

## FLOCKING AND INVARIANCE OF VELOCITY ANGLES

LE LI

College of Mathematics and Econometrics, Hunan University  
Changsha, Hunan, 410082, China

LIHONG HUANG

College of Mathematics and Econometrics  
Hunan University & Hunan Women's University  
Changsha, Hunan, 410004, China

JIANHONG WU

Laboratory for Industrial and Applied Mathematics  
Department of Mathematics and Statistics  
York University, Toronto, Ontario, M3J 1P3, Canada

(Communicated by Shigui Ruan)

**ABSTRACT.** Motsch and Tadmor considered an extended Cucker-Smale model to investigate the flocking behavior of self-organized systems of interacting species. In this extended model, a cone of the vision was introduced so that outside the cone the influence of one agent on the other is lost and hence the corresponding influence function takes the value zero. This creates a problem to apply the Motsch-Tadmor and Cucker-Smale method to prove the flocking property of the system. Here, we examine the variation of the velocity angles between two arbitrary agents, and obtain a monotonicity property for the maximum cone of velocity angles. This monotonicity permits us to utilize existing arguments to show the flocking property of the system under consideration, when the initial velocity angles satisfy some minor technical constraints.

**1. Introduction.** Flocking, where agents in a network adapt by their relative locations to achieve an uniform velocity, is a universal phenomenon in biological, social and economical systems. Examples include bird migration, fish schooling [12, 13, 4, 3, 2, 8] and emergent economic behavior including common belief in a price system in a complex market environment [5, 3].

Reynolds [10] gave three simple rules for flocking: Separation–avoid crowding neighbors (short range repulsion); Alignment–steer towards average heading of neighbors; Cohesion–steer towards average position of neighbors (long range attraction). Later, Vicsek[14] characterized flocking in terms of bounded distance–individuals stay at bounded distance from each other; and alignment–they all move in the same direction. There were substantial researches including modelling studies on flocking, but with models seemingly too complex to analyze. In 2007, Cucker

---

2010 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

*Key words and phrases.* Flocking, dynamic system, biomathematics.

The first author is supported by the China Scholarship Council for his visiting York University.

The second author is supported by National Natural Science Foundation of China (11371127).

and Smale [4, 3] developed a model, referred to as CS model [4, 3] here, that provides a basic framework to describe how agents interact with each other in order to achieve flocking and this model has inspired very intensive activities to explain self-organized behavior in various complex systems. See [1, 9, 11, 6] and references therein.

The CS model describes how agents interact with each other following the simple rule below[10]:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j=1}^N \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i), \quad (1)$$

where  $\mathbf{x}_i \in R^d$  and  $\mathbf{v}_i \in R^d$  are the location and velocity of the agent “ $i$ ”. In the model,  $\alpha$  is a positive constant, and  $\phi_{ij}$  quantifies the pairwise influence of agent “ $j$ ” on the alignment of agent “ $i$ ” as a function of their distance. More precisely, in the CS model, we have

$$\phi_{ij} = \frac{\phi(\|\mathbf{x}_j - \mathbf{x}_i\|)}{N}, \quad \phi(\|\mathbf{x}_j - \mathbf{x}_i\|) = \frac{1}{(\sigma^2 + \|\mathbf{x}_j - \mathbf{x}_i\|^2)^\beta},$$

where  $\phi$  is given above or, in general, is a strictly positive decreasing function, and  $\beta$  is a parameter. This influence function is symmetric, that is, agent “ $i$ ” and agent “ $j$ ” have the same influence on the alignment of each other ( $\phi_{ij} = \phi_{ji}$ ). Motsch and Tadmor [9] later introduced an influence function, which is non-symmetric and takes into account the relative distance between agents, as follows

$$\phi_{ij} = \frac{\phi(\|\mathbf{x}_j - \mathbf{x}_i\|)}{\sum_{k=1}^N \phi(\|\mathbf{x}_k - \mathbf{x}_i\|)}, \quad (2)$$

with  $\phi$  as defined in the CS model. In their celebrated work [9], Motsch and Tadmor also called attention to a more general situation in which signal transmission is via vision. In this configuration, it is possible that agent “ $i$ ” can see agent “ $j$ ”, but agent “ $j$ ” may fail to see agent “ $i$ ” outside a cone of vision.

Here we show that in this revised CS-model and MT-model, flocking is still achieved. We provide a proof based on the “cone invariance” which implies that self-organization does keep all agents within the cone of version and hence the influence remains once initiated. We will formulate the Motsch-Tadmor (MT) model in Section 2, and then establish the cone invariance and flocking in dimension 2 (section 3) and dimension 3 (Section 4).

**2. The model and some preliminaries.** We consider a self-organized group with  $N$  agents. For agent “ $i$ ”, its position is denoted by  $\mathbf{x}_i \in \mathbf{R}^d$  and its velocity by  $\mathbf{v}_i \in \mathbf{R}^d$ , where  $d > 1$  is an integer. Motsch and Tadmor proposed the revised CS and/or MT model which is incorporating a cone of vision[9]:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j=1}^N \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i). \quad (3)$$

Here,  $\alpha (\alpha > 0)$  measures the interaction strength,  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i)$  determines whether the agent “ $j$ ” can be “seen” by the agent “ $i$ ” who heads in direction  $\omega_i := \mathbf{v}_i / \|\mathbf{v}_i\|$ :

$$\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = \begin{cases} 1, & \text{if } \omega_i \cdot \frac{\mathbf{x}_j - \mathbf{x}_i}{\|\mathbf{x}_j - \mathbf{x}_i\|} \geq \gamma > -1 \\ 0, & \text{if otherwise} \end{cases}$$

with  $2\gamma$  being the angle of the cone of vision. The  $\phi'_{ij}$ s determine the pairwise alignment within the cone of vision [9], and is defined similarly to the CS or MT model. Namely, either

$$\phi_{ij} = \frac{1}{N_i} \phi(\|\mathbf{x}_j - \mathbf{x}_i\|), \quad N_i := \{j \neq i | \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = 1\},$$

or

$$\phi_{ij} = \frac{\phi(\|\mathbf{x}_j - \mathbf{x}_i\|)}{\sum_{k \in N_i} \phi(\|\mathbf{x}_k - \mathbf{x}_i\|)}. \quad j \in N_i.$$

Here the influence function  $\phi$  is defined as above.

We can rewrite the model as follows:

$$\frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i), \quad a_{ij} = \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) \phi_{ij}. \quad (4)$$

In the CS model, we let  $a_{ii} = 1 - \sum_{j \in N_i} a_{ij}$  and  $\widetilde{N}_i = \{i\} + N_i$ . Then

$$\begin{aligned} \frac{d\mathbf{v}_i}{dt} &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i) \\ &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i) + \alpha a_{ii}(\mathbf{v}_i - \mathbf{v}_i) \\ &= \alpha \sum_{j \in \widetilde{N}_i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i). \end{aligned}$$

From the equation  $\sum_{j \in \widetilde{N}_i} a_{ij} = 1$  and  $\sum_{j \in N_i} a_{ij} \leq 1$ , we can easily deduce  $a_{ii} \geq 0$  and

$$\sum_{j \in \widetilde{N}_i} \sum_{l \in \widetilde{N}_k} a_{ij} a_{kl} = \sum_{j \in \widetilde{N}_i} a_{ij} \left( \sum_{l \in \widetilde{N}_k} a_{kl} \right) = 1.$$

Then,

$$\begin{aligned} \frac{d\mathbf{v}_p}{dt} &= \alpha \sum_{j \in \widetilde{N}_p} a_{pj}(\mathbf{v}_j - \mathbf{v}_p) \\ &= \alpha \sum_{j \in \widetilde{N}_p} a_{pj} \mathbf{v}_j - \alpha \sum_{j \in \widetilde{N}_p} a_{ij} \mathbf{v}_p \\ &= \alpha \sum_{j \in \widetilde{N}_p} a_{pj} \mathbf{v}_j - \alpha \mathbf{v}_p \left( \sum_{j \in \widetilde{N}_p} a_{ij} \right) \\ &= \alpha \sum_{j \in \widetilde{N}_p} a_{pj} \mathbf{v}_j - \alpha \mathbf{v}_p. \end{aligned}$$

In the MT model, we can easily deduce  $\sum_{j \in \widetilde{N}_i} a_{ij} = 1$ , so we also have

$$\sum_{j \in \widetilde{N}_i} \sum_{l \in \widetilde{N}_k} a_{ij} a_{kl} = \sum_{j \in \widetilde{N}_i} a_{ij} \left( \sum_{l \in \widetilde{N}_k} a_{kl} \right) = 1,$$

and

$$\begin{aligned}
\frac{d\mathbf{v}_p}{dt} &= \alpha \sum_{j \in \tilde{N}_p} a_{pj} (\mathbf{v}_j - \mathbf{v}_p) \\
&= \alpha \sum_{j \in \tilde{N}_p} a_{pj} \mathbf{v}_j - \alpha \sum_{j \in \tilde{N}_p} a_{ij} \mathbf{v}_p \\
&= \alpha \sum_{j \in \tilde{N}_p} a_{pj} \mathbf{v}_j - \alpha \mathbf{v}_p \left( \sum_{j \in \tilde{N}_p} a_{ij} \right) \\
&= \alpha \sum_{j \in \tilde{N}_p} a_{pj} \mathbf{v}_j - \alpha \mathbf{v}_p.
\end{aligned}$$

**Definition 2.1.** (See [7] p. 416). Let  $\{(\mathbf{x}_i, \mathbf{v}_i)\}_{i=1}^N$  be a solution of system (3), and let  $d_X(t)$  and  $d_V(t)$  denote its diameters in position and velocity phase space, which are given by  $d_X(t) = \max\{\|x_i(t) - x_j(t)\|\}$  and  $d_V(t) = \max\{\|v_i(t) - v_j(t)\|\}$ . If for every such solution, we have

$$\lim_{t \rightarrow \infty} d_X(t) < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} d_V(t) = 0 \quad (5)$$

then we say system (3) converges to a flock. If the above property holds for a particular solution, then we say the solution flocks.

**3. Flocking behavior in 2-D spaces.** When agents under consideration are animals moving on the land such as wolf packs and elephant herds, we can consider the system posed in a 2-dimensional spaces. In this section, we consider the case where  $x_i \in R^2$ ,  $v_i \in R^2$  for a solution of the system (3).

**Lemma 3.1.** (Cone Invariance) Let  $(\mathbf{x}_i, \mathbf{v}_i)$  be the solution of system (3), where  $\mathbf{x}_i \in R^2$  and  $\mathbf{v}_i \in R^2$ . For the group of agents,  $\theta_{ij}(t) = \langle \mathbf{v}_i(t), \mathbf{v}_j(t) \rangle$  is the angle of velocity  $\mathbf{v}_i$  and  $\mathbf{v}_j$ . Denote by  $\bar{\theta}(t) = \max_{i,j} \langle \mathbf{v}_i(t), \mathbf{v}_j(t) \rangle$ . If  $\bar{\theta}(0) = \theta_{ij}(0) < \pi$  for some  $i, j$  and if  $\theta_{ij}(0) = \theta_{ik}(0) + \theta_{kj}(0)$  holds for all such  $i, j$  and for all  $k$ , then we have  $\bar{\theta}(t) \leq \bar{\theta}(0)$  for  $t > 0$ .

*Proof.* As  $\theta_{ij}(t) = \langle \mathbf{v}_i(t), \mathbf{v}_j(t) \rangle$ , we get  $\cos \theta_{ij}(t) = \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|}$ . Then we have

$$\theta_{ij}(t) = \arccos \frac{\mathbf{v}_i(t) \cdot \mathbf{v}_j(t)}{\|\mathbf{v}_i(t)\| \cdot \|\mathbf{v}_j(t)\|}, \quad \bar{\theta}(t) = \max_{i,j \in N} \arccos \frac{\mathbf{v}_i(t) \cdot \mathbf{v}_j(t)}{\|\mathbf{v}_i(t)\| \cdot \|\mathbf{v}_j(t)\|}.$$

For any given  $t$ , there exist two agents  $i_0 = i_0(t)$  and  $j_0 = j_0(t)$  such that  $\bar{\theta}(t) = \theta_{i_0 j_0}(t) = \langle \mathbf{v}_{i_0}(t), \mathbf{v}_{j_0}(t) \rangle$ . We have

$$\theta'_{i_0 j_0}(t) = \left( \arccos \frac{\mathbf{v}_{i_0}(t) \cdot \mathbf{v}_{j_0}(t)}{\|\mathbf{v}_{i_0}(t)\| \cdot \|\mathbf{v}_{j_0}(t)\|} \right)' = - \frac{1}{\sqrt{1 - \left( \frac{\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0}}{\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|} \right)^2}} \cdot g(t),$$

where

$$g(t) = \frac{(\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0})' \cdot (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|) - (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|)' \cdot (\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0})}{\|\mathbf{v}_{i_0}\|^2 \cdot \|\mathbf{v}_{j_0}\|^2}.$$

Here and in what follows,  $\theta'_{i_0 j_0}(t)$  means to fix  $i_0$  and  $j_0$  and take the derivative of the function  $\theta_{i_0 j_0}(t)$  with respect to  $t$ . In other words,  $\theta'_{i_0 j_0}(t) = \theta'_{i,j}(t)|_{i=i_0, j=j_0}$ .

In the case of the CS-model, we have

$$\begin{aligned} \frac{d\mathbf{v}_{i_0}}{dt} &= \alpha \sum_{k \in \widetilde{N}_{i_0}} a_{i_0 k} \mathbf{v}_k - \alpha \mathbf{v}_{i_0}, \\ \frac{d\mathbf{v}_{j_0}}{dt} &= \alpha \sum_{l \in \widetilde{N}_{j_0}} a_{j_0 l} \mathbf{v}_l - \alpha \mathbf{v}_{j_0}, \\ \frac{d\|\mathbf{v}_{i_0}\|^2}{dt} &= 2 \langle \dot{\mathbf{v}}_{i_0}, \mathbf{v}_{i_0} \rangle = 2\alpha \left( \sum_{k \in \widetilde{N}_{i_0}} a_{i_0 k} \mathbf{v}_k - \mathbf{v}_{i_0} \right) \cdot \mathbf{v}_{i_0}. \end{aligned}$$

Hence, we get

$$\begin{aligned} \frac{d\|\mathbf{v}_{i_0}\|}{dt} &= \alpha \frac{1}{\|\mathbf{v}_{i_0}\|} \left( \sum_{k \in \widetilde{N}_{i_0}} a_{i_0 k} \mathbf{v}_k - \mathbf{v}_{i_0} \right) \cdot \mathbf{v}_{i_0} = \alpha \left( \sum_{k \in \widetilde{N}_{i_0}} a_{i_0 k} \frac{\mathbf{v}_k \cdot \mathbf{v}_{i_0}}{\|\mathbf{v}_{i_0}\|} - \|\mathbf{v}_{i_0}\| \right), \\ \frac{d\|\mathbf{v}_{j_0}\|}{dt} &= \alpha \frac{1}{\|\mathbf{v}_{j_0}\|} \left( \sum_{l \in \widetilde{N}_{j_0}} a_{j_0 l} \mathbf{v}_l - \mathbf{v}_{j_0} \right) \cdot \mathbf{v}_{j_0} = \alpha \left( \sum_{l \in \widetilde{N}_{j_0}} a_{j_0 l} \frac{\mathbf{v}_l \cdot \mathbf{v}_{j_0}}{\|\mathbf{v}_{j_0}\|} - \|\mathbf{v}_{j_0}\| \right). \end{aligned}$$

Let

$$f(t) = (\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0})' \cdot (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|) - (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|)' \cdot (\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0}).$$

Then

$$\begin{aligned} f(t) &= \alpha \left( \sum_{k \in \widetilde{N}_{i_0}} a_{i_0 k} \mathbf{v}_k \mathbf{v}_{j_0} + \sum_{l \in \widetilde{N}_{j_0}} a_{j_0 l} \mathbf{v}_l \mathbf{v}_{i_0} - 2\mathbf{v}_{i_0} \mathbf{v}_{j_0} \right) \cdot \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| \\ &\quad - \alpha \left( \sum_{k \in \widetilde{N}_{i_0}} a_{i_0 k} \frac{\mathbf{v}_k \cdot \mathbf{v}_{i_0}}{\|\mathbf{v}_{i_0}\|} \|\mathbf{v}_{j_0}\| + \sum_{l \in \widetilde{N}_{j_0}} a_{j_0 l} \frac{\mathbf{v}_l \cdot \mathbf{v}_{j_0}}{\|\mathbf{v}_{j_0}\|} \|\mathbf{v}_{i_0}\| \right. \\ &\quad \left. - 2\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| \right) \cdot \mathbf{v}_{i_0} \mathbf{v}_{j_0}, \end{aligned}$$

and

$$\begin{aligned} &(a_{i_0 k} \mathbf{v}_k \mathbf{v}_{j_0}) \cdot \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| - a_{i_0 k} \frac{\mathbf{v}_k \cdot \mathbf{v}_{i_0}}{\|\mathbf{v}_{i_0}\|} \|\mathbf{v}_{j_0}\| \cdot \mathbf{v}_{i_0} \mathbf{v}_{j_0} \\ &= a_{i_0 k} \|\mathbf{v}_k\| \cdot \|\mathbf{v}_{j_0}\| \cdot \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| \cdot (\cos \theta_{k j_0} - \cos \bar{\theta} \cos \theta_{k i_0}). \end{aligned}$$

In what follows, we try to show that  $\bar{\theta}(t) \leq \bar{\theta}(0)$  for all  $t$  by proving that  $D^+ \bar{\theta}(t) \leq 0$  for all  $t \geq 0$ , where  $D^+$  is the Dini-upper derivative.

Recall that for any given  $t$ ,  $\exists i_0 = i_0(t) \in N$ ,  $j_0 = j_0(t) \in N$  such that  $\bar{\theta}(t) = \theta_{i_0 j_0}(t)$ . We prove the following

**Claim.**  $\bar{\theta}(t) = \theta_{i_0 k}(t) + \theta_{k j_0}(t)$  for every  $k \in N$ .

We prove this claim by contradiction. Note that for a given  $k$ , we have only two cases: either **Case 1:**  $\theta_{i_0 j_0}(t) = \theta_{i_0 k}(t) + \theta_{k j_0}(t)$ ; or **Case 2:**  $\theta_{i_0 j_0}(t) + \theta_{i_0 k}(t) + \theta_{k j_0}(t) = 2\pi$ . If Case 2 occurs, then  $\exists t_0 > 0$ ,  $i_0^* \in N$ ,  $j_0^* \in N$  such that  $\bar{\theta}(t_0) = \theta_{i_0^* j_0^*}(t_0)$  and  $\theta_{i_0^* j_0^*}(t_0) + \theta_{i_0^* k}(t_0) + \theta_{k j_0^*}(t_0) = 2\pi$  for some  $k \in N$ . Assume, without loss of generosity, that  $\theta_{i_0^* j_0^*}(0) = \max\{\theta_{i_0^* j_0^*}(0), \theta_{i_0^* k}(0), \theta_{k j_0^*}(0)\}$ . Since  $\bar{\theta}(0) < \pi$ , we must have  $\theta_{i_0^* j_0^*}(0) = \theta_{i_0^* k}(0) + \theta_{k j_0^*}(0) < \pi$ . Using the continuity of  $\theta_{ij}(t)$ , we conclude that there must be the first  $t_0^* \leq t_0$  such that  $\theta_{i_0^* j_0^*}(t_0^*) + \theta_{i_0^* k}(t_0^*) + \theta_{k j_0^*}(t_0^*) = 2\pi$ . Consequently, there must be the first  $t_1 > 0$  and integers  $i_1 \in N$  and  $j_1 \in N$  such that  $\theta_{i_1 k}(t_1) + \theta_{k j_1}(t_1) + \theta_{i_1 j_1}(t_1) = 2\pi$ .

Suppose  $\theta_{j_1 k}(t_1)$  is the largest angle among  $\theta_{i_1 k}(t_1)$ ,  $\theta_{k j_1}(t_1)$  and  $\theta_{i_1 j_1}(t_1)$ . As  $\theta_{k i_1}(0) \leq \bar{\theta}(0) < \pi$ ,  $\theta_{k j_1}(0) \leq \bar{\theta}(0) < \pi$  and  $\theta_{i_1 j_1}(0) \leq \bar{\theta}(0) < \pi$ , we have  $\bar{\theta}(t_1) =$

$\theta_{kj_1}(t_1) = \pi$ . Therefore, we can find time  $0 < t_2 < t_1$  such that  $\theta_{kj_1}(t) = \bar{\theta}(t) < \pi$  and  $\theta_{kj_1}(t) = \theta_{kl}(t) + \theta_{lj_1}(t)$  for all  $l \in N$  and  $t \in (t_2, t_1)$ .

We now prove that  $\theta'_{kj_1}(t) \leq 0$  for  $t \in (t_2, t_1)$  by showing that  $f(t) \geq 0$  on the interval. When  $0 \leq \theta_{kj_1}(t) \leq \frac{\pi}{2}$ ,  $\cos \theta_{lj_1}(t) - \cos \theta_{kj_1}(t) \cos \theta_{kl}(t) \geq 0$  holds obviously. We only need to consider the case where  $\frac{\pi}{2} < \theta_{kj_1}(t) < \pi$  and  $\frac{\pi}{2} \leq \theta_{lj_1}(t) < \theta_{kj_1}(t)$ . Denote by

$$h(\theta_{lj_1}) = \cos \theta_{lj_1} - \cos \theta_{kj_1} \cos \theta_{kl}.$$

We have  $(d/d\theta_{kj_1})h(\theta_{lj_1}) < 0$ . So we have  $h(\theta_{lj_1}(t)) > h(\theta_{kj_1}(t)) = 0$ . Therefore, when  $\frac{\pi}{2} < \theta_{kj_1}(t) < \pi$  and  $\theta_{kj_1}(t) - \frac{\pi}{2} \leq \theta_{lj_1}(t) < \frac{\pi}{2}$ , we have  $\cos \theta_{lj_1}(t), \cos \theta_{kl}(t) > 0$ , and  $h(\theta_{lj_1}(t)) > 0$ . When  $\frac{\pi}{2} < \theta_{kj_1}(t) < \pi$  and  $0 < \theta_{lj_1}(t) < \theta_{kj_1}(t) - \frac{\pi}{2}$ , we get  $\cos \theta_{lj_1}(t) > |\cos \theta_{kl}(t)|$  and hence,  $h(\theta_{lj_1}(t)) > 0$ . Based on the analysis above, we have  $\theta'_{kj_1}(t) \leq 0$  for  $t \in (t_2, t_1)$ , from which we conclude that  $\theta_{kj_1}(t) \leq \theta_{kj_1}(t_2) < \pi$ . Thus,  $\theta_{kj_1}(t_1) \neq \pi$ , a contradiction. This proves the claim.

Then, for any given  $t$ , there exists  $i_* \in N$  and  $j_* \in N$  such that  $\bar{\theta}(t) = \theta_{i_*j_*}(t)$  and that  $\theta_{i_*j_*}(t) = \theta_{i_*k}(t) + \theta_{kj_*}(t)$  for every  $k \in N$ . Using the same argument as above, we conclude that  $\theta'_{i_*j_*}(t) \leq 0$ .

Now we consider  $D^+\bar{\theta}(t)$ . For any given time  $t$ , there exist finitely many  $i \times j \in I \times J = \{i \times j | \theta_{ij}(t) = \bar{\theta}(t), i \in N, j \in N\}$ , infinitely  $h_n$  with  $h_n \geq 0$ ,  $\lim_{n \rightarrow \infty} h_n = 0$  and  $\bar{\theta}(t + h_n) = \theta_{ij}(t + h_n)$ . Then we can find a subsequence  $\{h_{m_n}\}$  of  $\{h_n\}_{n=1}^\infty$  and a fixed  $i^* \times j^* \in I \times J$  such that  $\bar{\theta}(t + h_{m_n}) = \theta_{i^*j^*}(t + h_{m_n})$  holds. Therefore,

$$\begin{aligned} D^+\bar{\theta}(t) &= \lim_{h_{m_n} \rightarrow 0} \sup \frac{\bar{\theta}(t + h_{m_n}) - \bar{\theta}(t)}{h_{m_n}} \\ &= \lim_{h_{m_n} \rightarrow 0} \sup \frac{\theta_{i^*j^*}(t + h_{m_n}) - \theta_{i^*j^*}(t)}{h_{m_n}} = \theta'_{i^*j^*}(t) \leq 0. \end{aligned}$$

This concludes that  $\bar{\theta}(t) \leq \bar{\theta}(0)$  for all  $t > 0$ , completing the proof.  $\square$

Lemma 3.1 ensures that monotonicity of the maximal cone of vision. As will be shown below, this ensures that agents will eventually move towards the same direction. To describe our arguments, for the  $i$ th-agent's position  $\mathbf{x}_i \in R^2$  and its velocity  $\mathbf{v}_i \in R^2$ , we introduce the rectangular coordinates:

$$\begin{aligned} \mathbf{v}_i &= \mathbf{v}_{1_i} + \mathbf{v}_{2_i}, \\ d_X &= \max_{i,j \in N} \|\mathbf{x}_i - \mathbf{x}_j\|, \\ d_{V_n} &= \max_{i,j \in N} \|\mathbf{v}_{n_i} - \mathbf{v}_{n_j}\|, \\ \frac{d\mathbf{v}_{n_i}}{dt} &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{n_j} - \mathbf{v}_{n_i}), \end{aligned}$$

for  $n = 1, 2$ . Here  $a_{ij} = \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i)\phi_{ij}$ .

For  $\mathbf{v}_i = \mathbf{v}_{1_i} + \mathbf{v}_{2_i}$ , we can get

$$\begin{aligned} \frac{d\mathbf{x}_i}{dt} &= \mathbf{v}_i, \\ \frac{d\mathbf{v}_i}{dt} &= \frac{d\mathbf{v}_{1_i}}{dt} + \frac{d\mathbf{v}_{2_i}}{dt} \\ &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{1_j} - \mathbf{v}_{1_i}) + \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{2_j} - \mathbf{v}_{2_i}) \end{aligned} \tag{6}$$

$$\begin{aligned}
 &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{1_j} + \mathbf{v}_{2_j} - \mathbf{v}_{1_i} - \mathbf{v}_{2_i}) \\
 &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i).
 \end{aligned}$$

**Lemma 3.2.** *Let  $\mathbf{x}_i, \mathbf{v}_i \in R^2$  and  $(\mathbf{x}_i, \mathbf{v}_i)$  be a solution of (6). Let  $\bar{\theta}(0) = \theta_{ij}(0)$ , and assume  $\theta_{ij}(0) = \theta_{ik}(0) + \theta_{kj}(0)$  for every other agent  $k \in N$ . If  $\bar{\theta}(0) < 2\gamma - \pi$  with  $\frac{\pi}{2} \leq \gamma < \pi$ , then we have*

$$\frac{d}{dt}d_{Vn} \leq -\alpha \frac{\phi^2(d_X)}{N^2}d_{Vn}, \quad n = 1, 2.$$

*Proof.* By Lemma 3.1, for any  $t \geq 0$ , we have

$$\bar{\theta}(t) \leq \bar{\theta}(0) < 2\gamma - \pi.$$

This implies that any agent “i” and agent “j” satisfy one of the following three situations

- 1).  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = \kappa(\omega_j, \mathbf{x}_i - \mathbf{x}_j) = 1$ ,
- 2).  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = 0$  and  $\kappa(\omega_j, \mathbf{x}_i - \mathbf{x}_j) = 1$ ,
- 3).  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = 1$  and  $\kappa(\omega_j, \mathbf{x}_i - \mathbf{x}_j) = 0$ .

When  $n = 1$ , we can choose the agents  $p$  and  $q$  which satisfy  $d_{V1} = \|\mathbf{v}_{1_p} - \mathbf{v}_{1_q}\|$  for any given  $t$ . We now consider  $\phi_{ij}$  defined in the MT model[9].

If case 1) occurs, we have  $a_{pq} \neq 0$  and  $a_{qp} \neq 0$ . Then

$$\begin{aligned}
 \frac{d}{dt}d_{V1}^2 &= 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \dot{\mathbf{v}}_{1_p} - \dot{\mathbf{v}}_{1_q} \rangle \\
 &= 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in \tilde{N}_p, j \in \tilde{N}_q} a_{pi}a_{qj}(\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle - 2\alpha d_{V1}^2 \\
 &= 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in \tilde{N}_p - \{q\}, j \in \tilde{N}_q - \{p\}} a_{pi}a_{qj}(\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle \\
 &\quad + 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha a_{pq}a_{qp}(\mathbf{v}_{1_q} - \mathbf{v}_{1_p}) \rangle - 2\alpha d_{V1}^2 \\
 &\leq 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in \tilde{N}_p - \{q\}, j \in \tilde{N}_q - \{p\}} a_{pi}a_{qj}(\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle - 2\alpha d_{V1}^2 \\
 &\leq 2\alpha \sum_{i \in \tilde{N}_p, j \in \tilde{N}_q} a_{pi}a_{qj} \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \mathbf{v}_{1_p} \\
 &\quad - \mathbf{v}_{1_q} \rangle - 2\alpha a_{pq}a_{qp} \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, (\mathbf{v}_{1_p} - \mathbf{v}_{1_q}) \rangle - 2\alpha d_{V1}^2 \\
 &= -2\alpha a_{pq}a_{qp} \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \mathbf{v}_{1_p} - \mathbf{v}_{1_q} \rangle \\
 &= -2\alpha a_{pq}a_{qp}d_{V1}^2.
 \end{aligned}$$

If case 2) occurs, then we have  $a_{pq} = 0$  and  $a_{qp} \neq 0$ , and

$$\frac{d}{dt}d_{V1}^2 = 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \dot{\mathbf{v}}_{1_p} - \dot{\mathbf{v}}_{1_q} \rangle$$

$$\begin{aligned}
&= 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in N_p, j \in N_q} a_{pi} a_{qj} (\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle - 2\alpha d_{V1}^2 \\
&= 2 \langle \mathbf{v}_{x_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in N_p - \{k\}, j \in N_q - \{p\}} a_{pi} a_{qj} (\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle \\
&\quad + 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha a_{pk} a_{qp} (\mathbf{v}_{1_k} - \mathbf{v}_{1_p}) \rangle - 2\alpha d_{V1}^2 \\
&\leq 2 \langle \mathbf{v}_{x_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in N_p - \{k\}, j \in N_q - \{p\}} a_{pi} a_{qj} (\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle - 2\alpha d_{V1}^2 \\
&= 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha \sum_{i \in N_p - \{k\}, j \in N_q - \{p\}} a_{pi} a_{qj} (\mathbf{v}_{1_i} - \mathbf{v}_{1_j}) \rangle \\
&\quad + 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha a_{pk} a_{qp} (\mathbf{v}_{1_p} - \mathbf{v}_{1_q}) \rangle \\
&\quad - 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha a_{pk} a_{qp} (\mathbf{v}_{1_p} - \mathbf{v}_{1_q}) \rangle - 2\alpha d_{V1}^2 \\
&\leq 2\alpha \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \mathbf{v}_{1_p} - \mathbf{v}_{1_q} \rangle \\
&\quad - 2 \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \alpha a_{pk} a_{qp} (\mathbf{v}_{1_p} - \mathbf{v}_{1_q}) \rangle - 2\alpha d_{V1}^2 \\
&\leq -2\alpha a_{pk} a_{qp} \langle \mathbf{v}_{1_p} - \mathbf{v}_{1_q}, \mathbf{v}_{1_p} - \mathbf{v}_{1_q} \rangle \\
&\leq -2\alpha a_{pk} a_{qp} d_{V1}^2.
\end{aligned}$$

A similar argument applies when case 3) occurs. So, we have

$$\frac{d}{dt} d_{V1}^2 \leq -2\alpha a_{pq} a_{ql} d_{V1}^2.$$

For the CS model, using a similar argument as that for MT model above, we have

$$\frac{d}{dt} d_{V1}^2 \leq -2\alpha a_{pk} a_{qp} d_{V1}^2.$$

When  $\phi_{ij}$  is defined in the CS model, we have  $a_{ij} = \frac{\phi(\|\mathbf{x}_j - \mathbf{x}_i\|)}{N_i} \geq \frac{\phi(d_X)}{N}$  for any pair  $(i, j)$ . When  $\phi_{ij}$  is defined in the MT model, we have  $a_{ij} = \frac{\phi(\|\mathbf{x}_j - \mathbf{x}_i\|)}{\sum_{k \in N_i} \phi(\|\mathbf{x}_k - \mathbf{x}_i\|)} \geq \frac{\phi(d_X)}{N}$  for any pair  $(i, j)$ . Thus, we have

$$\begin{aligned}
-2\alpha a_{pq} a_{qp} d_{V1}^2 &\leq -2\alpha \frac{\phi^2(d_X)}{N^2} d_{V1}^2, \\
-2\alpha a_{pk} a_{qp} d_{V1}^2 &\leq -2\alpha \frac{\phi^2(d_X)}{N^2} d_{V1}^2, \\
-2\alpha a_{pq} a_{ql} d_{V1}^2 &\leq -2\alpha \frac{\phi^2(d_X)}{N^2} d_{V1}^2.
\end{aligned}$$

Therefore, for both CS model and MT model, we have

$$\frac{d}{dt} d_{V1} \leq -\alpha \frac{\phi^2(d_X)}{N^2} d_{V1}, \quad t > 0.$$

Similarly, we can get

$$\frac{d}{dt} d_{V2} \leq -\alpha \frac{\phi^2(d_X)}{N^2} d_{V2}.$$

This completes the proof.  $\square$

**Theorem 3.3.** *Let  $(\mathbf{x}_i, \mathbf{v}_i)$  be a solution of (3), with  $\mathbf{x}_i, \mathbf{v}_i \in \mathbf{R}^2$ . Assume that the influence function  $\phi(r)$  satisfies  $\int_0^\infty \phi^2(r) = \infty$  and the angle of the cone of vision  $\gamma$  satisfies  $\frac{\pi}{2} \leq \gamma < \pi$ . Furthermore, assume the initial angle of velocities*



satisfies  $\bar{\theta}(0) < 2\gamma - \pi$ , and assume when  $\bar{\theta}(0) = \theta_{ij}(0)$  for some  $(i, j)$ , we have  $\theta_{ij}(0) = \theta_{ik}(0) + \theta_{kj}(0)$  for every  $k \in N$ . Then system (3) converges to a flock.

*Proof.* From Lemma 3.1 and Lemma 3.2 we have

$$\frac{d}{dt}d_{Vn} \leq -\alpha \frac{\phi^2(d_X)}{N^2} d_{Vn}, \quad n = 1, 2.$$

Let  $d_X = \max_{i,j \in N} \|\mathbf{x}_i - \mathbf{x}_j\| = \|\mathbf{x}_k - \mathbf{x}_l\|$  for some  $k$  and  $l$ . Then,

$$\begin{aligned} D^+ d_X^2 &= 2 \langle \mathbf{x}_k - \mathbf{x}_l, \dot{\mathbf{x}}_k - \dot{\mathbf{x}}_l \rangle \\ &= 2 \langle \mathbf{x}_k - \mathbf{x}_l, \mathbf{v}_k - \mathbf{v}_l \rangle \\ &\leq 2 \|\mathbf{x}_k - \mathbf{x}_l\| \|\mathbf{v}_k - \mathbf{v}_l\| \\ &\leq 2d_X d_V. \end{aligned}$$

Hence,  $D^+ d_X \leq d_V$ .

The argument below is similar to that in [9]. Namely, we introduce an energy function  $E(d_X, d_{V1}, d_{V2})(t) = d_{V1}(t) + d_{V2}(t) + \alpha \int_0^{d_X(t)} \Psi(r) dr$ , where  $\Psi(r) = \frac{\phi^2(r)}{N^2}$ . Then

$$\begin{aligned} D^+ E(d_X, d_{V1}, d_{V2})(t) &\leq \dot{d}_{V1}(t) + \dot{d}_{V2}(t) + \alpha \varphi(d_X) D^+ d_X(t) \\ &\leq -\alpha \varphi(d_X) d_{V1}(t) - \alpha \varphi(d_X) d_{V2}(t) + \alpha \varphi(d_X) d_V(t) \\ &\leq -\alpha \varphi(d_X) d_{V1}(t) - \alpha \varphi(d_X) d_{V2}(t) + \alpha \varphi(d_X) (d_{V1}(t) + d_{V2}(t)) \\ &= 0. \end{aligned}$$

So the function  $E(d_X, d_{V1}, d_{V2})(t) = d_{V1}(t) + d_{V2}(t) + \alpha \int_0^{d_X(t)} \Psi(r) dr$  is non-increasing along the pathway  $(d_X(t), d_{V1}(t), d_{V2}(t))$ , and we deduce

$$d_{V1}(t) + d_{V2}(t) + \alpha \int_0^{d_X(t)} \Psi(r) dr \leq d_{V1}(0) + d_{V2}(0) + \alpha \int_0^{d_X(0)} \Psi(r) dr. \quad (7)$$

As  $\int_0^\infty \Psi^2(r) = \infty$ , there must be a constant  $d^* < \infty$ , satisfying  $d_{X(0)} \leq d^*$ , such that  $d_{V1}(0) + d_{V2}(0) = \alpha \int_{d_{X(0)}}^{d^*} \Psi(r) dr$ . Inequality (7) can be rewritten as

$$d_{V1}(t) + d_{V2}(t) \leq \alpha \int_{d_{X(t)}}^{d^*} \Psi(r) dr.$$

Obviously, we have  $d_{X(t)} \leq d^*$  for all  $t \geq 0$ . As  $\phi(r)$  is decreasing on  $(0, +\infty)$ , we obtain

$$D^+ d_{Vn} \leq -\alpha \frac{\phi^2(d^*)}{N^2} d_{Vn}, \quad n = 1, 2.$$

By using the Gronwall's inequality, we easily get  $\lim_{t \rightarrow \infty} d_{Vn}(t) = 0$ ,  $n = 1, 2$ . Combining with  $d_V \leq d_{V1} + d_{V2}$ , we conclude  $\lim_{t \rightarrow \infty} d_V(t) = 0$ .  $\square$

**4. Flocking behavior in 3-D spaces and remarks.** Some of the arguments can be adopted to the case of phase spaces, as we outlined below. So, we now consider agent  $i$ , with its position  $\mathbf{x}_i \in R^3$  and its velocity  $\mathbf{v}_i \in R^3$ .

**Lemma 4.1.** *Let  $(\mathbf{x}_i, \mathbf{v}_i)$  be a solution of (3), where  $\mathbf{x}_i \in \mathbf{R}^3$ ,  $\mathbf{v}_i \in \mathbf{R}^3$ . For the group of agents, define  $\theta_{ij}(t) = \langle \mathbf{v}_i(t), \mathbf{v}_j(t) \rangle$  as the angle of velocity  $\mathbf{v}_i$  and  $\mathbf{v}_j$  and define  $\bar{\theta}(t) = \max_{i,j} \langle \mathbf{v}_i(t), \mathbf{v}_j(t) \rangle$ . Suppose  $\bar{\theta}(0) = \theta_{ij}(0) < \frac{\pi}{2}$ . Then we have  $\bar{\theta}(t) \leq \bar{\theta}(0)$ , for any  $t > 0$ .*

*Proof.* Like in the 2D-space, we have

$$\cos \theta_{ij}(t) = \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|}, \theta_{ij}(t) = \arccos \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|}, \bar{\theta}(t) = \max_{i,j \in N} \arccos \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|}.$$

For any given  $t$ , there exist two agents  $i_0$  and  $j_0$ , satisfying  $\bar{\theta}(t) = \langle \mathbf{v}_{i_0}(t), \mathbf{v}_{j_0}(t) \rangle$ . Let  $h_1(t) = \frac{(\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0})' \cdot (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|) - (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|)' \cdot (\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0})}{\|\mathbf{v}_{i_0}\|^2 \cdot \|\mathbf{v}_{j_0}\|^2}$ . Then we get

$$\theta'_{i_0 j_0}(t) = \left( \arccos \frac{\mathbf{v}_{i_0}(t) \cdot \mathbf{v}_{j_0}(t)}{\|\mathbf{v}_{i_0}(t)\| \cdot \|\mathbf{v}_{j_0}(t)\|} \right)' = - \frac{1}{\sqrt{1 - \left( \frac{\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0}}{\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|} \right)^2}} \cdot h_1(t).$$

By Lemma 3.1, we have

$$\begin{aligned} f_1(t) &:= (\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0})' \cdot (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|) - (\|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\|)' \cdot (\mathbf{v}_{i_0} \cdot \mathbf{v}_{j_0}) \\ &= \alpha \left( \sum_{k \in \bar{N}_{i_0}} a_{i_0 k} \mathbf{v}_k \mathbf{v}_{j_0} + \sum_{l \in \bar{N}_{j_0}} a_{j_0 l} \mathbf{v}_l \mathbf{v}_{i_0} - 2 \mathbf{v}_{i_0} \mathbf{v}_{j_0} \right) \cdot \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| \\ &\quad - \alpha \left( \sum_{k \in \bar{N}_{i_0}} a_{i_0 k} \frac{\mathbf{v}_k \cdot \mathbf{v}_{i_0}}{\|\mathbf{v}_{i_0}\|} \|\mathbf{v}_{j_0}\| + \sum_{l \in \bar{N}_{j_0}} a_{j_0 l} \frac{\mathbf{v}_l \cdot \mathbf{v}_{j_0}}{\|\mathbf{v}_{j_0}\|} \|\mathbf{v}_{i_0}\| \right. \\ &\quad \left. - 2 \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| \right) \cdot \mathbf{v}_{i_0} \mathbf{v}_{j_0}. \end{aligned}$$

Note that

$$\begin{aligned} & (a_{i_0 k} \mathbf{v}_k \mathbf{v}_{j_0}) \cdot \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| - a_{i_0 k} \frac{\mathbf{v}_k \cdot \mathbf{v}_{i_0}}{\|\mathbf{v}_{i_0}\|} \|\mathbf{v}_{j_0}\| \cdot \mathbf{v}_{i_0} \mathbf{v}_{j_0} \\ &= a_{i_0 k} \|\mathbf{v}_k\| \cdot \|\mathbf{v}_{j_0}\| \cdot \|\mathbf{v}_{i_0}\| \cdot \|\mathbf{v}_{j_0}\| \cdot (\cos \theta_{k j_0} - \cos \bar{\theta} \cos \theta_{k i_0}). \end{aligned}$$

If  $\theta_{i_1 k_1}, \theta_{k_1 j_1} \leq \theta_{i_1 j_1}$  and  $\theta_{k_1 j_1}, \theta_{i_1 k_1}, \theta_{i_1 j_1} \in [0, \frac{\pi}{2})$  for  $i_1, j_1, k_1 \in N$ , we get

$$\cos \theta_{k_1 j_1} - \cos \theta_{i_1 j_1} \cos \theta_{i_1 k_1} \geq 0.$$

Using a similar argument in Lemma 3.1, we claim that for any given  $t$ ,  $\exists i_* \in N$ ,  $j_* \in N$  such that  $\bar{\theta}(t) = \theta_{i_* j_*}(t) < \frac{\pi}{2}$  and  $D^+ \bar{\theta}(t) \leq 0$ . We then conclude that  $\bar{\theta}(t) \leq \bar{\theta}(0)$  for all  $t \geq 0$ .  $\square$

For the agent's position  $\mathbf{x}_i \in R^3$  and its velocity  $\mathbf{v}_i \in R^3$ , denote by  $d_X = \max_{i,j \in N} \|\mathbf{x}_i - \mathbf{x}_j\|$ ,  $d_{Vn} = \max_{i,j \in N} \|\mathbf{v}_{n_i} - \mathbf{v}_{n_j}\|$ ,  $n = x, y, z$ . We introduce the spatial rectangular coordinate system  $\mathbf{v}_i = \mathbf{v}_{x_i} + \mathbf{v}_{y_i} + \mathbf{v}_{z_i}$ . Then, we have

$$\begin{aligned} \frac{d\mathbf{v}_{x_i}}{dt} &= \alpha \sum_{j \in N_i} a_{ij} (\mathbf{v}_{x_j} - \mathbf{v}_{x_i}), \quad a_{ij} = \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) \phi_{ij}, \\ \frac{d\mathbf{v}_{y_i}}{dt} &= \alpha \sum_{j \in N_i} a_{ij} (\mathbf{v}_{y_j} - \mathbf{v}_{y_i}), \quad a_{ij} = \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) \phi_{ij}, \\ \frac{d\mathbf{v}_{z_i}}{dt} &= \alpha \sum_{j \in N_i} a_{ij} (\mathbf{v}_{z_j} - \mathbf{v}_{z_i}), \quad a_{ij} = \kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) \phi_{ij}. \end{aligned}$$

From  $\mathbf{v}_i = \mathbf{v}_{x_i} + \mathbf{v}_{y_i} + \mathbf{v}_{z_i}$ , we can get  $\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$ , and

$$\begin{aligned} \frac{d\mathbf{v}_i}{dt} &= \frac{d\mathbf{v}_{x_i}}{dt} + \frac{d\mathbf{v}_{y_i}}{dt} + \frac{d\mathbf{v}_{z_i}}{dt} \\ &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{x_j} - \mathbf{v}_{x_i}) + \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{y_j} - \mathbf{v}_{y_i}) \\ &\quad + \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{z_j} - \mathbf{v}_{z_i}) \\ &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_{x_j} + \mathbf{v}_{y_j} + \mathbf{v}_{z_j} - \mathbf{v}_{x_i} - \mathbf{v}_{y_i} - \mathbf{v}_{z_i}) \\ &= \alpha \sum_{j \in N_i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i). \end{aligned} \tag{8}$$

**Lemma 4.2.** *Let  $\mathbf{x}_i, \mathbf{v}_i \in \mathbf{R}^3$ , and  $(\mathbf{x}_i, \mathbf{v}_i)$  be a solution of (8). If*

- i).  $\frac{\pi}{2} \leq \gamma \leq \frac{3\pi}{4}$  and the initial condition  $\bar{\theta}(0)$  satisfies  $\bar{\theta}(0) \leq 2\gamma - \pi$ , or*
- ii).  $\frac{3\pi}{4} < \gamma < \pi$  and the initial condition  $\bar{\theta}(0)$  satisfies  $\bar{\theta}(0) < \frac{\pi}{2}$ ,*

*then*

$$\frac{d}{dt}dV_n \leq -\alpha \frac{\phi^2(d_X)}{N^2}dV_n \quad (n = x, y, z).$$

*Proof.* In both i) and ii), we have  $\bar{\theta}(0) \leq \frac{\pi}{2}$ . By Lemma 4.1, we obtain  $\bar{\theta}(t) \leq \bar{\theta}(0) \leq 2\gamma - \pi$  for any  $t \geq 0$ . Then, for any agent  $i$  and agent  $j$ , one of the following three situations can occur

- 1).  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = \kappa(\omega_j, \mathbf{x}_i - \mathbf{x}_j) = 1$ ;
- 2).  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = 0$  and  $\kappa(\omega_j, \mathbf{x}_i - \mathbf{x}_j) = 1$ ;
- 3).  $\kappa(\omega_i, \mathbf{x}_j - \mathbf{x}_i) = 1$  and  $\kappa(\omega_j, \mathbf{x}_i - \mathbf{x}_j) = 0$ .

We can then use a similar argument to that of Lemma 3.2 to prove

$$\frac{d}{dt}dV_n \leq -\alpha \frac{\phi^2(d_X)}{N^2}dV_n, \quad n = x, y, z.$$

□

**Theorem 4.3.** *Let  $(\mathbf{x}_i, \mathbf{v}_i)$  be a solution of (3) and  $\mathbf{x}_i, \mathbf{v}_i \in \mathbf{R}^3$ . Assume the influence function  $\phi(r)$  satisfies  $\int_0^\infty \phi^2(r) = \infty$  and the angle of the cone of vision  $\gamma$  satisfies  $\frac{\pi}{2} \leq \gamma < \pi$ . If the initial conditions satisfy one of i) and ii) in lemma 4.2, then the solution flocks.*

*Proof.* We consider the case where the initial condition satisfies i), and a similar argument applies to case ii). From Lemma 4.1 and Lemma 4.2, we have

$$\frac{d}{dt}dV_n \leq -\alpha \frac{\phi^2(d_X)}{N^2}dV_n, \quad n = x, y, z.$$

Introduce a function  $E(d_X, d_{V_x}, d_{V_y}, d_{V_z})(t) = d_{V_x}(t) + d_{V_y}(t) + d_{V_z}(t) + \alpha \int_0^{d_X(t)} \Psi(r)dr$ , where  $\Psi(r) = \frac{\phi^2(r)}{N^2}$ . We have

$$\begin{aligned} D^+E(d_X, d_{V_x}, d_{V_y}, d_{V_z})(t) &= D^+d_{V_x}(t) + \dot{d}_{V_y}(t) + D^+d_{V_z}(t) + \alpha\varphi(d_X)D^+d_X(t) \\ &\leq -\alpha\varphi(d_X)d_{V_x}(t) - \alpha\varphi(d_X)d_{V_y}(t) - \alpha\varphi(d_X)d_{V_z}(t) \\ &\quad + \alpha\varphi(d_X)d_V(t) \\ &\leq -\alpha\varphi(d_X)d_{V_x}(t) - \alpha\varphi(d_X)d_{V_y}(t) - \alpha\varphi(d_X)d_{V_z}(t) \end{aligned}$$

$$\begin{aligned}
& + \alpha \varphi(d_X)(d_{V_x} + d_{V_y} + d_{V_z})(t) \\
& = 0.
\end{aligned}$$

Using a similar argument as in the proof of Theorem 3.3, we can get a constant  $d^{**} < \infty$ , ( $d_{X(0)} \leq d^{**}$ ) such that  $d_{V_x}(t) + d_{V_y}(t) + d_{V_z}(t) = \alpha \int_{d_{X(0)}}^{d^{**}} \varphi(r) dr$ .

Then, we have  $d_{X(t)} \leq d^{**}$  for any  $t \geq 0$ . Hence, we have

$$\frac{d}{dt} d_{V_n} \leq -\alpha \frac{\phi^2(d^{**})}{N^2} d_{V_n}, \quad n = x, y, z.$$

By the Gronwall's inequality, we obtain  $\lim_{t \rightarrow \infty} d_{V_n}(t) = 0$ ,  $n = x, y, z$ . As  $d_V \leq d_{V_x} + d_{V_y} + d_{V_z}$ , we conclude that  $\lim_{t \rightarrow \infty} d_V(t) = 0$ .  $\square$

**Acknowledgments.** We would like to thank CSC for supporting Le Li's visiting study at the Laboratory for Industrial and Applied Mathematics (LIAM) at York University.

#### REFERENCES

- [1] J. A. Carrillo, M. Fornasier, J. Rosado and G. Toscani, [Asymptotic flocking dynamics for the kinetic Cucker-Smale model](#), *SIAM J. Math. Anal.*, **42** (2010), 218–236.
- [2] I. D. Couzin, J. Krause, N. R. Franks and S. Levin, [Effective leadership and decision making in animal groups on the move](#), *Nature*, **433** (2005), 513–516.
- [3] F. Cucker and S. Smale, [Lectures on emergence](#), *Japan J. Math.*, **2** (2007), 197–227.
- [4] F. Cucker and S. Smale, [Emergent behavior in flocks](#), *IEEE Trans. Automat. Control*, **52** (2007), 852–862.
- [5] F. Cucker, S. Smale and D. Zhou, [Modeling language evolution](#), *Found. Comput. Math.*, **4** (2004), 315–343.
- [6] S. Y. Ha and J. G. Liu, [A simple proof of the Cucker-Smale flocking dynamics and mean-field limit](#), *Commun. Math. Sci.*, **7** (2009), 297–325.
- [7] S. Y. Ha and E. Tadmor, [From particle to kinetic and hydrodynamic descriptions of flocking](#), *Kinet. Relat. Models*, **1** (2008), 415–435.
- [8] Y. Liu and K. Passino, [Stable social foraging swarms in a noisy environment](#), *IEEE Trans. Automat. Control*, **49** (2004), 30–44.
- [9] S. Motsch and E. Tadmor, [A new model for Self-organized dynamics and its flocking behavior](#), *J. Stat. Phys.*, **144** (2011), 923–947.
- [10] C. W. Reynolds, [Flocks, herds and schools: A distributed behavioral model](#), *In: ACM SIGGRAPH Computer Graphics*, **21** (1987), 25–34.
- [11] J. Shen, [Cucker-Smale flocking under hierarchical leadership](#), *SIAM J. Appl. Math.*, **68** (2008), 694–719.
- [12] C. M. Topaz and A. L. Bertozzi, [Swarming patterns in a two-dimensional kinematic model for biological groups](#), *SIAM J. Appl. Math.*, **65** (2004), 152–174.
- [13] C. M. Topaz, A. L. Bertozzi and M. A. Lewis, [A nonlocal continuum model for biological aggregation](#), *Bull. Math. Bio.*, **68** (2006), 1601–1623.
- [14] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, [Novel type of phase transition in a system of self-driven particles](#), *Phys. Rev. Lett.*, **75** (1995), 1226–1225.

Received November 05, 2014; Accepted May 09, 2015.

*E-mail address:* [lile19860221@gmail.com](mailto:lile19860221@gmail.com)

*E-mail address:* [lhhuang@hnu.edu.cn](mailto:lhhuang@hnu.edu.cn)

*E-mail address:* [wujh@mathstat.yorku.ca](mailto:wujh@mathstat.yorku.ca)