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Flocking and asymptotic velocity of the Cucker–Smale model with processing delay

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ABSTRACT

The processing delay is incorporated into the influence function of the well-known Cucker–Smale model for self-organized systems with multiple agents. Both symmetric and non-symmetric pairwise influence functions are considered, and a Lyapunov functional approach is developed to establish the existence of flocking solutions for the proposed delayed Cucker–Smale model. An analytic formula is given to calculate the asymptotic flocking velocity in terms of model parameters and the variation of the position during the initial time interval.

1. Introduction

Self-organized systems arise very naturally in artificial intelligence, and in physical, biological and social sciences. Such systems seem to have remarkable capability to regulate the flow of information from distinct and independent components to achieve prescribed performance. It is of particular interest, in both theories and applications, to understand how self-propelled individuals use only limited environmental information and simple rules to organize into ordered motions. These emerging behaviours such as flocking, herding and schooling have been observed in many self-organized systems, including fish swimming in schools (Pitcher et al. [13]), birds flying in flocks for the purpose of enhancing the foraging success (Camazine et al. [2]), and the flight guidance in honeybee swarms (Fetecau and Guo [6]).

The celebrated Cucker–Smale model [4] proposed in 2007 provides a framework to examine the emergent properties of flocks in order to explain self-organized behaviours in various complex systems. This Cucker–Smale model has since been extended to include asymmetric influence functions and multi-agent systems with hierarchical leadership, see [1,3,5,7,9,10] and the references therein. These emerging behaviours are typically described by the so-called flocking solutions.

A flocking solution of the Cucker–Smale model (that describes the evolution of the position and velocity of agents/individuals involved) is a solution for which the agents asymptotically reach a uniform velocity

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while remaining in finite distance from each other. In the literature, one talks about unconditional flocking (if flocking solutions exist for all initial data) and conditional flocking (if flocking solutions exist for only a certain class of initial configurations). A popular criterion to guarantee unconditional flocking is that the influence function slowly decays with a diverging tail [11].

In this study, we incorporate processing delay in the influence function into the Cucker–Smale model and its variations. We extend the aforementioned criterion for unconditional flocking to the delayed model, using an approach based on a Lyapunov functional and delay differential inequalities. We also examine the role of this processing delay and its impact on the asymptotic velocity. Our results show that this processing delay does not change the qualitative flocking behaviours but alters the flocking velocity in a complicated nonlinear fashion, involving both the initial velocity and the variation of the initial positions. This part of our work also answered an open problem proposed by Motsch and Tadmor [11].

After a brief introduction into the Cucker–Smale model, we justify the introduction of time delay in the influence function in Section 2, present an existence criterion for the flocking solution in Section 3, and then derive an asymptotic flocking velocity formula in Section 4.

2. Cucker–Smale model and delayed influence

We consider the motion of a self-organized group with N agents, with each agent i being characterized by its position $\mathbf{x}_i \in \mathbf{R}^d$ and velocity $\mathbf{v}_i \in \mathbf{R}^d$, where $d \geqslant 1$ is an integer. The Cucker–Smale model [4] is given by

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j \neq i}^{N} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i), \tag{1}$$

where α measures the interaction strength and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$. In the original formulation by Cucker and Smale, the function a_{ij} takes the following format

$$a_{ij}^{CS}(\mathbf{x}) = I\big(|\mathbf{x}_i - \mathbf{x}_j|\big)/N \tag{2}$$

to quantify the pairwise influence of agent j on the alignment of agent i, as a function of the (metric) distance. This influence function I is a strictly positive monotonically decreasing continuous function with a prototype given by $I(r) = (1 + r^2)^{-\beta}$ for $r \geqslant 0$, where β is a constant. In the recent study of Motsch and Tadmor [11], a non-symmetric pairwise influence function

$$a_{ij}^{MT}(\mathbf{x}) = \frac{I(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} I(|\mathbf{x}_i - \mathbf{x}_k|)} \tag{3}$$

is used to emphasize the importance of relative influence among agents.

We are interested in a more general setting by incorporating delay arguments in the pairwise influence due to the finite speed in processing the influence. It seems to be very natural to introduce time lags for most self-organized systems, and we will show that these time lags will not change the unconditional flocking property qualitatively, but alter the flocking velocity in a nonlinear way.

In general, the influence of agents on each other is realized in various fashions including smell, sound and vision. For example, the influence among honey bees is transferred mainly by a certain chemical material [12], while the influence among geese is mainly made through vision [11]. As such, the influence of an agent on another is naturally transferred with a finite speed. We will focus in this study on the case of delayed processing of the information about the location and velocity of neighbouring agents, resulting in the following modified Cucker–Smale model with delay:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j\neq i}^{N} a_{ij}\big(\mathbf{x}(t-\tau)\big)\big(\mathbf{v}_j(t-\tau) - \mathbf{v}_i(t)\big), \tag{4}$$

where τ denotes the communication time between agents i and j, which includes the response time of agent i, $a_{ij} = a_{ij}^{CS}$ or a_{ij}^{MT}.

To specify a solution for the self-organized system (4), we need to specify the initial conditions

$$\mathbf{x}_i(\theta) = \mathbf{f}_i(\theta), \qquad \mathbf{v}_i(\theta) = \mathbf{g}_i(\theta) \quad \text{for } \theta \in [-\tau, 0], \tag{5}$$

where $\mathbf{f}$ and $\mathbf{g}$ are given continuous vector-valued functions. It will be shown that the final flocking velocity will depend not only on the size of the time lag, but also on the variation of the agent positions at the initial time interval.

As usual, we let d_X and d_V denote the diameters in position and velocity spaces, namely,

$$d_X(\mathbf{x}) = \max_{i,j}\big\{|\mathbf{x}_j - \mathbf{x}_i|\big\}, \qquad d_V(\mathbf{v}) = \max_{i,j}\big\{|\mathbf{v}_j - \mathbf{v}_i|\big\}.$$

A solution $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1}^N$ of system (4) subject to the initial condition (5) is called a *flocking solution* if it converges to a flock in the sense that

$$\sup_{t\geqslant 0} d_X\big(\mathbf{x}(t)\big) < +\infty \quad \text{and} \quad \lim_{t\to+\infty} d_V\big(\mathbf{v}(t)\big) = 0. \tag{6}$$

We can simplify the model by using similar arguments used in [11]. Namely, using the boundedness of a_{ij} and by rescaling α if necessary, we may assume that a_{ij} are normalized so that $\sum_{j\neq i} a_{ij}(\mathbf{x}) < 1$ for all $\mathbf{x}$. Let

$$\tilde{a}_{ii}(t) = 1 - \sum_{j\neq i} a_{ij}\big(\mathbf{x}(t-\tau)\big), \qquad \tilde{a}_{ij}(t) = a_{ij}\big(\mathbf{x}(t-\tau)\big),$$

then we can rewrite system (4) in the form

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \alpha\big(\bar{\mathbf{v}}_i(t) - \mathbf{v}_i(t)\big), \tag{7}$$

where

$$\bar{\mathbf{v}}_i(t) = \sum_{j=1}^{N} \tilde{a}_{ij}(t)\mathbf{v}_j(t-\tau).$$

Remark 2.1. Due to the definitions of d_X and d_V with the super-norm, functions $d_X(\mathbf{x}(t))$ and $d_V(\mathbf{v}(t))$ are not in general C^1 smooth, so we will use upper Dini derivative in the discussion below. This is defined, for a given function w that is continuous at t, by

$$D^+ w(t) = \limsup_{h\to 0^+} h^{-1}\big[w(t+h) - w(t)\big].$$

If w is differentiable at t, then $D^+ w(t) = \frac{dw(t)}{dt} = \dot{w}(t)$. In general, we have a sequence $h_n \to 0^+$ so that $D^+ w(t) = \lim_{n\to\infty} h_n^{-1}[w(t+h_n) - w(t)]$. More importantly, for a given t and for given $\mathbf{x}$, there exist integers r and s so that $d_X(\mathbf{x}(t)) = |\mathbf{x}_r(t) - \mathbf{x}_s(t)|$ and a sequence $h_n \to 0^+$ so that

$$D^+ d_X\big(\mathbf{x}(t)\big) = \lim_{n\to\infty} h_n^{-1}\big[\big|\mathbf{x}_r(t+h_n) - \mathbf{x}_s(t+h_n)\big| - \big|\mathbf{x}_r(t) - \mathbf{x}_s(t)\big|\big] \leqslant \big|\dot{\mathbf{x}}_r(t) - \dot{\mathbf{x}}_s(t)\big|.$$

A similar observation applies to $D^+d_V(\mathbf{v}(t))$ and $D^+[d_V(\mathbf{v}(t))]^2$. This observation will be used in the next section.

3. The existence of flocking solutions

In order to establish the existence of the flocking solution of system (7) with the initial condition (5), we need the following

Lemma 3.1. *(See [11].) Let S be an antisymmetric matrix, $S_{ij} = -S_{ji}$ with $|S_{ij}| \leqslant M$ for a constant M and for $i \neq j$. Let $u, w \in R^N$ be two given real vectors with positive entries, and let $\bar{U}$ and $\bar{W}$ denote their respective sums: $\bar{U} = \sum_i u_i$ and $\bar{W} = \sum_j w_j$. Fix $\theta > 0$ and define*

$$\lambda(\theta) = \#\Lambda(\theta), \quad \Lambda(\theta) := \{j \mid u_j \geqslant \theta\bar{U} \text{ and } w_j \geqslant \theta\bar{W}\}.$$

Then

$$\left|\langle Su, w\rangle\right| \leqslant M\bar{U}\bar{W}\left(1 - \lambda^2(\theta)\theta^2\right).$$

We can now apply the above technical lemma to system (7) to obtain

Lemma 3.2. *Let $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}$ be a solution of the dynamical system (7). Fix $t \geqslant 0$ and $\theta > 0$, and let*

$$\lambda_{pq}(\theta) = \#\left\{j \mid \tilde{a}_{pj}(t) \geqslant \theta \text{ and } \tilde{a}_{qj}(t) \geqslant \theta\right\}.$$

Then the diameters of this solution, $d_X(t) := d_X(\mathbf{x}(t))$ and $d_V(t) := d_V(\mathbf{v}(t))$, satisfy

$$\begin{aligned} &D^+d_X(t) \leqslant d_V(t), \\ &D^+d_V(t) \leqslant \alpha\Big(1 - \min_{pq} \lambda_{pq}^2(\theta)\theta^2\Big)d_V(t-\tau) - \alpha d_V(t). \end{aligned}$$

Proof. We use Remark 2.1 at the end of the last section to find integers p, q, r and s such that

- $|\mathbf{v}_p(t) - \mathbf{v}_q(t)| = d_V(t)$, $|\mathbf{x}_r(t) - \mathbf{x}_s(t)| = d_X(t)$;
- $D^+d_X(t) \leqslant |\dot{\mathbf{x}}_r(t) - \dot{\mathbf{x}}_s(t)| = |\mathbf{v}_r(t) - \mathbf{v}_s(t)| \leqslant d_V(t)$;
- $D^+[d_V(t)]^2 \leqslant 2\langle \mathbf{v}_p(t) - \mathbf{v}_q(t), \dot{\mathbf{v}}_p(t) - \dot{\mathbf{v}}_q(t)\rangle$.

Therefore, we have

$$D^+\left[d_V(t)\right]^2 \leqslant 2\alpha\left\langle \mathbf{v}_p(t) - \mathbf{v}_q(t), \bar{\mathbf{v}}_p(t) - \bar{\mathbf{v}}_q(t)\right\rangle - 2\alpha\left|\mathbf{v}_p(t) - \mathbf{v}_q(t)\right|^2.$$

Noting that $\sum_j \tilde{a}_{pj}(t) = \sum_j \tilde{a}_{qj}(t) = 1$, we obtain

$$\begin{aligned} \bar{\mathbf{v}}_p(t) - \bar{\mathbf{v}}_q(t) &= \sum_j \tilde{a}_{pj}(t)\mathbf{v}_j(t-\tau) - \sum_i \tilde{a}_{qi}(t)\mathbf{v}_i(t-\tau) \\ &= \sum_i \tilde{a}_{qi}(t)\sum_j \tilde{a}_{pj}(t)\mathbf{v}_j(t-\tau) - \sum_j \tilde{a}_{pj}(t)\sum_i \tilde{a}_{qi}(t)\mathbf{v}_i(t-\tau) \\ &= \sum_{i,j} \tilde{a}_{pj}(t)\tilde{a}_{qi}(t)\big(\mathbf{v}_j(t-\tau) - \mathbf{v}_i(t-\tau)\big). \end{aligned}$$

This leads to

$$D^+\big[d_V(t)\big]^2 = 2\alpha\Bigg[\sum_{i,j}\tilde{a}_{pj}(t)\tilde{a}_{qi}(t)\big\langle \mathbf{v}_j(t-\tau)-\mathbf{v}_i(t-\tau), \mathbf{v}_p(t)-\mathbf{v}_q(t)\big\rangle - \big|\mathbf{v}_p(t)-\mathbf{v}_q(t)\big|^2\Bigg]. \tag{8}$$

Let $S_{i,j} = \langle \mathbf{v}_j(t-\tau)-\mathbf{v}_i(t-\tau), \mathbf{v}_p-\mathbf{v}_q\rangle$, $u_i = \tilde{a}_{qi}(t)$ and $w_j = \tilde{a}_{pj}(t)$. Then for the fix $\theta > 0$, by Lemma 3.1, we have

$$\begin{aligned}\big|\langle Su, w\rangle\big| &= \Bigg|\sum_{i,j}\tilde{a}_{pj}(t)\tilde{a}_{qi}(t)\big\langle \mathbf{v}_j(t-\tau)-\mathbf{v}_i(t-\tau), \mathbf{v}_p-\mathbf{v}_q\big\rangle\Bigg| \\ &\leqslant \Big(1-\min_{pq}\lambda_{pq}^2(\theta)\theta^2\Big)d_V(t-\tau)d_V(t).\end{aligned}$$

From this and (8) it follows that

$$D^+d_V(t) \leqslant \alpha\Big(\Big(1-\min_{pq}\lambda_{pq}^2(\theta)\theta^2\Big)d_V(t-\tau) - d_V(t)\Big),$$

completing the proof. □

Lemma 3.3. *Suppose the diameters $d_X(t)$ and $d_V(t)$ are governed, for $t \geqslant 0$, by the inequalities:*

$$\begin{aligned}&D^+d_X(t) \leqslant d_V(t),\\ &D^+d_V(t) \leqslant \alpha\big[1-\psi\big(d_X(t-\tau)\big)\big]d_V(t-\tau) - \alpha d_V(t),\end{aligned}$$

where $\psi(\cdot)$ is a positive continuous function with diverging tail in the sense that $\int_a^\infty \psi(r)\,dr = \infty$ for some $a > 0$, then $\sup_{t\geqslant 0} d_X(t) < \infty$ and $d_V(t) \to 0$ as $t \to \infty$.

Proof. We use a Lyapunov functional inspired by the work of Ha and Liu [8]. Consider

$$E(d_X, d_V)(t) := d_V(t) + \alpha\int_0^{d_X(t-\tau)}\psi(r)\,dr + \alpha\int_{-\tau}^0 d_V(t+\theta)\,d\theta.$$

We have for $t \geqslant \tau$ the following estimate

$$\begin{aligned}D^+E(d_X, d_V)(t) &= D^+d_V(t) + \alpha D^+\int_0^{d_X(t-\tau)}\psi(r)\,dr + \alpha D^+\int_{-\tau}^0 d_V(t+\theta)\,d\theta\\ &\leqslant \alpha\big[1-\psi\big(d_X(t-\tau)\big)\big]d_V(t-\tau) - \alpha d_V(t)\\ &\quad + \alpha d_V(t) - \alpha d_V(t-\tau) + \alpha\psi\big(d_X(t-\tau)\big)D^+d_X(t-\tau)\\ &\leqslant -\alpha\psi\big(d_X(t-\tau)\big)d_V(t-\tau) + \alpha\psi\big(d_X(t-\tau)\big)d_V(t-\tau)\\ &= 0.\end{aligned}$$

Thus the functional E is an energy functional in the sense that it is decreasing along the trajectory $(d_X(t), d_V(t))$. We deduce that

$$d_V(t) + \alpha\int_0^{d_X(t-\tau)}\psi(r)\,dr + \alpha\int_{-\tau}^0 d_V(t+\theta)\,d\theta \leqslant d_V(\tau) + \alpha\int_0^{d_X(0)}\psi(r)\,dr + \alpha\int_0^{\tau} d_V(\theta)\,d\theta.$$

It follows from the fact that ψ has a divergent tail that there must be a constant $d^* < \infty$ so that $d_X(t) \leqslant d^*$ for all $t \geqslant 0$. Consequently, we have

$$\begin{aligned} D^+ d_V(t) &\leqslant \alpha\big[1 - \psi\big(d_X(t-\tau)\big)\big] d_V(t-\tau) - \alpha d_V(t) \\ &\leqslant \alpha\big(1 - \psi^*\big) d_V(t-\tau) - \alpha d_V(t), \end{aligned}$$

where $\psi^* = \min_{0 \leqslant r \leqslant d^*} \psi(r) > 0$. We can then use a standard argument in the theory of delay differential equations to show that $d_V(t) \to 0$ as $t \to \infty$. This completes the proof. □

We can now state our main result for the existence of flocking solutions.

Theorem 3.1. *Assume that the influence function I satisfies $\int_a^\infty I^2(r)\,dr = \infty$ for some $a > 0$. Then the solution $\{(\mathbf{x}_i(t), \mathbf{v}_i(t))\}_{i=1}^N$ for the self-organized systems (4) with either $a_{ij} = a_{ij}^{CS}$ or a_{ij}^{MT} converges to a flock.*

Proof. Since I is decreasing, under the normalization condition, we have $I(|\mathbf{x}_j - \mathbf{x}_i|) \leqslant I(0) \leqslant 1$. Then we have

$$I\big(d_X(t)\big) \leqslant I\big(|\mathbf{x}_j - \mathbf{x}_i|\big) \leqslant 1.$$

If $a_{ij} = a_{ij}^{CS}$ in (4), then $\tilde{a}_{ij}(t) \geqslant \frac{I(d_X(t-\tau))}{N}$ for all $i \neq j$ and

$$\tilde{a}_{ii}(t) = 1 - \sum_{j \neq i} a_{ij}(t-\tau) \geqslant 1 - \frac{N-1}{N} I(0) \geqslant \frac{I(0)}{N} \geqslant \frac{I(d_X(t-\tau))}{N}.$$

If $a_{ij} = a_{ij}^{MT}$ in (4), then

$$\tilde{a}_{ij}(t) = \frac{I(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)}{\sum_{k=1}^N I(|\mathbf{x}_i(t) - \mathbf{x}_k(t)|)} \geqslant \frac{I(d_X(t-\tau))}{N}$$

for all $i \neq j$ and

$$\tilde{a}_{ii}(t) = 1 - \sum_{j \neq i} a_{ij}(t-\tau) = \frac{I(0)}{\sum_{k=1}^N I(|\mathbf{x}_i(t) - \mathbf{x}_k(t)|)} \geqslant \frac{I(d_X(t-\tau))}{N}.$$

Thus all the alignment coefficients $\tilde{a}_{ij}(t)$ in (7) are lower-bounded by $\tilde{a}_{ij}(t) \geqslant \frac{I(d_X(t-\tau))}{N}$.

Let $\theta(t) := \frac{I(d_X(t-\tau))}{N}$, then $\min_{pq} \lambda_{pq}(\theta) = N$ in Lemma 3.2. Thus the diameters $d_X(t)$ and $d_V(t)$ are governed by the inequalities:

$$\begin{aligned} &D^+ d_X(t) \leqslant d_V(t), \\ &D^+ d_V(t) \leqslant \alpha\big[1 - I^2\big(d_X(t-\tau)\big)\big] d_V(t-\tau) - \alpha d_V(t). \end{aligned}$$

The result follows from Lemma 3.3 with $\psi(r) = I^2(r)$. □

We note that with $\tau = 0$, Theorem 3.1 reduces to the main result in [11, Theorem 4.1]. To obtain this extension of Theorem 4.1 in [11] for the case with delay, we relied heavily on the use of a Lyapunov functional.

4. Flocking velocity and the impact of processing delay

Using the variation-of-constants formula, we can rewrite the solution of system (7) with the initial value (5) as

$$\mathbf{x}_i(t) = \left(1 - e^{-\alpha t}\right)\frac{\mathbf{g}_i(0)}{\alpha} + \mathbf{f}_i(0) + \int_0^t \left(1 - e^{-\alpha(t-s)}\right)\sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\mathbf{v}_j(s-\tau)\,ds, \tag{9}$$

$$\mathbf{v}_i(t) = e^{-\alpha t}\mathbf{g}_i(0) + \alpha\int_0^t e^{-\alpha(t-s)}\sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\mathbf{v}_j(s-\tau)\,ds. \tag{10}$$

For the asymptotic flocking velocity, we start with a general result as follows.

Theorem 4.1. *Assume that the influence function I satisfies $\int_a^\infty I^2(r)\,dr = \infty$ for some $a > 0$. Then the asymptotic flocking velocity $\mathbf{v}_\infty$ is given by*

$$\lim_{t\to+\infty}\mathbf{v}_i(t) = \mathbf{v}_\infty = \frac{\mathbf{g}_i(0)}{1+\alpha\tau} + \frac{\alpha}{1+\alpha\tau}\left[\mathbf{w}_i + \mathbf{f}_i(0) - \mathbf{f}_i(-\tau)\right],$$

where

$$\mathbf{w}_i = \lim_{t\to\infty}\int_0^t \sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\left(\mathbf{v}_j(s-\tau) - \mathbf{v}_i(s-\tau)\right)ds.$$

Proof. By Theorem 3.1, we have $\lim_{t\to\infty}|\mathbf{v}_j(t) - \mathbf{v}_i(t)| = 0$ for all i, j. Thus there is an asymptotic flocking velocity $\mathbf{v}_\infty$ such that $\lim_{t\to\infty}\mathbf{v}_i(t) = \mathbf{v}_\infty$ for all i.

On the other hand, since $\mathbf{x}_i(t) - \mathbf{x}_i(t-\tau) = \int_{t-\tau}^t \mathbf{v}_i(s)\,ds = \tau\mathbf{v}_i(\eta)$ for some $\eta \in (t-\tau, t)$, we conclude $\lim_{t\to+\infty}(\mathbf{x}_i(t) - \mathbf{x}_i(t-\tau)) = \tau\mathbf{v}_\infty$. A direct computation, using Eqs. (9) and (10), yields that

$$\begin{aligned}
\mathbf{x}_i(t) &= \left(1 - e^{-\alpha t}\right)\frac{\mathbf{g}_i(0)}{\alpha} + \mathbf{f}_i(0) + \int_0^t \left(1 - e^{-\alpha(t-s)}\right)\sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\mathbf{v}_j(s-\tau)\,ds \\
&= \frac{\mathbf{g}_i(0)}{\alpha} + \mathbf{f}_i(0) - \frac{\mathbf{v}_i(t)}{\alpha} + \int_0^t \sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\mathbf{v}_i(s-\tau)\,ds \\
&\quad + \int_0^t \sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\left(\mathbf{v}_j(s-\tau) - \mathbf{v}_i(s-\tau)\right)ds \\
&= \frac{\mathbf{g}_i(0)}{\alpha} + \mathbf{f}_i(0) - \frac{\mathbf{v}_i(t)}{\alpha} + \mathbf{x}_i(t-\tau) - \mathbf{f}_i(-\tau) + \int_0^t \sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\left(\mathbf{v}_j(s-\tau) - \mathbf{v}_i(s-\tau)\right)ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{t\to+\infty}\int_0^t \sum_{j=1}^N a_{ij}\left(\mathbf{x}(s-\tau)\right)\left(\mathbf{v}_j(s-\tau) - \mathbf{v}_i(s-\tau)\right)ds \\
&\quad = \lim_{t\to+\infty}\left[\mathbf{x}_i(t) - \mathbf{x}_i(t-\tau) - \frac{\mathbf{g}_i(0)}{\alpha} - \mathbf{f}_i(0) + \frac{\mathbf{v}_i(t)}{\alpha} + \mathbf{f}_i(\tau)\right]
\end{aligned}$$

$$= \tau \mathbf{v}_\infty - \frac{\mathbf{g}_i(0)}{\alpha} - \mathbf{f}_i(0) + \frac{\mathbf{v}_\infty}{\alpha} + \mathbf{f}_i(\tau) := \mathbf{w}_i.$$

Then we have

$$\lim_{t \to +\infty} \mathbf{v}_i(t) = \mathbf{v}_\infty = \frac{\mathbf{g}_i(0)}{1+\alpha\tau} + \frac{\alpha}{1+\alpha\tau}\big[\mathbf{w}_i + \mathbf{f}_i(0) - \mathbf{f}_i(-\tau)\big],$$

completing the proof. □

Remark 4.1. In Theorem 4.1, if $a_{ij} = a_{ji}$ for all i, j then $\sum_{i=1}^N \mathbf{w}_i = 0$. Thus

$$\mathbf{v}_\infty = \frac{\sum_{i=1}^N \mathbf{g}_i(0)}{N(1+\alpha\tau)} + \frac{\alpha}{N(1+\alpha\tau)} \sum_{i=1}^N \big[\mathbf{f}_i(0) - \mathbf{f}_i(-\tau)\big].$$

Remark 4.2. Theorem 4.1 gives a positive answer to the problem posed in [11, Remark 4.2] about whether $\mathbf{v}_\infty$ can be computed from the initial configuration. We also note that the time delay impacts on the final flocking velocity in a nonlinear way, and the variation of the initial position during the delay interval may also contribute to the determination of the final velocity.

5. Conclusions and discussions

It is of particular interest, in both theories and applications, to understand how self-propelled individuals use only limited environmental information and simple rules to organize into ordered motions. In this paper, we extended the Cucker–Smale model by incorporating the communication time lag between agents. It was shown, using some Lyapunov functional and delay differential inequalities, that the communication delay does not affect the existence of flocking solutions when the influence function has a divergent tail. We also gave a positive answer to the open problem posed by Motsch and Tadmor, that relates the final flocking velocity to the time lag, and the variation of the initial position during the delay interval.

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