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Brief paper

Stability of nonlinear differential systems with state-dependent delayed impulses[☆]

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ABSTRACT

We consider nonlinear differential systems with state-dependent delayed impulses (impulses which involve the delayed state of the system for which the delay is state-dependent). Such systems arise naturally from a number of applications and the stability issue is complex due to the state-dependence of the delay. We establish general and applicable results for uniform stability, uniform asymptotic stability and exponential stability of the systems by using the impulsive control theory and some comparison arguments. We show how restrictions on the change rates of states and impulses should be imposed to achieve system's stability, in comparison with general impulsive delay differential systems with state-dependent delay in the nonlinearity, or the differential systems with constant delays. In our approach, the boundedness of the state-dependent delay is not required but derives from the stability result obtained. Examples are given to demonstrate the sharpness and applicability of our general results and the proposed approach.

1. Introduction

Impulsive delay differential systems have been used for modelling natural phenomena in many areas for many years, and there have been significant studies of such systems, as indicated by Churilov and Medvedev (2014), Dashkovskiy, Kosmykov, Mironchenko, and Naujok (2012), Lakshmikantham, Bainov, and Simeonov (1989), Li, Bohner, and Wang (2015), Sakthivel, Mahmudov, and Kim (2009), Sakthivel, Ren, and Mahmudov (2010) and Samoilenko and Perestyuk (1995) and references therein. Of current interest is the delayed impulses of differential systems arising in such applications as automatic control, secure communication and population dynamics (Akca, Alassar, Covachev, Covacheva, & Al-Zahrani, 2004; Akhmet & Yilmaz, 2014; Chen, Wei, & Lu, 2013; Chen & Zheng, 2011, 2009; Khadra, Liu, & Shen, 2005, 2009; Liu, Teo, & Xu, 2005), here and in what follows, *a delayed impulse* describes a phenomenon where impulsive transients depend on not only their current but also historical states of the system. For instance, in communication security systems based on impulsive synchronization, there exist transmission and sampling delays during the information transmission process, where the sampling delay created from sampling the impulses at some discrete instances causes the impulsive transients depend on their historical states (Chen et al., 2013; Khadra et al., 2005). The existing studies, however, such as those in Akca et al. (2004), Akhmet and Yilmaz (2014), Chen et al. (2013), Chen and Zheng (2011, 2009), Khadra et al. (2005, 2009) and Liu et al. (2005), assume the delays in impulsive perturbations are either fixed as constants or given by integrals with state-independent distributed kernels. For example, Khadra et al. (2005) considered the impulsive synchronization of chaotic systems with transmission delay and sampling delay, and then applied the results to the design of communication security scheme. Chen and Zheng (2011) studied the nonlinear time-delay systems with two kinds of delayed impulses, that is, destabilizing delayed impulses and stabilizing delayed impulses, and derived some interesting results for exponential stability. But in both results, the delays in impulses are given constants. Akca et al. (2004) derived some results for global stability of Hopfield-type neural networks with delayed impulses, where the delays in impulses are in integral forms with state-independent distributed kernels. However, in many cases it is important to consider state-dependent delays in impulsive perturbations. For example, the sampling delay varies with the change of state variables since it is natural to consider sending control signals less frequently when

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the state is small and more frequently when the state is large (Hespanha, Naghshtabrizi, & Xu, 2007); while in some other impulsive models arising from disease control, financial options, and population dynamics, it is also natural to introduce state-dependent delays into the impulses. There have already been some results in the literature on the existence, uniqueness and controllability for some classes of differential systems with impulses involving state-dependent delay (see Chang, Nieto, & Zhao, 2010, Hernandez, Sakthivel, & Aki, 2008, Liu & Ballinger, 2001, Sakthivel & Anandhi, 2010), but little seems to have been done for the stability.

We should have mentioned that state-dependent delay incorporated in the differential system has also found increasing applications in a variety of fields, such as control systems (Hespanha et al., 2007; Liberis & Krstic, 2013b; Niemeyer & Slotine, 2001), turning processes (Insperger, Stepan, & Turi, 2007), complex networks (Sterman, 2000; Witrant, Carlos, Georges, & Alamir, 2007), and biological systems (Adimy, Crauste, Hbid, & Qesmi, 2010; Aiello, Freedman, & Wu, 1992). Many interesting and important results for state-dependent delay systems have been recently reported (see Ecimovic & Wu, 2002, Hartung, Krisztin, Walther, & Wu, 2006, Paret & Nussbaum, 2011, Sakthivel & Ren, 2013, Walther, 2008 and references therein) including stability analysis (Cooke & Huang, 1996; Gyori & Hartung, 2007; Hartung & Turi, 1995; Liberis & Krstic, 2013a; Verriest, 2002). However, many classical methods for stability analysis of delay systems, including delay decomposition approach, free-weighting matrix method, and Leibniz–Newton formula have not been extended to differential systems with state-dependent delay in general, and differential systems with state-dependent delayed impulses in particular.

In this study, we focus on stability problem of nonlinear differential systems with impulses involving state-dependent delay based on Lyapunov methods. As is well known, in systems with time delays, there exist two main Lyapunov methods for stability analysis: the Krasovskii method of Lyapunov functionals and the Razumikhin method of Lyapunov functions. However, when the time delays exist in impulses and moreover is state-dependent, there are substantial difficulties to apply either method. In fact, due to the existence of state delay in impulses, it is hard to know exactly a priori how far in the history the information is needed, and is hard to determine the historical states at impulsive instances. Moreover, it is possible that function V along a solution can be increasing at certain impulses points due to the state-dependence of the delay. In this study, we provide some new insights on the features of systems with impulses involving state-dependent delay, and give an estimate of Lyapunov functions which is coupled with the effect of state delay based on impulsive control theory and some comparison arguments. Then we establish (in Section 3) some general results for (Lyapunov) uniform stability, uniform asymptotic stability and exponential stability, where the necessary constraint on state-dependent delay is specified of boundedness of the state-dependent delay is not required a priori. We will also provide, in Section 4, numerical examples to demonstrate the effectiveness of the proposed approach and our established results.

2. Preliminaries

Notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the n-dimensional and $n \times m$-dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$, respectively, $\mathbb{Z}_+$ the set of positive integer numbers, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the maximum and minimum eigenvalues of symmetric matrix A, respectively. $A > 0$ or $A < 0$ denotes that the matrix A is a symmetric and positive or negative definite matrix. I the identity matrix with appropriate dimensions. $\mathscr{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+)\,|\, a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0 \text{ and } a \text{ is increasing in } s\}$.

Consider the following impulsive differential system

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0,\ t \neq t_k, \\ x(t_k) = I_k(t_k^- - \tau, x(t_k^- - \tau)), \quad \tau = \tau(t_k, x(t_k^-)), \\ x_{t_0} = \phi, \end{cases} \tag{1}$$

where $\phi \in \mathbb{C}_\alpha$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $I_k \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $k \in \mathbb{Z}_+$, $\tau \in C(\mathbb{R}_+ \times \mathbb{R}^n, [0, \alpha])$, $x_{t_0} = \{x(t_0 + s) : s \in [-\alpha, 0]\}$, $0 \leq \alpha \leq +\infty$, especially when $\alpha = \infty$, the interval $[s - \alpha, s]$ is understood to be replaced by $(-\infty, s]$ for any $s \in \mathbb{R}$. $\mathbb{C}_\alpha \doteq C([-\alpha, 0], \mathbb{R}^n) = \{\phi : [-\alpha, 0] \to \mathbb{R}^n \text{ is continuous}\}$ with the norm $\|\phi\|_\alpha = \sup_{-\alpha \leq \theta \leq 0} \|\phi(\theta)\|$ for $\phi \in \mathbb{C}_\alpha$. Given a constant $\mathcal{M} > 0$, set $\mathbb{C}_\alpha^{\mathcal{M}} = \{\phi \in \mathbb{C}_\alpha : 0 < \|\phi\| \leq \mathcal{M}\}$. The impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k \to +\infty$ as $k \to \infty$.

Note that the continuity of f, I_k and τ, and a fact that system (1) is an ODE which is continuous on each interval $[t_{k-1}, t_k)$. We assume that the vector field f satisfies suitable conditions so the solutions exist in relevant time intervals. These conditions can be formulated using standard conditions such as conditions (H_1)–(H_3) in Liu and Ballinger (2001) (or Lakshmikantham et al., 1989). Denote by $x(t) \doteq x(t, t_0, \phi)$ the solution of the system (1). In addition, we always assume that $f(t, 0) \equiv 0$, $t \geq t_0$, and $I_k(t, x) = 0$ if and only if $x = 0$, $t \geq t_0$, $k \in \mathbb{Z}_+$. Thus system (1) admits a trivial solution $x(t) \equiv 0$. Some definitions are given in the following.

Definition 1. The function $V : [t_0 - \alpha, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to class υ_0 if

(1) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and $\lim_{(t,u)\to(t_k^-,v)} V(t, u) = V(t_k^-, v)$ exists;
(2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2. Let $V \in \upsilon_0$, D^+V is defined as

$$D^+V(t, x(t)) = \limsup_{h\to 0^+} \frac{1}{h}\Big\{V(t + h, x(t) + hf(t, x(t))) - V(t, x(t))\Big\}.$$

Definition 3. System (1) is said to be

(1) locally uniformly stable (LUS) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, if there exists a constant $\mathcal{M} > 0$, and if for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon, \mathcal{M}) \in (0, \mathcal{M}]$ such that $\phi \in \mathbb{C}_\alpha^{\delta}$ implies $|x(t, t_0, \phi)| < \varepsilon$, $t \geq t_0$;
(2) locally uniformly asymptotically stable (LUAS) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, if it is uniformly stable and uniformly attractive;
(3) locally exponentially stable (LES) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, if there exist constants $\lambda > 0$, $M^\star \geq 1$, $\mathcal{M} > 0$ such that

$$\|x(t)\| \leq M^\star \|\phi\|_\alpha\, e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

for any initial value $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

3. Main results

Theorem 1. *Assume that there exist constants $\gamma > 0$, $\theta \in (0, 1)$, $\mathcal{M} > 0$, $\rho_k \geq 1$, $k \in \mathbb{Z}_+$ functions $\omega_1, \omega_2 \in \mathscr{K}$, $H \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and $V \in \upsilon_0$ such that*

(i) $\omega_1(\|x\|) \leq V(t, x) \leq \omega_2(\|x\|)$ *for all* $(t, x) \in [t_0 - \alpha, \infty) \times \mathbb{R}^n$;
(ii) $D^+V(t, x(t)) \leq -H\Big(t, V(t, x(t))\Big)$, $t \in [t_{k-1}, t_k)$;
(iii) $V(t_k, x(t_k)) \leq \rho_k V(t_k^- - \tau, x(t_k^- - \tau))$, $\tau = \tau(t_k, x(t_k^-))$, $k \in \mathbb{Z}_+$, *where* $x(t) = x(t, t_0, \phi)$ *is a solution of* (1);
(iv) $|\tau(s, \mathbf{u}) - \tau(s, \mathbf{0})| \leq \gamma\|u\|$ *for any* $s \in \mathbb{R}_+$, $\mathbf{u} \in \mathbb{R}^n$;
(v) $\tau^\star \doteq \sup_{t\geq t_0} \tau(t, \mathbf{0}) < \infty$;

(vi) *for any* $k \in \mathbb{Z}_+$,

$$\ln \rho_k + \int_{t_k-\mathscr{A}}^{t_k} \sup_{u\in(0,\omega_2(\mathcal{M}))} \frac{H(s,u)}{u} ds \leq \theta \int_{t_k-\mu}^{t_k} \inf_{u\in(0,\omega_2(\mathcal{M}))} \frac{H(s,u)}{u} ds,$$

where $\mathscr{A} = \gamma\omega_1^{-1}[\omega_2(\mathcal{M})] + \tau^\star < \mu$, $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > 0$.

Then system (1) *is LUS in the region* $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$. *Furthermore, if for any* $\kappa > 0$, *there exists* $\mathcal{T} = \mathcal{T}(\kappa) > 0$ *such that*

$$\int_{t_0}^{t} \inf_{u\in(0,\omega_2(\mathcal{M}))} \frac{H(s,u)}{u} ds \geq \kappa, \quad t \geq t_0 + \mathcal{T}, \tag{2}$$

then system (1) *is LUAS in the region* $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

Proof. Given initial value $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, $t_0 \geq 0$, let $x(t) \doteq x(t, t_0, \phi)$ be a solution of system (1) through (t_0, ϕ). Define $V(t) = V(t, x(t))$. First we claim that $V(t) > 0$, $t \geq t_0 - \alpha$. Since $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, then $V(t) > 0$, $[t_0 - \alpha, t_0]$. And consider system (1), $\phi(0) \neq 0$ implies that $V(t) > 0$, $t \in [t_0, t_1)$. If $V(t_1^+) = 0$, that is, $x(t_1^+) = I_1(t_1^- - \tau_1, x(t_1^- - \tau_1)) = 0$, where $\tau_1 = \tau(t_1, x(t_1^-))$, then $x(t_1^- - \tau_1) = 0$, that is, $V(t_1^- - \tau_1) = 0$, which is a contradiction and thus $V(t_1^+) > 0$, implies that $V(t) > 0$, $t \in [t_1, t_2)$. In this way, it can be deduced that $V(t) > 0$, $t \geq t_0 - \alpha$.

Next we claim that, for $t \geq t_0$,

$$V(t) \leq \prod_{t_0<t_k\leq t} \rho_k\omega_2(\|\phi\|_\alpha) \exp\Big(-\int_{t_0}^{t} \frac{H(s,V(s))}{V(s)} ds\Big) \times \exp\Big(\sum_{t_0<t_k\leq t} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \frac{H(s,V(s))}{V(s)} ds\Big). \tag{3}$$

For convenience, set $\mathbb{H}(s,V) \doteq \frac{H(s,V(s))}{V(s)}$ in the following. First, it is obvious that

$$V(t) \leq V(t_0) \leq \omega_2(\|\phi\|_\alpha), \quad t \in [t_0, t_1)$$

and thus

$$V(t) \leq V(t_0) \exp\Big(-\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big) \leq \omega_2(\|\phi\|_\alpha) \exp\Big(-(1-\theta)\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big)$$

for $t \in [t_0, t_1)$, which implies that (3) holds for $t \in [t_0, t_1)$. Moreover, we obtain $V(t) \leq \omega_2(\mathcal{M})$ and

$$\|x(t)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)] \leq \omega_1^{-1}[\omega_2(\mathcal{M})], \quad t \in [t_0, t_1),$$

which together with (iv) yields

$$\tau_1 = \tau(t_1, x(t_1^-)) - \tau(t_1, \mathbf{0}) + \tau(t_1, \mathbf{0}) \leq \gamma\|x(t_1^-)\| + \tau^\star \leq \mathscr{A}.$$

Then it is easy to derive that

$$\rho_1 \exp\Big(\int_{t_1-\tau_1}^{t_1} \mathbb{H}(s,V)ds\Big) \leq \rho_1 \exp\Big(\int_{t_1-\mathscr{A}}^{t_1} \mathbb{H}(s,V)ds\Big) \leq \exp\Big(\theta\int_{t_1-\mu}^{t_1} \mathbb{H}(s,V)ds\Big) \leq \exp\Big(\theta\int_{t_0}^{t_1} \mathbb{H}(s,V)ds\Big). \tag{4}$$

Considering condition (iii), we get

$$\begin{aligned} V(t_1) &\leq \rho_1 V(t_1^- - \tau_1) \\ &\leq \rho_1 \begin{cases} V(t_0)\exp\Big(-\int_{t_0}^{t_1-\tau_1} \mathbb{H}(s,V)ds\Big), & t_1-\tau_1 \geq t_0, \\ \omega_2(\|\phi\|_\alpha), & t_1-\tau_1 < t_0 \end{cases} \\ &\leq \rho_1 \begin{cases} V(t_0)\exp\Big(-\int_{t_0}^{t_1-\tau_1} \mathbb{H}(s,V)ds\Big), & t_1-\tau_1 \geq t_0, \\ \omega_2(\|\phi\|_\alpha)\exp\Big(-\int_{t_0}^{t_1-\tau_1} \mathbb{H}(s,V)ds\Big), & t_1-\tau_1 < t_0 \end{cases} \\ &\leq \rho_1\omega_2(\|\phi\|_\alpha)\exp\Big(-\int_{t_0}^{t_1-\tau_1} \mathbb{H}(s,V)ds\Big) \\ &\leq \rho_1\omega_2(\|\phi\|_\alpha)\exp\Big(-\int_{t_0}^{t_1} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\int_{t_1-\tau_1}^{t_1} \mathbb{H}(s,V)ds\Big), \end{aligned}$$

which leads to

$$V(t) \leq V(t_1^+)\exp\Big(-\int_{t_1}^{t} \mathbb{H}(s,V)ds\Big) \leq \rho_1\omega_2(\|\phi\|_\alpha)\exp\Big(-\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\int_{t_1-\tau_1}^{t_1} \mathbb{H}(s,V)ds\Big), \quad t \in [t_1, t_2),$$

where $\tau_1 = \tau(t_1, x(t_1^-))$. Thus, (3) holds for $t \in [t_1, t_2)$. Then note that $V(t) \leq \omega_2(\mathcal{M})$, $t \in [t_0, t_1)$, from (4) it holds that

$$V(t) \leq \omega_2(\|\phi\|_\alpha)\exp\Big(-(1-\theta)\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big)$$

and $V(t) \leq \omega_2(\mathcal{M})$, $\|x(t)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)]$, $t \in [t_1, t_2)$.

Now assume that (3) holds for $t \in [t_{l-1}, t_l)$, $l \leq N$, $N \geq 2$, that is, the following inequalities hold

$$\begin{cases} V(t) \leq \prod_{k=1}^{l-1} \rho_k\omega_2(\|\phi\|_\alpha)\exp\Big(-\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\sum_{k=1}^{l-1}\int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big), \\ V(t) \leq \omega_2(\|\phi\|_\alpha)\exp\Big(-(1-\theta)\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big), \\ \|x(t)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)], \quad t \in [t_{l-1}, t_l),\ l \leq N. \end{cases} \tag{5}$$

Next we prove that

$$\begin{cases} V(t) \leq \prod_{k=1}^{N} \rho_k\omega_2(\|\phi\|_\alpha)\exp\Big(-\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\sum_{k=1}^{N}\int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big), \\ V(t) \leq \omega_2(\|\phi\|_\alpha)\exp\Big(-(1-\theta)\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big), \\ \|x(t)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)], \quad t \in [t_N, t_{N+1}). \end{cases} \tag{6}$$

First, it follows from (5) that

$$V(t_N) \leq \rho_N V(t_N^- - \tau_N)$$

$$\leq \rho_N \begin{cases} \prod_{k=1}^{N-1} \rho_k \omega_2(\|\phi\|_\alpha) \exp\Big(-\int_{t_0}^{t_N-\tau_N} \mathbb{H}(s,V)ds\Big) \\ \quad \times \exp\Big(\sum_{k=1}^{N-1} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big), \\ \quad t_N - \tau_N > t_{N-1}, \\ \prod_{k=1}^{N-2} \rho_k \omega_2(\|\phi\|_\alpha) \exp\Big(-\int_{t_0}^{t_N-\tau_N} \mathbb{H}(s,V)ds\Big) \\ \quad \times \exp\Big(\sum_{k=1}^{N-2} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big), \\ \quad t_{N-2} < t_N - \tau_N \leq t_{N-1}, \\ \vdots \\ \omega_2(\|\phi\|_\alpha), \quad t_N - \tau_N < t_0 \end{cases}$$

$$\leq \prod_{k=1}^{N} \rho_k \omega_2(\|\phi\|_\alpha) \exp\Big(-\int_{t_0}^{t_N-\tau_N} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\sum_{k=1}^{N-1} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big)$$

$$\leq \prod_{k=1}^{N} \rho_k \omega_2(\|\phi\|_\alpha) \exp\Big(-\int_{t_0}^{t_N} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\sum_{k=1}^{N} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big),$$

where $\tau_N = \tau(t_N, x(t_N^-))$. Consequently, for $t \in [t_N, t_{N+1})$,

$$V(t) \leq V(t_N^+) \exp\Big(-\int_{t_N}^{t} \mathbb{H}(s,V)ds\Big) \leq \prod_{k=1}^{N} \rho_k \omega_2(\|\phi\|_\alpha) \exp\Big(-\int_{t_0}^{t} \mathbb{H}(s,V)ds\Big) \times \exp\Big(\sum_{k=1}^{N} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big).$$

Since $\|x(t_N^-)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)]$, then $\tau_N = \tau(t_N, x(t_N^-)) - \tau(t_N, \mathbf{0}) + \tau(t_N, \mathbf{0}) \leq \gamma\|x(t_N^-)\| + \tau^\star \leq \mathscr{A}$. Also note that $V(t) \leq \omega_2(\|\phi\|_\alpha)$, $t \in [t_0, t_N)$, using condition (vi) again, we get

$$\prod_{k=1}^{N} \rho_k \exp\Big(\sum_{k=1}^{N} \int_{t_k-\tau(t_k,x(t_k^-))}^{t_k} \mathbb{H}(s,V)ds\Big) \leq \prod_{k=1}^{N} \rho_k \exp\Big(\sum_{k=1}^{N} \int_{t_k-\mathscr{A}}^{t_k} \mathbb{H}(s,V)ds\Big) \leq \exp\Big(\theta \int_{t_0}^{t_N} \mathbb{H}(s,V)ds\Big),$$

which yields

$$V(t) \leq \omega_2(\|\phi\|_\alpha) \exp\Big(-(1-\theta) \int_{t_0}^{t} \mathbb{H}(s,V)ds\Big)$$

for $t \in [t_N, t_{N+1})$. Moreover, it is easy to check that $\|x(t)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)]$, $t \in [t_N, t_{N+1})$. Hence, we have proven that (6) holds, which also completes the proof of (3). From (3), (5) and (6), it is easy to derive that

$$\|x(t)\| \leq \omega_1^{-1}[\omega_2(\|\phi\|_\alpha)], \quad t \geq t_0.$$

Then for any $\varepsilon > 0$, choose $\delta = \min\{\omega_2^{-1}[\omega_1(\varepsilon)], \mathcal{M}\}$, $\phi \in \mathbb{C}_\alpha^\delta$ implies $\|x(t)\| < \varepsilon$, which implies the local uniform stability of system (1) in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$. In addition, if condition (2) holds, note that

$$V(t) \leq \omega_2(\|\phi\|_\alpha) \exp\Big(-(1-\theta) \int_{t_0}^{t} \mathbb{H}(s,V)ds\Big), \quad t \geq t_0,$$

it is obvious that system (1) is LUAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$. The proof of Theorem 1 is complete. ■

Remark 1. From the proof of Theorem 1, condition (vi) can be replaced by the following condition: for any $u \in (0, \omega_2(\mathcal{M}))$, $k \in \mathbb{Z}_+$,

$$\ln \rho_k + \int_{t_k-\mathscr{A}}^{t_k} \frac{H(s,u)}{u} ds \leq \theta \int_{t_k-\mu}^{t_k} \frac{H(s,u)}{u} ds.$$

Remark 2. Theorem 1 presents some conditions for uniform stability and uniform asymptotic stability of systems with impulses involving state-dependent delay. One may observe from the proof that these kinds of impulses are more complicated than the ones in Akca et al. (2004), Akhmet and Yilmaz (2014), Chen et al. (2013), Chen and Zheng (2011, 2009), Khadra et al. (2005) and Liu et al. (2005) that are only dependent on current states or past states in given time interval. Even if $\rho_k \leq 1$, it is possible that function V has state-dependent increase at different impulse points. Thus more conditions such as restrictions (iv) and (v) on state-dependent delay τ must be imposed on these kinds of impulsive systems. In fact, one may note that the local stability of system (1) implies the local boundedness of system states, which leads to the boundedness of the state delay. Thus the time delay in Theorem 1 actually is bounded, but it is not required a priori. In other words, we can utilize the stability criteria in this paper to know the boundedness of the state-dependent delay, but we do not assume the boundedness of the delay a priori. Moreover, due to the existence of impulse effects, especially for persistent impulsive perturbations ($\rho_k > 1$), it is possible that system (1) is unbounded if the impulse interval is small enough. This problem will be shown in the following by Example 1.

If we drop the effect of state delay and consider a special case that $\tau = 0$, then one may derive the following result.

Corollary 1. *Assume that there exist constants $\theta \in (0,1)$, $\mathcal{M} > 0$, $\rho_k \geq 1$, $k \in \mathbb{Z}_+$ functions $\omega_1, \omega_2 \in \mathscr{K}$, $H \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and $V \in \upsilon_0$ such that*

(i$_0$) *$\omega_1(\|x\|) \leq V(t,x) \leq \omega_2(\|x\|)$ for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$;*

(ii$_0$) *$D^+V(t,x(t)) \leq -H\big(t, V(t,x(t))\big)$, $t \in [t_{k-1}, t_k)$;*

(iii$_0$) *$V(t_k, x(t_k)) \leq \rho_k V(t_k^-, x(t_k^-))$, $k \in \mathbb{Z}_+$, where $x(t) = x(t,t_0,\phi)$ is a solution of (1);*

(iv$_0$)

$$\frac{\ln \rho_k}{\theta} \leq \int_{t_k-\mu}^{t_k} \inf_{u\in(0,\omega_2(\mathcal{M}))} \frac{H(s,u)}{u} ds, \quad k \in \mathbb{Z}_+,$$

where $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > 0$.

Then system (1) with $\tau = 0$ is LUS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$. Furthermore, if for any $\kappa > 0$, there exists $\mathcal{T} = \mathcal{T}(\kappa) > 0$ such that (2) holds. Then system (1) with $\tau = 0$ is LUAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

Remark 3. Note that Corollary 1 as the special case of Theorem 1 has been partially derived by Samoilenko and Perestyuk (1995). But there are some difference between them. For example, Corollary 1 can be applied to the case that $H(t,V) = h(t)C(V)$, where h and C are two given functions; while the results in Samoilenko and Perestyuk (1995) can be applied to the case of state-dependent impulses. Thus they are different but complementary with each other. In addition, if we exclude explicit dependence on time t, then Corollary 1 becomes a special case of Dashkovskiy et al. (2012).

If function H is specialized, based on Theorem 1 and Remark 1, we can derive the following results which are easy to check in real applications.

Corollary 2. *Under the conditions* (i), (iii), (iv), (v) *in Theorem 1, if there exists function $h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $D^+V(t, x(t)) \le -h(t)V$, $t \in [t_{k-1}, t_k)$, where h satisfies*

$$\ln \rho_k + \int_{t_k-\mathscr{A}}^{t_k} h(s)ds \le \theta \int_{t_k-\mu}^{t_k} h(s)ds,$$

where $\mathscr{A} = \gamma\omega_1^{-1}[\omega_2(\mathcal{M})] + \tau^\star < \mu$, $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > 0$. Then the system (1) is LUS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$. Furthermore, if for any $\kappa > 0$, there exists $\mathcal{T} = \mathcal{T}(\kappa) > 0$ such that

$$\int_{t_0}^{t} h(s)ds \ge \kappa, \quad t \ge t_0 + \mathcal{T},$$

then system (1) is LUAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

Corollary 3. *Under the conditions* (i), (iii), (iv), (v) *in Theorem 1, assume that $\theta\mu > \tau^\star$ and there exists function $W(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $D^+V(t, x(t)) \le -W(V)$, $t \in [t_{k-1}, t_k)$, and*

$$\frac{\sup_{k\in\mathbb{Z}_+} \ln \rho_k}{\theta\mu - \tau^\star} < \inf_{u\in(0,\omega_2(\mathcal{M}))} \frac{W(u)}{u}.$$

Then system (1) is LUAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, where

$$0 < \mathcal{M} \le \omega_2^{-1}\left[\omega_1\left(\frac{\theta\mu - \tau^\star}{\gamma} - \frac{\sup_{k\in\mathbb{Z}_+} \ln \rho_k}{\gamma \inf_{u\in(0,\omega_2(\mathcal{M}))} \frac{W(u)}{u}}\right)\right].$$

Corollary 4. *Under the conditions* (iv), (v) *in Theorem 1, assume that there exist constants $\gamma > 0$, $\theta \in (0, 1)$, $\rho \ge 1$, $c_1 > 0$, $c_2 > 0$, $h > 0$ and $V \in \upsilon_0$ such that*

(i$_a$) $c_1\|x\| \le V(t, x) \le c_2\|x\|$, $t \ge t_0$, $x \in \mathbb{R}^n$;
(i$_b$) $D^+V(t, x(t)) \le -hV(t, x(t))$, $t \in [t_{k-1}, t_k)$;
(i$_c$) $V(t_k, x(t_k)) \le \rho V(t_k^- - \tau, x(t_k^- - \tau))$, $\tau = \tau(t_k, x(t_k^-))$, $k \in \mathbb{Z}_+$, *where $x(t) = x(t, t_0, \phi)$ is a solution of (1);*
(i$_d$) $\rho < \exp\Big(h\theta\mu - h\tau^\star\Big)$, *where $\mu \doteq \inf\{t_k - t_{k-1}\} > 0$.*

Then the system (1) is LES in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$ with Lyapunov exponent $(1 - \theta)h$. Moreover the solution $x(t) \doteq x(t, t_0, \phi)$ of system (1) satisfies

$$\|x(t)\| \le \frac{c_2}{c_1}\|\phi\|_\alpha \exp\Big(-(1-\theta)h(t - t_0)\Big), \quad t \ge t_0,$$

where $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, $\mathcal{M} = \frac{c_1\left(\theta h\mu - \ln\rho - \tau^\star h\right)}{h\gamma c_2}$.

Next we shall apply the previous results to the following nonlinear differential system:

$$\dot{x}(t) = -C(t)x(t) + B(t)g(x(t)), \quad t > 0,\ t \ne t_k \tag{7}$$

subject to impulses:

$$x(t_k) = \Gamma_k x(t_k^- - \tau), \quad \tau = \tau(x(t_k^-)),\ k \in \mathbb{Z}_+, \tag{8}$$

where $x \in \mathbb{R}^n$, $C = (c_{ij}(t))$, $B = (b_{ij}(t)) \in C(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $\tau \in C(\mathbb{R}^n, [0, \alpha])$, $\Gamma_k = (\chi_{ij}^{(k)}) \in \mathbb{R}^{n\times n}$, $g(x) = (g_1(x_1), \ldots, g_n(x_n))^T$ satisfies $|g_j(u)| \le l_j|u|$, $u \in \mathbb{R}$, $j = 1, \ldots, n$, l_j are some positive constants.

Obviously, considering (7) and (8), one may derive the following impulsive differential equations:

$$\begin{cases} \dot{x}_i(t) = -\sum_{j=1}^n c_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)g_j(x_j(t)), & t \ne t_k, \\ x_i(t_k) = \sum_{j=1}^n \chi_{ij}^{(k)} x_j(t_k^- - \tau), & \tau = \tau(x(t_k^-)). \end{cases} \tag{9}$$

Theorem 2. *Assume that $c_{ii}(t) > 0$, $i = 1, \ldots, n$, and there exist constants $\theta \in (0, 1)$, $\mathcal{M} > 0$, $\gamma > 0$, $\Xi_k \ge 1$, and function $\mathscr{F} \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $|\tau(u) - \tau(\mathbf{0})| \le \gamma\|u\|$, $u \in \mathbb{R}^n$ and the following conditions hold:*

$$\sum_{i=1}^n\sum_{j=1}^n (\chi_{ij}^{(k)})^2 \le \Xi_k, \quad k \in \mathbb{Z}_+$$

and

$$\ln \Xi_k + \int_{t_k-\gamma\mathcal{M}-\tau(\mathbf{0})}^{t_k} \mathscr{F}(s)ds \le \theta\int_{t_k-\mu}^{t_k}\mathscr{F}(s)ds, \quad k \in \mathbb{Z}_+,$$

where $\mu \doteq \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} > 0$ and

$$\begin{aligned}\mathscr{F}(t) \le{}& 2\min_i c_{ii}(t) - \max_i \sum_{j\ne i}^n |c_{ij}(t)| - \sum_{i=1}^n \max_{j\ne i}|c_{ij}(t)| \\ &- \max_i \sum_{j=1}^n |b_{ij}(t)|l_j - \sum_{i=1}^n \max_j |b_{ij}(t)|l_j, \quad t > 0.\end{aligned}$$

Then the system (9) is LUS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$. Furthermore, if for any $\kappa > 0$, there exists $\mathcal{T} = \mathcal{T}(\kappa) > 0$ such that

$$\int_{t_0}^{t} \mathscr{F}(s)ds \ge \kappa, \quad t \ge t_0 + \mathcal{T},$$

then system (9) is LUAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

Remark 4. Let $V = \|x\|^2$, then it is not difficult to derive the above result and thus the proof is omitted here. By Corollary 3, we can obtain some results for exponential stability as follows.

Corollary 5. *Assume that $c_{ii}(t) > 0$, $i = 1, \ldots, n$, and there exist constants $\theta \in (0, 1)$, $\gamma > 0$, $\Xi \ge 1$ such that $|\tau(u) - \tau(\mathbf{0})| \le \gamma\|u\|$, $u \in \mathbb{R}^n$ and $\Xi < \exp\Big(\mathscr{F}\theta\mu - \mathscr{F}\tau(\mathbf{0})\Big)$, $\mu \doteq \inf\{t_k - t_{k-1}\} > 0$, where*

$$\sup_{k\in\mathbb{Z}_+} \Xi_k \le \Xi, \qquad \mathscr{F} \doteq \inf_{t\ge 0} \mathscr{F}(t) > 0.$$

Then the system (9) is LES in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$ with Lyapunov exponent $(1 - \theta)\mathscr{F}$. Moreover the solution $x(t) \doteq x(t, 0, \phi)$ of system (9) satisfies

$$\|x(t)\| \le \|\phi\|_\alpha \exp\Big(-(1-\theta)\mathscr{F}t\Big), \quad t \ge 0, \tag{10}$$

where $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, $\mathcal{M} = \frac{\mathscr{F}\theta\mu - \ln\Xi - \mathscr{F}\tau(\mathbf{0})}{\mathscr{F}\gamma}$.

Corollary 6. *Assume that*

$$\sup_{k\in\mathbb{Z}_+} \Xi_k \le 1, \qquad \mathscr{F} \doteq \inf_{t\ge 0}\mathscr{F}(t) > 0, \qquad c_{ii}(t) > 0,$$

$i = 1, \ldots, n$,

and there exist constants $\theta \in (0, 1)$, $\gamma > 0$ such that $|\tau(u) - \tau(\mathbf{0})| \le \gamma\|u\|$, $u \in \mathbb{R}^n$ and $\inf\{t_k - t_{k-1}\} > \frac{\tau(\mathbf{0})}{\theta}$. Then the system (9) is LES in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$ with Lyapunov exponent $(1 - \theta)\mathscr{F}$. Moreover, each solution $x(t) \doteq x(t, 0, \phi)$ satisfies (10), where $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, $\mathcal{M} = \frac{\theta\mu - \tau(\mathbf{0})}{\gamma}$, $\mu \doteq \inf\{t_k - t_{k-1}\}$.

If (7) is an autonomous system, that is,

$$\dot{x}(t) = -Cx(t) + Bg(x(t)), \quad t > 0, t \neq t_k \tag{11}$$

subject to impulses (8), where $C = (c_{ij}),\ B = (b_{ij}) \in \mathbb{R}^{n\times n}$ and other conditions are the same as previous. Based on Lyapunov function $V = x^T Px$, the following LMI-based result can be derived.

Corollary 7. *Assume that there exist an $n \times n$ matrix $P > 0$, an $n \times n$ diagonal matrix $Q > 0$ and constants $\gamma > 0,\ \sigma > 0,\ \theta \in (0, 1),\ \rho_k \geq 1$ such that $|\tau(u) - \tau(\mathbf{0})| \leq \gamma\|u\|,\ u \in \mathbb{R}^n$,*

$$\rho < \exp\Big(\sigma\theta\mu - \sigma\tau(\mathbf{0})\Big), \tag{12}$$

and the following LMIs hold:

$$\rho_k P - \Gamma_k^T P\Gamma_k \geq 0, \quad k \in \mathbb{Z}_+,$$

$$\begin{bmatrix} -\sigma P + C^T P + PC - L^g QL^g & PB \\ B^T P & Q \end{bmatrix} > 0, \tag{13}$$

where $\rho = \sup_{k\in\mathbb{Z}_+} \rho_k$, $\mu \doteq \inf\{t_k - t_{k-1}\} > 0$, $L^g = \mathrm{diag}(l_1^g, \ldots, l_n^g)$. Then the system (11) with (8) is LES in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$ with Lyapunov exponent $0.5(1-\theta)\sigma$. Moreover, the solution $x(t) = x(t, 0, \phi)$ satisfies

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|\phi\|_\alpha e^{-\frac{(1-\theta)\sigma}{2}t}, \quad t \geq 0,$$

where $\phi \in \mathbb{C}_\alpha^{\mathcal{M}},\ \mathcal{M} = \frac{\left(\theta\sigma\mu - \ln\rho - \sigma\tau(\mathbf{0})\right)}{\sigma\gamma}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$.

Remark 5. Note that Corollary 7 is given based on the LMI technique. The main advantages of such approach include that first it only needs tuning of parameters and/or matrices, and second it can be solved numerically using the LMI control toolbox in MATLAB. But here it should be mentioned that the choice of constant σ plays a very important role for the feasibility of LMIs in (13). In particular, if $\Gamma_k = \omega_k I$ is a diagonal matrix satisfying $\omega_k \leq 1,\ k \in \mathbb{Z}_+$, then (12) becomes $\theta\mu - \tau(\mathbf{0}) > 0$, which is independent of constant σ. In this case, the choice of σ should satisfy the LMI in (13). In addition, the method of free-weighting matrix or decomposition coupled with LMI technique is very popular in stability analysis of delay system, which could potentially improve the system performance. Unfortunately, to the best of our knowledge, these methods have not been applied to systems with state delay or impulses involving state-dependent delays.

4. Examples

Example 1. Consider the following 1D impulsive system:

$$\begin{cases} \dot{x}(t) = -x^{\frac{1}{3}}\exp(-x^2), & t \geq 0, t \neq t_k, \\ x(t_k) = \Big(1.2 + \dfrac{1}{10k}\Big)x(t_k^- - \tau), & k \in \mathbb{Z}_+, \\ x(s) = \phi(s), & s \leq 0, \end{cases} \tag{14}$$

where $\tau = 0.2 + 0.1\sin t + |x|,\ t_k = k,\ k \in \mathbb{Z}_+$. In the case, let $V(t) = |x|$, then $\omega_1 = \omega_2 = s$. Clearly, $W(u) = u^{\frac{1}{3}}\exp(-u^2),\ \mu = 1,\ \tau^\star = 0.3,\ \gamma = 1$ and $\max\rho_k = 1.3$. Choose $\theta = 0.95 < 1$, then by Corollary 3, it can be deduced that (14) is LUAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, where $\mathcal{M} \leq 0.96$.

Remark 6. Since system (14) is UAS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, its state is locally bounded, which implies the boundedness of state delay τ. But for such kind of delay, we cannot pre-assume its boundedness because the boundedness of system (14) cannot be derived a priori due to the effect of impulsive perturbations. In fact, it is possible that system (14) is unbounded if the impulse interval is small enough. Thus the stability criteria such as Corollary 3 is presented to ensure the stability (boundedness) of system (14), and then one may derive the boundedness of the state delay.

Example 2. Consider 2D impulsive system (9) with $g_j(s) = \sin s,\ j = 1, 2,\ \tau(x) = |x_1| + |x_2| + \eta,\ t_k = 2\eta k$, where $\eta > 0$ is a given constant; Functions $C(t)$, $B(t)$ and matrix Γ_k are given by

$$C(t) = \begin{bmatrix} 4 + 0.5e^{-t} & 0 \\ 1 + \cos t & 4 + e^{-0.5t} \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 2 + \sin t & -\cos t \\ \sin t & 2 - \cos t \end{bmatrix},$$

$$\Gamma_k = \frac{\sqrt{2}}{2}\begin{bmatrix} \sin 2^k & \cos 2^k \\ \cos\dfrac{1}{3k} & \sin\dfrac{1}{3k} \end{bmatrix}.$$

Note that $l_j = 1,\ \gamma = \sqrt{2},\ \mu = 2\eta,\ \tau(\mathbf{0}) = \eta,\ \Xi_k = 1$. Choose $\theta = \frac{2}{e^\eta + 1} < 1,\ \mathscr{F} = e^{-t}$, By Theorem 2, we know that (9) with the above parameters is LUS in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, where $\mathcal{M} \leq \frac{\sqrt{2}}{2}\ln(e^\eta - 1)$. But note that $\int_0^\infty \mathscr{F}ds = 1$, the LUAS cannot be guaranteed by our development results.

Example 3. Consider 2D impulsive system (11) with $g_j(s) = \tanh s,\ j = 1, 2,\ \tau = \sqrt{x^T x},\ t_k = 2k$, matrices C and B are given by

$$C = \begin{bmatrix} 2.4 & -0.2 \\ -0.48 & 2.3 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.15 & 0.2 \\ -0.5 & 0.6 \end{bmatrix}$$

and

$$\Gamma_k = \begin{bmatrix} \sqrt{2 + e^{-k}} & 0 \\ 0 & \sqrt{2 + e^{-k}} \end{bmatrix}.$$

It is clear that $l_j = \gamma = 1,\ \rho = 3,\ \tau(\mathbf{0}) = 0$ and $\mu = 2$. Choose $\theta = 0.9,\ \sigma = 1$, we obtain that the LMI (13) has solution

$$P = \begin{bmatrix} 28.2939 & 3.8461 \\ 3.8461 & 26.9744 \end{bmatrix}, \qquad Q = 49.8989I.$$

Then by Corollary 7, (11) with the above parameters is LES in the region $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$, where $\mathcal{M} \leq 0.608$. Moreover, the solution $x(t) = x(t, 0, \phi)$ satisfies $\|x(t)\| \leq 1.1528\|\phi\|_\alpha e^{-0.05t},\ t \geq 0$, where $\phi \in \mathbb{C}_\alpha^{\mathcal{M}}$.

5. Conclusion

Nonlinear differential systems with impulses involving state-dependent delay were considered. Some stability results including uniform stability, uniform asymptotic stability and exponential stability were established by employing techniques from the impulsive control theory. In our results, no presupposition is made on the boundedness of the state-dependent delay. Our main idea is to fetch the information of state-dependent delay on impulses and then integrate it into the constraint on the Lyapunov function.

Due to the presence of state-dependent delay on the impulses, our results apply only when the initial conditions are constrained in a bounded domain. In other words, we are not able to extend our ideas to address the global stability issue. In addition, here we restrict ourselves to impulsive perturbations rather than impulsive controls; the latter requires further studies. We also noted that many classical methods for stability analysis of delay systems, including delay decomposition approach, free-weighting matrix method, and Leibniz–Newton formula have not been extended to differential systems with state-dependent delays or impulses involving state-dependent delays.

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