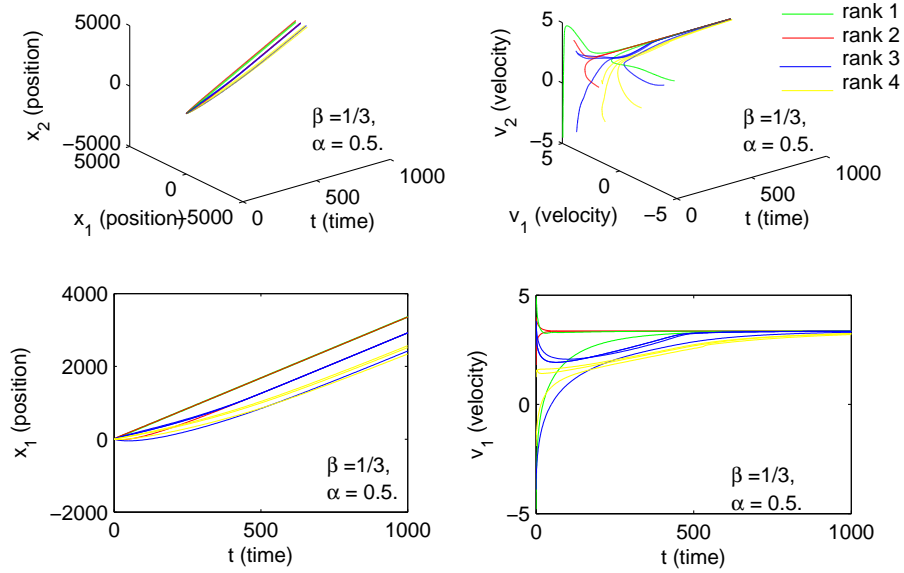


FIGURE 6. Flocking in a general HR model with free will: the parameter values are $\alpha = 0.5, \beta = 1/3$.

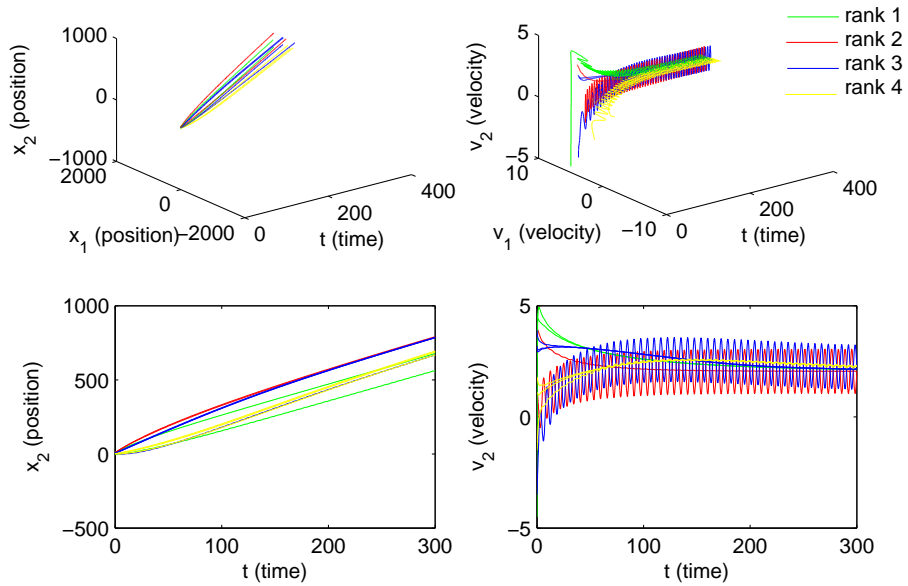


FIGURE 7. Non-flocking of the system, but agents remain bounded away from others: the parameter values are $\alpha = 0.5, \beta = 1/3$.

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