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Abstract algebraic-delay differential systems and age structured population dynamics

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ABSTRACT

We consider the abstract algebraic-delay differential system,

$$x'(t) = Ax(t) + F\big(x(t), a(t)\big),$$

$$a(t) = H(x_t, a_t).$$

Here A is a linear operator on $D(A) \subset X$ satisfying the Hille–Yosida conditions, $x(t) \in \overline{D(A)} \subset X$, and $a(t) \in \mathbf{R}^n$, where X is a real Banach space. With a global Lipschitz condition on F and an appropriate hypothesis on the function H, we show that the corresponding initial value problem gives rise to a continuous semiflow in a subset of the space of continuous functions. We establish the positivity of the x-component and give some examples arising from age structured population dynamics. The examples come from situations where the age of maturity of an individual at a given time is determined by whether or not the resource concentration density, which depends on the *immature population*, reaches a prescribed threshold within that time.

1. Introduction

The motivation for this paper comes from the curiosity for analyzing a hyperbolic *partial differential equation* (PDE) with a state dependent delay, since not much work has been done on this subject. The works of Rezounenko, e.g. [11,12], indicate that it is difficult to give a general theory for any kind

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of PDEs with state dependent delays, because the nonlinear term is not Lipschitz on the usual space of continuous functions and the corresponding initial value problem is not well posed. In the case of *ordinary differential equations* (ODEs) with state dependent delays, an alternative is to work on the space of Lipschitz or C^1 functions, as in [16]. The reason that this approach is difficult to carry over to PDEs is a recurring difficulty which stems from the fact that, in general, it is difficult to construct a solution of a PDE which is *a priori* regular. This is why the existing works for PDEs with state dependent delays only study very special classes of equations.

This paper is about a class of first order hyperbolic equations, different from those studied in [11, 12]. The example in mind comes from a model for an age structured population, with the special feature that the age of maturity at a given time of an individual is determined by whether or not the resource concentration density, which depends on the *immature population*, reaches a prescribed threshold value within that time. In case of a single population, this takes the form of a first order hyperbolic PDE coupled to a scalar algebraic-delay term. To the best of our knowledge, this system has not previously been studied without being reduced to some kind of ODE or integral equation containing a delay. See [3,2] and references therein.

We now give a mechanistic derivation for the example we have in mind below, to motivate the main results. We make a comparison with relevant existing results and outline our contribution.

1.1. Mechanistic derivation

The derivation below is adapted from [3].

Consider some abstract habitat and some population of individuals living in this habitat. Let $u(t,a)$ be the density of individuals of age a at time t. Let the immature population at time t be given by $I(t)$. Let $S(t)$ denote the concentration density of some resource per unit volume in the habitat at time t. To derive a deterministic model we need to make some assumptions.

First, we assume that $S(t)$ satisfies $S'(t) = S_0 - (\gamma_i I(t) + C)S(t)$. Here $S_0 > 0$ is a constant rate of food recruited in the habitat, $\gamma_i > 0$ is the rate of food consumption of the immature population per unit time, and $C > 0$ represents the resource consumption rate by anything else in the habitat. Since the resource consumption happens on a much faster time scale than that of life of the population, we can make a simplifying assumption. If we hold the immature population fixed, we get the equation, $S'(t) = S_0 - (\gamma_i I + C)S(t)$. The steady state is given by the formula $S = \frac{S_0}{\gamma_i I + C}$. Since this steady state is globally stable, the quasi steady state approximation gives

$$S(t) = \frac{S_0}{\gamma_i I(t) + C}. \tag{1}$$

For further details see [9].

Second we assume that the age of maturity at time t, $\tau(t)$, is defined by the condition

$$\int_{t-\tau(t)}^{t} S(\sigma)\,d\sigma = T > 0, \tag{2}$$

where $T > 0$ is a "size" threshold. This represents the difference between an individual's size at birth and their size τ units of time after birth. Combining (1) with (2) gives us

$$\int_{t-\tau(t)}^{t} \frac{S_0}{\gamma_i I(\sigma) + C}\,d\sigma = T \quad \text{with } I(\sigma) = \int_{0}^{\tau(\sigma)} u(\sigma,a)\,da$$

or equivalently

$$\int_{t-\tau(t)}^{t} S_0\left[\gamma_i \int_0^{\tau(\sigma)} u(\sigma,a)\,da + C\right]^{-1} d\sigma = T.$$

For convenience, we set $S_0 = \gamma_i = 1$.

Finally we assume that the individuals have maximum age $0 < m \leqslant \infty$ and $u(t,a)$ satisfies the standard first order hyperbolic PDE,

$$\begin{aligned}
\partial_t u(t,a) + \partial_a u(t,a) &= -d(a)u(t,a), \quad t \geqslant 0 \text{ and } 0 \leqslant a \leqslant m;\\
u(t,0) &= b\left(\int_{\tau(t)}^{m} \beta(\xi)u(t,\xi)\,d\xi\right),
\end{aligned}$$

where $\tau(t)$ is given by

$$\int_{-\tau(t)}^{0} \left[\int_0^{\tau(t+\sigma)} u(t+\sigma,a)\,da + C\right]^{-1} d\sigma = T. \tag{3}$$

Note that we ignore technicalities concerning whether $\tau(t)$ is well defined by (3) at this stage. To have solutions for $t \geqslant 0$ we must specify the initial conditions,

$$\tau(t) = \varphi(t) \quad \text{for } -a_m \leqslant t \leqslant 0$$

and

$$u(t,a) = \psi(t,a) \quad \text{for } -a_m \leqslant t \leqslant 0 \text{ and } 0 \leqslant a \leqslant m.$$

Here $a_m \in (0,m)$ is the maximal age of maturity.

A look at (3) reveals that $\tau(t)$ depends on the history at time t of the population density, u_t, and the history of itself, τ_t. As usual, $u_t(\theta)(\cdot) = u(t+\theta)(\cdot)$ and $\tau_t(\theta) = \tau(t+\theta)$ for $\theta \in [-a_m, 0]$. We assume naively that $\tau(t)$ is a function of u_t and τ_t, $\tau(t) = H(u_t, \tau_t)$ (see Sections 2.1 and 5).

To summarize, we have obtained the initial value problem,

$$\begin{aligned}
\partial_t u(t,a) + \partial_a u(t,a) &= -d(a)u(t,a),\\
u(t,0) &= b\left(\int_{\tau(t)}^{m} \beta(\xi)u(t,\xi)\,d\xi\right),\\
\tau(t) &= H(u_t, \tau_t)
\end{aligned} \tag{4}$$

for $t \geqslant 0$ and $0 \leqslant a \leqslant m$ with initial conditions

$$\tau(t) = \varphi(t) \quad \text{and} \quad u(t,a) = \psi(t,a) \quad \text{for } -a_m \leqslant t \leqslant 0 \text{ and } 0 \leqslant a \leqslant m. \tag{5}$$

Note that for each $t \geqslant 0$, $\binom{u_t}{\tau_t} \in M_0$, where

$$M_0 = \left\{ \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \in \text{some subset of } C\big([-a_m, 0], L^1[0,m) \times \mathbf{R}\big) \;\middle|\; \varphi(0) = H(\psi, \varphi) \right\}.$$

The precise definitions of H and M_0 are given in Sections 2 and 5.

We can rewrite the initial value problem (4)–(5) abstractly as

$$\frac{d}{dt}\begin{pmatrix} 0 \\ u(t,\cdot) \end{pmatrix} = \begin{pmatrix} -u(t,0) \\ -u_a(t,\cdot) \end{pmatrix} + \begin{pmatrix} b(\int_{\tau(t)}^{m} \beta(\xi)u(t,\xi)\,d\xi) \\ -d(\cdot)u(t,\cdot) \end{pmatrix},$$
$$\tau(t) = H(u_t, \tau_t),$$
$$\begin{pmatrix} x_0 \\ \tau_0 \end{pmatrix} = \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \in M_0.$$

This mechanistic derivation shows how we are naturally led to considering the more general algebraic-delay differential system, the initial value problem (6) in Section 2, which is the object of study in this paper.

1.2. Comparison with existing results

Similar types of structured population models were already considered by Smith in [14]. The models considered by Smith were reduced to a single retarded functional differential equation whose nonlinear term is Lipschitz on the usual phase space of continuous functions. Due to the dependence of the age of maturity on the immature population, this is not possible for the case we are considering. This is because the second component of the system we are studying, which describes the age of maturity in the mechanistic derivation above, depends not only on the history of the population density but also on the history of itself. For a related work on threshold type delay differential equations, see [4].

More recently, Hbid et al. [2] considered a stage structured population model with the same feature determining the age of maturity that we have here. However, based on a simplifying assumption, they reduced the model to an integral equation containing a state dependent delay, for which the immature population depends only on the history of the state variable, and consequently, does not need to be initialized.

The nonlinear semigroups approach we are using here was motivated by the works of Thieme [15], and Magal and Ruan [6,5], specifically for the case of structured population models. For another semigroup approach for age structured models, see [19]. The basic reference for semigroup theory is [10].

A unification of various fundamental results for PDE with ordinary delay is given in [13], which uses a more general class of operators than we have here. It would be nice to see if the results presented here can find such generalizations. For a treatment of reaction diffusion systems with ordinary delay, see [20].

Also closely related is the recent work of Walther [17] on ODE algebraic-delay differential systems. Walther considered systems of the form

$$x'(t) = f\bigl(x_t, r(t)\bigr),$$
$$0 = \Delta\bigl(r(t), x_t\bigr),$$

where $x(t) \in \mathbf{R}^k$ and $r(t)$ is defined implicitly by the history of the state, x_t. As long as the derivative of Δ in the first component is nonsingular, such systems will be locally uniquely solvable thanks to the implicit function theorem. Unfortunately, we cannot apply the implicit function theorem for the case we are considering, so instead we impose a special Lipschitz condition on the function H given in the next section.

1.3. Outline and main results

In Section 2, we state the relevant technical preliminaries and hypotheses, including the appropriate notion of mild solutions in the subset M_0 of the ambient linear space of continuous functions. In Section 3 we prove the existence and uniqueness of local mild solutions in M_0, in Theorem 1.

In Section 4, we discuss the corresponding semiflow and show that it is continuous in Theorem 2. In Section 5, we give an application of the general theory in Proposition 3. Finally, in Section 6, we briefly discuss the upcoming sequel of this work.

2. Technical preliminaries and hypotheses

In this section we state the relevant technical preliminaries and hypotheses. All Banach spaces are assumed to be over the real numbers. Whenever a product of Banach spaces is considered, we view it as a Banach space equipped with the corresponding product norm.

2.1. The ambient linear space of initial data

Let $\delta > 0$ and $I = [-\delta, 0]$. For $F \subset E$, where E is a Banach space, $C(I, F)$ denotes the set of continuous functions mapping I into F. For $\psi \in C(I, F)$, we let $\|\psi\|$ be the supremum norm of ψ. Then $(C(I, E), \|\cdot\|)$ is a Banach space.

Suppose that $0 < T < \infty$ and $y : I \cup [0, T] \to F$ is some map. As usual in the literature on delay equations, for each $t \in [0, T]$, we define $y_t : I \to F$ by $y_t(\theta) = y(t + \theta)$ for $\theta \in I$ and call y_t the history of y at time t. If $T = \infty$ then the same definition applies with $t \in [0, T]$ being replaced with $t \in [0, T)$.

2.2. Hypotheses

(H1) Let $(X, |\cdot|)$ denote a Banach space and suppose that $A : D(A) \to X$ with $D(A) \subset X$ is a linear operator satisfying the estimates of the Hille–Yosida theorem. That is, there is some $M \geqslant 1$ and some $\omega \in \mathbf{R}$ such that the ray $(\omega, \infty) \subset \rho(A)$ and $\|(A - \lambda I)^{-n}\| \leqslant \frac{M}{(\lambda-\omega)^n}$ for $\lambda > \omega$ and for each positive integer n.

We let $X_0 = \overline{D(A)}$ and A_0 denote the part of A in X_0. Actually this class of operators falls under a more general class of well known operators as pointed out in [13]. Set $R_\lambda = (A - \lambda I)^{-1}$. Without loss of generality, assume that $\omega > 0$. It follows from (H1) that A_0 generates a C^0-semigroup of linear operators on X_0, $\{T(t)\}_{t\geqslant 0}$, and that $\|T(t)\| \leqslant Me^{\omega t}$.

(H2) Let $n > 0$ be given. Suppose that K is some compact subset of $\mathbf{R}^n$ such that K is contained in the closed ball of radius $h > 0$ centered at the origin. Set $I = [-h, 0] \subset \mathbf{R}$ and let C_0 be some closed and convex subset of X_0. Assume that $R_0 > 0$ and $f : \mathbf{R}^+ \to \mathbf{R}^+$ is a strictly increasing function with $f(R_0) = 1$. Let $D(H) = \left\{\binom{\psi}{\varphi} \in C(I, C_0 \times K) \mid \|\psi\| \leqslant R_0\right\}$ and suppose $H : D(H) \to K$ is a function which satisfies the following Lipschitz condition: for each $Q > 0$ there is some $L_Q > 0$ such that, for $\binom{\psi_1}{\varphi_1}$, $\binom{\psi_2}{\varphi_2} \in D(H)$ with $\|\psi_i\| \leqslant Q$ $(i = 1, 2)$, we have

$$\left|H(\psi_1, \varphi_1) - H(\psi_2, \varphi_2)\right| \leqslant f(Q)\|\varphi_1 - \varphi_2\| + L_Q\|\psi_1 - \psi_2\|.$$

For simplicity of notation, $|\cdot|$ has been used to denote the norm on X and also the norm on $\mathbf{R}^n$. This will not cause any confusion.

(H3) Let $M_0 = \left\{\binom{\psi}{\varphi} \in D(H) \mid \varphi(0) = H(\psi, \varphi) \text{ and } \|\psi\| < R_0\right\}$. Assume $M_0 \neq \emptyset$.

(H4) Suppose $F : C_0 \times K \to X$ is a globally Lipschitz function, i.e., there is some $D > 0$ such that, for $c_1, c_2 \in C_0$ and $k_1, k_2 \in K$, we have $|F(c_1, k_1) - F(c_2, k_2)| \leqslant D(|c_1 - c_2| + |k_1 - k_2|)$.

(H5) (*Subtangential condition*) We assume that, for each $(c, k) \in C_0 \times K$,

$$\lim_{h\downarrow 0} \frac{\operatorname{dist}(T(h)c + \lim_{\mu\to\infty} \int_0^h T(s)\mu R_\mu F(c, k)\, ds, C_0)}{h} = 0$$

holds. Here, $\operatorname{dist}(x, B) = \inf_{b\in B} |x - b|$ for $x \in X$ and $B \subset X$. (H5) is a well known condition which ensures the invariance of a closed and convex set, sometimes referred to as positivity. We refer readers to [8,13,15] for more detail.

Definition. Consider the following initial value problem,

$$\begin{cases} x'(t) = Ax(t) + F\big(x(t), a(t)\big), \\ a(t) = H(x_t, a_t), \\ \begin{pmatrix} x_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \in M_0. \end{cases} \tag{6}$$

By a mild solution of (6) on $I \cup [0, T]$ in M_0 with $T < \infty$, we mean a pair of functions $\binom{x(t)}{a(t)}$ with the following properties:

(i) $a : I \cup [0, T] \to K$ is continuous.
(ii) $x : I \cup [0, T] \to C_0$ is continuous such that, for each $t \in [0, T]$, $\int_0^t x(s)\,ds \in D(A)$ and

$$x(t) = A \int_0^t x(s)\,ds + \int_0^t F\big(x(s), a(s)\big)\,ds.$$

(iii) For $0 \leqslant t \leqslant T$, $\binom{x_t}{a_t} \in M_0$, i.e., $a(t) = H(x_t, a_t)$ and $\|x_t\| < R_0$.
(iv) $\binom{x_0}{a_0} = \binom{\psi}{\varphi}$.

We similarly define mild solutions in M_0 on $I \cup [0, T)$ for $T = \infty$.
Note that (H1) implies that (ii) is equivalent to

$$x(t) = T(t)\psi(0) + \lim_{\mu \to \infty} \int_0^t T(t-s)\mu R_\mu F\big(x(s), a(s)\big)\,ds \quad \text{for } t \in [0, T]$$

(see [15]).

3. Local solutions in M_0

In this section we establish the existence and uniqueness of local mild solutions for (6) in M_0.

Theorem 1. *Suppose $A : D(A) \to X$, $H : D(H) \to K$, $F : C_0 \times K \to X$, and M_0 are as in Section 2. Assume* (H1)–(H5) *hold. Then the initial value problem* (6) *has a unique mild solution $\binom{x(t)}{a(t)}$ in M_0 on $I \cup [0, \tau]$ for some $0 < \tau < \infty$.*

Proof. We establish the existence and uniqueness of a local mild solution of (6) in M_0 by constructing a net of approximate solutions using a discrete approximation scheme. This is done in such a way that the histories of the approximate solutions lie in M_0. We show that the net constructed converges to a local mild solution of (6) in M_0. This method is a well known approach. See [15,8,13], for example. The difference between our version and others is that we must work on a "nonlinear submanifold" of the ambient space.

Step 1: Constructing an approximate solution of (6) in M_0.

We choose $R_1 > 0$ such that $\|\psi\| < R_1 < R_0$. Set $R = R_1 - \|\psi\|$. Then $0 < R < R_0$. Moreover, $f(R_1) < 1$. By (H2) we can find $J > 0$ such that if $\|\gamma_1\|$, $\|\gamma_2\| \leqslant R_1$ then, for $\varphi_1, \varphi_2 \in C(I, K)$, we have

$$\big|H(\gamma_1, \varphi_1) - H(\gamma_2, \varphi_2)\big| \leqslant f(R_1)\|\varphi_1 - \varphi_2\| + J\|\gamma_1 - \gamma_2\|.$$

Let $\psi(0) = x^0$ and $\varphi(0) = a^0$. Fix some number $\epsilon \in (0,1)$. Pick some $0 < \tau \leqslant R_1$ (another upper bound on τ independent of ϵ will be imposed later). Using (H5), the strong continuity of $T(t)$, and the uniform continuity of ψ and φ, we can find some $0 < t_1 \leqslant \min\{\epsilon, 2\tau\}$ such that

$$\frac{\operatorname{dist}(G(t_1), C_0)}{t_1} < \frac{\epsilon}{2},$$
$$\text{if } s \in [0, t_1) \quad \text{then } \left|T(s)\left(x^0\right) - x^0\right| \leqslant \epsilon,$$
$$\text{if } s_1, s_2 \in I \text{ with } |s_1 - s_2| < t_1 \quad \text{then } \left|\varphi(s_1) - \varphi(s_2)\right|, \left|\psi(s_1) - \psi(s_2)\right| \leqslant \epsilon, \tag{7}$$

where $G(t_1) = T(t_1)x^0 + \lim_{\mu\to\infty} \int_0^{t_1} T(t_1 - s)\mu R_\mu F(x^0, a^0)\, ds$. Choose $x^1 \in C_0$ such that

$$\left|x^1 - G(t_1)\right| \leqslant \frac{\epsilon t_1}{2} + \operatorname{dist}\left(G(t_1), C_0\right) \leqslant \epsilon t_1.$$

It follows that

$$\begin{aligned}\left|x^1 - T(t_1)x^0\right| &\leqslant \epsilon t_1 + \int_0^{t_1} M^2 e^{\omega t_1}\left|F\left(x^0, a^0\right)\right| ds \\ &\leqslant 2\tau + M^2 2\tau e^{\omega 2\tau}\left|F\left(x^0, a^0\right)\right|.\end{aligned}$$

This, combined with $|x^1 - x^0| \leqslant |x^1 - T(t_1)(x^0)| + |T(t_1)(x^0) - x^0|$, tells us that we can choose τ independently of ϵ and t_1 so that $|x^1 - x^0| \leqslant R$.

We define a function $x^1 : I \cup [0, t_1] \to C_0$ by

$$x^1(t) = \begin{cases} \psi(t) & \text{if } t \in I, \\ \frac{t}{t_1}x^1 + \frac{t_1 - t}{t_1}x^0 & \text{if } t \in [0, t_1]. \end{cases}$$

Then, for $t \in [0, t_1]$, $x^1(t)$ is a parameterization of the straight line segment joining x^0 and x^1, meaning that $x^1(t) \in C_0 \cap B_R(x^0)$, where $B_R(x^0)$ denotes the closed ball of radius R in X_0 about x_0, which is convex. Consequently, for $t \in [0, t_1]$, $x_t^1 \in C(I, C_0)$ and $\|x_t^1\| \leqslant R_1 < R_0$.

To find a corresponding approximation for the second component of the system, we wish to solve the equation

$$a^1(t) = \begin{cases} \varphi(t) & \text{if } t \in I, \\ H(x_t^1, a_t^1) & \text{if } t \in [0, t_1]. \end{cases} \tag{8}$$

To show that (8) has a unique solution, we construct an appropriate contraction on $C(I \cup [0, t_1], K)$ which is a closed subset of the Banach space $C(I \cup [0, t_1], \mathbf{R}^n)$ since K is closed. Note that $\{x_t^1\} \times C(I, K) \subset D(H)$ for $t \in [0, t_1]$. So let $\mathcal{A} : C(I \cup [0, t_1], K) \to C(I \cup [0, t_1], \mathbf{R}^n)$ be given by the right-hand side of (8). It follows from (H2) that

$$(\mathcal{A}a)(s) \in K \quad \text{for each } s \in I \cup [0, t_1] \quad \text{and that} \quad \mathcal{A}a \text{ is continuous on } I \cup [0, t_1]$$

and

$$\|\mathcal{A}a - \mathcal{A}b\| \leqslant W\|a - b\| \quad \text{for some } W < 1.$$

Therefore, Eq. (8) has a unique solution, denoted by a^1.

This concludes the first step of our recursion and we have obtained appropriate functions $x^1 : I \cup [0,t_1] \to C_0$ and $a^1 : I \cup [0,t_1] \to K$. By relabeling if necessary, we assume that t_1 is chosen maximally in the following way:

Let $S_1 = \sup\{s \in [0,2\tau] \mid 0 < s \leqslant \epsilon,\ \xi \in [0,s) \Rightarrow |T(\xi)x^0 - x^0| \leqslant \epsilon$, if $s_1, s_2 \in I$ and $|s_1 - s_2| < s$ then $|\varphi(s_1) - \varphi(s_2)|$ and $|\psi(s_1) - \psi(s_2)| \leqslant \epsilon$, $\operatorname{dist}(T(s)x^0 + \lim_{\mu\to\infty} \int_0^s T(s-\xi)\mu R_\mu F(x(0), a(0))\,ds, C_0) \leqslant \epsilon s/2\}$. Clearly, $S_1 \neq \emptyset$. By a standard continuity argument, it is easy to see that $\sup(S_1) \in S_1$ and we set $t_1 = \max(S_1)$.

Let $t_0 = 0$. Suppose that $k \geqslant 1$ and that we are granted a sequence of mesh points $(t_j, x^j, a^j(t_j))$, and corresponding functions, $x^j \in C(I \cup [0,t_j], C_0)$ and $a^j \in C(I \cup [0,t_j], K)$ such that, for each $1 \leqslant j < k$, the following properties hold:

If $t_{j-1} < \tau$ then (P1)–(P7) hold and if $t_{j-1} \geqslant \tau$ then $t_j = t_{j-1}$.

(P1) $t_j \leqslant 2\tau$ and $0 < t_j - t_{j-1} \leqslant \epsilon$.

(P2) If $s \in [0, t_j - t_{j-1})$ then $|T(s)x^{j-1} - x^{j-1}| \leqslant \epsilon$. Moreover, for $s_1, s_2 \in I \cup [0, t_{j-1}]$, if $|s_1 - s_2| < t_j - t_{j-1}$ then $|a^{j-1}(s_1) - a^{j-1}(s_2)|$, $|x^{j-1}(s_1) - x^{j-1}(s_2)| \leqslant \epsilon$.

(P3) $\operatorname{dist}(T(t_j - t_{j-1})x^{j-1} + \lim_{\mu\to\infty} \int_{t_{j-1}}^{t_j} T(t_j - s)\mu R_\mu F(x^{j-1}, a^{j-1}(t_{j-1}))\,ds, C_0) \leqslant \epsilon(t_j - t_{j-1})/2$.

(P4) t_j is chosen maximally with respect to (P1)–(P3). Namely, $t_j = \max_{\xi\in[0,2\tau]}\{$(P1)–(P3) hold with '$\xi$' in place of '$t_j$'$\}$.

(P5) $|x^j - T(t_j - t_{j-1})x^{j-1} - \lim_{\mu\to\infty} \int_{t_{j-1}}^{t_j} T(t_j - s)\mu R_\mu F(x^{j-1}, a^{j-1}(t_{j-1}))\,ds| \leqslant \epsilon(t_j - t_{j-1})$.

(P6) $x^j \in B_R(x^0)$.

(P7)

$$x^j(t) = \begin{cases} x^{j-1}(t) & \text{if } t \leqslant t_{j-1}, \\ \frac{t-t_{j-1}}{t_j - t_{j-1}} x^j + \frac{t_j - t}{t_j - t_{j-1}} x^{j-1} & \text{if } t \in [t_{j-1}, t_j] \end{cases}$$

and

$$a^j(t) = \begin{cases} a^{j-1}(t) & \text{if } t \leqslant t_{j-1}, \\ H(x_t^j, a_t^j) & \text{if } t \in [t_{j-1}, t_j]. \end{cases}$$

Note that we denote by 'x^j' and 'a^j' both members of C_0 and K, respectively, and the corresponding functions since this should not cause any confusion.

In order to complete the recursion, we show that (P1)–(P7) hold for $j = k$ whenever τ is small enough. It should be noted that τ has not yet been chosen.

If it happens that $t_{k-1} \geqslant \tau$ then we set $t_k = t_{k-1}$, and we are done. Otherwise, by the same procedure as in the first step of the recursion, we can find some $t_k \leqslant 2\tau$ and $x_k \in C_0$ such that (P1)–(P5) hold. We need to verify (P6), then (P7) will follow exactly as in the first step of the recursion when (8) was solved using the contraction mapping principle. The purpose of the tedious estimates below is to show that τ can in fact be chosen *a priori* depending only on the initial data. These calculations are essentially those given in [15], but we repeat them here for completion. It should be noted that we use the hypothesis $\omega > 0$ from (H1) to establish (9).

For $j \leqslant k$, it follows from (P5) that $|x^j - T(t_j - t_{j-1})x^{j-1}| \leqslant \epsilon(t_j - t_{j-1}) + |\lim_{\mu\to\infty} \int_{t_{j-1}}^{t_j} T(t_j - s) \times \mu R_\mu F(x^{j-1}, a^{j-1}(t_{j-1}))\,ds|$. Then

$$\begin{aligned} \left|F\left(x^{j-1}, a^{j-1}(t_{j-1})\right)\right| &\leqslant \left|F\left(x^{j-1}, a^{j-1}(t_{j-1})\right) - F\left(x^0, a^0\right)\right| + \left|F\left(x^0, a^0\right)\right| \\ &\leqslant D\left(\left|x^{j-1} - x^0\right| + \left|a^{j-1}(t_{j-1}) - a^0\right|\right) + \left|F\left(x^0, a^0\right)\right| \\ &\leqslant D(R + 2h) + \left|F\left(x^0, a^0\right)\right| := P. \end{aligned}$$

Clearly P depends only on the initial data. Thus,

$$\left|x^j - T(t_j - t_{j-1})x^{j-1}\right| \leqslant Z(t_j - t_{j-1}), \quad \text{where } Z = \left(1 + M^2 e^{\omega 2\tau} P\right).$$

Having this at our disposal, we next show that, for each $j \leqslant k$,

$$\left|x^j - T(t_j)x^0\right| \leqslant MZe^{\omega t_j} t_j. \tag{9}$$

In fact, we have

$$\begin{aligned}
&\left|x^j - T(t_j - t_{j-2})x^{j-2}\right| \\
&\quad \leqslant \left|x^j - T(t_j - t_{j-1})x^{j-1}\right| + \left|T(t_j - t_{j-1})x^{j-1} - T(t_j - t_{j-2})x^{j-2}\right| \\
&\quad \leqslant Z(t_j - t_{j-1}) + \left|T(t_j - t_{j-1})(x^{j-1} - T(t_{j-1} - t_{j-2})x^{j-2}\right| \\
&\quad \leqslant Z(t_j - t_{j-1}) + Me^{\omega(t_j - t_{j-1})} Z(t_{j-1} - t_{j-2}) \\
&\quad \leqslant MZe^{\omega(t_j - t_{j-2})}(t_j - t_{j-2}).
\end{aligned}$$

Continuing in this way, we can prove (9). It follows form (9) that

$$\left|x^k - x^0\right| \leqslant \left|x^k - T(t_k)x^0\right| + \left|T(t_k)x^0 - x^0\right| \leqslant MZe^{\omega 2\tau} 2\tau + \left|T(t_k)x^0 - x^0\right|. \tag{10}$$

Then we can choose $\tau > 0$ such that $x^k \in B_R(x^0)$ and note that, by virtue of (10) and the strong continuity of $T(t)$, this choice is independent of ϵ.

This completes the recursion and we conclude that, for each positive integer j, we can find appropriate mesh points and functions such that (P1)–(P7) hold if $t_{j-1} < \tau$ and otherwise $t_j = t_{j-1}$.

To obtain an approximate solution in M_0, we need to show that this process ends after a finite number of steps. That is, we want to see that, for some positive integer j, $t_j \geqslant \tau$. We assume, by way of contradiction, that $t_j < \tau$ for each j. So there is some $0 < t \leqslant \tau$ such that $t_j \uparrow t$ and $t > t_j$. By the same calculations as those on pp. 32–33 of [15], we deduce that $x^j \to x$ for some $x \in C_0$. Now we define the function $x : I \cup [0, t] \to C_0$ by

$$x(s) = \begin{cases} x^j(s) & \text{if } -h \leqslant s \leqslant t_j, \\ x & \text{if } s = t. \end{cases} \tag{11}$$

Clearly, x is continuous. Since for each $s \in [0, t]$, $\|x_s\| \leqslant R_1$, the Lipschitz estimate for H with respect to R_1 and the contraction mapping principle give us a unique continuous solution to the equation

$$a(s) = \begin{cases} \varphi(s) & \text{if } s \in I, \\ H(x_s, a_s) & \text{if } s \in [0, t], \end{cases}$$

where x is given by (11). By uniqueness, it follows that $a(s) = a^j(s)$ for $s \in I \cup [0, t_j]$. By exploiting uniform continuity of x and a on $I \cup [0, t]$, and of the map $[0, t] \ni s \mapsto |T(s)x - x|$ we can find $0 < \delta < \epsilon$ such that $t + \delta \leqslant 2\tau$, and $|s_1 - s_2| < \delta \Rightarrow |x(s_1) - x(s_2)|, |a(s_1) - a(s_2)| < \epsilon$, and $0 \leqslant s < \delta \Rightarrow |T(s)x - x| < \epsilon/3$. Fix $\alpha \in (0, \delta)$. Since $t + \alpha > t_j$, by maximality, we see that for each j, one of (P1)–(P3) is not satisfied when 't_j' is replaced by '$t + \alpha$'. It is clear that (P1) is not satisfied for at most finitely many j when t_j is replaced with $t + \alpha$, and similarly for (P2). Therefore, there are infinitely many j such that

$$\operatorname{dist}\left(T(t+\alpha-t_{j-1})x^{j-1}+\lim_{\mu\to\infty}\int\limits_{t_{j-1}}^{t+\alpha}T(t+\alpha-s)\mu R_\mu F\big(x^{j-1},a^{j-1}(t_{j-1})\big)\,ds,C_0\right)$$
$$>\epsilon(t+\alpha-t_{j-1})/2.$$

Letting j tend to infinity and exploiting continuity shows that the subtangential condition, (H5), is violated, a contradiction.

Step 2: Estimates for the ϵ-approximate solution between mesh points.

The procedure in Step 1 granted us for each $0<\epsilon<1$ an approximate solution, which we denote by $\binom{x(t)}{a(t)}$, where $x: I\cup[0,\tau]\to C_0$ and $a: I\cup[0,\tau]\to K$ satisfy

$$x(t)=x^j(t)\quad\text{and}\quad a(t)=a^j(t)\quad\text{if } s\leqslant t_j$$

and $[0,\tau]\subset\bigcup_{1\leqslant j\leqslant k(\epsilon)}[t_{j-1},t_j]$ for some $k(\epsilon)=k<\infty$ such that $t_{k-1}<\tau$ and $t_k\geqslant\tau$. Before moving on to Step 3, we obtain crucial estimates for the components of our approximate solutions.

First we note that by (P2), (P5), and (P7) if $s\in[t_{j-1},t_j]$ then $|x(s)-x(t_{j-1})|\leqslant|x^j-x^{j-1}|\leqslant c_1\epsilon$ for some constant c_1 independent of ϵ. Similarly, we wish to show that there is some $c>0$ independent of ϵ such that

$$\big|a(s)-a(t_{j-1})\big|\leqslant c\epsilon\quad\text{for } s\in[t_{j-1},t_j]. \tag{12}$$

We achieve this by showing that $\|a_s-a_{t_{j-1}}\|\leqslant c\epsilon$ for each $s\in[t_{j-1},t_j]$. Let $\theta\in[-h,0]$ be given. If $s+\theta\leqslant t_{j-1}$ then $|a(s+\theta)-a(t_{j-1}+\theta)|\leqslant\epsilon$ by (P2) from Step 1. Otherwise, $s+\theta\geqslant t_{j-1}$. In this case, we have, by (P2), by the definition of a, and by the Lipschitz estimate for H with respect to R_1 in Step 1, that

$$\begin{aligned}\big|a(s+\theta)-a(t_{j-1}+\theta)\big|&\leqslant\big|a(s+\theta)-a(t_{j-1})\big|+\big|a(t_{j-1})-a(t_{j-1}+\theta)\big|\\&\leqslant\big|a(s+\theta)-a(t_{j-1})\big|+\epsilon\\&\leqslant J\|x_{s+\theta}-x_{t_{j-1}}\|+f(R_1)\|a_{s+\theta}-a_{t_{j-1}}\|+\epsilon.\end{aligned}$$

Using (P2) and $\xi\in[t_{j-1},t_j]\Rightarrow|x(\xi)-x(t_{j-1})|\leqslant c_1\epsilon$, it is easy to see that $\|x_{s+\theta}-x_{t_{j-1}}\|\leqslant g\epsilon$ for some constant $g>0$ independent of ϵ. Therefore, we have that, for each $s\in[t_{j-1},t_j]$,

$$\|a_s-a_{t_{j-1}}\|\leqslant Jg\epsilon+f(R_1)\sup_{\theta\in I\cap[t_{j-1}-s,0]}\|a_{s+\theta}-a_{t_{j-1}}\|+\epsilon. \tag{13}$$

The function $(s,\theta)\mapsto\|a_{s+\theta}-a_{t_{j-1}}\|$ defined on the compact set $K_0:=\{(s,\theta)\mid s\in[t_{j-1},t_j],\ \theta\in I\cap[t_{j-1}-s,0]\}$ is continuous and hence attains its maximum for some $(s^*,\theta^*)\in K_0$. By (13), we get

$$\|a_{s^*+\theta^*}-a_{t_{j-1}}\|\leqslant Jg\epsilon+f(R_1)\|a_{s^*+\theta^*}-a_{t_{j-1}}\|+\epsilon.$$

This, combined with the fact that $f(R_1)<1$, gives us

$$\|a_{s^*+\theta^*}-a_{t_{j-1}}\|\leqslant(Jg+1)\big(1-f(R_1)\big)^{-1}\epsilon. \tag{14}$$

Then (14) and (13) together tell us that (12) holds with $c=(Jg+1)+f(R_1)(Jg+1)(1-f(R_1))^{-1}>0$. Clearly c depends only on the initial data.

Step 3: The net of approximate solutions converges to a solution as $\epsilon \downarrow 0$.

Using (P1), (P2), (P5), and the estimate (12) from Step 2, then proceeding exactly as on p. 34 in [15], we obtain

$$\left| x^j - T(t_j)x^0 - \lim_{\mu\to\infty} \int_0^{t_j} T(t_j - s)\mu R_\mu F\big(x(s), a(s)\big)\, ds \right| \leqslant d\epsilon e^{\omega t_j} t_j \tag{15}$$

for some constant $d > 0$ independent of ϵ. With the help of (15), we can argue in the same way as in [15] to get the critical estimate

$$\left| x(t) - T(t)x^0 - \lim_{\mu\to\infty} \int_0^{t} T(t - s)\mu R_\mu F\big(x(s), a(s)\big)\, ds \right| \leqslant d\epsilon,$$

which holds for each $t \in [0, \tau]$. The constant d is larger than before (we relabeled) but still independent of ϵ. To complete this step, we must show that the net $\binom{x^\epsilon(t)}{a^\epsilon(t)}$ for $\epsilon \in (0, 1)$ of approximate solutions converges to a solution of (6).

First we show that $\{\binom{x^\epsilon}{a^\epsilon}\}$ is Cauchy in the complete metric space $C(I \cup [0, \tau], C_0 \times K)$. If $\binom{x^\epsilon(t)}{a^\epsilon(t)}$ and $\binom{y^\delta(t)}{b^\delta(t)}$ for ϵ, $\delta \in (0, 1)$ are approximate solutions, then (dropping the superscripts) we get

$$\begin{aligned} \big|x(t) - y(t)\big| &\leqslant (\epsilon + \delta)d + \left| \lim_{\mu\to\infty} \int_0^{t} T(t - s)\mu R_\mu (F\big(x(s), a(s)\big) - F\big(y(s), b(s)\big)\, ds \right| \\ &\leqslant (\epsilon + \delta)d + \int_0^{t} M^2 e^{\omega(t-s)} D\big(\big|x(s) - y(s)\big| + \big|a(s) - b(s)\big|\big)\, ds. \end{aligned} \tag{16}$$

Since $\|x_t\|$, $\|y_t\| \leqslant R_1$ for $t \in [0, \tau]$, we get

$$\big|a(t) - b(t)\big| \leqslant J\|x_t - y_t\| + f(R_1)\|a_t - b_t\|$$

and hence

$$\sup_{-t-h\leqslant\theta\leqslant 0} \big|a(t+\theta) - b(t+\theta)\big| \leqslant \big(1 - f(R_1)\big)^{-1} J \sup_{-t-h\leqslant\theta\leqslant 0} \big|x(t+\theta) - y(t+\theta)\big|. \tag{17}$$

Then, by (16), (17), and an application of Gronwall's inequality, we have

$$\sup_{-t-h\leqslant\theta\leqslant 0} \big|x(t+\theta) - y(t+\theta)\big| \downarrow 0 \quad \text{uniformly with respect to } t \in [0, \tau] \text{ as } \epsilon, \delta \downarrow 0.$$

It follows that $\|x - y\|_\infty \downarrow 0$ and $\|a - b\|_\infty \downarrow 0$ as $\epsilon \downarrow 0$ and $\delta \downarrow 0$. Therefore, $\{\binom{x^\epsilon(t)}{a^\epsilon(t)}\}$ converges uniformly to a mild solution of (6) on $I \cup [0, \tau]$ in M_0 as $\epsilon \downarrow 0$.

The uniqueness deserves a few remarks. We suppose that $\binom{x(t)}{a(t)}$ and $\binom{y(t)}{b(t)}$ are two mild solutions of (6) respectively on $I \cup A_1$ and $I \cup A_2$ in M_0 with the same initial data. Here $A_i = [0, \tau_i]$ or $A_i = [0, \tau_i)$ for $0 < \tau_i \leqslant \infty$, for each $i = 1, 2$. Let $A = A_1 \cap A_2$. We will show that the two solutions agree on A. Assume first that $x \neq y$. Let $\alpha := \inf\{t \in A \mid x(t) \neq y(t)\}$. Then $x(t) = y(t)$ for $t \leqslant \alpha$. Choose $\delta > 0$ such that $(\alpha, \alpha + \delta] \subset A$ and $R_2 > 0$ such that for $t \in (\alpha, \alpha + \delta]$, $\|x_t\|$, $\|y_t\| < R_2$ for some $R_2 < R_0$. By (H2) we have that

$$\bigl|a(t) - b(t)\bigr| \leqslant L_{R_2}\|x_t - y_t\| + f(R_2)\|a_t - b_t\| \quad \text{for } t \in (\alpha, \alpha + \delta]. \tag{18}$$

Now we are in a position to repeat the arguments for (17) and conclude by (18) and Gronwall's inequality, that $x(t) = y(t)$ for $t \in (\alpha, \alpha + \delta]$, violating the minimality of α. This shows that $x = y$ on A. Using (H2) it is easily seen that $a = b$ on A.

This completes the proof of Theorem 1. □

4. Maximal solutions and a semiflow on M_0

Given $\Psi = \binom{\psi}{\varphi} \in M_0$, the local solution granted in the previous section can be extended to a unique maximal solution $\binom{x^\Psi}{a^\Psi}$ of (6) in M_0 defined for $t \in I \cup [0, t_e)$ for some $0 < t_e \leqslant \infty$ which depends on Ψ. Namely, $t_e = \sup\bigl\{\tau \in (0, \infty) \mid (6)$ has a solution $\binom{x}{a}$ on $I \cup [0, \tau]$ in M_0, with $\binom{x_0}{a_0} = \binom{\psi}{\varphi}\bigr\}$. In this section we discuss the semiflow on M_0 formed by these maximal solutions of (6) in M_0.

We first introduce some notations. Let $\Omega = \{(t, \Psi) \in [0, \infty) \times M_0 \mid t \in [0, t_e(\Psi))\}$. For $t \geqslant 0$, let $\Omega_t = \{\Psi \in M_0 \mid t < t_e(\Psi)\} \subset M_0$. Then $\Omega \subset \mathbf{R} \times C(I, X_0 \times \mathbf{R})$ and $\Omega_t \subset C(I, X_0 \times \mathbf{R})$. Both Ω and Ω_t are equipped with the relative topology. Define $S : \Omega \to M_0$ as

$$S(t, \Psi) = \begin{pmatrix} x_t^\Psi \\ a_t^\Psi \end{pmatrix} \quad \text{for } (t, \Psi) \in \Omega.$$

Theorem 2. *The map S is a continuous semiflow on M_0. That is, S is continuous and satisfies the following two properties*:

(i) $S(0, \Psi) = \Psi$ *for* $\Psi \in M_0$.
(ii) *For each* $s, t \geqslant 0$ *with* $s < t_e(\Psi)$ *and* $t < t_e(S(s, \Psi))$, *we have* $t + s < t_e(\Psi)$ *and* $S(t, S(s, \Psi)) = S(t + s, \Psi) \in M_0$.

Proof. Properties (i) and (ii) are straightforward. It suffices to show that S is continuous. This is done in three steps, where Step 2 and Step 3 are merely adapting the corresponding proofs in [17] to our framework.

Step 1: Let $\Psi \in M_0$. We show that there is $\tau > 0$ and a neighborhood U of Ψ in M_0 such that $[0, \tau] \times U \subset \Omega$ and the restriction $S|_{[0,\tau]\times U}$ is continuous.

We take $0 < R_2 < R_1 < R_0$ such that $\|\Psi\| < R_2$. Denote $R = (R_1 - R_2)/M$, where $M \geqslant 1$ is as in (H1). Let $\Phi = \binom{\phi^1}{\phi^2} \in M_0$ such that $\|\Phi - \Psi\| < R$. Denote the corresponding mild solution of Φ in M_0 as $\binom{x(t)}{a(t)}$. Then, for $t \in [0, t_e(\Phi))$,

$$x(t) = T(t)\phi^1(0) + \lim_{\mu\to\infty} \int_0^t T(t - s)\mu R_\mu F\bigl(x(s), a(s)\bigr)\, ds.$$

It follows that

$$\bigl|x(t)\bigr| \leqslant \bigl|T(t)\bigl(\phi^1(0) - \psi^1(0)\bigr)\bigr| + \bigl|T(t)\psi^1(0)\bigr| + \int_0^t M^2 e^{\omega(t-s)}\bigl|F\bigl(x(s), a(s)\bigr)\bigr|\, ds.$$

Observing that

$$\begin{aligned}\left|F\big(x(s),a(s)\big)\right| &\leqslant \left|F\big(x(s),a(s)\big)-F\big(\psi^1(0),\psi^2(0)\big)\right|+\left|F\big(\psi^1(0),\psi^2(0)\big)\right| \\ &\leqslant 2D(R_0+h)+\left|F\big(\psi^1(0),\psi^2(0)\big)\right|\end{aligned}$$

and setting $C(\Psi)=2D(R_0+h)+|F(\psi^1(0),\psi^2(0))|$, we get

$$\left|x(t)\right| \leqslant (R_1-R_2)e^{\omega t}+\left|T(t)\psi^1(0)\right|+tM^2e^{\omega t}C(\Psi).$$

By continuity, there is some $\tau>0$ such that, for any $\Phi\in M_0$ with $\|\Phi-\Psi\|<R$, we have $|x(t)|<R_1$ for each $t\in[0,\tau]\cap[0,t_e(\Phi))$. Thus if $t\in[0,\tau]\cap[0,t_e(\Phi))$ then $\|x_t\|<R_1$. In particular, this shows that $\tau<t_e(\Phi)$. Let U be the open ball of radius $R>0$ about Ψ in M_0. We have shown $[0,\tau]\times U\subset\Omega$.

Now suppose we are given $(t_0,\Phi_0)\in[0,\tau]\times U$. Then for each $(t,\Phi)\in[0,\tau]\times U$, $\|S(t,\Phi)-S(t_0,\Phi_0)\|\leqslant\|S(t,\Phi)-S(t,\Phi_0)\|+\|S(t,\Phi_0)-S(t_0,\Phi_0)\|$. To complete the proof of Step 1, it is now clear that it suffices to show that the first term on the right-hand side of the latter inequality is bounded by $c\|\Phi-\Phi_0\|$ for some constant $c>0$ uniformly for $t\in[0,\tau]$.

Let $x(t),a(t)$ correspond to Φ_0 and $y(t),b(t)$ correspond to Φ. Then we have that for each $t\in[0,\tau]$

$$\left|x(t)-y(t)\right|\leqslant Me^{\omega\tau}\|\Phi-\Phi_0\|+\int_0^t M^2e^{\omega(t-s)}D\big(\left|x(s)-y(s)\right|+\left|a(s)-b(s)\right|\big)\,ds$$

and $|a(t)-b(t)|\leqslant L_{R_1}\|x_t-y_t\|+f(R_1)\|a_t-b_t\|$. It is not difficult to see that the latter inequality implies

$$\sup_{-h\leqslant t+\theta\leqslant t}\left|a(t+\theta)-b(t+\theta)\right|\leqslant c\Big(\sup_{-h\leqslant t+\theta\leqslant t}\left|x(t+\theta)-y(t+\theta)\right|+\|\Phi-\Phi_0\|\Big)$$

for some constant $c>0$ depending on R_1. This information combined with a Gronwall's inequality argument completes the proof of Step 1.

Step 2: Let $\Psi\in M_0$ and $t\in[0,t_e(\Psi))$. We show that $\Omega_t\subset M_0$ is open and the map $\Omega_t\ni\Phi\mapsto S(t,\Phi)$ is continuous at Ψ.

By continuity, we see that the set $K_1=\{S(s,\Psi)\mid s\in[0,t]\}\subset M_0$ is compact. Therefore, applying Step 1, we find some $u>0$ and some open subset N in M_0 containing K_1 such that $[0,u]\times N\subset\Omega$ and $S|_{[0,u]\times N}$ is continuous. Let J be the smallest positive integer such that $t/J<u$. Obviously, $(J-1)u\leqslant t<Ju$. Given $\epsilon>0$, we find $\delta_1>0$ such that

$$\begin{gathered}\text{if } \big\|\gamma-S\big((J-1)u,\Psi\big)\big\|<\delta_1\quad\text{then } \gamma\in N\quad\text{and}\\ \big\|S\big(t-(J-1)u,S\big((J-1)u,\Psi\big)\big)-S\big(t-(J-1)u,\gamma\big)\big\|<\epsilon.\end{gathered}\tag{19}$$

Recursively we can find $\delta_j>0$ for $j=2,\dots,J$ such that

$$\begin{gathered}\text{if } \big\|\gamma-S\big((J-j)u,\Psi\big)\big\|<\delta_j\quad\text{then } \gamma\in N\quad\text{and}\\ \big\|S(u,\gamma)-S\big(u,S\big((J-j)u,\Psi\big)\big)\big\|<\delta_{j-1}.\end{gathered}\tag{20}$$

Using (19), (20), the semigroup property, and induction, we see that if $\Phi\in M_0$ with $\|\Phi-\Psi\|<\delta_J$ then $\Phi\in\Omega_t$ and $\|S(t,\Phi)-S(t,\Psi)\|<\epsilon$. This completes the proof of Step 2.

Step 3: We prove that the map $S : \Omega \to M_0$ is continuous.

For $(t_0, \Psi_0) \in \Omega$, let U be a neighborhood of $S(t_0, \Psi_0)$ in M_0. We want to find a neighborhood $W \subset \Omega$ of (t_0, Ψ_0) such that $S(W) \subset U$. If $t_0 = 0$, by Step 1, we are done. Otherwise, $t_0 > 0$. By Step 1, we find some $0 < u < t_0$ and a neighborhood W_1 of Ψ_0 in M_0 such that $[0, u] \times W_1 \subset \Omega$ and $S|_{[0,u]\times W_1}$ is continuous. Let $0 < u_1 < u$. It follows from $S(t_0, \Psi_0) = S(t_0 - u_1, S(u_1, \Psi_0))$ that $S(u_1, \Psi_0) \in \Omega_{t_0-u_1}$. By Step 2, we can find a neighborhood W_2 of $S(u_1, \Psi_0)$ in M_0 such that $S(t_0 - u_1, W_2) \subset U$. Take $0 < \delta < u_1$ such that $(u_1 - \delta, u_1 + \delta) \subset (0, u)$ and choose a neighborhood W_3 of Ψ_0 in M_0 with $S((u_1 - \delta, u_1 + \delta) \times W_3) \subset W_2$. If $s \in (t_0 - \delta, t_0 + \delta)$ then $s = (t_0 - u_1) + (s - t_0 + u_1)$ and therefore the semigroup property gives $S((t_0 - \delta, t_0 + \delta) \times W_3) \subset U$, which completes the proof. □

5. An application

In this section we present an application of the general theory. We will see that in practice, it is nontrivial to check that all of the relevant hypotheses are satisfied.

Consider the following class of scalar age structured models with threshold dependent age of maturity,

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -d(a)u(t,a), \\ u(t,0) = b\left(\displaystyle\int_{\tau(t)}^{m} \beta(\xi)u(t,\xi)\,d\xi \right), \\ \displaystyle\int_{t-\tau(t)}^{t} \left[\int_0^{\tau(\sigma)} u(\sigma,a)\,da + C \right]^{-1} d\sigma = T, \\ \begin{pmatrix} u_0 \\ \tau_0 \end{pmatrix} = \begin{pmatrix} \hat{\psi} \\ \hat{\varphi} \end{pmatrix} \in C\big([-a_m, 0], \big(L^1{}_+[0,m)\big) \times \mathbf{R}^+\big), \end{cases} \tag{21}$$

where $t \geqslant 0$, $0 \leqslant a < m$, and $a_m < m \leqslant \infty$. Here m represents the maximum age and a_m stands for the maximum juvenile age. We make the following assumptions:

(A1) $d : [0, m) \to \mathbf{R}^+$ and $\beta : [0, m) \to \mathbf{R}^+$ are essentially bounded.
(A2) $b : \mathbf{R}^+ \to \mathbf{R}^+$ is bounded, globally Lipschitz, and $0 < \max_{x \in \mathbf{R}^+} b(x) \leqslant \theta$ for some $\theta > 0$.
(A3) $a_m = (R_0 + C)T < m \leqslant \infty$, where $R_0 = C(\frac{1}{\sqrt{T\theta}} - 1) > 0$.

In order to apply Theorem 1, we rewrite (21) as follows. Let $X = \mathbf{R} \times L^1([0, m), \mathbf{R})$ and define $A : D(A) \to X$ by

$$A \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} -x(0) \\ -x' \end{pmatrix} \quad \text{for } \begin{pmatrix} 0 \\ x \end{pmatrix} \in D(A) = \{0\} \times W^{1,1}\big([0, m), \mathbf{R}\big).$$

Note that $X_0 = \overline{D(A)} = \{0\} \times L^1[0, m)$. It is well known that A satisfies (H1) (see, for instance, [15,5]). Denote

$$C_0 = \left\{ \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \in 0 \times L^1[0, m) \;\middle|\; 0 \leqslant \gamma(a) \leqslant \theta \text{ a.e. } a \in [0, m) \right\}$$

and

$$D(H) = \left\{ \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \in C\big([-a_m, 0], C_0 \times K\big) \;\middle|\; \|\psi\| \leqslant R_0 \right\},$$

where $K = [0, a_m] \subset \mathbf{R}$. We prove that our "age of maturity function" is well defined in the following result.

Lemma 1. *The relation $H : D(H) \to K$, which is given by $(\psi, \varphi, \alpha) \in H$ if and only if $\int_{-\alpha}^{0} [\int_0^{\varphi(\sigma)} \psi(\sigma, \xi)\, d\xi + C]^{-1}\, d\sigma = T$, is a function.*

Proof. Given $\binom{\psi}{\varphi} \in D(H)$, it suffices to show that there exists a unique $\alpha \in K$ such that $(\psi, \phi, \alpha) \in H$. In fact, note that the map $\alpha \mapsto \int_{-\alpha}^{0} [\int_0^{\varphi(\sigma)} \psi(\sigma, \xi)\, d\xi + C]^{-1}\, d\sigma$ defined for $\alpha \in [0, a_m]$ is strictly increasing and continuous. Moreover, $\int_{-a_m}^{0} [\int_0^{\varphi(\sigma)} \psi(\sigma, \xi)\, d\xi + C]^{-1}\, d\sigma \geqslant a_m/(R_0 + C) = T$. Now the result follows immediately. □

Define $f : \mathbf{R}^+ \to \mathbf{R}^+$ by $f(Q) = \frac{(Q+C)^2 T}{C^2}\theta$. The coming result tells us that H satisfies an appropriate Lipschitz condition.

Lemma 2. *For any $Q > 0$ there is some $L_Q > 0$ such that, for $\binom{\psi_1}{\varphi_1}, \binom{\psi_2}{\varphi_2} \in D(H)$ with $\|\psi_i\| \leqslant Q$ $(i = 1, 2)$, we have*

$$\left|H(\psi_1, \varphi_1) - H(\psi_2, \varphi_2)\right| \leqslant f(Q)\|\varphi_1 - \varphi_2\| + L_Q \|\psi_1 - \psi_2\|.$$

Proof. Let $t_1 = H(\psi_1, \varphi_1)$ and $t_2 = H(\psi_2, \varphi_2)$. Without loss of generality, assume that $t_1 \leqslant t_2$. Then we have

$$\int_{-t_1}^{0} \left[\int_0^{\varphi_1(\sigma)} \psi_1(\sigma, \xi)\, d\xi + C \right]^{-1} d\sigma - \int_{-t_2}^{0} \left[\int_0^{\varphi_2(\sigma)} \psi_2(\sigma, \xi)\, d\xi + C \right]^{-1} d\sigma = 0$$

or

$$\begin{aligned} &\int_{-t_2}^{-t_1} \left[\int_0^{\varphi_2(\sigma)} \psi_2(\sigma, \xi)\, d\xi + C \right]^{-1} d\sigma \\ &\quad = \int_{-t_1}^{0} \left[\left(\int_0^{\varphi_1(\sigma)} \psi_1(\sigma, \xi)\, d\xi + C \right)^{-1} - \left(\int_0^{\varphi_2(\sigma)} \psi_2(\sigma, \xi)\, d\xi + C \right)^{-1} \right] d\sigma. \end{aligned}$$

Using the fact that the function $u \mapsto 1/(u + C)$ is globally Lipschitz on $(0, \infty)$ with Lipschitz constant $1/C^2$, we get

$$\frac{|t_1 - t_2|}{Q + C} \leqslant \frac{1}{C^2} \int_{-t_1}^{0} \left| \int_0^{\varphi_1(\sigma)} \psi_1(\sigma, \xi)\, d\xi - \int_0^{\varphi_2(\sigma)} \psi_2(\sigma, \xi)\, d\xi \right| d\sigma.$$

It follows that $|t_1 - t_2| \leqslant (Q + C)t_1(\|\psi_1 - \psi_2\| + \theta\|\varphi_1 - \varphi_2\|)/C^2$. Since $t_1 \leqslant (Q + C)T$, we obtain

$$\left|H(\psi_1, \varphi_1) - H(\psi_2, \varphi_2)\right| \leqslant \frac{(Q + C)^2 T}{C^2} \|\psi_1 - \psi_2\| + \frac{(Q + C)^2 T}{C^2} \theta \|\varphi_1 - \varphi_2\|.$$

This completes the proof. □

Define $F: C_0 \times K \to X$ by $F(x,a) = \binom{b(\int_a^m \beta(\xi)x(\xi)\,d\xi)}{-d(\cdot)x(\cdot)}$. By (A1) and (A2) it is clear that F is Lipschitz on $C_0 \times K$. The verification of the subtangential condition (H6) with respect to C_0, K, and F follows exactly as on pp. 12–14 of the examples in [15]. Therefore, by Proposition 1, we have the following result.

Proposition 3. *In addition to* (A1)–(A3), *assume that* $\binom{\hat{\psi}}{\hat{\varphi}} \in C([-a_m, 0], L^1_+[0,m) \times \mathbf{R}^+)$ *satisfies the following two conditions*:

(i) *For each* $\sigma \in [-a_m, 0]$, $0 \leqslant \hat{\psi}(\sigma)(a) \leqslant \theta$ *a.e.* $a \in [0,m)$ *and* $\hat{\varphi}(\sigma) \in [0, a_m]$.
(ii) *For each* $\sigma \in [-a_m, 0]$, $\int_0^m \hat{\psi}(\sigma)(a)\,da < C(\frac{1}{\sqrt{T\theta}} - 1)$ *and* $\int_{-\hat{\varphi}(0)}^0 [\int_0^{\hat{\varphi}(\sigma)} \hat{\psi}(\sigma,\xi)\,d\xi + C]^{-1}\,d\sigma = T$.

Then the initial value problem (21) *has a unique maximal solution* $\binom{u}{\tau} \in C([-a_m, t_e), L^1_+[0,m) \times [0, a_m])$ *on* $[-a_m, t_e)$ $(t_e > 0)$ *with* $\binom{u_0}{\tau_0} = \binom{\hat{\psi}}{\hat{\varphi}}$ *in the following sense*:

(i) *For* $0 \leqslant t < t_e$, $a \mapsto \int_0^t u(s,a)\,ds$ *is absolutely continuous, and for a.e.* $a \in [0,m)$,

$$u(t,a) = u(0,a) - \partial_a \int_0^t u(s,a)\,ds - \int_0^t d(a)u(s,a)\,ds,$$

$$\int_0^t u(s,0)\,ds = \int_0^t b\left(\int_{\tau(s)}^m \beta(a)u(s,a)\,da\right) ds.$$

(ii) *For* $0 \leqslant t < t_e$, $\int_{t-\tau(t)}^t [\int_0^{\tau(\sigma)} u(\sigma,a)\,da + C]^{-1}\,d\sigma = T$.
(iii) *For* $t \in [0, t_e)$ *the "total population" satisfies* $\int_0^m u(t,a)\,da < C(\frac{1}{\sqrt{T\theta}} - 1)$ *and* $0 \leqslant u(t,a) \leqslant \theta$ *for a.e.* $a \in [0,m)$.

Finally we note that, by Theorem 2, the corresponding semiflow is continuous.

6. Future work

To summarize, in Section 1 we have motivated an abstract algebraic-delay differential system arising from threshold phenomena in age structured population dynamics, with emphasis on the immature population. In Section 2 we gave the relevant technical preliminaries and hypotheses. In Sections 3 and 4 we obtained the existence of a continuous semiflow formed by maximal solutions of the system, and in Section 5 we applied the results to the model motivated in the introduction.

There are two issues which require further study. First of all, in the upcoming sequel of this work, we will investigate appropriate sufficient conditions for the differentiability with respect to time of mild solutions of (6) in M_0. Our abstract results will be used to infer about some specific examples studied in the work by Magpantay et al. [7].

Secondly, we note that the results of the present work require strong hypotheses which imply uniform boundedness of the age of maturity, and the total population in the model discussed in Section 5. This is because in our hypotheses in Section 2, we never made any assumptions on the differentiability of the function H from (H2). In the future work, we will investigate to what extent a special property of the derivative of H enables us to set up the problem motivated in the introduction, as a *locally bounded adaptive state dependent delay* initial value problem (see [1, p. 450] and [18]), on a submanifold of the space of *continuously differentiable* functions. We eventually plan to address some dynamical properties of the corresponding global semiflow.

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