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AN AGE-STRUCTURED POPULATION MODEL WITH STATE-DEPENDENT DELAY: DERIVATION AND NUMERICAL INTEGRATION*

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Abstract. We present an age-structured population model that accounts for complex life cycles and competition for resources limiting the transition to maturity. Taking all of these into account leads to a new mathematical model with a state-dependent delay that cannot be analyzed using the established framework of functional differential equations or directly simulated by standard numerical schemes for age-structured populations. In this paper we present the derivation of the model and a numerical scheme to integrate the equations. Convergence of the method is proven by verifying first consistency and then stability. The difficulties involved in this proof, as well as in the implementation of the scheme, stem from the state-dependent delay term. The numerical scheme is shown to be of order 2 in the given norm, and at least of order $\frac{3}{2}$ in the supremum norm over the mesh point values.

Key words. state-dependent delay differential equations, age-structured population models, method of characteristics, numerical stability

AMS subject classifications. 65N12, 92B05

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1. Introduction.

Delay differential equations (DDEs) generate infinite dimensional dynamical systems [4] and can lead to complex dynamics even in the scalar case. Many new mathematical models that include delays usually employ constant or time-dependent delays since those are simpler to analyze. Studies of these models have led to much insight into the complicated dynamics of DDE systems. However, Bocharov and Hadeler [3] opine that in some cases the delays were "unscrupulously" introduced to the equation. In many models it certainly seems like systems are assigned constant delays even when these should be state-dependent. We refer to Hartung et al. [6] for examples of systems where the delays are naturally state-dependent delays.

DDEs have been used to describe age-structured population dynamics [3, 4, 7, 9]. In this paper we propose a population model in which the delay, corresponding to the age of maturity, is carefully derived under reasonable biological assumptions. The model accounts for complex life cycles in which there are distinct juvenile and adult stages (e.g., the larval and adult stages of insects). The two stages are assumed to consume different food sources, and juveniles transition to the adult stage after consuming enough nutrition. As a result of this the age of maturity is affected by the competition within the immature population and this leads to a system of equations with state-dependent delay given by (2.1)–(2.5). This is a new system of equations, and we introduced an abstract framework for analyzing such systems in the paper [10]. The state-dependence makes it difficult to predict the behavior of solutions so we need a robust scheme to numerically experiment with the model.

We present a method to numerically integrate the model equations based on the

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method of characteristics approach by Chiu [5] and Abia and Lopez-Marcos [2]. This technique was extended to accommodate a changing delay. At every time step the proposed scheme first calculates the value of the population densities along characteristics, then solves a two-dimensional system of equations for the population density at the boundary and the value of the delay. This system is highly nonlinear due to the state-dependence, and it involves values from the current and past time steps. Furthermore, the propagation of discontinuities in the solution needs to be taken into account. These issues make it nontrivial to correctly set up a numerical scheme and provide significant challenges in the proof of convergence.

We briefly present a derivation of the model equations (2.1)–(2.5) in section 2. Properties of the model solutions are examined in section 3 and in the supplementary material. The numerical scheme and its suggested implementation are described in section 4. Second order convergence in the discrete L2 norm (found in Definition 5.1), and at least $\frac{3}{2}$ order convergence in the supremum norm (defined in (5.22)) are established in Theorem 5.18 in section 5. We summarize our results and give some final remarks in section 6.

2. Derivation of the model. Let $u(t,a)$ be the density of individuals of age a at time t. Let $\tau(t)$ be the age of maturity at time t. The model equations are

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -d(a)\,u(t,a), \qquad t>0, \tag{2.1}$$

$$u(t,0) = B\left(t, \int_{\xi=\tau(t)}^{M} \beta(\xi)\,u(t,\xi)\,d\xi\right), \qquad t>0, \tag{2.2}$$

$$\int_{t-\tau(t)}^{t} \frac{d\sigma}{\int_0^{\tau(\sigma)} u(\sigma,\xi)\,d\xi + C} = T, \qquad t>0, \tag{2.3}$$

$$u(t,a) = \psi(t,a), \qquad t\leqslant 0, \tag{2.4}$$

$$\tau(t) = \varphi(t), \qquad t\leqslant 0. \tag{2.5}$$

The constants and functions involved are required to satisfy Assumption 1, which is discussed in the next section. Equation (2.1) is the McKendrick equation [15] describing the evolution of the population density. The boundary condition (2.2) at $a = 0$ describes the nonlinear dependence of the number of newborns on the population of ages ranging from the age of maturity $\tau(t)$ to a maximum reproductive age $M \in (0,\infty)$. The threshold condition (2.3) is an equation for the age of maturity which we will derive in the next paragraph. Due to the form of (2.2) and (2.3), both $u(t,a)$ and $\tau(t)$ need to be initialized for $t \leqslant 0$. The initializations are given by (2.4)–(2.5). The functions $\psi(t,a)$ and $\varphi(t)$ are the histories of the population density and age of maturity, respectively.

To derive the threshold condition (2.3), we consider the juvenile population at time t given by $I(t) = \int_0^{\tau(t)} u(t,a)\,da$. Let $S(t)$ be the concentration of the food that the juvenile individuals consume at time t and assume that

$$S'(t) = S_0 - \gamma I(t)S(t) - cS(t), \tag{2.6}$$

where $S_0 > 0$ is the constant rate that food is recruited into the habitat, $\gamma > 0$ is the constant rate of food consumption by the immature population, and $c > 0$ reflects both the consumption rate by other populations and rate of decay of resources. As in Korvasova [9], we assume that food consumption happens at a much faster time scale than population growth. We set $S'(t) = 0$ and derive a quasi-steady state approximation for $S(t)$ given by $S(t) = \frac{S_0}{\gamma I(t)+c}$. Assuming that T^* units of food needs to be consumed to reach maturity, an individual born at time $t-\tau(t)$ becomes mature at time t if

$$\int_{t-\tau(t)}^{t} S(\sigma)\, d\sigma = \int_{t-\tau(t)}^{t} \frac{S_0}{\gamma I(\sigma)+c} d\sigma = T^*. \tag{2.7}$$

Substituting in $I(t) = \int_0^{\tau(t)} u(t,a)\, da$ in this equation and rescaling yields the proposed threshold condition (2.3). We note that (2.3) is a state-dependent threshold condition similar to those considered in Korvasova [9] and Hbid, Louihi, and Sanchez [7]. A comparison with these models is available in the supplementary material.

3. Properties of the model. In this section we state some of the properties of the model and a new formulation of the equations using the differentiated form of the threshold condition (see (3.4)–(3.8)). A more comprehensive discussion is available in the supplementary material where we outline results on the existence, uniqueness, boundedness, and regularity of solutions. The supplementary material also contains a proof of the equivalence of the two formulations and an explanation of why it is not possible to apply the method of steps to the system (and therefore it is not possible to apply the method of steps numerically either). More analyses of the model are available in the papers [10, 11].

By considering the evolution of (2.1) along characteristics $t-a =$ constant we can formally derive the formula

$$u(t,a) = \begin{cases} \psi(0, a-t) \exp\left(-\int_{\theta=a-t}^{a} d(\theta)\, d\theta\right), & t \leqslant a, \\ u(t-a, 0) \exp\left(-\int_{\theta=0}^{a} d(\theta)\, d\theta\right), & t > a. \end{cases} \tag{3.1}$$

Equation (3.1) shows that if $t > a$, the solution depends on $u(t-a, 0)$, which is given by (2.2)–(2.3). This coupling of the boundary and threshold conditions with integration along the characteristics will also appear in our numerical scheme. The solution will have a discontinuity at $t = a = 0$ if the following matching conditions do not hold:

$$\psi(0,0) = B\left(0, \int_{\xi=\varphi(0)}^{M} \beta(a)\, \psi(0,\xi)\, d\xi\right), \tag{3.2}$$

$$\int_{-\varphi(0)}^{0} \frac{d\sigma}{\int_0^{\varphi(\sigma)} \psi(\sigma,\xi)\, d\xi + C} = T. \tag{3.3}$$

If (3.2) is not satisfied, then the population density will have a discontinuity along the line $t = a$. As for the other condition (3.3), this requires the initializations to be consistent with the threshold condition (2.3) defining the age of maturity. We can get around the difficulty of requiring (3.3) by differentiating (2.3) and removing the need to give a value for T. This leads to our new formulation of the model equations,

$$\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -d(a)\,u(t,a), \qquad t > 0, \tag{3.4}$$

$$u(t,0) = B\left(t, \int_{\xi=\tau(t)}^{M} \beta(\xi)\,u(t,\xi)\,d\xi\right), \qquad t > 0, \tag{3.5}$$

$$\tau'(t) = 1 - \frac{\int_0^{\tau(t-\tau(t))} u(t-\tau(t),\xi)\,d\xi + C}{\int_0^{\tau(t)} u(t,\xi)\,d\xi + C}, \qquad t > 0, \tag{3.6}$$

$$u(t,a) = \psi(t,a), \qquad t \leqslant 0, \tag{3.7}$$

$$\tau(t) = \varphi(t), \qquad t \leqslant 0. \tag{3.8}$$

From (3.6) and the boundedness of solutions, we easily derive that $\tau'(t) < 1$. It follows from this that individuals that are already mature cannot become immature at a later time. Another property of the delay term is $\tau(t) \geqslant TC$ (from (2.3) and the positivity of solutions) so TC is a lower bound on the age of maturity.

For the remainder of this paper we will work with (3.4)–(3.8). We also make the following assumptions which are discussed in the supplementary material and [10].

Assumption 1.

1. *The functions* $\beta : [0,M] \to \mathbb{R}^+$ *and* $d : [0,M] \to \mathbb{R}^+$ *are bounded, differentiable functions.*
2. $B : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ *is bounded, smooth, and has a bounded derivative.*
3. *The initializations* $\psi : [-M,0] \times [0,M] \to \mathbb{R}^+$ *and* $\varphi : [-M,0] \to [TC, M]$ *are continuous and* (3.3) *holds for some* $T > 0$.
4. *There exist* $T_f > 0$, $u_{\max} > 0$, $\tau_{\max} \in (0,M)$, *and a classical solution to* (3.4)–(3.8) *such that* $u : [0,T_f] \times [0,M] \to [0,u_{\max}]$ *and* $\tau : [0,T_f] \to [TC, \tau_{\max}]$.
5. *The initializations* ψ *and* φ *are smooth enough such that the first, second, and third order partial derivatives of* $u(t,a)$ *exist and are bounded on each of the following sets:*

$$\{(t,a)\,|0 \leqslant a \leqslant M, a < t \leqslant T_f\} \quad \text{and} \quad \{(t,a)\,|0 \leqslant t \leqslant T_f, t \leqslant a \leqslant M\}.$$

6. *The first and second derivatives of* $\tau(t)$ *exist and are continuous on* $[0,T_f]$.

4. Numerical method. In this section we describe the scheme we developed to numerically solve the model equations (3.4)–(3.8). In section 4.1 we discuss the implementation of the scheme, and in section 4.2 we show some sample simulations.

4.1. Implementation of the numerical scheme. Our proposed numerical scheme is based on the method of characteristics approach that is also implemented in Chiu [5] and Abia and Lopez-Marcos [2]. To apply this technique to (3.4)–(3.8), we first consider integration along a characteristic (see (3.1)) by one time step of h which yields the equation $u(t+h,a+h) = u(t,a)\exp(-\int_0^h d(a+\theta)\,d\theta)$. Set the age step to be the same as the time step h. Let U_j^n be an approximation to $u(t_n,a_j)$, and χ^n be one for $\tau(t_n)$. A second order approximation to one-step integration along a characteristic is given by

$$U_{j+1}^{n+1} = U_j^n \exp\Big(-\frac{h}{2}\left(d(a_j) + d(a_{j+1})\right)\Big).$$

The discretization scheme using the method of characteristics is illustrated in Figure 1. The reason for using the method of characteristics instead of a finite difference scheme, such as the upwind scheme described in [1] and [12], is the possibility

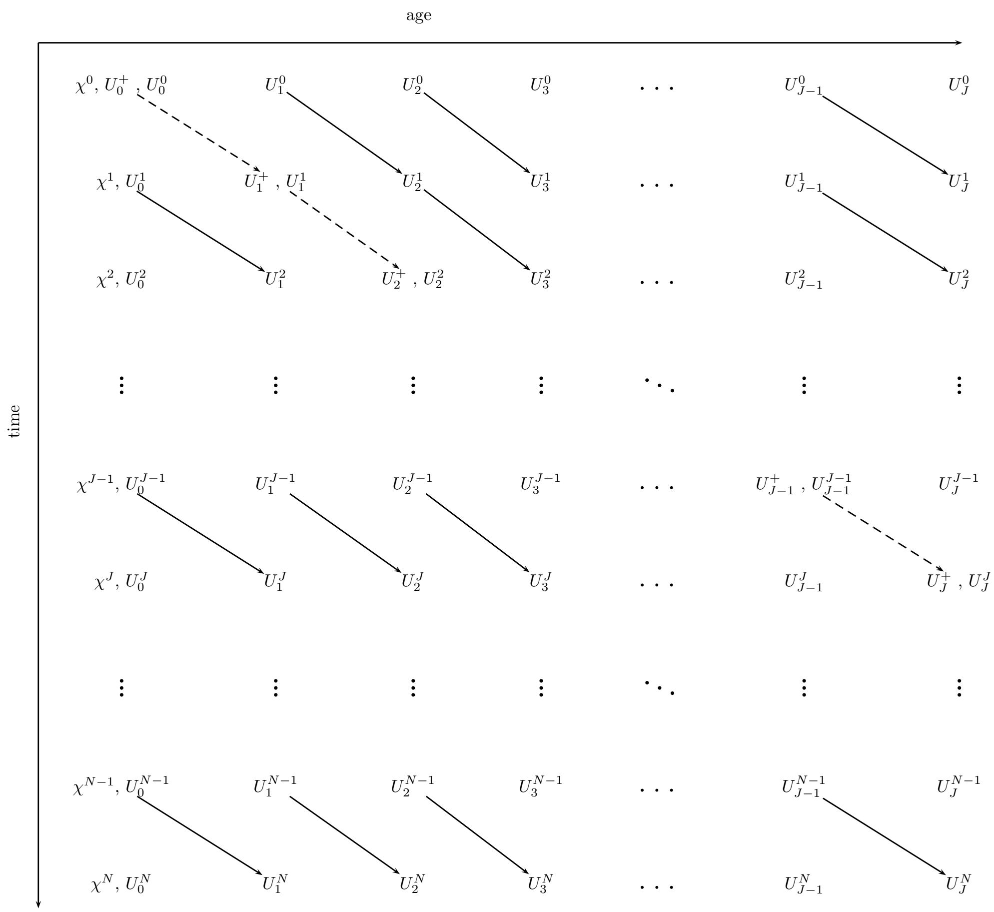


FIG. 1. *The method of characteristics described in Algorithm* 1. *The dashed lines show the characteristic* $t = a$ $(n = j)$ *along which discontinuities in the solution will propagate.*

of discontinuities in $u(t,a)$ and its derivatives propagating along the line $t = a$. The method of characteristics guarantees that the effects of the discontinuity will also propagate diagonally along the mesh points where $n = j$. In the case when the matching condition (3.2) is not met, then we may have two values for $u(t,a)$ at $t = a$. Thus we need to define a second value for U_j^n whenever $n = j$.

$$U_{j+1}^{j+1} = U_j^j \exp\Big(-\frac{h}{2}\left(d(a_j) + d(a_{j+1})\right)\Big), \quad U_{j+1}^{+} = U_j^{+} \exp\Big(-\frac{h}{2}\left(d(a_j) + d(a_{j+1})\right)\Big),$$

where $U_0^0 = \psi(0,0)$ and U_0^+ is an approximation of $B(0, \int_{a=\varphi(0)}^{M} \beta(a)\,\psi(0,a)\,da)$. In order to determine U_0^+ we require a quadrature formula (denoted by Q_h) which is given in Definition 4.2. The method of characteristics approach is summarized in Algorithm 1.

In order to solve for the boundary conditions, at each step we need to introduce continuous functions which approximate the values of $u(t,a)$ and $\tau(t)$ using interpo-

Algorithm 1. Main program.

Require: T_f, M, J, ψ, φ
  $h \leftarrow \frac{M}{J}$, $N \leftarrow \lceil \frac{T_f}{h} \rceil$, $a \leftarrow [0, h, 2h, \ldots, Jh]$.
  for $j = 1$ to $J - 1$ **do**
    $U_j^0 \leftarrow \psi(0, a_j)$.
  end for
  $U_0^+ \leftarrow B(0, Q_h(\beta(a)\psi(0,a), a = 0, M))$.
  $\chi^0 = \varphi(0)$.
  for $n = 0$ to $N - 1$ **do**
    for $j = 0$ to $J - 1$ **do**
      $U_{j+1}^{n+1} \leftarrow U_j^n \exp\left(-\frac{h}{2}(d(a_j) + d(a_{j+1}))\right)$.
    end for
    if $J < N$ **then**
      $U_{j+1}^+ \leftarrow U_j^+ \exp\left(-\frac{h}{2}(d(a_j) + d(a_{j+1}))\right)$.
    end if
    Solve for U_0^{n+1} and χ^{n+1} using Algorithm 2.
  end for

lation on the mesh values $\{U_j^+\}$, $\{U_j^n\}$, and $\{\chi^n\}$.

Definition 4.1. *At the $(n+1)$st-step, let*

$$m = \left\lfloor \frac{t}{h} \right\rfloor, \quad \theta_t = \frac{t - mh}{h}, \quad j = \left\lfloor \frac{a}{h} \right\rfloor, \quad \theta_a = \frac{a - jh}{h}.$$

At the $(n+1)$st-step we already have the values of U_j^m, U_m^+, and τ^m for $m = 0, \ldots, n$ and $j = 0, \ldots, J$. Due to the order of the operations in Algorithm 1, just as we get to Algorithm 2 we also have the values of U_{n+1}^+ and U_j^{n+1} for $j = 1, \ldots, J$. Define $\eta_{(U)}^n(t, a, \bar{U})$ to be a function that interpolates between the currently known mesh values of U and assumes a value of $\bar{U}$ in place of U_0^{n+1}.

1. *If $m < 0$, then $\eta_{(U)}^n(t, a, \bar{U}) = \psi(t, a)$.*
2. *If $0 \leqslant m < n$, then consider the following cases:*
 (i) *If $m \neq j$,*

$$\eta_{(U)}^n(t, a, \bar{U}) = [1 - \theta_t, \theta_t] \begin{bmatrix} U_j^m & U_{j+1}^m \\ U_j^{m+1} & U_{j+1}^{m+1} \end{bmatrix} \begin{bmatrix} 1 - \theta_a \\ \theta_a \end{bmatrix}.$$

 (ii) *If $m = j$ and $\theta_t > \theta_a$,*

$$\eta_{(U)}^n(t, a, \bar{U}) = (1 - \theta_t) U_m^+ + \theta_a U_{m+1}^+ + (\theta_t - \theta_a) U_m^{m+1}.$$

 (iii) *If $m = j$ and $\theta_t \leqslant \theta_a$,*

$$\eta_{(U)}^n(t, a, \bar{U}) = (1 - \theta_a) U_m^m + \theta_t U_{m+1}^{m+1} + (\theta_a - \theta_t) U_{m+1}^m.$$

3. *If $m = n$, then consider the following cases:*
 (i) *If $j > 0$, use the same formula as in the $m = j$ case.*
 (ii) *If $j = 0$, use the same formula as in the $m = j$ case but replace U_j^{m+1} by $\bar{U}$.*

We also define $\zeta_{(\chi)}^n(t, \bar{\chi})$ using linear interpolation between the known mesh values of χ and substituting $\bar{\chi}$ in place of χ^{n+1}:

$$\zeta_{(\chi)}^n(t, \bar{\chi}) = \begin{cases} \varphi(t) & \text{if } m \leqslant 0, \\ (1 - \theta_t)\chi^m + \theta_t \chi^{m+1} & \text{if } m < n, \\ (1 - \theta_t)\chi^n + \theta_t \overline{\chi} & \text{if } m = n. \end{cases}$$

The definition of $\eta^n_{(U)}\left(t,a,\bar{U}\right)$ is derived from applying bilinear interpolation over a square when $m \neq j$, and over a triangle when $m = j$ [16]. By Definition 4.1, $\zeta^n_{(\chi)}\left(t,\bar{\chi}\right)$ is continuous over $[t_0,t_f]$ and is linear over each interval $[t_m,t_{m+1}]$. The function $\eta^n_{(U)}\left(t,a,\bar{U}\right)$ is continuous on each set $\{(t,a)\,|0 \leqslant a \leqslant M, a \leqslant t \leqslant t_f\}$ and $\{(t,a)\,|0 \leqslant t \leqslant t_f, t \leqslant a \leqslant M\}$ with a possible discontinuity along the characteristic $t = a$.

The function $\eta^n_{(U)}(t,a,\bar{U})$ is also piecewise linear for fixed $a \in [0,M]$. Let $j = \lfloor \frac{a}{h} \rfloor$. If $j \geqslant n$, then $\eta^n_{(U)}(t,a,\bar{U})$ is linear over each interval $[t_m,t_{m+1}]$. Otherwise, $\eta^n_{(U)}\left(t,a,\bar{U}\right)$ is linear over each interval $[t_0,t_1],\ldots,\ [t_{j-1},t_j]$, $[t_j,a]$, $[a,t_{j+1}]$, $[t_{j+1},t_{j+2}],\ldots,\ [t_{n-1},t_n]$.

Also, for any fixed $t \in [0,t_n]$, let $m = \left\lfloor \frac{t}{h} \right\rfloor$. If $m \geqslant J$, then $\eta^n_{(U)}\left(t,a,\bar{U}\right)$ is linear over each interval $[a_j,a_{j+1}]$, $j = 0,\ldots,J$. Otherwise, $\eta^n_{(U)}\left(t,a,\bar{U}\right)$ is linear over each of $[a_0,a_1],\ldots,\ [a_{m-1},a_m]$, $[a_m,t]$, $[t,a_{m+1}]$, $[a_{m+1},a_{m+2}],\ldots,\ [a_{J-1},a_J]$.

We also define a quadrature function Q_h such that $Q_h\left(f(t,a),a = A_0,A_f\right)$ yields the exact value of $\int_{a=A_0}^{A_f} f(t,a)\,da$ whenever $f\left(t,a\right)$ is a function like $\eta^n_{(U)}\left(t,a,\bar{U}\right)$ (i.e., a piecewise linear function with a possible discontinuity at $a = t$).

Definition 4.2. *Let* $j_0 = \lceil \frac{A_0}{h} \rceil$, $j_f = \lfloor \frac{A_f}{h} \rfloor$, $j_* = \lfloor \frac{t}{h} \rfloor$. *If* $A_0 < t < A_f$,

$$\begin{aligned} Q_h\left(f(t,a),a = A_0,A_f\right) &= \frac{1}{2}\left(a_{j_0} - A_0\right) f\left(t,A_0\right) + \frac{1}{2}\left(h + a_{j_0} - A_0\right) f\left(t,a_{j_0}\right) \\ &+ h\sum_{i=j_0+1}^{j_*-1} f\left(t,a_i\right) + \frac{1}{2}\left(h + t - a_{j_*}\right) f\left(t,a_{j_*}\right) + \frac{1}{2}\left(t - a_{j_*}\right) \lim_{a\to a_{j_*}^-} f\left(t,a\right) \\ &+ \frac{1}{2}\left(h - \left(t - a_{j_*}\right)\right) \lim_{a\to a_{j_*}^+} f\left(t,a\right) + \frac{1}{2}\left(2h - \left(t - a_{j_*}\right)\right) f\left(t,a_{j_*+1}\right) \\ &+ h\sum_{i=j_*+2}^{j_f-1} f\left(t,a_i\right) + \frac{1}{2}\left(h + A_f - a_{j_f}\right) f\left(t,a_{j_f}\right) + \frac{1}{2}\left(A_f - a_{j_f}\right) f\left(t,A_f\right). \end{aligned}$$

Otherwise,

$$\begin{aligned} Q_h\left(f(t,a),a = A_0,A_f\right) &= \frac{1}{2}\left(a_{j_0} - A_0\right) f\left(t,A_0\right) + \frac{1}{2}\left(h + a_{j_0} - A_0\right) f\left(t,a_{j_0}\right) \\ &+ h\sum_{i=j_0+1}^{j_f-1} f\left(t,a_i\right) + \frac{1}{2}\left(h + A_f - a_{j_f}\right) f\left(t,a_{j_f}\right) + \frac{1}{2}\left(A_f - a_{j_f}\right) f\left(t,A_f\right). \end{aligned}$$

Observe that Q_h integrates a function like $\eta^n_{(U)}\left(t,a\right)$ exactly. For other functions that are at least twice differentiable everywhere except possibly at $a = t$, Q_h is a second order method.

4.2. Numerical simulations. Simulations of the model system (3.4)–(3.8) are shown in Figures 2–3. For these simulations we set $\beta\left(a\right) = 1$, $B\left(t,x\right) = \alpha x e^{-x}$, $d(a) = 0.3$, $C = 10$, and $M = 50$, where the value of α is indicated in the figures. We use an initial population density function of $\psi\left(t,a\right) = \left(1 - \frac{a}{M}\right)^2$ for $t \leqslant 0$ and $a \in [0,M]$. The initial delay function $\varphi(t)$ for $t \leqslant 0$ is assumed to be a constant function that matches the required condition for continuity in (3.2)–(3.3). It is not necessary to compute T, but it can be derived from ψ and φ using (3.3). The steady

Algorithm 2. Solving for the U_0^{n+1} and χ^{n+1}.

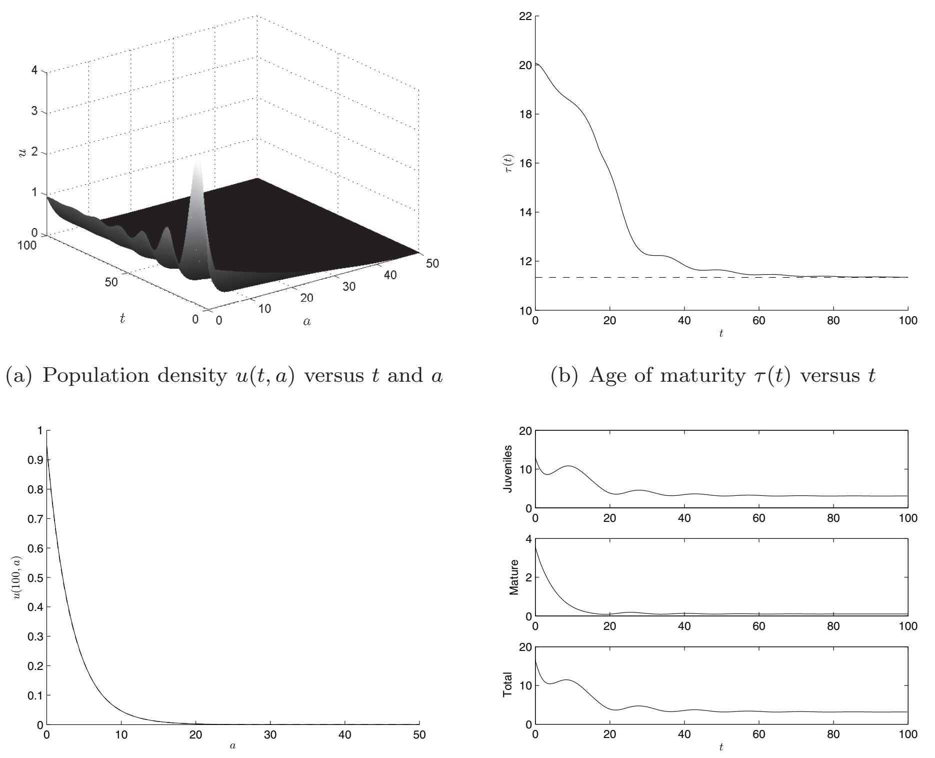
Require: $\eta^n_{(U)}, \zeta^n_{(\chi)}$

Using an iterative solver, find the solution $\left(\bar{U}, \bar{\chi}\right)$ to

$$\begin{cases} \bar{U} = Q_h\Big(B(\xi)\eta^n_{(U)}\left(t_{n+1},\xi,\bar{U}\right), \xi=\bar{\chi}, M\Big), \\ \bar{\chi} = \chi_n + h\Big[1 - \frac{1}{2}\frac{Q_h\left(\eta^n_{(U)}\left(t_{n+1}-\bar{\chi},\xi,\bar{U}\right),\xi=0,\zeta^n_{(\chi)}\left(t_{n+1}-\bar{\chi},\bar{\chi}\right)\right)+C}{Q_h\left(\eta^n_{(U)}\left(t_{n+1},\xi,\bar{u}\right),\xi=0,\chi^{n+1}\right)+C} \\ \qquad\qquad -\frac{1}{2}\frac{Q_h\left(\eta^n_{(U)}\left(t_n-\chi^n,\xi,\bar{U}\right),\xi=0,\zeta^n_{(\chi)}\left(t_n-\chi^n,\bar{\chi}\right)\right)+C}{Q_h\left(\eta^n_{(U)}\left(t_n,\xi,\bar{U}\right),\xi=0,\chi^n\right)+C}\Big]. \end{cases}$$

if the iteration converged and $\bar{\chi} \in [0, M]$ **then**
  $U_0^{n+1} \leftarrow \bar{U}$ and $\chi^{n+1} \leftarrow \bar{\chi}$.
end if

state population distribution $v(a)$ and steady state maturation delay $\bar{\tau}$ determined by the system of equations (derived in the supplementary material) are also shown in dashed lines.



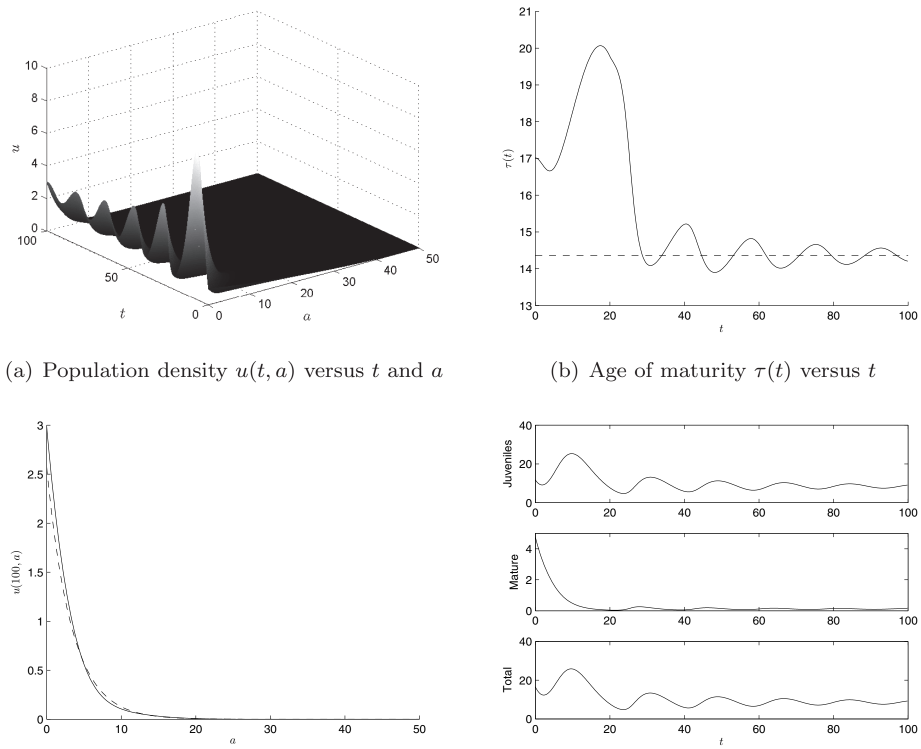
(a) Population density $u(t, a)$ versus t and a

(b) Age of maturity $\tau(t)$ versus t

(c) Age profile $u(t = 100, a)$

(d) Juvenile/Mature/Total Population versus t

Fig. 2. *Simulation with $\alpha = 10$. See section* 4.2 *for a full description of the simulations.*



(a) Population density $u(t,a)$ versus t and a (b) Age of maturity $\tau(t)$ versus t

(c) Age profile $u(t=100,a)$ (d) Juvenile/Mature/Total Population versus t

FIG. 3. *Simulation with* $\alpha = 25$. *In this simulation the solution has not yet settled into a steady state by* $t = 100$. *See section* 4.2 *for a full description of the simulations.*

5. Convergence of the numerical method. We present the proof of convergence of the numerical scheme in section 5.1 and the results of numerical tests on the order of convergence in section 5.2. The proof is based on a theorem by López-Marcos and Sanz-Serna [13], restated here as Theorem 5.4. This was also applied by López-Marcos [12] and Abia and López-Marcos [2] to prove convergence of methods for integro-partial differential equations (without delay) in population dynamics.

5.1. Proof of convergence. Let $H = \left\{h | h > 0, h = \frac{M}{J}, J \in \mathbb{N}\right\}$. Let $h \in H$ be the fixed step-size, $N = \lfloor \frac{T_f}{h} \rfloor$, $J = \lfloor \frac{M}{h} \rfloor$, $t_n = nh$ for $n = 0, \ldots, N$, and $a_j = jh$ for $j = 0, \ldots, J$. Define the spaces X_h and Y_h by $X_h = Y_h = \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{R}^{J+1} \times \left(\mathbb{R}^J\right)^{N+1}$.

Let $\chi \in \mathbb{R}^{n+1}$, $U_0 \in \mathbb{R}^{n+1}$, $U^+ \in \mathbb{R}^{J+1}$, and $U^n \in \mathbb{R}^J$ for $n = 0, \ldots, N$. Then

$$\left(\chi, U_0, U^+, U^0, U^1, \ldots, U^N\right) \in X_h. \tag{5.1}$$

Define the mapping $\Phi_h : X_h \to Y_h$ by

$$\Phi_h \left(\chi, U_0, U^+, U^0, U^1, \ldots, U^N\right) = \left(p, P_0, P^+, P^0, P^1, \ldots, P^N\right), \tag{5.2}$$

where $p = (p^0, p^1, \ldots, p^N)^T \in \mathbb{R}^{n+1}$, $P_0 = (P_0^0, P_0^1, \ldots, P_0^N)^T \in \mathbb{R}^{N+1}$, $P^+ = (P_1^+, P_2^+, \ldots, P_J^+)^T \in \mathbb{R}^{J+1}$, $P^n = (P_1^n, P_2^n, \ldots, P_J^n)^T \in \mathbb{R}^J$, and for $j = 1, \ldots, J$,

$n = 1, \ldots, N$, we have

$$\begin{aligned}
p^0 &= \chi^0 - \varphi(0), \\
p^n &= \frac{\chi^n - \chi^{n-1}}{h} - \left[1 - \frac{1}{2}\frac{Q_h\left(\eta_{(U)}\left(t_n - \chi^n, a\right), a = 0, \zeta_{(\chi)}(t_n - \chi^n)\right) + C}{Q_h\left(\eta_{(U)}\left(t_n, a\right), a = 0, \chi^n\right) + C}\right. \\
&\qquad \left. - \frac{1}{2}\frac{Q_h\left(\eta_{(U)}\left(t_{n-1} - \chi^{n-1}, a\right), a = 0, \zeta_{(\chi)}(t_{n-1} - \chi^{n-1})\right) + C}{Q_h\left(\eta_{(U)}\left(t_{n-1}, a\right), a = 0, \chi^{n-1}\right) + C}\right], \\
P_0^n &= U_0^n - B\left(t_n, Q_h\left(\beta(a)\eta_{(U)}(t_n, a), a = \chi^n, M\right)\right), \\
P_0^+ &= U_0^+ - B\left(t_0, Q_h\left(\beta(a)\psi(0, a), \chi^0, M\right)\right), \\
P_j^+ &= \frac{U_j^+ - U_{j-1}^+ \exp\left(-\frac{h}{2}\left(d(a_{j-1}) + d(a_j)\right)\right)}{h}, \\
P_j^0 &= U_j^0 - \psi(t_0, a_j), \\
P_j^n &= \frac{U_j^n - U_{j-1}^{n-1} \exp\left(-\frac{h}{2}\left(d(a_{j-1}) + d(a_j)\right)\right)}{h}.
\end{aligned}$$

In this section $\eta_{(U)}$ and $\zeta_{(\chi)}$ are the final $\eta_{(U)}^n$ and $\zeta_{(\chi)}^n$ functions in Definition 4.1, i.e.,

$$\eta_{(U)}(t, a) = \eta_{(U)}^{N-1}\left(t, a, U_0^N\right) \quad \text{and} \quad \zeta_{(\chi)}(t) = \zeta_{(\chi)}^{N-1}\left(t, \chi^N\right).$$

The numerical scheme described in section 4 is equivalent to solving for $W_h = (\chi, U_0, U^+, U^0, \ldots, U^N) \in X_h$ such that

$$\Phi_h(W_h) = 0 \in Y_h. \tag{5.3}$$

Definition 5.1. *Define the norms on X_h and Y_h to be*

$$\begin{aligned}
&\left\|\left(\chi, U_0, U^+, U^0, U^1, \ldots, U^N\right)\right\|_{X_h} \\
&\qquad = \max\left\{\|\chi\|_*, \|U_0\|_*, \|\left(U, U^+\right)^0\|, \|\left(U, U^+\right)^1\|, \ldots, \|\left(U, U^+\right)^N\|\right\}, \\
&\left\|\left(p, P_0, P^+, P^0, P^1, \ldots, P^N\right)\right\|_{Y_h} = \left(\|p\|_*^2 + \|P_0\|_*^2 + \sum_{n=0}^N h\|\left(P, P^+\right)^n\|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where $\|Z\|_* = \left(\sum_{n=0}^N h\left|Z^n\right|^2\right)^{\frac{1}{2}}$ *and for* $n = 0, \ldots, N$,

$$\left\|\left(Z, Z^+\right)^n\right\| = \begin{cases} \left(\sum_{j=1}^J h\left|Z_j^n\right|^2 + h\left|Z_n^+\right|^2\right)^{\frac{1}{2}} & \text{if } 0 \leqslant n \leqslant J, \\ \left(\sum_{j=1}^J h\left|Z_j^n\right|^2\right)^{\frac{1}{2}} & \text{if } n > J. \end{cases}$$

Definition 5.2. *Let $w_h = (\tau, u_0, u^+, u^0, \ldots, u^n) \in X_h$ be a discrete representation of the classical solution to* (3.4)–(3.8). *This means*

$$\tau^0 = \varphi(0), \quad \tau^n = \tau(t_n), \quad u_j^0 = \psi(0, a_j), \quad u_j^n = u(t_n, a_j) \text{ for } j \neq n,$$
$$u_j^j = \lim_{a \to a_j^+} u(t_j, a), \quad \text{and} \quad u_j^+ = \lim_{a \to a_j^-} u(t_j, a).$$

By the lower bound on $\tau(t)$, $\tau^n \geqslant TC$ for $n = 0, \ldots, N$. We also let $\eta_{(u)} = \eta_{(u)}^{N-1}(t, a, u_0^N)$ and $\zeta_{(\tau)} = \zeta_{(\tau)}^{N-1}(t, \tau^N)$, the interpolation of the w_h values at the mesh points using Definition 4.1.

Before proving the convergence of the numerical scheme we review some concepts in consistency, stability, and convergence of numerical methods from López-Marcos and Sanz-Serna [13].

DEFINITION 5.3 (from López-Marcos [12] and López-Marcos and Sanz-Serna [13]). *Define the global discretization error of the method* (5.2)–(5.3) *to be* $\|w_h - W_h\|_{X_h}$. *The method* (5.2)–(5.3) *is convergent if there exists* $h_0 > 0$ *such that if* $h \in (0, h_0]$, *then* (5.2)–(5.3) *has a solution* W_h *for which* $\lim_{h\to 0} \|w_h - W_h\|_{X_h} = 0$.

Define the local discretization error of the method (5.2)–(5.3) *to be* $\|\Phi_h(w_h)\|_{Y_h}$. *The method is consistent if there exists* $h_0 > 0$ *such that if* $h \in (0, h_0]$, *then* (5.2)–(5.3) *has a solution* W_h *for which* $\lim_{h\to 0} \|\Phi_h(w_h)\|_{Y_h} = 0$.

For every $h > 0$ *let* $K_h \in (0, \infty]$. *The method is stable restricted to thresholds* K_h *if there exist* $h_0 > 0$ *and* $S > 0$ *such that if* $h \in (0, h_0]$ *and* $v_h, z_h \in B(w_h, K_h)$, *then*

$$\|v_h - z_h\|_{X_h} \leqslant S\|\Phi_h(v_h) - \Phi_h(z_h)\|_{Y_h}.$$

This definition of stability was extended from a definition by Keller [8].

THEOREM 5.4 (from López-Marcos and Sanz-Serna [13]). *Assume that the numerical scheme* (5.3) *is consistent and stable with threshold* K_h. *If* Φ_h *is continuous in the ball* $B(w_h, K_h)$ *and* $\|\Phi_h(w_h)\|_{Y_h} = o(K_h)$ *as* $h \to 0$, *then the following hold:*

1. *For* h *small enough, the system* (5.2)–(5.3) *has a unique solution in* $B(w_h, K_h)$.
2. *As* $h \to 0$ *the solution converges and the order of convergence is not smaller than the order of consistency.*

In the succeeding proof of convergence we refer to (3.4)–(3.8) as the initial value problem. When we say that a constant depends on the initial value problem, we mean that it depends on the model parameters and initializations which determine the properties of the analytic solution to the initial value problem.

LEMMA 5.5. *Suppose that Assumption* 1 *is satisfied. Let* $t \in [0, T_f]$ *and* $0 \leqslant A_i \leqslant A_f \leqslant M$. *Then*

$$\left|u(t,a) - \eta_{(u)}(t,a)\right| \leqslant C_1 h^2, \qquad \left|\tau(t) - \zeta_{(\tau)}(t)\right| \leqslant C_2 h^2,$$

$$\left|\int_{A_i}^{A_f} u(t,a)\,da - Q_h\left(\eta_{(u)}(t,a), a = A_i, A_f\right)\right| \leqslant C_3 h^2,$$

$$\left|\int_{A_i}^{A_f} \beta(a)\,u(t,a)\,da - Q_h\left(\beta(a)\,\eta_{(u)}(t,a), a = A_i, A_f\right)\right| \leqslant C_4 h^2,$$

where the constants $C_1, \ldots, C_4$ *depend only on the initial value problem.*

Proof. The proof follows from the choice of interpolation schemes [16] and the assumption that the solutions are at least second order differentiable everywhere except possibly along the line $t = a$. Note that we can choose $C_3 = MC_1$. ∎

LEMMA 5.6. *Let* A_1, A_2, B_1, $B_2 \geqslant -\frac{C}{2}$. *Then*

$$\left|\frac{A_1 + C}{B_1 + C} - \frac{A_2 + C}{B_2 + C}\right| \leqslant 4\left(C + \max\{|A_1|, |A_2|, |B_1|, |B_2|\}\right) \frac{|A_1 - A_2| + |B_1 - B_2|}{C^2}. \quad ∎$$

LEMMA 5.7. *Suppose that Assumption* 1 *is satisfied. Then for small enough* h,

$$|p^n| = O\left(h^2\right).$$

Proof. If $n=0$, $p^0=0$. If $n=1,\dots,N$,

$$
\begin{aligned}
|p^n| \leqslant & \left| \frac{\tau^n-\tau^{n-1}}{h} - \left(1-\frac{1}{2}\frac{\int_0^{\tau(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)da + C}{\int_0^{\tau^n} u\left(t_n,a\right)da+C}\right.\right.\\
& \left.\left. -\frac{1}{2}\frac{\int_0^{\tau(t^*)} u\left(t^*,a\right)da+C}{\int_0^{\tau^{n-1}} u\left(t_{n-1},a\right)da+C}\right)\right|\\
& +\frac{1}{2}\left|\frac{\int_0^{\tau(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)da+C}{\int_0^{\tau^n} u\left(t_n,a\right)da+C} - \frac{\int_0^{\zeta_{(\tau)}(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)da+C}{\int_0^{\tau^n} u\left(t_n,a\right)da+C}\right|\\
& +\frac{1}{2}\left|\frac{\int_0^{\tau(t^*)} u\left(t^*,a\right)da+C}{\int_0^{\tau^{n-1}} u\left(t_{n-1},a\right)da+C} - \frac{\int_0^{\zeta_{(\tau)}(t^*)} u\left(t^*,a\right)da+C}{\int_0^{\tau^{n-1}} u\left(t_{n-1},a\right)da+C}\right|\\
& +\frac{1}{2}\left|\frac{\int_0^{\zeta_{(\tau)}(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)da+C}{\int_0^{\tau^n} u\left(t_n,a\right)da+C}\right.\\
& \left. -\frac{Q_h\left(\eta_{(u)}\left(t_n-\tau^n,a\right),a=0,\zeta_{(\tau)}(t_n-\chi^n)\right)+C}{Q_h\left(\eta_{(u)}\left(t_n,a\right),a=0,\tau^n\right)+C}\right|\\
& +\frac{1}{2}\left|\frac{\int_0^{\zeta_{(\tau)}(t^*)} u\left(t^*,a\right)da+C}{\int_0^{\tau^{n-1}} u\left(t_{n-1},a\right)da+C} - \frac{Q_h\left(\eta_{(u)}\left(t^*,a\right),a=0,\zeta_{(\tau)}(t^*)\right)+C}{Q_h\left(\eta_{(u)}\left(t_{n-1},a\right),a=0,\tau^{n-1}\right)+C}\right|,
\end{aligned} \tag{5.4}
$$

where $t^*=t_{n-1}-\tau^{n-1}$. By the truncation error of the trapezoidal rule, the first term is of order h^2. By Lemma 5.5, the second term in (5.4) satisfies

$$
\begin{aligned}
&\left|\frac{\int_0^{\tau(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)+C}{\int_0^{\tau^n} u\left(t_n,a\right)+C} - \frac{\int_0^{\zeta_{(\tau)}(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)+C}{\int_0^{\tau^n} u\left(t_n,a\right)+C}\right|\\
&\qquad \leqslant \frac{u_{\max}}{C}\left|\tau\left(t_n-\tau^n\right)-\zeta_{(\tau)}\left(t_n-\tau^n\right)\right| \leqslant \frac{u_{\max}}{C}C_2h^2.
\end{aligned} \tag{5.5}
$$

By Lemmas 5.5 and 5.6 we can bound the fourth term in (5.4) by

$$
\begin{aligned}
&\left|\frac{\int_0^{\zeta_{(\tau)}(t_n-\tau^n)} u\left(t_n-\tau^n,a\right)+C}{\int_0^{\tau^n} u\left(t_n,a\right)+C} - \frac{Q_h\left(\eta_{(u)}\left(t_n-\tau^n,a\right),a=0,\zeta_{(\tau)}\left(t_n-\tau^n\right)\right)+C}{Q_h\left(\eta_{(u)}\left(t_n,a\right),a=0,\tau^n\right)+C}\right|\\
&\qquad \leqslant 8\frac{u_{\max}M+C}{C^2}C_3h^2.
\end{aligned} \tag{5.6}
$$

The third and fifth terms in (5.4) can be bound in a similar manner to (5.5) and (5.6), respectively. Using all of these bounds in (5.4) proves that $|p^n|=O\left(h^2\right)$. ∎

LEMMA 5.8. *Suppose that Assumption* 1 *is satisfied. For small enough* h,

$$\left|P_0^+\right|=O\left(h^2\right),\quad \left|P_0^n\right|=O\left(h^2\right),\quad \left|P_j^+\right|=O\left(h^2\right),\quad \left|P_j^n\right|=O\left(h^2\right)$$

for $n=0,\dots,N$ *and* $j=1,\dots,J$.

Proof. The first result follows from Lemma 5.5 and 5.7. By definition of P^0, $\left|P_j^0\right|=0$. The rest follow from integration along characteristics and truncation error of trapezoidal rule. ∎

LEMMA 5.9. *Suppose that Assumption* 1 *is satisfied. For small enough* h, *the local error satisfies* $\|\Phi_h(w_h)\|_{Y_h}=O\left(h^2\right)$.

Proof. This result follows from Lemmas 5.7 and 5.8. ∎

We have now proven the consistency of the numerical method in Lemma 5.9. Let us now consider stability. Suppose v_h, $z_h \in B\left(w_h, Kh^{3/2}\right)$, where $K > 0$ is a constant:

$$\begin{aligned} v_h &= \left(\chi, V_0, V^+, V^0, \dots, V^N\right), \quad \Phi_h\left(v_h\right) = \left(\ell, L_0, L^+, L^1, \dots, L^N\right), \\ z_h &= \left(\sigma, Z_0, Z^+, Z^0, \dots, Z^N\right), \quad \Phi_h\left(z_h\right) = \left(r, R_0, R^+, R^1, \dots, R^N\right). \end{aligned} \tag{5.7}$$

Denote by $\eta_{(v)}$ and $\zeta_{(\chi)}$ the η and ζ continuations corresponding to v_h, respectively (recall the construction in Definition 4.1). Similarly we let $\eta_{(z)}$ and $\zeta_{(\sigma)}$ be the continuations corresponding to z_h.

LEMMA 5.10. *Let $v_h \in X_h$ with notation given in* (5.7). *Then for $i = 1, \dots, N$,*

$$\max\left\{\left|V_n^+\right|, \left|V_1^n\right|, \dots, \left|V_J^n\right|\right\} \leqslant \frac{1}{\sqrt{h}} \left\|\left(V, V^+\right)^n\right\|,$$

$$\max\left\{\left|V_0^0\right|, \dots, \left|V_0^N\right|\right\} \leqslant \frac{1}{\sqrt{h}} \left\|V_0\right\|_*, \quad \textit{and} \quad \max\left\{\left|\chi^0\right|, \dots, \left|\chi^N\right|\right\} \leqslant \frac{1}{\sqrt{h}} \left\|\chi\right\|_*.$$

Furthermore, for any $a_1 \leqslant A_i \leqslant A_f \leqslant M$ we have

$$\left|Q_h\left(\eta_{(v)}\left(t_n, a\right), a = A_i, A_f\right)\right| \leqslant \sqrt{M} \left\|\left(V, V^+\right)^n\right\|,$$

and for any $0 \leqslant A_i \leqslant A_f \leqslant M$, we have

$$\left|Q_h\left(\eta_{(v)}\left(t_n, a\right), a = A_i, A_f\right)\right| \leqslant \sqrt{M} \left\|\left(V, V^+\right)^n\right\| + \frac{1}{2} h \left|V_0^n\right|.$$

Proof. This follows from the definitions of the norms $\|\cdot\|$, $\|\cdot\|_*$ and Q_h. ∎

LEMMA 5.11. *Suppose that the conditions of Assumption* 1 *are satisfied. Let $v_h, z_h \in B\left(w_h, Kh^{3/2}\right)$ be as given in* (5.7). *Then,*

$$\begin{aligned} \left|\eta_{(v)}\left(t, a\right) - \eta_{(z)}\left(t, a\right)\right| &\leqslant 2Kh, \\ \left|\zeta_{(\chi)}\left(t\right) - \zeta_{(\sigma)}\left(t\right)\right| &\leqslant 2Kh, \\ \left|\eta_{(v)}\left(t, a\right) - u\left(t, a\right)\right| &\leqslant 2Kh + C_1 h^2, \\ \left|\zeta_{(\chi)}\left(t\right) - \tau\left(t\right)\right| &\leqslant 2Kh + C_2 h^2. \end{aligned}$$

Also, for any $t \in [0, T_f]$ and $0 \leqslant A_i \leqslant A_f \leqslant M$,

$$\begin{aligned} \left|\int_{A_i}^{A_f} \left(\eta_{(v)}\left(t, a\right) - \eta_{(z)}\left(t, a\right)\right) da\right| &\leqslant 2KMh, \\ \left|\int_{A_i}^{A_f} \left(\eta_{(v)}\left(t, a\right) - u\left(t, a\right)\right) da\right| &\leqslant 2KMh + C_3 h^2. \end{aligned}$$

Proof. Since $v_h, z_h \in B(w_h, Kh^{3/2})$, then $\|v_h - z_h\|_{X_h} \leqslant 2Kh^{3/2}$. By the convexity of the interpolation and Lemma 5.10, for any $t \in [0, T_f]$ and $a \in [0, M]$ we have

$$\begin{aligned} \left|\eta_{(v)}\left(t, a\right) - \eta_{(z)}\left(t, a\right)\right| &\leqslant \frac{1}{\sqrt{h}} \max\left\{\max_{n=1,\dots,N} \left\|\left(V - Z, V^+ - Z^+\right)^n\right\|, \left\|V_0 - Z_0\right\|_*\right\} \\ &\leqslant \frac{1}{\sqrt{h}} \left\|v_h - z_h\right\|_{X_h} \leqslant 2Kh. \end{aligned}$$

The proof of the second inequality is similar to this. The proof of the third and fourth inequalities follows from the first two and Lemma 5.5. The last two inequalities follow from the first four (or using Lemma 5.10). ∎

LEMMA 5.12. *Suppose that the conditions of Assumption* 1 *are satisfied. Let* $v_h, z_h \in B\left(w_h, Kh^{3/2}\right)$ *be as given in* (5.7). *Then it is possible to choose* h *small enough such that* $\chi^n, \sigma^n \geqslant TC - Kh \geqslant h$, *and* $V_j^n, V_j^+, Z_j^n, Z_j^+ \geqslant 0 - Kh \geqslant -\frac{C}{2M}$ *for all* $n = 0, \ldots, N$ *and* $j = 0, \ldots, J$.

Proof. The conclusion follows from Lemma 5.10. ∎

To make the succeeding analysis easier, define e_n and E_n to be

$$e_n = \max_{i=0,\ldots,n} h|\chi^i - \sigma^i|^2, \qquad E_n = \max_{i=0,\ldots,n} \|(V - Z, V^+ - Z^+)^n\|^2. \tag{5.8}$$

By the convexity of the interpolation schemes, for any $t \in [t_0, t_n]$ and any $a_1 \leqslant A_i \leqslant A_f \leqslant M$ we have

$$\begin{aligned}\left|Q_h\left(\eta_{(v)}(t,a) - \eta_{(z)}(t,a), a = A_i, A_f\right)\right| \\ = \left|\int_{A_i}^{A_f} \left(\eta_{(v)}(t,a) - \eta_{(z)}(t,a)\right) da\right| \leqslant \sqrt{ME_n},\end{aligned} \tag{5.9}$$

by Lemma 5.10. Furthermore, for any $0 \leqslant A_i \leqslant A_f \leqslant M$ and $t \leqslant t_n$ we have

$$\left|Q_h\left(\eta_{(v)}(t,a) - \eta_{(z)}(t,a), a = A_i, A_f\right)\right| \leqslant \sqrt{ME_n} + \frac{1}{2}h \max_{i=0,\ldots,n} \left|V_0^i - Z_0^i\right|. \tag{5.10}$$

LEMMA 5.13. *Suppose that the conditions of Assumption* 1 *are satisfied. Let* $v_h, z_h \in B\left(w_h, Kh^{3/2}\right)$ *be as given in* (5.7). *Then*

$$|V_0^n - Z_0^n|^2 \leqslant C_5\left(E_n + he_n + |L_0^n - R_0^n|^2\right),$$

where C_5 *depends only on the initial value problem.*

Proof. From (5.2),

$$\begin{aligned}|V_0^n - Z_0^n| \leqslant \Big|B\left(t_n, Q_h\left(\beta(a)\eta_{(v)}(t_n,a), a = \chi^n, M\right)\right) \\ - B\left(t_n, Q_h\left(\beta(a)\eta_{(z)}(t_n,a), a = \sigma^n, M\right)\right)\Big| + |L_0^n - R_0^n|.\end{aligned}$$

Without loss of generality, assume $\chi^n \leqslant \sigma^n$. By Lemma 5.12, we can choose h to be small enough such that $h < TC - 2Kh < \chi^n$. If $\text{Lip}(B)$ is the Lipschitz constant of B and $\bar{\beta}$ is an upper bound on β, then

$$\begin{aligned}|V_0^n - Z_0^n| \leqslant \text{Lip}(B)\,\bar{\beta}\Big(\left|Q_h\left(\eta_{(v)}(t_n,a) - \eta_{(z)}(t_n,a), a = \sigma^n, M\right)\right| \\ + \left|Q_h\left(\eta_{(v)}(t_n,a) - \eta_{(z)}(t_n,a), a = \chi^n, \sigma^n\right)\right|\Big) + |L_0^n - R_0^n|.\end{aligned}$$

By (5.9) and Lemma 5.11,

$$|V_0^n - Z_0^n| \leqslant \text{Lip}(B)\,\bar{\beta}\left(\sqrt{ME_n} + 2Kh\,|\chi^n - \sigma^n|\right) + |L_0^n - R_0^n|.$$

By squaring and applying $xy \leqslant \frac{1}{2}x^2 + \frac{1}{2}y^2$ we derive the result

$$|V_0^n - Z_0^n|^2 \leqslant C_5\left(E_n + h^2|\chi^n - \sigma^n|^2 + |L_0^n - R_0^n|^2\right) \leqslant C_5\left(E_n + he_n + |L_0^n - R_0^n|^2\right),$$

where C_5 depends on Lip (B), M, and K. ∎

LEMMA 5.14. *Suppose that the conditions of Assumption* 1 *are satisfied. Let* $v_h, z_h \in B(w_h, Kh^{3/2})$ *be as given in* (5.7). *Then*

$$E_n \leqslant (1 + C_6 h)E_{n-1} + C_6 h^2 e_{n-1} + C_6 h\big(\|(L - R, L^+ - R^+)^n\|^2 + |L_0^{n-1} - R_0^{n-1}|^2\big),$$

where C_6 *depends only on the initial value problem.*

Proof. Assume first that $n < J$. The proof for $n > J$ is similar. Since $d(a) \geqslant 0$ for $a \in [0, M]$, then

$$|V_j^n - Z_j^n| \leqslant |V_{j-1}^{n-1} - Z_{j-1}^{n-1}| + h|L_j^n - R_j^n|, \quad |V_n^+ - Z_n^+| \leqslant |V_{n-1}^+ - Z_{n-1}^+| + h|L_n^+ - R_n^+|.$$

It follows that

$$\begin{aligned}
&\|(V - Z, V^+ - Z^+)^n\|^2 = \sum\nolimits_{j=1}^{J} h|V_j^n - Z_j^n|^2 + h|V_n^+ - Z_n^+|^2,\\
&\leqslant \sum\nolimits_{j=1}^{J} h|V_{j-1}^{n-1} - Z_{j-1}^{n-1}|^2 + h|V_{n-1}^+ - Z_{n-1}^+|^2\\
&\qquad + h^3\left(\sum\nolimits_{j=1}^{N} |L_j^n - R_j^n|^2 + |L_n^+ - R_n^+|^2\right),\\
&\qquad + 2h^2 \sum\nolimits_{j=1}^{N} |V_{j-1}^{n-1} - Z_{j-1}^{n-1}||L_j^n - R_j^n| + 2h^2|V_{n-1}^+ - Z_{n-1}^+||L_n^+ - R_n^+|,\\
&\leqslant \|(V - Z, V^+ - Z^+)^{n-1}\|^2 + h|V_0^{n-1} - Z_0^{n-1}|^2 + h^2\|(L - R, L^+ - R^+)^n\|^2\\
&\qquad + 2h^2\left(\sum\nolimits_{j=1}^{N} |V_{j-1}^{n-1} - Z_{j-1}^{n-1}||L_j^n - R_j^n| + |V_{n-1}^+ - Z_{n-1}^+||L_n^+ - R_n^+|\right),\\
&\leqslant (1 + h)\|(V - Z, V^+ - Z^+)^{n-1}\|^2 + (1 + h)h\|(L - R, L^+ - R^+)^{n-1}\|^2\\
&\qquad + (1 + h)h|V_0^{n-1} - Z_0^{n-1}|^2,
\end{aligned}$$

where the last line is due to the Cauchy–Schwarz inequality. By applying Lemma 5.13 and (5.8) this inequality becomes

$$\begin{aligned}
&\|(V - Z, V^+ - Z^+)^n\|^2 \leqslant (1 + h)\|(V - Z, V^+ - Z^+)^{n-1}\|^2\\
&+ (1 + h)h\|(L - R, L^+ - R^+)^{n-1}\|^2 + C_5(1 + h)h(E_{n-1} + he_{n-1} + |L_0^{n-1} - R_0^{n-1}|^2).
\end{aligned}$$

By the definition of E_n we can derive from this

$$E_n \leqslant (1 + C_6 h)E_{n-1} + C_6 h^2 e_{n-1} + C_6 h(\|(L - R, L^+ - R^+)^{n-1}\|^2 + |L_0^{n-1} - R_0^{n-1}|^2)$$

for some C_6 independent of h (e.g., $C_6 = C_5(1 + M)$). ∎

LEMMA 5.15. *Suppose that the conditions of Assumption* 1 *are satisfied. Let* $z_h \in B\left(w_h, Kh^{3/2}\right)$ *with notation given in* (5.7). *Then*

$$\begin{aligned}
\left|\zeta_{(\sigma)}(t_1) - \zeta_{(\sigma)}(t_2)\right| &\leqslant C_7 |t_1 - t_2|,\\
\left|\int_{A_i}^{A_f} \left(\eta_{(z)}(t_1, a) - \eta_{(z)}(t_2, a)\right) da\right| &\leqslant C_8 |t_1 - t_2|,
\end{aligned}$$

where C_7 *and* C_8 *depend only on the initial value problem.*

Proof. Let $\tau'_{\max}$ be the bound on $|\tau'(t)|$. Then,

$$\begin{aligned}
|\sigma^n - \sigma^{n-1}| &\leqslant |\sigma^n - \tau^n| + |\tau^n - \tau^{n-1}| + |\tau^{n-1} - \sigma^{n-1}|\\
&\leqslant Kh + \tau'_{\max} h + Kh = (2K + \tau'_{\max})\, h.
\end{aligned}$$

Let $C_7 = 2K + \tau'_{\max}$. Then $\frac{|\sigma^n - \sigma^{n-1}|}{h} \leqslant C_7$ for all n. So $|\zeta_{(\sigma)}(t_1) - \zeta_{(\sigma)}(t_2)| \leqslant C_7|t_1 - t_2|$ follows. This is the first result of the lemma.

Similarly let $u'_{\max}$ be the bound on $|\frac{\partial}{\partial t} u(t,a)|$ everywhere where $t \neq a$. Then,

$$\begin{aligned}|Z_j^n - Z_j^{n-1}| &\leqslant |Z_j^n - u_j^n| + |u_j^n - u_j^{n-1}| + |u_j^{n-1} - Z_j^{n-1}| \\ &\leqslant Kh + u'_{\max} + Kh = (2K + u'_{\max})\, h.\end{aligned}$$

Let $\tilde{C}_8 = 2K + u'_{\max}$. Then $\frac{|Z_j^n - Z_j^{n-1}|}{h} \leqslant \tilde{C}_8$ for all $n \neq j+1$. Similarly, $\frac{|Z_j^{j+1} - Z_j^+|}{h} \leqslant \tilde{C}_8$. Without loss of generality, assume $t_1 \leqslant t_2$. Then $|\eta_{(z)}(t_1,a) - \eta_{(z)}(t_2,a)| \leqslant \tilde{C}_8|t_1 - t_2|$ as long as $\operatorname{sign}(t_1 - a) = \operatorname{sign}(t_2 - a)$. Thus if we integrate the functions from A_i to A_f where either $0 \leqslant A_i \leqslant A_f \leqslant t_1 \leqslant t_2$ or $0 \leqslant t_1 \leqslant t_2 \leqslant A_i \leqslant A_f$, then it follows that $|\int_{A_i}^{A_f} \left(\eta_{(z)}(t_1,a) - \eta_{(z)}(t_2,a)\right) da| \leqslant M\tilde{C}_8|t_1 - t_2|$.

If $0 \leqslant A_i \leqslant t_1 \leqslant t_2 \leqslant A_f \leqslant M$, then

$$\begin{aligned}&\left|\int_{A_i}^{A_f} \left(\eta_{(z)}(t_1,a) - \eta_{(z)}(t_2,a)\right) da\right| \\ &\leqslant \left|\int_{A_i}^{t_1} \left(\eta_{(z)}(t_1,a) - \eta_{(z)}(t_2,a)\right) da\right| + \left|\int_{t_1}^{t_2} \left(\eta_{(z)}(t_1,a) - \eta_{(z)}(t_2,a)\right) da\right| \\ &\quad + \left|\int_{t_2}^{A_f} \left(\eta_{(z)}(t_1,a) - \eta_{(z)}(t_2,a)\right) da\right|, \\ &\leqslant M\tilde{C}_8|t_1 - t_2| + \max\left\{\eta_{(z)}(t,a)\right\}|t_1 - t_2| + M\tilde{C}_8|t_1 - t_2|, \\ &\leqslant (2M\tilde{C}_8 + u_{\max} + 2Kh + C_1h^2)|t_1 - t_2|.\end{aligned}$$

The other cases can be bounded similarly. Thus we can choose C_8 that depends only on the model parameters such that the second inequality of the lemma holds. ∎

LEMMA 5.16. *Suppose that the conditions of Assumption* 1 *are satisfied. Let* $v_h, z_h \in B\left(w_h, Kh^{3/2}\right)$ *be as given in* (5.7) *with all components of* v_h *and* z_h *being nonnegative. Also restrict* χ^n *and* $\sigma^n \in [0, M]$. *Then,*

$$\begin{aligned}e_n \leqslant C_9h^2E_{n-1} + (1 + C_9h)\, e_{n-1} + C_9h^2|\ell^n - r^n|^2 + C_9h^2\|L_0 - R_0\|_*^2 \\ + C_9h^3\|(L - R, L^+ - R^+)^{n-1}\|^2,\end{aligned}$$

where C_9 *depends only on the initial value problem.*

Proof. We use integrals instead of Q_h since they are the same for $\eta_{(v)}$ and $\eta_{(z)}$:

$$\begin{aligned}|\chi^n - \sigma^n| &\leqslant h|\ell^n - r^n| + |\chi^{n-1} - \sigma^{n-1}| \\ &\quad + \frac{h}{2}\left|\frac{\int_0^{\zeta_{(\chi)}(t_n - \chi^n)} \eta_{(v)}(t_n - \chi^n, a)\, da + C}{\int_0^{\chi^n} \eta_{(v)}(t_n, a)\, da + C} - \frac{\int_0^{\zeta_{(z)}(t_n - \sigma^n)} \eta_{(z)}(t_n - \sigma^n, a)\, da + C}{\int_0^{\sigma^n} \eta_{(z)}(t_n, a)\, da + C}\right| \\ &\quad + \frac{h}{2}\left|\frac{\int_0^{\zeta_{(v)}(t_{n-1} - \chi^{n-1})} \eta_{(v)}\left(t_{n-1} - \chi^{n-1}, a\right) da + C}{\int_0^{\chi^{n-1}} \eta_{(v)}(t_{n-1}, a)\, da + C}\right. \\ &\quad \left. - \frac{\int_0^{\zeta_{(z)}(t_{n-1} - \sigma^{n-1})} \eta_{(z)}\left(t_{n-1} - \sigma^{n-1}, a\right) da + C}{\int_0^{\sigma^{n-1}} \eta_{(z)}(t_{n-1}, a)\, da + C}\right|.\end{aligned} \tag{5.11}$$

Consider the third term on the right-hand side of (5.11). By the bounds on $u(t,a)$ and Lemma 5.11, $\int_{A_i}^{A_f} \eta_{(v)}(t,a)\,da \leqslant u_{\max}M + 4KMh + C_3h^2$. Let $k_1 = \frac{4}{C^2}(C + u_{\max}M + 4KM^2 + C_3M^2)$. By Lemma 5.6 we have

$$\left|\frac{\int_0^{\zeta_{(v)}(t_n-\chi^n)} \eta_{(v)}(t_n-\chi^n,a)\,da + C}{\int_0^{\chi^n} \eta_{(v)}(t_n,a)\,da + C} - \frac{\int_0^{\zeta_{(z)}(t_n-\sigma^n)} \eta_{(z)}(t_n-\sigma^n,a)\,da + C}{\int_0^{\sigma^n} \eta_{(z)}(t_n,a)\,da + C}\right| \tag{5.12}$$
$$\leqslant k_1\left|\int_0^{\zeta_{(v)}(t_n-\chi^n)} \eta_{(v)}(t_n-\chi^n,a)\,da - \int_0^{\zeta_{(z)}(t_n-\sigma^n)} \eta_{(z)}(t_n-\sigma^n,a)\,da\right|$$
$$+ k_1\left|\int_0^{\chi^n} \eta_{(v)}(t_n,a)\,da - \int_0^{\sigma^n} \eta_{(z)}(t_n,a)\,da\right|.$$

Consider the first term on the right-hand side of (5.12). We can derive

$$\left|\int_0^{\zeta_{(v)}(t_n-\chi^n)} \eta_{(v)}(t_n-\chi^n,a)\,da - \int_0^{\zeta_{(z)}(t_n-\sigma^n)} \eta_{(z)}(t_n-\sigma^n,a)\,da\right|$$
$$\leqslant \left|\int_0^{\zeta_{(v)}(t_n-\chi^n)} \eta_{(v)}(t_n-\chi^n,a)\,da - \int_0^{\zeta_{(v)}(t_n-\chi^n)} \eta_{(z)}(t_n-\chi^n,a)\,da\right| \tag{5.13}$$
$$+ \left|\int_0^{\zeta_{(v)}(t_n-\chi^n)} \eta_{(z)}(t_n-\chi^n,a)\,da - \int_0^{\zeta_{(z)}(t_n-\chi^n)} \eta_{(z)}(t_n-\chi^n,a)\,da\right|$$
$$+ \left|\int_0^{\zeta_{(z)}(t_n-\chi^n)} \eta_{(z)}(t_n-\chi^n,a)\,da - \int_0^{\zeta_{(z)}(t_n-\sigma^n)} \eta_{(z)}(t_n-\chi^n,a)\,da\right|$$
$$+ \left|\int_0^{\zeta_{(z)}(t_n-\sigma^n)} \eta_{(z)}(t_n-\chi^n,a)\,da - \int_0^{\zeta_{(z)}(t_n-\sigma^n)} \eta_{(z)}(t_n-\sigma^n,a)\,da\right|.$$

Without loss of generality assume again that $\chi^n \leqslant \sigma^n$. Recall from Lemma 5.12 that we can choose h to be small enough such that $h < TC - 2Kh < \chi^n \leqslant \sigma^n$. Then $t_n - \sigma^n \leqslant t_n - \chi^n \leqslant t_{n-1}$. Using (5.10) and Lemma 5.13, we have

$$\begin{aligned}\text{first term in (5.13)} &\leqslant \sqrt{ME_{n-1}} + \frac{h}{2}\max\nolimits_{i=1,\ldots,n-1}|V_0^i - Z_0^i| \\ &\leqslant \sqrt{ME_{n-1}} + \frac{h}{2}\sqrt{C_5\left(E_{n-1} + he_{n-1} + \max\nolimits_{i=1,\ldots,n-1}|L_0^i - R_0^i|^2\right)} \\ &\leqslant \left(\sqrt{M} + \frac{h}{2}\sqrt{C_5}\right)\sqrt{E_{n-1}} + \frac{h^{3/2}}{2}\sqrt{C_5}e_{n-1} \\ &+ \frac{h^{1/2}}{2}\sqrt{C_5}\|L_0 - R_0\|_*.\end{aligned}$$

By Lemma 5.11, the convexity of the interpolation scheme for ζ, we derive

$$\begin{aligned}\text{second term in (5.13)} &\leqslant \max_{a\in[0,M]}|\eta_{(z)}(t_n-\chi^n,a)||\zeta_{(\chi)}(t_n-\chi^n) - \zeta_{(\sigma)}(t_n-\chi^n)| \\ &\leqslant (u_{\max} + 2Kh + C_1h^2)\max\nolimits_{i=0,\ldots,n-1}|\chi^i - \sigma^i| \\ &= (u_{\max} + 2Kh + C_1h^2)h^{-1/2}\sqrt{e_{n-1}}.\end{aligned}$$

By Lemma 5.15 we can find bounds on the third and fourth terms in (5.13) so that

$$\begin{aligned}\text{third term in (5.13)} &\leqslant \left(u_{\max} + 2Kh + C_1h^2\right)\left|\zeta_{(z)}\left(t_n - \chi^n\right) - \zeta_{(z)}\left(t_n - \sigma^n\right)\right| \\ &\leqslant \left(u_{\max} + 2Kh + C_1h^2\right)C_7|\chi^n - \sigma^n|, \\ \text{fourth term in (5.13)} &\leqslant M\left|\eta_{(z)}\left(t_n - \chi^n\right) - \eta_{(z)}\left(t_n - \sigma^n\right)\right| \\ &\leqslant MC_8|\chi^n - \sigma^n|.\end{aligned}$$

From these terms, we get a bound for (5.13). Using simpler estimates we can bound the second term of (5.12) (involves E_n and e_n). Similar estimates bound the fourth term of (5.11). Keeping track of leading order terms lead to

$$\begin{aligned}|\chi^n - \sigma^n| \leqslant (1 + k_2h)|\chi^{n-1} - \sigma^{n-1}| + k_2h|\chi^n - \sigma^n| + h|\ell^n - r^n| \\ + k_2h^{3/2}\|L_0 - R_0\|_* + k_2h\sqrt{E_n} + k_2h^{1/2}\sqrt{e_{n-1}}\end{aligned}$$

for some k_2 independent of h. Multiplying both sides by $|\chi^n - \sigma^n|$, expanding using $xy \leqslant \frac{1}{2}x^2 + \frac{1}{2}y^2$, and collecting the $|\chi^n - \sigma^n|^2$ terms on the right, we get

$$\begin{aligned}&\left(1 - k_2h - \frac{1}{2}\left(1 + k_2h + h + 3k_2h\right)\right)|\chi^n - \sigma^n|^2 \\ &\qquad \leqslant \frac{1}{2}\left(|\chi^{n-1} - \sigma^{n-1}|^2 + h|\ell^n - r^n|^2 + k_2h^2\|L_0 - R_0\|_*^2 + k_2hE_n + k_2e_{n-1}\right).\end{aligned}$$

Thus we can find a constant $\tilde{C}_9$ independent of h such that

$$\begin{aligned}|\chi^n - \sigma^n|^2 \leqslant (1 + \tilde{C}_9h)|\chi^{n-1} - \sigma^{n-1}|^2 + \tilde{C}_9h|\ell^n - r^n|^2 \\ + \tilde{C}_9h^2\|L_0 - R_0\|_*^2 + \tilde{C}_9hE_n + \tilde{C}_9e_{n-1}.\end{aligned}$$

Multiplying everything by h and using the definition of e_n yields

$$e_n \leqslant \tilde{C}_9h^2E_n + (1 + 2\tilde{C}_9h)e_{n-1} + \tilde{C}_9h^2|\ell^n - r^n|^2 + \tilde{C}_9h^3\|L_0 - R_0\|_*^2.$$

Lemma 5.14, combining terms, and defining a constant C_9 yields the final result. ◻

LEMMA 5.17. *Suppose that the conditions of Assumption* 1 *are satisfied. Then the discretization is stable with threshold* $K_h = Kh^{3/2}$.

Proof. Let $K_h = Kh^{3/2}$. Let v_h and $z_h \in B\left(w_h, K_h\right)$. By Lemmas 5.14 and 5.16, we can pick a constant k such that

$$\begin{aligned}E_n &\leqslant (1 + kh)E_{n-1} + kh^2e_{n-1} + kh\|(L - R, L^+ - R^+)^n\|^2 + kh\left|L_0^n - R_0^n\right|^2, \\ e_n &\leqslant kh^2E_{n-1} + (1 + kh)e_{n-1} + kh^2\left|\ell^n - r^n\right|^2 \\ &\qquad + kh^2\|L_0 - R_0\|_*^2 + kh^3\|(L - R, L^+ - R^+)^n\|^2.\end{aligned}$$

Define A, f_n, g_n, $\bar{E}_N$, and $\bar{e}_N$ as

$$\begin{aligned}A &= \begin{bmatrix} 1 + kh & kh^2 \\ kh & 1 + kh^2 \end{bmatrix}, \\ f_n &= kh\|(L - R, L^+ - R^+)^n\|^2 + kh|L_0^n - R_0^n|^2, \\ g_n &= kh^2|\ell^n - r^n|^2 + kh^2\|L_0 - R_0\|_*^2 + kh^3\|(L - R, L^+ - R^+)^n\|^2,\end{aligned}$$

$$\begin{bmatrix} \bar{E}_N \\ \bar{e}_N \end{bmatrix} = A^N \begin{bmatrix} E_0 \\ e_0 \end{bmatrix} + \sum_{i=n}^{N} A^{N-n} \begin{bmatrix} f_n \\ g_n \end{bmatrix}. \tag{5.14}$$

By induction it is easy to show that

$$E_N \leqslant \bar{E}_N \quad \text{and} \quad e_N \leqslant \bar{e}_N. \tag{5.15}$$

Consider the eigenvalues of A: $\lambda_1 = 1 + kh(1+h)$ and $\lambda_2 = 1 + kh(1-h)$. For small h, $\lambda_2^N \leqslant \lambda_1^N = (1+kh(1+h))^N \leqslant (1+2kh)^{\frac{T_f}{h}} \leqslant \exp(2kT_f)$. It follows that A^N is bounded for all $N \in \mathbb{Z}^+$. Thus there exists a $\bar{k}$ such that for any $N \in \mathbb{Z}^+$,

$$E_N \leqslant \bar{k}\left(E_0 + e_0 + \sum_{n=0}^{N} f_n + \sum_{n=0}^{N} g_n\right), \quad e_N \leqslant \bar{k}\left(E_0 + e_0 + \sum_{n=0}^{N} f_n + \sum_{n=0}^{N} g_n\right). \tag{5.16}$$

For small h, by Lemma 5.13 we obtain

$$\begin{aligned} \|V_0 - Z_0\|_* &\leqslant C_5(ME_N + Mhe_N + \|L_0 - R_0\|_*) \\ &\leqslant C_5\left[2\bar{k}M\left(E_0 + e_0 + \sum_{n=0}^{N} f_n + \sum_{n=0}^{N} g_n\right) + \|L_0 - R_0\|_*\right]. \end{aligned} \tag{5.17}$$

We also derive that

$$e_0 = h|\chi^0 - \sigma^0|^2 = h|\ell^0 - r^0|^2 \leqslant \|\ell - r\|_*^2, \tag{5.18}$$

$$E_0 = \|(V - Z, V^+ - Z^+)^0\|^2 = \|(L - R, L^+ - R^+)^0\|^2, \tag{5.19}$$

$$\begin{aligned} \sum_{n=0}^{N} f_n &= k\left(\sum_{n=0}^{N} h\|(L - R, L^+ - R^+)^n\|^2 + h|L_0^n - R_0^n|^2\right), \\ &\leqslant k\sum_{n=0}^{N} h\|(L - R, L^+ - R^+)^n\|^2 + k\|L_0 - R_0\|_*^2, \end{aligned} \tag{5.20}$$

$$\begin{aligned} \sum_{n=0}^{N} g_n &= kh\left(\sum_{n=0}^{N} h|\ell^n - r^n|^2 + \sum_{n=0}^{N} h\|L_0 - R_0\|_*^2\right) + kh^2\sum_{n=0}^{N} h\|(L - R, L^+ - R^+)^n\|^2, \\ &\leqslant k\|\ell - r\|_*^2 + kM\|L_0 - R_0\|_*^2 + k\sum_{n=0}^{N} h\|(L - R, L^+ - R^+)^n\|^2 \end{aligned} \tag{5.21}$$

for small enough h. By the definition of the X_h norm we get

$$\begin{aligned} \|v_h - z_h\|_{X_h} = \max\Big\{&\|\chi - \sigma\|_*, \|V_0 - Z_0\|_*, \|(V - Z, V^+ - Z^+)^0\|, \\ &\|(V - Z, V^+ - Z^+)^1\|, \ldots, \|(V - Z, V^+ - Z^+)^N\|\Big\}, \\ \leqslant \max\Big\{&\sqrt{Me_N}, \|V_0 - Z_0\|_*, \sqrt{E_N}\Big\}. \end{aligned}$$

Using this, (5.16)–(5.21), and the definition of Y_h, we can pick an S that depends only on the initial value problem such that

$$\|v_h - z_h\|_{X_h} \leqslant S\|\Phi(v_h) - \Phi(z_h)\|_{Y_h}. \qquad \square$$

We are now in a position to prove the convergence of the numerical scheme. By Theorem 5.4, the order of convergence will be the same as the order of consistency. We would also like to find the order of convergence using the supremum norm over the mesh point values,

$$\|w_h - W_h\|_\infty \equiv \max\big\{\|\chi - \tau\|_\infty, \|U_0 - u_0\|_\infty, \|U^+ - u^+\|_\infty, \dots, \|U^0 - u^0\|_\infty, \|U^1 - u^1\|_\infty, \dots, \|U^N - u^N\|\big\}. \tag{5.22}$$

THEOREM 5.18. *The numerical scheme given by Algorithms* 1–2 *to solve the system* (3.4)–(3.8) *is convergent with the order of convergence given by* $\|w_h - W_h\|_{X_h} = O\left(h^2\right)$. *Furthermore,* $\|w_h - W_h\|_\infty$ *is at least* $O\left(h^{3/2}\right)$.

Proof. Using similar techniques above we can show that the mapping Φ_h is Lipschitz continuous close to u_h. We have shown that the scheme is consistent of order 2 in the X_h norm (Lemma 5.9) and stable in $B\left(u_h, Kh^{3/2}\right)$ (Lemma 5.17). Thus from Stetter [17], the inverse Φ_h^{-1} exists in $B\left(\Phi_h(u_h), \frac{K}{S}h^{3/2}\right)$ and the zero in Y_h is in this ball. As in Theorem 5.4 (López-Marcos and Sanz-Serna [13]), existence of the numerical solution W_h and second order convergence in the X_h norm follow.

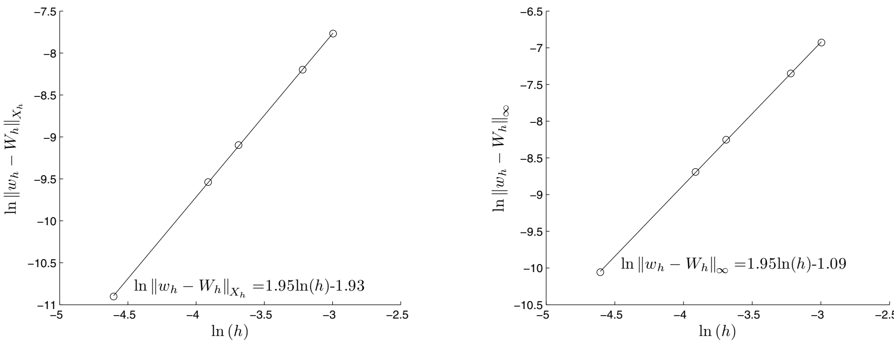
As for the order of convergence with respect to the $\|\cdot\|_\infty$ norm defined in (5.22), recall from Lemma 5.10 that

$$\begin{aligned}\|w_h - W_h\|_\infty &\leqslant \frac{1}{\sqrt{h}}\max\big\{\|\chi - \tau\|_*, \|U_0 - u_0\|_*, \|\left(U - u, U^+ - u^+\right)^0\|, \dots, \\ &\qquad \|(U - u, U^+ - u^+)^1\|, \dots, \|(U - u, U^+ - u^+)^N\|\big\}, \\ &\leqslant \frac{1}{\sqrt{h}}\|W_h - w_h\|_{X_h} = O\big(h^{3/2}\big). \qquad \square\end{aligned}$$

This last theorem shows that $\|w_h - W_h\|_\infty$ has at least $\frac{3}{2}$ order. As we will see in the numerical tests of convergence shown in Figure 4, $\|w_h - W_h\|_\infty$ appears to also be of order 2, which is consistent with it being at least of order $\frac{3}{2}$.

5.2. Numerical verification of convergence order. The complicated nature of the state-dependent delay in the system (3.4)–(3.8) makes it difficult, if not impossible, to derive analytic solutions to the system except in trivial cases such as at steady state (where both the density and delay are constant in time). Since most of the difficulty in our proof of convergence involved the state-dependent delay, it is important to verify the convergence of the numerical scheme when the delay is changing. A description of the test for convergence that we designed is given in the supplementary material. A log plot of the error versus stepsize is shown in Figure 4 and displays a slope of 1.95 which supports second order convergence.

6. Conclusions and future work. We presented an age-structured population model that describes populations in which immature individuals compete among themselves for food sources, but not with the matured individuals. From this main assumption we derived a new mathematical model given in (2.1)–(2.5) which is analyzed in papers [10, 11]. The equations were reformulated as (3.4)–(3.8) for easier numerical integration. This model was derived to describe complex populations such



(a) Convergence in X_h norm

(b) Convergence in the supremum norm

FIG. 4. *Numerical verification of the second order convergence of the method in the X_h norm and at least $\frac{3}{2}$ order convergence in the supremum norm. The second figure indicates that the convergence may be second order as well in the supremum norm, at least for the test problem.*

as those of insect and amphibian populations, but it may also be used to describe cellular populations with threshold dynamics.

In order to do reliable numerical experimentation with the model, we derived a new numerical scheme and proved its convergence assuming that the conditions given in Assumption 1 hold. This numerical scheme is based on the technique of integrating along characteristics introduced by Chiu [5] adapted for a system with state-dependent delay. The extension required defining continuous extensions of the density and delay terms over the mesh, deriving at each time step a system of equations to solve for the boundary value and delay term, and accommodating discontinuities that may propagate. The convergence of the numerical scheme is proven in section 5 (see Theorem 5.18). The convergence is shown to be of order 2 in the X_h norm (given in Definition 5.1) and at least of order $\frac{3}{2}$ in the supremum norm (defined in (5.22)). Figure 4 shows the results of a convergence test and indicates second order convergence in both norms.

We also presented some numerical experiments indicating that this system may exhibit the well-known sequence of bifurcations from transcritical to Hopf described in [14]. Some of these experiments are shown in Figures 2–3.

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