J. Differential Equations 254 (2013) 3538–3572

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Unimodal dynamical systems: Comparison principles, spreading speeds and travelling waves

Taishan Yi [a,*], Yuming Chen [b], Jianhong Wu [c]

[a] *School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, PR China*
[b] *Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada*
[c] *Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada*

ARTICLE INFO

Article history:
Received 20 July 2012
Revised 18 January 2013

Keywords:
Unimodal nonlinearity
Delayed reaction diffusion equation
Nonlocal time-delayed lattice differential system
Non-monotone dynamical system
Spreading speed
Travelling wave

ABSTRACT

Reaction diffusion equations with delayed nonlinear reaction terms are used as prototypes to motivate an appropriate abstract formulation of dynamical systems with unimodal nonlinearity. For such non-monotone dynamical systems, we develop a general comparison principle and show how this general comparison principle, coupled with some existing results for monotone dynamical systems, can be used to establish results on the asymptotic speeds of spread and travelling waves. We illustrate our main results by an integral equation which includes a nonlocal delayed reaction diffusion equation and a nonlocal delayed lattice differential system in an unbounded domain, with the non-monotone nonlinearities including the Ricker birth function and the Mackey–Glass hematopoiesis feedback.

1. Introduction

Incorporating time delay into a mathematical model for a certain dynamical process may destroy the order-preserving (monotonicity) property of the semiflow which is satisfied by the corresponding model without delay. Such a delay induced non-monotonicity of the solution semiflow has been observed in a large number of unimodal delay differential equations such as the Mackey–Glass equation and the Nicholson blowflies equation. The global dynamics of such equations can be quite complicated and chaotic behaviors are possible, as shown in the Mackey–Glass attractor. A notable feature of such a unimodal delay differential equation is the existence of two ordered equilibria, between which the nonlinearity changes monotonicity once (from being monotonically increasing to decreasing).

* Corresponding author.
E-mail addresses: yitaishan76@yahoo.com (T. Yi), ychen@wlu.ca (Y. Chen), wujh@mathstat.yorku.ca (J. Wu).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.01.031

A particular issue is the existence of connecting orbits between these two ordered equilibria, which is automatically guaranteed by the monotone dynamical systems theory if the nonlinearity is monotone between these two equilibria. When spatial diffusion is further incorporated, the delayed nonlinearity may involve nonlocal interaction. It is then very important for both theoretical advance and practical applications to know whether there are travelling wavefronts, representing the transition from one equilibrium to the other, and how they are related to the global dynamics of the reaction diffusion equations with nonlocal delayed nonlinearity.

Travelling waves (wavefronts) play an important role in the theory of reaction diffusion equations, because of their theoretical relevance to the global dynamics of such equations and because of their practical significance to the (chemical, physical, ecological and epidemiological) transition processes of systems modeled by these equations. In [1,2], Aronson and Weinberger introduced the concept of the spreading speed (i.e., asymptotic speed of spread) and showed that it coincides with the minimal wave speed for travelling waves under appropriate assumptions including the monotonicity. Weinberger [32] and Lui [19] later established a parallel theory of spreading speeds and monostable travelling waves for monotone discrete dynamical systems. Recent studies [13–16,31,33] have extended such a theory for monotone semiflows in great generality so that it can be applied to various discrete-time and continuous-time evolution equations admitting some sort of comparison principles. In particular, Liang and Zhao [15] established the theory of asymptotic speeds of spread and monotone travelling waves for monotone discrete-time and continuous-time semiflows with monostable nonlinearities which can be applied to evolution systems with time delays and reaction diffusion equations in cylinders with monotone nonlinearity.

Many dynamical systems however do not possess the order-preserving or monotonicity property, and there are various attempts to establish similar results on the spreading speeds and the existence of monostable travelling waves for such systems [5–7,11,20,22,24,29,30,36]. Unfortunately, most of these results have been established for specific type of evolution equations (especially delayed reaction diffusion equations) and the arguments used are pretty much equation-specific. One of the objectives of this paper is to develop a generic framework and relevant techniques for a class of non-monotone discrete-time and continuous-time dynamical systems on C_+, where $C_+ \triangleq BC([-\tau, 0] \times \mathcal{H}, \mathbb{R}_+)$ and $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Specifically, we present an abstract formulation (UM) of unimodal semiflows in the aforementioned phase spaces and develop a very general comparison principle, based on the concept of upper- and lower-systems which enjoy certain order-preserving properties. We then introduce a linear positive operator, satisfying conditions (AL1)–(AL3) formulated in terms of asymptotic behaviors of the semiflow trajectories near the trivial equilibrium, and use it to impose a certain global coupling condition (GC) between the upper- and lower-systems so that the well-established global dynamics of the semiflows generated by the (order-preserving) upper- and lower-systems can be used to draw conclusions about the asymptotic behaviors of the unimodal dynamical system. Critical importance of our approach is the possibility of this global coupling between the upper- and lower-systems, roughly in the sense that (a) both upper- and lower-systems are globally controlled by the same linear semiflow, obtained by linearizing the nonlinear system at the trivial equilibrium, and (b) both upper- and lower-systems are convergent to the nontrivial equilibrium at infinite.

This general framework is motivated by some of our recent studies [37,38], using ideas and arguments different from those of [4–7,11,20,22,24,29,30,36]. Our main results and principal arguments are

- the global attractivity of the nontrivial equilibrium for semiflows in an unbounded domain of $C_+ \setminus \{0\}$ under the *compact open topology* in [37], as a natural consequence of the comparison principle and the imposed global coupling between upper- and lower-systems;
- the existence of travelling waves and the coincidence of the minimal wave speed and the spread speed, as an application of a limiting argument to the sequence obtained by using the Schauder fixed point theorem and the corresponding results for monotone dynamical systems by Liang and Zhao [15].

Some preliminaries and standard hypotheses are given in Section 2 while the main general results are developed in Sections 3 and 4: Section 3 is devoted to the global asymptotic behaviors and the spreading speed, and Section 4 is devoted to travelling waves by using the Schauder fixed point theorem and the general $R_{c,k}$ operators approach instead of the classical upper–lower solution method. Finally, we apply our main results to an integral equation which includes, as special examples, a nonlocal delayed reaction diffusion equation and a nonlocal delayed lattice differential system in an unbounded domain. The general results are illustrated with two well-known examples (Ricker nonlinearity and Mackey–Glass equation) in Section 5, where we also show how to construct the upper- and lower-systems and explain the global coupling conditions. Comparisons of our results for these two examples with existing works [5–7,11,20,22,24] are also given which show that our general framework can yield optimal results for some specific systems with well-justified minimal hypotheses.

2. Preliminaries and basic hypotheses

We first introduce some notations. Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}$, and $\mathbb{R}_+$ be the sets of all integers, nonnegative integers, reals, and nonnegative reals, respectively. Let $X = BC(\mathcal{H}, \mathbb{R})$ be the normed vector space of all bounded and continuous functions from $\mathcal{H}$ to $\mathbb{R}$ with the norm $\|\phi\|_X \triangleq \sum_{n=1}^{\infty} 2^{-n} \sup\{|\phi(x)|:\ x \in \mathcal{H} \text{ with } |x| \leqslant n\}$ for all $\phi \in X$, where $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Let $X_+ = \{\phi \in X:\ \phi(x) \geqslant 0 \text{ for all } x \in \mathcal{H}\}$ and $X_+^o = \{\phi \in X:\ \phi(x) > 0 \text{ for all } x \in \mathcal{H}\}$.

For a given $\tau \in \mathbb{R}_+$, let $C = C([-\tau, 0], X)$ be the normed vector space of all continuous functions from $[-\tau, 0]$ into X with the norm $\|\varphi\|_C \triangleq \max\{\|\varphi(\theta)\|_X:\ \theta \in [-\tau, 0]\}$ for all $\varphi \in C$, $C_+ = C([-\tau, 0], X_+)$ and $C_+^o = C([-\tau, 0], X_+^o)$. It follows that C_+ is a closed cone in the normed vector space C. But $C_+^o \neq \mathrm{Int}(C_+)$ due to the non-compactness of the spatial domain $\mathcal{H}$. Also, let $Y = C([-\tau, 0], \mathbb{R})$ be the normed vector space of all continuous functions from $[-\tau, 0]$ into $\mathbb{R}$ with the norm $\|\beta\|_Y \triangleq \max\{|\beta(\theta)|:\ \theta \in [-\tau, 0]\}$ for all $\beta \in Y$ and $Y_+ = C([-\tau, 0], \mathbb{R}_+)$.

For the sake of convenience, we identify an element $\varphi \in C$ as a bounded and continuous function from $[-\tau, 0] \times \mathcal{H}$ into $\mathbb{R}$. For $a \in \mathbb{R}$, $\hat{a} \in X$ is defined as $\hat{a}(x) = a$ for all $x \in \mathcal{H}$. Similarly, $\hat{\hat{a}} \in C$ is defined as $\hat{\hat{a}}(\theta) = \hat{a}$ for all $\theta \in [-\tau, 0]$. Moreover, for any $\phi \in X$ and $\beta \in Y$, we define $\tilde{\phi} \in C$ and $\tilde{\beta} \in C$ respectively by $\tilde{\phi}(\theta, x) = \phi(x)$ and $\tilde{\beta}(\theta, x) = \beta(\theta)$ for all $(\theta, x) \in [-\tau, 0] \times \mathcal{H}$. In the following, we identify $\hat{a}$ or $\hat{\hat{a}}$ with a for $a \in \mathbb{R}$. Furthermore, we identify $\phi \in X$ and $\beta \in Y$ with $\tilde{\phi} \in C$ and $\tilde{\beta} \in C$, respectively. As a result, we can regard X and Y as subspaces of C.

For any $\xi, \eta \in X$, we write $\xi \geqslant_X \eta$ if $\xi - \eta \in X_+$, $\xi >_X \eta$ if $\xi \geqslant_X \eta$ and $\xi \neq \eta$, $\xi \gg_X \eta$ if $\xi - \eta \in X_+^o$. Similarly, for any $\varphi, \psi \in C$, we write $\varphi \geqslant_C \psi$ if $\varphi - \psi \in C_+$, $\varphi >_C \psi$ if $\varphi \geqslant_C \psi$ and $\varphi \neq \psi$, $\varphi \gg_C \psi$ if $\varphi - \psi \in C_+^o$; for any $\varrho, \rho \in Y$, we write $\varrho \geqslant_Y \rho$ if $\varrho - \rho \in Y_+$, $\varrho >_Y \rho$ if $\varrho \geqslant_Y \rho$ and $\varrho \neq \rho$, $\varrho \gg_Y \rho$ if $\varrho - \rho \in \mathrm{Int}(Y_+)$, where $\mathrm{Int}(Y_+) = \{\beta \in Y:\ \beta(\theta) > 0 \text{ for all } \theta \in [-\tau, 0]\}$. For simplicity of notations, we write $\geqslant$, $>$, $\gg$ and $\|\cdot\|$ respectively for $\geqslant_*$, $>_*$, $\gg_*$ and $\|\cdot\|_*$, where $*$ stands for one of X, C and Y.

For $r \in (0, \infty)$, $\sigma \in Y$ and $\varphi \in C$, we say $\varphi(\cdot, x) \geqslant \sigma$ for x on an interval of length $2r$ if there exists $a \in \mathbb{R}$ such that $\varphi(\cdot, x) \geqslant \sigma$ for all $x \in [a - r, a + r] \cap \mathcal{H}$.

For given numbers $r, s > 0$, define $C_r = \{\varphi \in C:\ 0 \leqslant \varphi \leqslant r\}$ and $C_{r,s} = \{\varphi \in C:\ r \leqslant \varphi \leqslant s\}$. Also, for $\varphi \in C_+$, define $C_\varphi = \{\psi \in C:\ 0 \leqslant \psi \leqslant \varphi\}$.

For a given $y \in \mathcal{H}$, define the translation operator T_y by $T_y[u](\theta, x) = u(\theta, x - y)$ and define the reflection operator $\mathcal{R}$ by $\mathcal{R}[u](\theta, x) = u(\theta, -x)$, where $u \in C$, $\theta \in [-\tau, 0]$ and $x \in \mathcal{H}$. A subset $W \subseteq C$ is said to be T-invariant if $T_y W = W$ for all $y \in \mathcal{H}$.

For given $\theta_0 \in [-\tau, 0]$, $x_0 \in \mathcal{H}$, and $W \subseteq C$, we denote $W(\cdot, x_0) := \{\varphi(\cdot, x_0) \in Y:\ \varphi \in W\}$ and $W(\theta_0, \cdot) := \{\varphi(\theta_0, \cdot) \in X:\ \varphi \in W\}$.

We will need a few fundamental results on monotone maps formulated in Liang and Zhao [15]. So, we start with the following list of standing assumptions about the reflection- and translation-equivariance, the continuity and compactness of a given map Q from C_+ to C_+ with two equilibria satisfying certain ejectivity and attractivity:

(A1) $Q[\mathcal{R}[u]] = \mathcal{R}[Q[u]]$ and $T_y[Q[u]] = Q[T_y[u]]$ for all $u \in C_+$ and $y \in \mathcal{H}$.
(A2) For any given number $r > 0$, $Q|_{C_r} : C_r \to C_+$ is continuous.

(A3) For any given number $r > 0$, one of the following two properties holds.
(a) $\{Q[u](\cdot, x)\colon\ u \in C_r,\ x \in \mathcal{H}\}$ is a precompact subset of Y.
(b) The set $Q[C_r](0, \cdot)$ is precompact in X. Moreover, there is a positive number $\varsigma \leqslant \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for all $\theta \in [-\tau, -\varsigma)$ and the operator

$$S[u](\theta, x) := \begin{cases} u(0, x) & \text{if } -\tau \leqslant \theta < -\varsigma, \\ Q[u](\theta, x) & \text{if } -\varsigma \leqslant \theta \leqslant 0 \end{cases}$$

has the property that $S[D](\cdot, 0)$ is precompact in Y for any T-invariant set $D \subseteq C_r$ with $D(0, \cdot)$ being a precompact subset of X.

(A4) There is a positive number $r^* > 0$ such that (a) $Q[0] = 0$, $Q[r^*] = r^*$; and (b) there is some positive integer n_0 such that for any $n \in \mathbb{Z}_+ \cap [n_0, \infty)$, $Q^n[\alpha] \gg \alpha$ for all $\alpha \in (0, r^*)$ and $Q^n[\alpha] \ll \alpha$ for all $\alpha \in (r^*, \infty)$.

In this section, we always assume that Q satisfies hypotheses (A1)–(A4). Note that assumptions (A1)–(A3) are just the corresponding ones in Liang and Zhao [15] with C_+ replacing C_β for given $\beta \in \mathrm{Int}(Y_+)$. We should point out that all conclusions in [15] still hold when the assumption (A5) in [15] is replaced with the above assumption (A4).

Let $\widetilde{C}_+$ be the set of all bounded and continuous functions from $[-\tau, 0] \times \mathbb{R}$ to $\mathbb{R}_+$. In the case where $\mathcal{H} = \mathbb{R}$, we have $\widetilde{C}_+ = C_+$. In the case where $\mathcal{H} = \mathbb{Z}$, we define an operator $\widetilde{Q}$ on the set $\widetilde{C}_+$ by

$$\widetilde{Q}[v](\theta, s) := Q\big[v(\cdot, \cdot + s)\big](\theta, 0) \quad \text{for all } \theta \in [-\tau, 0] \text{ and } s \in \mathbb{R}.$$

By the definition of $\widetilde{Q}$ and the proof of Lemma 2.1 in [15], it is easy to check that $\widetilde{Q}$ satisfies hypotheses (A1), (A2), (A4) with $\mathcal{H} = \mathbb{R}$.

In the remaining part of this section, we assume that $Q|_{C_{r^*}}$ is monotone (order-preserving) in the sense that $Q[u] \geqslant Q[v]$ whenever $u \geqslant v$ in C_{r^*}.

To conclude this section, we follow Liang and Zhao [15] to define the spreading speed c^*.

Choose a checking function φ such that $\varphi \in C([-\tau, 0] \times \mathbb{R}, [0, r^*])$ and it satisfies the following property.

(P) For any fixed $\theta \in [-\tau, 0]$, $\varphi(\theta, \cdot)$ is a non-increasing function on $\mathbb{R}$ with $\varphi([-\tau, 0] \times \mathbb{R}_+) = \{0\}$, $\varphi(\cdot, \infty) = 0$, and $\varphi(\cdot, -\infty) \in (0, r^*)$.

Given a real number c and $\varphi \in C([-\tau, 0] \times \mathbb{R}, [0, r^*])$ with the property (P), we define the operator R_c associated with Q by

$$R_c[a](\theta, s) = \max\big\{\varphi(\theta, s), T_{-c}\big[\widetilde{Q}[a]\big](\theta, s)\big\},$$

where $(\theta, s) \in [-\tau, 0] \times \mathbb{R}$ and $a \in C([-\tau, 0] \times \mathbb{R}, [0, r^*])$. We now define a sequence of functions $a_n(c; \theta, s)$ by the recursion

$$a_0 = \varphi \quad \text{and} \quad a_{n+1} = R_c[a_n] \quad \text{for } n \in \mathbb{Z}_+.$$

Following the statements of Section 2 in [15], there is a function a such that $a(c; \theta, s) = \lim_{n\to\infty} a_n(c; \theta, s)$, $a(c; \cdot, -\infty) = r^*$ and $a(c; \cdot, s)$ is non-increasing in c and s. Obviously, the sequence $\{a_n\}$ depends upon the choice of function φ. But Lemma 2.8 in [15] implies that $a(c; \cdot, \infty)$ is independent of the choice of φ. We now define the spreading speed c^* with Q,

$$c^* \equiv \sup\big\{c\colon\ a(c; \cdot, \infty) = r^*\big\}.$$

If $a(c; \cdot, \infty) = r^*$ for all c, we set $c^* = \infty$.

Note that explicit formulae of c^* have been obtained for abstract monotone semiflows in [14,15,32,33] and for some special non-monotone evolution equations in [5,6,11]. As a result, the focus of this paper is not to find the explicit computation formula for c^*. Instead, we present a direct approach to show that $c^*>0$ under proper conditions which automatically hold for a wide range of evolution equations.

Proposition 2.1. *Suppose that there exist a number $\delta \in (0,r^*)$ and a linear operator $Q_L : C \to C$ such that Q and Q_L satisfy the following assumptions.*

(L1) *Q_L satisfies* (A1) *and* (A2).
(L2) *$Q[u] \geqslant Q_L[u]$ for all $u \in C_\delta$ and for any $u \in C_+ \setminus \{0\}$ there is an integer $N_u \geqslant 1$ such that $(Q_L)^{N_u}[u] \in C_+^o$.*
(L3) *There exists $\eta_0 \in \mathrm{Int}(Y_+)$ such that $\|\eta_0\| = 1$ and $Q_L[\eta_0] \gg \eta_0$.*

If $Q|_{C_{r^*}}$ satisfies assumption (A3*) *in Section* 4 *by replacing C_+ with C_{r^*}* (*see also* (A6)(a) *and* (A6)(b′) *of* [15] *for this assumption*), *then $c^*>0$.*

Proof. By way of contradiction, suppose that $c^* \leqslant 0$. Then, by Theorem 4.2 in [15], there is a non-increasing function $U^* \in C_{r^*}$ such that $U^*(\cdot,\infty)=0$, $U^*(\cdot,-\infty)=r^*$ and $U^*=Q[U^*]$. It follows from (L2) that $U^* \in C_+^o$.

By assumptions (L1) and (L3), there is $\lambda_0 \in (1,\infty)$ such that $Q_L(\eta_0) \geqslant \lambda_0\eta_0$. This, combined with the continuity of Q and Q_L, implies that there exist an integer $n^*>0$ and a number $\delta^* \in (0,\delta)$ such that $\lambda_0^{n^*} > \frac{2}{\inf\{\eta_0(\theta):\ \theta\in[-\tau,0]\}}$ and $Q^n[C_{\delta^*}] \cup Q_L^n[C_{\delta^*}] \subseteq C_\delta$ for all positive integers $n \leqslant n^*$.

We may assume, without loss of generality, that $\epsilon^* \triangleq \inf\{U^*(\theta,0):\ \theta \in [-\tau,0]\} \in (0,\delta^*]$. By choices of n^*, δ^* and ϵ^* and by assumption (L2), we obtain that $U^* = Q^n[U^*] \geqslant Q^n[U^* \wedge \epsilon^*\eta_0] \geqslant Q_L^n[U^* \wedge \epsilon^*\eta_0]$ for any positive integers $n \leqslant n^*$, where $[U^* \wedge \epsilon^*\eta_0](\theta,x) = \min\{U^*(\theta,x), \epsilon^*\eta_0(\theta)\}$ for all $(\theta,x) \in [-\tau,0]\times\mathcal{H}$. In particular, $U^*(\cdot,0) \geqslant Q_L^{n^*}[U^* \wedge \epsilon^*\eta_0](\cdot,0)$.

We claim that $Q_L^{n^*}[U^* \wedge \epsilon^*\eta_0](\cdot,0) \geqslant \frac{1}{2}Q_L^{n^*}[\epsilon^*\eta_0](\cdot,0)$. Indeed, by $\mathcal{R}[U^* \wedge \epsilon^*\eta_0] = \mathcal{R}[U^*] \wedge \epsilon^*\eta_0$ and (L1), we have $Q_L^n[\mathcal{R}[U^*] \wedge \epsilon^*\eta_0] = \mathcal{R}[Q_L^n[U^* \wedge \epsilon^*\eta_0]]$ for all $n \in \mathbb{Z}_+$. Particularly, $Q_L^n[\mathcal{R}[U^*] \wedge \epsilon^*\eta_0](\cdot,0) = Q_L^n[U^* \wedge \epsilon^*\eta_0](\cdot,0)$ for all $n \in \mathbb{Z}_+$. Since $U^*(\cdot,x)$ is a non-increasing function in x, we easily know that $[\mathcal{R}[U^*] \wedge \epsilon^*\eta_0] + [U^* \wedge \epsilon^*\eta_0] \geqslant \epsilon^*\eta_0$. Then the linearity of Q_L forces $Q_L^n[U^* \wedge \epsilon^*\eta_0](\cdot,0) \geqslant \frac{1}{2}Q_L^n[\epsilon^*\eta_0](\cdot,0)$ for all $n \in \mathbb{Z}_+$. This proves the above claim. The claim together with the fact that $U^*(\cdot,0) \geqslant Q_L^{n^*}[U^* \wedge \epsilon^*\eta_0](\cdot,0)$ implies $U^*(\cdot,0) \geqslant \frac{1}{2}Q_L^{n^*}[\epsilon^*\eta_0](\cdot,0)$. It follows that $U^*(\cdot,0) \geqslant \frac{\epsilon^*}{2}Q_L^{n^*}[\eta_0](\cdot,0) \geqslant \frac{\epsilon^*}{2}\lambda_0^{n^*}\eta_0 > \epsilon^*$, a contradiction to the choice of ϵ^*. Therefore, $c^*>0$ and the proof is complete. □

3. The asymptotic behavior and the spreading speed

We now develop some generic comparison principles involving a pair of upper- and lower-trajectories of maps $Q^\pm$. In this and the next section, we always assume that for any given number $r>0$, $Q|_{C_r}$, $Q^-|_{C_r}$ and $Q^+|_{C_r} : C_r \to C_+$ are continuous, and Q and $Q^\pm$ satisfy the following properties.

(UL1) $Q^-[0]=Q^+[0]=0$, $Q^-[r^-]=r^-$ and $Q^+[r^+]=r^+$ for two given positive numbers $r^+ \geqslant r^-$;
(UL2) Q^- and Q^+ are monotone (order-preserving) in the sense that $Q^\pm[u] \geqslant Q^\pm[v]$ whenever $u \geqslant v$ in C_+;
(UL3) Q^- and Q^+ satisfy assumptions (A1)–(A4) where r^* is r^- and r^+, respectively;
(UL4) Q satisfies assumptions (A1)–(A3) and (A4)(a) with r^*.

For given $c \in \mathbb{R}$ and $n \in \mathbb{Z}_+$, let $\mathcal{A}_{n,c} = \{(\theta,x) \in [-\tau,0]\times\mathcal{H}:\ |x| \leqslant nc\}$ and $\mathcal{A}_{n,c}^+ = \{(\theta,x) \in [-\tau,0]\times\mathcal{H}:\ |x| \geqslant nc\}$.

Theorem 3.1. *Suppose that $Q[\varphi] \leqslant Q^+[\varphi]$ for all $\varphi \in C_+$. Define the spreading speed c_+^* with Q^+ similarly as c^* in Section 2. If $u_0 \in C_+$ has compact support with $u_0 \ll r^+$, then for any $c > c_+^*$ we have*

$$\lim_{n\to\infty} \max_{(\theta,x)\in\mathcal{A}_{n,c}^+} Q^n[u_0](\theta,x) = 0. \tag{3.1}$$

Proof. We can obtain $\lim_{n\to\infty}\max_{(\theta,x)\in\mathcal{A}_{n,c}^+}(Q^+)^n[u_0](\theta,x) = 0$ by applying Theorem 2.11 in [15] to $Q^+|_{C_{r^+}}$. This, combined with Proposition 2.3 in [15] and the fact that $Q[\varphi] \leqslant Q^+[\varphi]$ for all $\varphi \in C_+$, gives the conclusion. $\square$

Proposition 3.2. *For any $u \geqslant r^\pm$, we have*

$$\lim_{n\to\infty} \sup_{(\theta,x)\in[-\tau,0]\times\mathcal{H}} \left|\left(Q^\pm\right)^n[u](\theta,x) - r^\pm\right| = 0. \tag{3.2}$$

In particular,

$$\lim_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c}} \left(Q^\pm\right)^n[u](\theta,x) = \lim_{n\to\infty} \max_{(\theta,x)\in\mathcal{A}_{n,c}} \left(Q^\pm\right)^n[u](\theta,x) = r^\pm, \tag{3.3}$$

where $c < c_\pm^*$ and the spreading speeds $c_\pm^*$ with $Q^\pm$ are defined similarly as c^* in Section 2.

Proof. It suffices to prove (3.2) for Q^- as the proof for Q^+ is similar. Obviously, there exists a positive number α^* such that $r^- \leqslant u \leqslant \alpha^*$. By the monotonicity of Q^-, we have $r^- \leqslant (Q^-)^n[u] \leqslant (Q^-)^n[\alpha^*]$ for all $n \in \mathbb{Z}_+$. Then, by assumptions (A1), (A3) and (A4) we know that $(Q^-)^n[\alpha^*](\cdot,x) = (Q^-)^n[\alpha^*](\cdot,0)$ for all $x \in \mathcal{H}$, $\{(Q^-)^n[\alpha^*](\cdot,0)\colon n \in \mathbb{Z}_+\}$ is precompact in Y and $\lim_{n\to\infty}(Q^-)^n[\alpha^*](\cdot,0) = r^-$. Therefore, $\lim_{n\to\infty}\sup_{(\theta,x)\in[-\tau,0]\times\mathcal{H}}|(Q^-)^n[u](\theta,x) - r^-| = 0$, i.e., (3.2) holds for Q^-. This completes the proof. $\square$

Proposition 3.3. *Define the spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. For any $c < c_\pm^*$ and any $\sigma \in \mathrm{Int}(Y_+)$, there exists a positive number $r_\sigma > 0$ such that*

$$\lim_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c}} \left(Q^\pm\right)^n[u](\theta,x) = \lim_{n\to\infty} \max_{(\theta,x)\in\mathcal{A}_{n,c}} \left(Q^\pm\right)^n[u](\theta,x) = r^\pm, \tag{3.4}$$

where $u \in C_+$ with $u(\cdot,x) \geqslant \sigma$ on an interval of length $2r_\sigma$.

Proof. It suffices to prove the result for Q^- as the proof for Q^+ is similar. Define $\sigma^-\colon[-\tau,0]\to\mathbb{R}$ by $\sigma^-(\theta) = \min\{\sigma(\theta), \frac{r^-}{2}\}$ for $\theta \in [-\tau,0]$. For any $c < c_-^*$, by applying Theorem 2.15 in [15] to $Q^-|_{C_{r^-}}$, there exists a positive number $r_\sigma > 0$ such that $\lim_{n\to\infty}\min_{(\theta,x)\in\mathcal{A}_{n,c}}(Q^-)^n[v](\theta,x) = \lim_{n\to\infty}\max_{(\theta,x)\in\mathcal{A}_{n,c}}(Q^-)^n[v](\theta,x) = r^-$, where $v \in C_{r^-}$ with $v(\cdot,x) \geqslant \sigma^-$ on an interval of length $2r_\sigma$.

Now suppose that $u \in C_+$ with $u(\cdot,x) \geqslant \sigma$ on an interval of length $2r_\sigma$. Define $u^-, u^+\colon[-\tau,0]\to\mathbb{R}$ respectively by $u^-(\theta,x) = \min\{u(\theta,x), \frac{r^-}{2}\}$, $u^+(\theta,x) = \max\{u(\theta,x), r^-\}$ for $(\theta,x) \in [-\tau,0]\times\mathcal{H}$. Then $u^- \in C_{r^-}$ and $u^-(\cdot,x) \geqslant \sigma^-$ on an interval of length $2r_\sigma$. It follows that

$$\lim_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c}} \left(Q^-\right)^n\left[u^-\right](\theta,x) = \lim_{n\to\infty} \max_{(\theta,x)\in\mathcal{A}_{n,c}} \left(Q^-\right)^n\left[u^-\right](\theta,x) = r^-.$$

By definitions of $u^\pm$, we have $u^- \leqslant u \leqslant u^+$ and $u^+ \geqslant r^-$. These, together with the monotonicity of Q^- and (3.3) of Proposition 3.2, show that (3.4) holds for Q^-. This completes the proof. $\square$

We now introduce the abstract formulation of unimodality of the nonlinearity Q.

(UM) If $s > r > 0$, then there is a positive integer $N(r,s)$ such that $I_{r,s} > r$ or $S_{r,s} < s$, where $I_{r,s} \triangleq \inf\{Q^{N(r,s)}[u](\theta,x)\colon (\theta,x) \in [-\tau,0] \times \mathcal{H},\ u \in C_{r,s}\}$ and $S_{r,s} \triangleq \sup\{Q^{N(r,s)}[u](\theta,x)\colon (\theta,x) \in [-\tau,0] \times \mathcal{H},\ u \in C_{r,s}\}$.

Theorem 3.4. *Suppose that $Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define the spreading speed c_-^* with Q^- similarly as c^* in Section 2. For any $c < c_-^*$ and any $\sigma \in \mathrm{Int}(Y_+)$ there exists a positive number $r_\sigma > 0$ with the property that if $u_0, v_0 \in C_+$ with $v_0 \geqslant u_0$ and $u_0(\cdot,x) \geqslant \sigma$ on an interval of length $2r_\sigma$ then*

$$\liminf_{n\to\infty} \min\bigl\{Q^n[\varphi](\theta,x)\colon (\theta,x) \in \mathcal{A}_{n,c} \text{ and } u_0 \leqslant \varphi \leqslant v_0\bigr\} \geqslant r^-. \tag{3.5}$$

Moreover, suppose that $Q[\varphi] \leqslant Q^+[\varphi]$ for all $\varphi \in C_+$. Define the spreading speed c_+^* with Q^+ similarly as c^* in Section 2. If Q satisfies (UM), *then*

$$\lim_{n\to\infty} \max\bigl\{\bigl|Q^n[\varphi](\theta,x) - r^*\bigr|\colon (\theta,x) \in \mathcal{A}_{n,c} \text{ and } u_0 \leqslant \varphi \leqslant v_0\bigr\} = 0, \tag{3.6}$$

where $u_0, v_0 \in C_+$, $v_0 \geqslant u_0$ and $u_0(\cdot,x) \geqslant \sigma$ on an interval of length $2r_\sigma$.

Proof. By Proposition 3.3, for any $c < c_-^*$ and any $\sigma \in \mathrm{Int}(Y_+)$, there exists a positive number $r_\sigma > 0$ such that $\lim_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c}} (Q^-)^n[u_0](\theta,x) = r^-$, where $u_0(\cdot,x) \geqslant \sigma$ on an interval of length $2r_\sigma$. Then (3.5) follows immediately from Proposition 2.3 in [15] and the fact that $v_0 \geqslant \varphi \geqslant u_0$ and $Q \geqslant Q^-$ in C_+.

We now prove (3.6). Since $c < c_-^* \leqslant c_+^*$, by a similar argument as that in the proof of (3.5), we have $\limsup_{n\to\infty} \max\{Q^n[\varphi](\theta,x)\colon (\theta,x) \in \mathcal{A}_{n,c}$ and $u_0 \leqslant \varphi \leqslant v_0\} \leqslant r^+$.

Take a positive number $c_0 \in (c, c_-^*)$ and let $\varepsilon_0 = c_0 - c$.

For any $\varepsilon \geqslant 0$, define

$$V_-(\varepsilon) = \liminf_{n\to\infty} \min\bigl\{Q^n[\varphi](\theta,x)\colon (\theta,x) \in \mathcal{A}_{n,c+\varepsilon} \text{ and } u_0 \leqslant \varphi \leqslant v_0\bigr\}$$

and

$$V_+(\varepsilon) = \limsup_{n\to\infty} \max\bigl\{Q^n[\varphi](\theta,x)\colon (\theta,x) \in \mathcal{A}_{n,c+\varepsilon} \text{ and } u_0 \leqslant \varphi \leqslant v_0\bigr\}.$$

Clearly, $V_\pm(\varepsilon) \in [r^-, r^+]$ for any $\varepsilon \in [0,\varepsilon_0]$, $V_-(\varepsilon)$ is non-increasing and $V_+(\varepsilon)$ is non-decreasing in $\varepsilon \in [0,\varepsilon_0]$. Due to monotonicity of $V_\pm$, we easily see that $V_-(\varepsilon)$ and $V_+(\varepsilon)$ are continuous in $\varepsilon \in [0,\varepsilon_0]$ except for a countable subset of $[0,\varepsilon_0]$.

If $V_-(\varepsilon) = V_+(\varepsilon)$ for some $\varepsilon \in [0,\varepsilon_0]$, then by definitions of $V_\pm(\varepsilon)$ and the continuity of Q, we have $Q^n[u_0] \to V_+(\varepsilon)$ as $n \to \infty$. Thus, $Q[V_+(\varepsilon)] = V_+(\varepsilon) \triangleq r^{**}$. We claim that $r^{**} = r^*$; otherwise, $r^{**} \neq r^*$. Let $r = \min\{r^*, r^{**}\}$ and $s = \max\{r^*, r^{**}\}$. Then $s > r$. By (A4)(a) and choices of r, s, we have $Q[r] = r$ and $Q[s] = s$. In particular, $r, s \in Q[C_{r,s}]$. But, the assumption (UM) implies $r \notin Q[C_{r,s}]$ or $s \notin Q[C_{r,s}]$, a contradiction. This shows $V_-(0) = V_+(0) = r^*$ and thus the conclusion follows.

Now, suppose that $V_-(\varepsilon) < V_+(\varepsilon)$ for any $\varepsilon \in [0,\varepsilon_0]$. In view of the continuity of $V_\pm$, we may obtain that for some $\varepsilon \in (0,\varepsilon_0)$, V_-, V_+ are continuous at ε. By (UM), we may without loss of generality assume that there is a positive integer $N(\varepsilon)$ such that $I_{V_-(\varepsilon),V_+(\varepsilon)} > V_-(\varepsilon)$, where

$$I_{V_-(\varepsilon),V_+(\varepsilon)} \triangleq \inf\bigl\{Q^{N(\varepsilon)}[u](\theta,x)\colon (\theta,x) \in [-\tau,0] \times \mathcal{H},\ u \in C_{V_-(\varepsilon),V_+(\varepsilon)}\bigr\}.$$

According to the definition of $V_-(\cdot)$, for any $\sigma \in (0,\varepsilon)$, there exist sequences $n_k \to \infty$ as $k \to \infty$, $(\theta_k, x_k) \in [-\tau,0] \times ([-n_k c - n_k\sigma, n_k c + n_k\sigma] \cap \mathcal{H})$ and φ_k with $u_0 \leqslant \varphi_k \leqslant v_0$ such that $\lim_{k\to\infty} Q^{n_k}[\varphi_k](\theta_k, x_k) = V_-(\sigma)$. By $\sigma < \varepsilon$, we know that for any bounded subset $\mathcal{B}$ of $\mathcal{H}$, $x_k + \mathcal{B} \subseteq (n_k - N(\varepsilon))[-c-\varepsilon, c+\varepsilon]$ for all large k, which implies $\liminf_{k\to\infty} \min\{Q^{n_k - N(\varepsilon)}[\varphi_k](\theta, x_k + y)\colon$

$(\theta, y) \in [-\tau, 0] \times \mathcal{B}\} \in [V_-(\varepsilon), V_+(\varepsilon)]$ and $\limsup_{k\to\infty} \max\{Q^{n_k-N(\varepsilon)}[\varphi_k](\theta, x_k + y)\colon (\theta, y) \in [-\tau, 0] \times \mathcal{B}\} \in [V_-(\varepsilon), V_+(\varepsilon)]$. These, combined with (A1) and (A2), imply

$$\begin{aligned} V_-(\sigma) &= \lim_{k\to\infty} Q^{n_k}[\varphi_k](\theta_k, x_k) \\ &= \lim_{k\to\infty} Q^{N(\varepsilon)}\bigl[Q^{n_k-N(\varepsilon)}[\varphi_k](\cdot, \cdot + x_k)\bigr](\theta_k, 0) \\ &\geqslant I_{V_-(\varepsilon), V_+(\varepsilon)} \\ &> V_-(\varepsilon). \end{aligned}$$

By the continuity of V_- at ε and letting $\sigma \to \varepsilon$, we have $V_-(\varepsilon) \geqslant I_{V_-(\varepsilon), V_+(\varepsilon)} > V_-(\varepsilon)$, a contradiction. This completes the proof. □

By applying Theorem 3.4 with $u_0 = v_0$, we easily obtain the following result.

Theorem 3.5. *Suppose that $Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define the spreading speed c_-^* with Q^- similarly as c^* in Section 2. For any $c < c_-^*$ and any $\sigma \in \operatorname{Int}(Y_+)$ there exists a positive number $r_\sigma > 0$ with the property that if $u_0(\cdot, x) \geqslant \sigma$ on an interval of length $2r_\sigma$ then*

$$\liminf_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c}} Q[u_0](\theta, x) \geqslant r^-.$$

Moreover, suppose that $Q[\varphi] \leqslant Q^+[\varphi]$ for all $\varphi \in C_+$. Define the spreading speed c_+^* with Q^+ similarly as c^* in Section 2. If Q satisfies (UM)*, then*

$$\lim_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c}} u_n(\theta, x) = \lim_{n\to\infty} \max_{(\theta,x)\in\mathcal{A}_{n,c}} u_n(\theta, x) = r^*, \tag{3.7}$$

where $u_0(\cdot, x) \geqslant \sigma$ on an interval of length $2r_\sigma$.

Again, by Theorem 3.4, we shall get the following result which will play a key role in the next section.

Corollary 3.6. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. Assume that $0 < c < c_-^*$ and there is a sequence $\{\varphi_k\}$ in C_{r^+} such that $\lim_{k\to\infty} \varphi_k = \varphi_0$ and $\lim_{n\to\infty} \max\{|Q^n[\varphi_0](\theta, x) - r^*|\colon (\theta, x) \in \mathcal{A}_{n,c}\} = 0$. If Q satisfies* (UM)*, then there is $K_0 \in \mathbb{Z}_+$ such that $\lim_{n\to\infty} \max\{|Q^n[\varphi_k](\theta, x) - r^*|\colon (\theta, x) \in \mathcal{A}_{n,c}$ and $k \in \mathbb{Z} \cap [K_0, \infty)\} = 0$.*

Proof. Take $c_0 \in (c, c_-^*)$. By (3.6) of Theorem 3.4, there exists a positive number $r_0 \triangleq r_{\frac{r^*}{3}} > 0$ with the property that if $u_0 \in C_{r^+}$ with $u_0(\cdot, x) \geqslant \frac{r^*}{3}$ on an interval of length $2r_0$ then $\lim_{n\to\infty} \max\{|Q^n[\varphi](\theta, x) - r^*|\colon (\theta, x) \in \mathcal{A}_{n,c_0}$ and $u_0 \leqslant \varphi \leqslant r^+\} = 0$. Since $\lim_{k\to\infty} \varphi_k = \varphi_0$ and $\lim_{n\to\infty} \max\{|Q^n[\varphi_0](\theta, x) - r^*|\colon (\theta, x) \in \mathcal{A}_{n,c}\} = 0$, there exists an integer $K_0 > 0$ such that $K_0 c > r_0$ and $Q^{K_0}[\varphi_k](\theta, x) > \frac{r^*}{3}$ for all $(\theta, x) \in [-\tau, 0] \times ([-K_0 c, K_0 c] \cap \mathcal{H})$ and $k \in \mathbb{Z} \cap [K_0, \infty)$. This shows that, for any $k \in \mathbb{Z} \cap [K_0, \infty)$, $Q^{K_0}[\varphi_k](\cdot, x) \geqslant \frac{r^*}{3}$ on an interval of length $2r_0$. Thus $\lim_{n\to\infty} \max\{|Q^n[Q^{K_0}[\varphi_k]](\theta, x) - r^*|\colon (\theta, x) \in \mathcal{A}_{n,c_0}$ and $k \in \mathbb{Z} \cap [K_0, \infty)\} = 0$. In view of $c < c_0$, we easily see that $\lim_{n\to\infty} \max\{|Q^n[\varphi_k](\theta, x) - r^*|\colon (\theta, x) \in \mathcal{A}_{n,c}$ and $k \in \mathbb{Z} \cap [K_0, \infty)\} = 0$. This completes the proof. □

In the following, we assume that $\mathcal{M}: C \to C$ is a linear operator such that $\mathcal{M}(C_+ \setminus \{0\}) \subseteq C_+ \setminus \{0\}$ and $\mathcal{M}(Y) \subseteq Y$. Let $\mathbf{M} \equiv \mathcal{M}|_Y$. Then $\mathbf{M}$ is a positive linear operator, that is, $\mathbf{M}(Y_+) \subseteq Y_+$. We make the following hypotheses on $\mathcal{M}$ and $\mathbf{M}$.

(AL1) $\mathcal{M}$ satisfies (A1), (A2) and (A3).
(AL2) (a) There is a positive integer n_0 such that $\mathbf{M}^{n_0}$ is compact and strongly positive on Y, that is, $\mathbf{M}^{n_0}(Y_+ \setminus \{0\}) \subseteq \mathrm{Int}(Y_+)$ where $\mathbf{M}^{n_0} = \underbrace{\mathbf{M} \circ \cdots \circ \mathbf{M}}_{n_0}$; and (b) the spectral radius λ_0 of $\mathbf{M}$ is larger than 1.
(AL3) For $\varphi \in C_+ \setminus \{0\}$, there is a positive integer N_φ such that $\mathcal{M}^{N_\varphi}[\varphi] \in C_+^o$.

If (AL2) holds, then by Lemma 3.1 in [15], the spectral radius λ_0 of M is the simple eigenvalue of M such that there is a strongly positive eigenvector associated with λ_0 and the modulus of any other eigenvalue is less than λ_0. Let ζ_0 be a strongly positive eigenvector of M associated with the simple eigenvalue λ_0.

For any positive number μ and for any $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$, define $\mathcal{M}_\mu^\varepsilon : C \to C$ by $\mathcal{M}_\mu^\varepsilon[\varphi](\theta, x) = \min\{\mu\zeta_0(\theta), (1-\varepsilon)\mathcal{M}[\varphi](\theta, x)\}$ for all $\varphi \in C$. If the linear operator $\mathcal{M}$ satisfies hypotheses (AL1) and (AL2), then by Lemma 3.3 in [15], $\mathcal{M}_\mu^\varepsilon$ has exactly two fixed points 0 and $\mu\zeta_0$ due to $(1-\varepsilon)\lambda_0 > 1$. Obviously, all the conclusions of Theorem 2.11 and Corollary 3.4 in [15] for $\mathcal{M}_\mu^\varepsilon|_{C_{\mu\zeta_0}}$ hold. Thus $\mathcal{M}_\mu^\varepsilon$ has a spreading speed $c^*_{\mu,\varepsilon}$ which is just the spreading speed with $\mathcal{M}_\mu^\varepsilon$ defined similarly as c^* with Q in Section 2 and is independent of the choice of $\mu > 0$, which follows from the following lemma.

Lemma 3.7. *Assume that* (AL1)–(AL3) *hold. If $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$ and $\mu_1 > \mu_2 > 0$, then $c^*_{\mu_1,\varepsilon} = c^*_{\mu_2,\varepsilon} > 0$.*

Proof. Clearly, by the definition of $\mathcal{M}_\mu^\varepsilon$, we have $\mathcal{M}_{\mu_1}^\varepsilon \geqslant \mathcal{M}_{\mu_2}^\varepsilon$ for all $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$ and $\mu_1 > \mu_2 > 0$. This and Lemma 2.9 in [15] imply that for given $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$, $c^*_{\mu,\varepsilon}$ is non-decreasing in $\mu \in (0, \infty)$. By Proposition 2.1 and the fact that $(1-\varepsilon)\mathcal{M}[\zeta_0] = (1-\varepsilon)\lambda_0\zeta_0 \gg \zeta_0$ for all $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$, we know that $c^*_{\mu,\varepsilon} > 0$ for all $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$ and $\mu > 0$.

By way of contradiction, we may assume that there exist $\varepsilon \in [0, 1 - \frac{1}{\lambda_0})$ and $\mu_1 > \mu_2 > 0$ such that $c^*_{\mu_1,\varepsilon} > c^*_{\mu_2,\varepsilon} > 0$. Take two rational numbers $c, \tilde{c} \in (c^*_{\mu_2,\varepsilon}, c^*_{\mu_1,\varepsilon})$ with $c < \tilde{c}$. Applying Corollary 3.4 in [15] to $\mathcal{M}_{\mu_1}^\varepsilon$, we find $r > 0$ such that

$$\lim_{n\to\infty} \min_{|x|\leqslant n\tilde{c}} \left(\mathcal{M}_{\mu_1}^\varepsilon\right)^n[\varphi](\cdot, x) = \lim_{n\to\infty} \max_{|x|\leqslant n\tilde{c}} \left(\mathcal{M}_{\mu_1}^\varepsilon\right)^n[\varphi](\cdot, x) = \mu_1\zeta_0,$$

where $\varphi \in C_{\mu_1\zeta_0}$ with $\varphi(\cdot, x) \geqslant \frac{\mu_1\zeta_0}{3}$ on an interval of length $2r$.

Choose $\psi \in C_{\frac{\mu_1\zeta_0}{3}}$ such that $\psi(\cdot, x) = \frac{\mu_1\zeta_0}{3}$ for all $x \in [-r, r] \cap \mathcal{H}$ and $\psi(\cdot, x) = 0$ for all $x \notin [-1-r, 1+r]$. Then

$$\lim_{n\to\infty} \min_{|x|\leqslant n\tilde{c}} \left(\mathcal{M}_{\mu_1}^\varepsilon\right)^n[\psi](\cdot, x) = \lim_{n\to\infty} \max_{|x|\leqslant n\tilde{c}} \left(\mathcal{M}_{\mu_1}^\varepsilon\right)^n[\psi](\cdot, x) = \mu_1\zeta_0.$$

Thus, by $c < \tilde{c}$, there exists a positive integer k_0 such that $T_y[(\mathcal{M}_{\mu_1}^\varepsilon)^{k_0}[\psi]] \geqslant \psi$ for all $y \in [-k_0c, k_0c] \cap \mathcal{H}$.

Let $\mathcal{M}^\varepsilon = (1-\varepsilon)\mathcal{M}$, $\mu_2^* = \frac{\mu_2^*}{(1-\varepsilon)\lambda_0}$ and $\varepsilon^* = \frac{\mu_2 \inf\{\zeta_0(\theta):\ \theta \in [-\tau, 0]\}}{\delta}$, and $\psi^* = \varepsilon^*\psi$, where $\delta = \max\{(\mathcal{M}^\varepsilon)^k[\psi](\theta, x):\ (\theta, x, k) \in [-\tau, 0] \times \mathcal{H} \times ([0, k_0] \cap \mathbb{Z})\}$. Obviously, due to $\mathcal{M}^\varepsilon \geqslant \mathcal{M}_{\mu_1}^\varepsilon$, we have $T_y[(\mathcal{M}^\varepsilon)^{k_0}[\psi]] \geqslant \psi$ for all $y \in [-k_0c, k_0c] \cap \mathcal{H}$. By the choice of ψ^*, we easily verify that $T_y[(\mathcal{M}^\varepsilon)^{k_0}[\psi^*]] \geqslant \psi^*$ for all $y \in [-k_0c, k_0c] \cap \mathcal{H}$ and $(\mathcal{M}^\varepsilon)^k[\psi^*] \in C_{\mu_2^*\zeta_0}$ for all $k \in [0, k_0] \cap \mathbb{Z}$. In view of $\mathcal{M}_{\mu_2}^\varepsilon = \mathcal{M}_{\mu_1}^\varepsilon = \mathcal{M}^\varepsilon$ in $C_{\mu_2^*\varepsilon}$, we have

$$T_y\left[\left(\mathcal{M}_{\mu_2}^\varepsilon\right)^{k_0}\left[\psi^*\right]\right] \geqslant \psi^* \quad \text{for all } y \in [-k_0c, k_0c] \cap \mathcal{H}. \tag{3.8}$$

By (A1) and the monotonicity of $\mathcal{M}^{\varepsilon}_{\mu_2}$, we apply (3.8) repeatedly to obtain that, for any positive integer l,

$$T_y\big[\big(\mathcal{M}^{\varepsilon}_{\mu_2}\big)^{lk_0}\big[\psi^*\big]\big] \geqslant \psi^* \quad \text{for all } y \in [-lk_0c, lk_0c] \cap \mathcal{H}.$$

In particular, there exists a sequence $\{l_j\}$ of $\mathbb{Z}_+$ such that $\lim_{j\to\infty} l_j = \infty$, $l_jk_0c \in \mathcal{H}$ and $\liminf_{j\to\infty}(\mathcal{M}^{\varepsilon}_{\mu_2})^{l_jk_0}[\psi^*](0, l_jk_0c) \geqslant \psi^*(0,0) = \frac{\varepsilon^*\mu_1\zeta_0(0)}{3} > 0$. But, applying Theorem 2.11 in [15] to $\mathcal{M}^{\varepsilon}_{\mu_2}$, we have $\lim_{n\to\infty}\max_{|x|\geqslant nc}(\mathcal{M}^{\varepsilon}_{\mu_2})^n[\psi^*](\cdot,x) = 0$ since $c > c^*_{\mu_2,\varepsilon}$ and ψ^* has compact support with $\psi^* \in C_{\mu_2\zeta_0}$. It follows that $\lim_{j\to\infty}(\mathcal{M}^{\varepsilon}_{\mu_2})^{l_jk_0}[\psi^*](0, l_jk_0c) = 0$, a contradiction. Therefore, $c^*_{\mu_1,\varepsilon} = c^*_{\mu_2,\varepsilon}$. This completes the proof. $\square$

For convenience, we call $c^*(\varepsilon) \triangleq c^*_{\mu,\varepsilon}$ the spreading speed of $(1-\varepsilon)\mathcal{M}$. Clearly, Lemma 2.9 in [15] tells us that $c^*(\varepsilon)$ is non-increasing in $\varepsilon \in [0, 1-\frac{1}{\lambda_0})$.

In the remaining of this section and also in the next section, we always assume that the linear operator $\mathcal{M}$ satisfies hypotheses (AL1)–(AL3).

We can now state the global coupling condition.

(GC) (a) $Q^+[u] \leqslant \mathcal{M}[u]$ for all $u \in C_+$; and (b) for any $\varepsilon \in (0,1)$ there is a number $\delta \in (0, r^-)$ such that $Q^-[u] \geqslant (1-\varepsilon)\mathcal{M}[u]$ for all $u \in C_\delta$.

Lemma 3.8. *If* (GC)(b) *holds, then $c^*_- > 0$, where the spreading speed c^*_- with Q^- is defined similarly as c^* in Section 2.*

Proof. Let $\varepsilon = \frac{\lambda_0-1}{2\lambda_0}$ and $Q_L = (1-\varepsilon)\mathcal{M}$. It follows that $Q_L[\zeta_0] = (1-\varepsilon)\lambda_0\zeta_0 \gg \zeta_0$. Then, by (GC)(b) and Proposition 2.1, we have $c^*_- > 0$. This completes the proof. $\square$

Theorem 3.9. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c^*_\pm$ with $Q^\pm$ similarly as c^* in Section 2. If the global coupling condition* (GC) *holds, then the following statements are true.*

(i) $0 < c^*_- = c^*_+ = c^*(0) \triangleq c^*$.
(ii) *If $u_0 \in C$ has compact support with $u_0 \ll r^+$, then* (3.1) *holds for any $c > c^*$.*
(iii) *If* (UM) *and* (AL3) *hold, then for any $c < c^*$ and $u_0 \in C_+ \setminus \{0\}$ we have* (3.7).

Proof. We first prove (i). Firstly, by (GC)(b), for any $\varepsilon \in (0, 1-\frac{1}{\lambda_0})$, there is a number $\delta \in (0, r^-)$ such that $Q^- \geqslant (1-\varepsilon)\mathcal{M}$ in C_δ. It follows that $\mathcal{M}^{\varepsilon}_{\delta} \leqslant Q^- \leqslant Q^+ \leqslant \mathcal{M}$ in C_+. This, combined with Lemma 2.9 in [15] and Lemma 3.8, implies that $0 < c^*(\varepsilon) \leqslant c^*_- \leqslant c^*_+ \leqslant c^*(0)$ for all $\varepsilon \in (0, 1-\frac{1}{\lambda_0})$.

By way of contradiction, we may assume that $c^*_- < c^*(0)$. Then, for any $\varepsilon \in (0, 1-\frac{1}{\lambda_0})$, $c^*(\varepsilon) \leqslant c^*(0^+) \leqslant c^*_- < c^*(0)$, where $c(0^+) = \lim_{\varepsilon\to 0} c(\varepsilon)$. Take two rational numbers $c_1, c_2 \in (c^*(0^+), c^*(0))$ with $c_1 < c_2$. Applying Theorem 2.15 in [15] to $\mathcal{M}^0_1$, we find $r > 0$ such that

$$\lim_{n\to\infty} \min_{|x|\leqslant nc_2} \big(\mathcal{M}^0_1\big)^n[\varphi](\cdot,x) = \lim_{n\to\infty} \max_{|x|\leqslant nc_2} \big(\mathcal{M}^0_1\big)^n[\varphi](\cdot,x) = \zeta_0,$$

where $\varphi \in C_{\zeta_0}$ with $\varphi(\cdot,x) \geqslant \frac{\zeta_0}{3}$ on an interval of length $2r$.

Choose $\psi \in C_{\zeta_0}$ such that $\psi(\cdot,x) = \frac{\zeta_0}{3}$ for all $x \in [-r,r] \cap \mathcal{H}$ and $\psi(\cdot,x) = 0$ for all $x \notin [-1-r, 1+r]$. Then $\lim_{n\to\infty}\min_{|x|\leqslant nc_2}(\mathcal{M}^0_1)^n[\psi](\cdot,x) = \lim_{n\to\infty}\max_{|x|\leqslant nc_2}(\mathcal{M}^0_1)^n[\psi](\cdot,x) = \zeta_0$. Since $c_1 < c_2$ and ψ has compact support, there exists a positive integer k_0 such that $T_y[(\mathcal{M}^0_1)^{k_0}[\psi]] \geqslant 2\psi$ and $(\mathcal{M}^0_1)^{k_0}[\psi](\cdot,y) \geqslant \frac{2\zeta_0}{3}$ for all $y \in [-k_0c_1, k_0c_1] \cap \mathcal{H}$. Take $\varepsilon \in (0, 1-\frac{1}{\lambda_0})$ such that $(1-\varepsilon)^{k_0} > \frac{1}{2}$. Note that $\mathcal{M}_1^{\varepsilon}[\varphi] \geqslant (1-\varepsilon)\mathcal{M}_1^0[\varphi]$ for all $\varphi \in C_+$ due to definitions of $\mathcal{M}_1^{\varepsilon}$ and $\mathcal{M}_1^0$. These, combined with the monotonicity of $\mathcal{M}_1^{\varepsilon}$ and $\mathcal{M}_1^0$, imply that

$$T_y\big[\big(\mathcal{M}_1^{\varepsilon}\big)^{k_0}[\psi]\big] \geqslant \psi \quad \text{and} \quad \big(\mathcal{M}_1^{\varepsilon}\big)^{k_0}[\psi](\cdot, y) \geqslant \frac{\zeta_0}{3} \quad \text{for all } y \in [-k_0c_1, k_0c_1] \cap \mathcal{H}. \tag{3.9}$$

By (A1) and the monotonicity of $\mathcal{M}_1^{\varepsilon}$, we apply (3.9) repeatedly to obtain that, for any positive integer l,

$$T_y\big[\big(\mathcal{M}_1^{\varepsilon}\big)^{lk_0}[\psi]\big] \geqslant \psi \quad \text{and} \quad \big(\mathcal{M}_1^{\varepsilon}\big)^{lk_0}[\psi](\cdot, y) \geqslant \frac{\zeta_0}{3} \quad \text{for all } y \in [-lk_0c_1, lk_0c_1] \cap \mathcal{H}.$$

In particular, there exists a sequence $\{l_j\}$ of $\mathbb{Z}_+$ such that $\lim_{j\to\infty} l_j = \infty$, $l_jk_0c_1 \in \mathcal{H}$ and

$$\liminf_{j\to\infty}\big(\mathcal{M}_1^{\varepsilon}\big)^{l_jk_0}[\psi](0, l_jk_0c_1) \geqslant \frac{\zeta_0(0)}{3} > 0.$$

But, applying Theorem 2.11 in [15] to $\mathcal{M}_1^{\varepsilon}$, we have $\lim_{n\to\infty}\max_{|x|\geqslant nc_1}(\mathcal{M}_1^{\varepsilon})^n[\psi](\cdot, x) = 0$ due to $c_1 > c^*(\varepsilon)$ and the choice of ψ. It follows that $\lim_{j\to\infty}(\mathcal{M}_1^{\varepsilon})^{l_jk_0}[\psi](0, l_jk_0c_1) = 0$, a contradiction. Therefore, $c_-^* = c_+^* = c^*(0)$. This proves (i).

(ii) follows from statement (i) and Theorem 3.1.

We finally prove (iii). Suppose that (UM) holds. By applying similar arguments as those in the proof of Theorem 3.5 in [15] and (3.6), we find that for any $c < c^*$ there exists a positive number r_0 such that (3.7) holds whenever $u_0(\cdot, x) \gg 0$ on an interval of length $2r_0$.

Take $\delta \in (0, c^* - c)$. By statement (ii), there exists $r_0 > 0$ such that if $\varphi(\cdot, x) > 0$ on an interval of length $2r_0$ then

$$\lim_{n\to\infty} \max\big\{\big|Q^n[\varphi](\theta, x) - r^*\big|: (\theta, x) \in \mathcal{A}_{n,c+\delta}\big\} = 0.$$

On the other hand, assumptions (AL3) and (GC)(b) imply that $Q^{n_1}[u_0](\cdot, x) > 0$ on an interval of length $2r_0$ for some positive integer n_1. Hence,

$$\lim_{n\to\infty} \max\big\{\big|Q^{n+n_1}[u_0](\theta, x) - r^*\big|: (\theta, x) \in \mathcal{A}_{n,c+\delta}\big\} = 0.$$

Since $|x| \leqslant nc$ implies $|x| \leqslant (n - n_1)(c + \delta)$ for all large n, we have

$$\lim_{n\to\infty} \max\big\{\big|Q^n[u_0](\theta, x) - r^*\big|: (\theta, x) \in \mathcal{A}_{n,c}\big\} = 0.$$

So, statement (iii) holds. This completes the proof. $\square$

If (GC) in Theorem 3.9 is replaced by (GC)(b) only, then by using an argument similar to the proof of Theorem 3.9, we can still obtain the following results.

Theorem 3.10. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. If the global coupling condition* (GC)(b) *holds, then the following statements are true.*

(i) $0 < c_-^* \leqslant c_+^*$.
(ii) *If $u_0 \in C$ has compact support with $u_0 \ll r^+$, then* (3.1) *holds for any $c > c_+^*$.*
(iii) *If* (UM) *holds, then for any $c < c_-^*$ and $u_0 \in C_+ \setminus \{0\}$ we have* (3.7).

For convenience, we call c^* in Theorem 3.9 the spreading speed of Q.

Finally, we extend our results on spreading speeds and asymptotic behavior to a continuous-time semiflow on C_+. In this paper, the map $Q:\mathbb{R}_+\times C_+\to C_+$ is said to be a continuous-time semiflow on C_+ if for any given number $r>0$, $Q|_{\mathbb{R}_+\times C_r}:\mathbb{R}_+\times C_r\to C_+$ is continuous such that $Q_0=\mathrm{Id}|_{C_+}$ and $Q_t\circ Q_s=Q_{t+s}$ for all $t,s\in\mathbb{R}_+$, where $Q_t\triangleq Q(t,\cdot)$ for all $t\in\mathbb{R}_+$.

Theorem 3.11. *Let r^* be a given positive number and $\{Q_t\}_{t=0}^{\infty}$ be a continuous-time semiflow on C_+ with $Q_t[0]=0$ and $Q_t[r^*]=r^*$ for all $t\geqslant 0$. Suppose that Q_t satisfies* (A1) *for any $t\geqslant 0$, and there exists $\varrho>0$ such that Q_ϱ satisfies* (A3) *and* (GC), *and $Q^+[\varphi]\geqslant Q_\varrho[\varphi]\geqslant Q^-[\varphi]$ for all $\varphi\in C_+$. Assume that $\{Q_t[v]: t\in\mathbb{R}_+\}$ is uniformly bounded in C for any $v\in C_+$. Let c_ϱ^* be the spreading speed of Q_ϱ as in Theorem 3.9*(i) *and let $c^*=\frac{c_\varrho^*}{\varrho}$. Then $c^*>0$ and the following statements are true.*

(i) *For any $c>c^*$, $\lim_{t\to\infty}\max\{Q_t[v](\theta,x):\ \theta\in[-\tau,0]$ and $x\notin(-tc,tc)\}=0$, where $v\in C_{r^*}$ has compact support and $v\ll r^*$.*
(ii) *If Q_ϱ satisfies* (UM), *then $\lim_{t\to\infty}\max\{\|Q_t[v](\cdot,x)-r^*\|:\ x\in[-tc,tc]\cap\mathcal{H}\}=0$ for any $c<c^*$ and $v\in C_+\setminus\{0\}$.*

Proof. Obviously, Lemma 3.8 gives $c_\varrho^*>0$, and thus $c^*>0$.

We only prove statement (ii) since the other statement can be proved similarly. By applying Theorem 3.9(iii) to Q_ϱ, for any $c<c^*$, there exists a positive number r_0 such that

$$\lim_{n\to\infty}\max\left\{\left\|Q_{n\varrho}[v](\cdot,x)-r^*\right\|:\ x\in\left[-\frac{n\varrho(c+c^*)}{2},\frac{n\varrho(c+c^*)}{2}\right]\cap\mathcal{H}\right\}=0,\tag{3.10}$$

where $v\in C_+\setminus\{0\}$.

Let $v\in C_+\setminus\{0\}$. Note that there is a positive number $r^{**}\geqslant r^*$ such that $Q_t[v]\in C_{r^{**}}$ for all $t\in\mathbb{R}_+$. By the continuity of Q, we know that $Q|_{C_{r^{**}}\times[0,\varrho]}$ is continuous at r^* uniformly for $t\in[0,\varrho]$. In particular, for any $\varepsilon>0$, there exist $\delta=\delta(\varepsilon)>0$ and $d=d(\varepsilon)>0$ such that if $u\in C_{r^{**}}$ with $\|u(\cdot,x)-r^*\|<\delta$ for all $x\in[-d,d]\cap\mathcal{H}$ then $\|Q_t[u](\cdot,0)-r^*\|<\varepsilon$ for all $t\in[0,\varrho]$. Thus it follows from (3.10) that there is an integer $n_1>0$ such that $\|Q_{n\varrho}[v](\cdot,x)-r^*\|<\delta$ for all $x\in[-\frac{n\varrho(c+c^*)}{2},\frac{n\varrho(c+c^*)}{2}]\cap\mathcal{H}$ when $n>n_1$. Let $n_2=\max\{n_1,\frac{2(d+\varrho c)}{\varrho(c^*-c)}\}$. Then for any $n>n_2$ and $y\in[-(1+n)\varrho c,(1+n)\varrho c]\cap\mathcal{H}$, we have $T_{-y}[Q_{n\varrho}[v]]\in C_{r^{**}}$ with $\|T_{-y}[Q_{n\varrho}[v]](\cdot,x)-r^*\|<\delta$ for all $x\in[-d,d]\cap\mathcal{H}$. Therefore, by the fact that Q_t satisfies (A1) for any $t\geqslant 0$, we obtain that, for any $n>n_2$ and $y\in[-(1+n)\varrho c,(1+n)\varrho c]\cap\mathcal{H}$,

$$\left\|Q_{t+n\varrho}[v](\cdot,y)-r^*\right\|=\left\|Q_t\left[T_{-y}\left[Q_{n\varrho}[v]\right]\right](\cdot,0)-r^*\right\|<\varepsilon$$

for all $t\in[0,\varrho]$. In particular, $\|Q_t[v](\cdot,y)-r^*\|<\varepsilon$ for all $t>\varrho+n_2\varrho$ and $y\in[-tc,tc]\cap\mathcal{H}$. This completes the proof. □

4. Travelling waves

In this section, we show that under appropriate assumptions, the spreading speed for a nonmonotone semiflow may coincide with the minimal wave speed of its monostable travel wave fronts.

For any real number c, we define $D_c:=\{x-mc:\ x\in\mathcal{H},\ m\in\mathbb{Z}\}$. Note that when $\mathcal{H}=\mathbb{Z}$, D_c is discrete if and only if c is a rational number. If c is a rational number, then we easily see that there exists a positive integer l_c such that $D_c=\bigcup_{l=0}^{l_c-1}(\frac{l}{l_c}+\mathcal{H})$, where $\frac{l}{l_c}+\mathcal{H}=\{\frac{l}{l_c}+x:\ x\in\mathcal{H}\}$.

Let $Y_{c,\sigma}=BC([-\sigma,0]\times D_c,\mathbb{R})$ be the normed vector space of all bounded and continuous functions from $[-\sigma,0]\times D_c$ into $\mathbb{R}$ with the norm

$$\|a\|_{Y_{c,\sigma}}\triangleq\sum_{n\geqslant 1}2^{-n}\sup\left\{\left|a(\theta,s)\right|:\ (\theta,s)\in[-\sigma,0]\times D_c\text{ with }|s|\leqslant n\right\}$$

for all $a \in Y_{c,\sigma}$, and let $Y^+_{c,\sigma} = BC([-\sigma, 0] \times D_c, \mathbb{R}_+)$ and $Y^o_{c,\sigma} = BC([-\sigma, 0] \times D_c, (0, \infty))$. It follows that $Y^+_{c,\sigma}$ is a closed cone in the normed vector space $Y_{c,\sigma}$.

For any $y \in D_c$ and $a \in Y_{c,\tau}$, let us define $T_y[a](\theta, x) = a(\theta, x - y)$ for all $(\theta, x) \in [-\tau, 0] \times D_c$. $A \subseteq Y_{c,\tau}$ is said to be T-invariant in $Y_{c,\tau}$ if $T_y[A] = A$ for all $y \in D_c$. Clearly, $\bigcup_{y \in D_c} T_y[A]$ is T-invariant in $Y_{c,\tau}$ where $A \subseteq Y_{c,\tau}$; if A is T-invariant in $Y_{c,\tau}$, then A_d is T-invariant in C for all $d \in D_c$, where $A_d = \{a(\cdot, \cdot + d) \triangleq a(\cdot, \cdot + d)|_{[-\tau,0]\times\mathcal{H}}\colon a \in A\}$. For convenience, let us denote $Y_{c,\tau}$ and $Y^+_{c,\tau}$ by Y_c and Y^+_c, respectively.

We say that W is a travelling wave of the map Q with the wave speed c if $W : [-\tau, 0] \times D_c \to \mathbb{R}$ is bounded and nonconstant, $W(\cdot, \cdot + d)|_{[-\tau,0]\times\mathcal{H}} \in C_+$ for all $d \in D_c$ and $Q[W(\cdot, \cdot - nc)](\theta, x) = W(\theta, x - (1 + n)c)$ for all $n \in \mathbb{Z}$. We say that $W(\theta, x - nc)$ connects r^* to 0 if $W(\cdot, -\infty) = r^*$ and $W(\cdot, \infty) = 0$.

The following result gives the travelling waves of Q by applying some results of Liang and Zhao [15]. We should point out that other methods have also been used to get the travelling waves of some special evolution equations in [4–7,11,20,22,24,29,30,36].

Given a checking function $\varphi^* \in C([-\tau, 0] \times \mathbb{R}, [0, r^-])$ with the property (P) in Section 2, where r^* is replaced with r^-. For any $k \in (0, 1)$ and any real number c, we define the operators $R_{c,k} : Y^+_c \to Y^+_c$ and $R^{\pm}_{c,k} : Y^+_c \to Y^+_c$ respectively by

$$R_{c,k}[a](\theta, s) = \max\bigl\{k\varphi^*(\theta, s),\, Q\bigl[a(\cdot, \cdot + s + c)\bigr](\theta, 0)\bigr\}$$

and

$$R^{\pm}_{c,k}[a](\theta, s) = \max\bigl\{k\varphi^*(\theta, s),\, Q^{\pm}\bigl[a(\cdot, \cdot + s + c)\bigr](\theta, 0)\bigr\},$$

where $a \in Y^+_c$ and $(\theta, s) \in [-\tau, 0] \times D_c$.

In order to obtain the existence of a travelling wave with the wave speed c, following Liang and Zhao [15], we strengthen hypothesis (A3) as follows.

(A3*) For any given number $r > 0$, one of the following two properties holds.

(a) $Q[C_r]$ is precompact in C.

(b) The set $Q[C_r](0, \cdot)$ is precompact in X. Moreover, there is a positive number $\varsigma \leqslant \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for all $\theta \in [-\tau, -\varsigma)$ and the operator

$$S[u](\theta, x) := \begin{cases} u(0, x) & \text{if } -\tau \leqslant \theta < -\varsigma, \\ Q[u](\theta, x) & \text{if } -\varsigma \leqslant \theta \leqslant 0 \end{cases}$$

has the property that $S[D]$ is precompact in C for any T-invariant set $D \subset C_r$ with $D(0, \cdot)$ being a precompact subset of X.

In the remaining of this section, we always assume that Q and $Q^{\pm}$ all satisfy assumption (A3*).

Lemma 4.1. *The following statements are true.*

(i) *If $r \in (0, \infty)$, $n \in (1 + \frac{\tau}{\varsigma}, \infty) \cap \mathbb{Z}$ and $A \subseteq C_+$ such that $Q[A] \subseteq A$, then $Q^n[A]$ is precompact in C.*

(ii) *Let $\{W_\alpha\}_{\alpha \in \mathcal{J}}$ be a family of travelling waves of Q with the wave speed c such that $\sup\{W_\alpha(\theta, s)\colon (\theta, s) \in [-\tau, 0] \times D_c$ and $\alpha \in \mathcal{J}\} < \infty$. Then $\{W_\alpha(\cdot, \cdot + s)|_{[-\tau,0]\times\mathcal{H}}\colon s \in D_c$ and $\alpha \in \mathcal{J}\}$ is precompact in C, and particularly $\{W_\alpha(\cdot, s)|_{[-\tau,0]}\colon s \in D_c$ and $\alpha \in \mathcal{J}\}$ is precompact in Y.*

Proof. (i) follows from assumption (A3*).

(ii) Let $r^{**} = \sup\{W_\alpha(\theta, s)\colon (\theta, s) \in [-\tau, 0] \times D_c$ and $\alpha \in \mathcal{J}\}$ and take a positive integer $n^* > 1 + \frac{\tau}{\varsigma}$. It follows from (A1) and the definition of the travelling wave that $W_\alpha(\cdot, \cdot + s)|_{[-\tau,0]\times\mathcal{H}} = Q[W_\alpha(\cdot, \cdot + s + c)|_{[-\tau,0]\times\mathcal{H}}] \in C_{r^{**}}$ for all $(\alpha, s) \in \mathcal{J} \times D_c$. Let $A = \{W_\alpha(\cdot, \cdot + s)|_{[-\tau,0]\times\mathcal{H}}\colon s \in D_c$ and $\alpha \in \mathcal{J}\}$. Then

$Q[A] \subseteq A \subseteq C_{r^{**}}$. This and (i) produce that $\{W_\alpha(\cdot,\cdot+s)|_{[-\tau,0]\times\mathcal{H}}\colon s \in D_c \text{ and } \alpha \in \mathcal{J}\}$ is precompact in C. This completes the proof. $\square$

Given $c \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ with $\alpha > 0$, let $Y_c^\alpha = \{\phi \in Y_c^+\colon \phi \leqslant \alpha\}$ and $\Phi_c = \{k\varphi^*(\cdot,\cdot+d)|_{[-\tau,0]\times D_c}\colon d \in D_c,\ k \in (0,1)\}$. Then by Dini's Theorem and the choice of φ^*, Φ_c is T-invariant and precompact in Y_c.

Define $f_c : Y_c^+ \times Y_c^+ \to Y_c^+$, $g_c : Y_c^+ \to Y_c^+$, $h_c : Y_c^+ \times Y_c^+ \to Y_c^+$ by $f_c[\phi,a](\theta,s) = \max\{\phi(\theta,s), a(\theta,s)\}$ for all $\phi, a \in Y_c^+$ and $(\theta,s) \in [-\tau,0]\times D_c$, $g_c[a](\theta,s) = Q[a(\cdot,\cdot+s+c)](\theta,0)$ for all $a \in Y_c^+$ and $(\theta,s) \in [-\tau,0]\times D_c$, $h_c[\phi,a] = f_c[\phi, g_c[a]]$ for all $\phi, a \in Y_c^+$, respectively. Clearly, $f_c|_{Y_c^r\times Y_c^r}$, $g_c|_{Y_c^r}$ and $h_c|_{Y_c^r\times Y_c^r}$ are all continuous, where r is a given positive number.

Let w be a function from $[-\tau,0]\times D_c$ to $\mathbb{R}$. We denote $\liminf_{s\to\pm\infty}\min\{w(\theta,s)\colon \theta \in [-\tau,0]\}$ and $\limsup_{s\to\pm\infty}\max\{w(\theta,s)\colon \theta\in[-\tau,0]\}$ by $\liminf_{s\to\pm\infty} w(\cdot,s)$ and $\limsup_{s\to\pm\infty} w(\cdot,s)$, respectively.

Lemma 4.2. *Let $\alpha \geqslant r^+$, c be a given rational number and $A \subseteq Y_c^\alpha$. Moreover, assume that $g_c[Y_c^\alpha] \subseteq Y_c^\alpha$. Then the following statements are true.*

(i) *There exists a positive integer l_c such that $D_c = \bigcup_{l=0}^{l_c-1}(\frac{l}{l_c} + \mathcal{H})$.*
(ii) *For any $\sigma \in [0,\tau]$, $A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$ if and only if $A(\cdot,\cdot+\frac{l}{l_c})|_{[-\sigma,0]\times\mathcal{H}}$ is precompact in $Y_{0,\sigma}$ for all integers $l \in [0, l_c-1]$.*
(iii) *$h_c[\Phi_c \times Y_c^\alpha] \subseteq Y_c^\alpha$, in particular, $h_c[\Phi_c\times A] \subseteq Y_c^\alpha$.*
(iv) *If A is T-invariant in Y_c, then $\overline{\text{co}}\,A$, $g_c[A]$ and $h_c[\Phi_c\times A]$ are T-invariant in Y_c, where $\overline{\text{co}}\,A$ represents the convex closure of the subset A in Y_c.*
(v) *If $A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$, then $\overline{\text{co}}\,A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$ and $h_c[\Phi_c\times A](\cdot-\varsigma,\cdot)|_{[-\min\{\sigma,\tau-\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\sigma,\tau-\varsigma\}}$, where $\sigma\in[0,\tau]$.*
(vi) *$h_c[\Phi_c\times A](0,\cdot)$ and $\bigcup_{y\in D_c} T_y[h_c[\Phi_c\times A]](0,\cdot)$ are precompact in $Y_{c,0}$.*
(vii) *If A is T-invariant in Y_c and $A(0,\cdot)$ is precompact in $Y_{c,0}$, then $h_c[\Phi_c\times A]|_{[-\varsigma,0]\times D_c}$ is precompact in $Y_{c,\varsigma}$.*
(viii) *If $\sigma\in(0,\tau)$, $A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$, and $\bigcup_{y\in D_c}T_y[A](0,\cdot)$ is precompact in $Y_{c,0}$, then $h_c[\Phi_c\times A]|_{[-\min\{\tau,\varsigma+\sigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,\varsigma+\sigma\}}$.*
(ix) *$(p_c)^n[A]$ and $(q_c)^n[A]$ are precompact in Y_c for any integer $n > 1+\frac{\tau}{\varsigma}$, where $p_c[A] = h_c[\Phi_c\times A]$ and $q_c[A] = \overline{\text{co}}(h_c[\Phi_c\times A])$.*
(x) *For any integer $n > 1+\frac{\tau}{\varsigma}$, $\bigcup_{k\in(0,1)}(R_{c,k})^n[A] \subseteq (p_c)^n[A]$ is precompact in Y_c. In particular, $(R_{c,k})^n[A]$ is precompact in Y_c for any integer $n > 1+\frac{\tau}{\varsigma}$ and $k\in(0,1)$.*
(xi) *For $k\in(0,1)$, $(r_{c,k})^n[A]$ is precompact in Y_c for any integer $n > 1+\frac{\tau}{\varsigma}$ where $r_{c,k}[A] = \overline{\text{co}}(R_{c,k}[A])$.*

Proof. Statement (i) follows from the definition of D_c and the fact that c is a given rational number.

Statement (ii) follows easily from statement (i) and definitions of $Y_{c,\sigma}$ and $Y_{0,\sigma}$.

Statement (iii) follows from the definition of h_c and the fact that $g_c[Y_c^\alpha]\subseteq Y_c^\alpha$.

Statement (iv) follows from definitions of h_c and the T-invariant properties of A and Φ_c.

In the following, let $A_d = \{a(\cdot,\cdot+d) \triangleq a(\cdot,\cdot+d)|_{[-\tau,0]\times\mathcal{H}}\colon a\in A\}$, where $d\in D_c$. Then $A_d\subseteq C_\alpha$ for $d\in D_c$, and $Q[C_\alpha]\subseteq C_\alpha$ follows from $g_c[Y_c^\alpha]\subseteq Y_c^\alpha$ and the definition of g_c.

To finish the proof, we first assume that (A3*)(a) holds for $r=\alpha$. Hence $Q[C_\alpha]$ is precompact in C, and in particular $Q[A_d]$ is precompact in C for all integers $d\in D_c$. This implies that $g_c[A](\cdot,\cdot+\frac{l}{l_c})|_{[-\tau,0]\times\mathcal{H}}$ is precompact in $Y_{0,\tau}\equiv C$ for all integers $l\in[0,l_c-1]$. Then, by statement (ii), we know that $g_c[A]$ is precompact in Y_c. This, together with the definition of h_c and the compactness of Φ_c, implies that $h_c[\Phi_c\times A]$ is precompact in Y_c. Therefore, statements (v)–(xi) hold.

We next assume that (A3*)(b) holds for $r=\alpha$. Then $Q[C_\alpha](0,\cdot)$ is precompact in X and hence $Q[A_d](0,\cdot)$ is precompact in X for all $d\in D_c$. We shall prove the other statements one by one.

For (v), it is easy to observe that $\overline{\text{co}}\,A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$. To finish the proof of (v), we only prove that $g_c[A](\cdot,\cdot-\varsigma)|_{[-\min\{\sigma,\tau-\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\sigma,\tau-\varsigma\}}$. Indeed, by the representation of Q in assumption (A3*)(b), we have $g_c[A](\cdot,\cdot-\varsigma)|_{[-\min\{\sigma,\tau-\varsigma\},0]\times D_c} = A|_{[-\min\{\sigma,\tau-\varsigma\},0]}$. It follows that $g_c[A](\cdot,\cdot-\varsigma)|_{[-\min\{\sigma,\tau-\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\sigma,\tau-\varsigma\}}$.

For (vi), it suffices to prove that $\bigcup_{y\in D_c} T_y[g_c[A]](0,\cdot)$ is precompact in $Y_{c,0}$ due to the definition of h_c and the compactness of Φ_c. Note that $Q[C_\alpha](0,\cdot)$ is precompact in X. This and the definition of g_c imply that $g_c[Y_c^\alpha](0,\cdot+\frac{l}{l_c})|_{\mathcal{H}}$ is precompact in $X\equiv Y_{0,0}$ for all integers $l\in[0,l_c-1]$, which combined with statements (i) and (ii) yields that $g_c[Y_c^\alpha](0,\cdot)$ is precompact in $Y_{c,0}$. With the help of statement (iv), we know that $g_c[Y_c^\alpha]$ is T-invariant in Y_c. This, together with $\bigcup_{y\in D_c} T_y[g_c[A]]\subseteq g_c[Y_c^\alpha]$, yields that $\bigcup_{y\in D_c} T_y[g_c[A]](0,\cdot)$ is precompact in $Y_{c,0}$.

For (vii), assume that A is T-invariant and $A(0,\cdot)$ is precompact in $Y_{c,0}$. By (A3*)(b), we obtain that $Q[A_{\frac{l}{l_c}+c}]|_{[-\varsigma,0]\times\mathcal{H}}$ is precompact in $Y_{0,\varsigma}$ for any integer $l\in[0,l_c-1]$. This, together with statement (ii) and the definition of g_c, shows that $g_c[A]$ is precompact in $Y_{c,\varsigma}$. Then statement (vii) follows immediately from the definition of h_c and the compactness of Φ_c.

For (viii), suppose that $\sigma\in(0,\tau)$, $A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$, and $\bigcup_{y\in D_c} T_y[A](0,\cdot)$ is precompact in $Y_{c,0}$. Clearly, $h_c[\Phi_c\times A]|_{[-\varsigma,0]\times D_c}$ is precompact in $Y_{0,\varsigma}$ due to statement (vii) and $A\subseteq\bigcup_{y\in D_c} T_y[A]$. This, combined with statement (v) and the fact that $A|_{[-\sigma,0]\times D_c}$ is precompact in $Y_{c,\sigma}$, implies statement (viii).

For (ix), by the representation of Q in assumption (A3*)(b) and statements (iv) and (v), it suffices to show that $(p_c)^n[A]$ is precompact in Y_c for any integer $n>1+\frac{\tau}{\varsigma}$. This follows from the claim that $(p_c)^n[A]|_{[-\min\{\tau,(n-1)\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,(n-1)\varsigma\}}$ for all integers $n\geqslant 1$. We prove this claim by mathematical induction. Obviously, the claim with $n=1$ follows from statement (vi). Moreover, $\bigcup_{y\in D_c} T_y[(p_c)^j[A]](0,\cdot)$ is precompact in $Y_{c,0}$ for all positive integers j. Suppose that $(p_c)^j[A]|_{[-\min\{\tau,(j-1)\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,(j-1)\varsigma\}}$. First, assume that $\min\{\tau,(j-1)\varsigma\}=\tau$. Then $(p_c)^j[A]$ is precompact in Y_c. It follows from the continuity of h_c that $(p_c)^{j+1}[A]$ is precompact in Y_c, that is, $(p_c)^{j+1}[A]|_{[-\min\{\tau,j\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,j\varsigma\}}$. Next, assume that $\sigma\triangleq\min\{\tau,(j-1)\varsigma\}<\tau$. Since $\bigcup_{y\in D_c} T_y[(p_c)^j[A]](0,\cdot)$ is precompact in $Y_{c,0}$ and $(p_c)^j[A]|_{[-\min\{\tau,(j-1)\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,(j-1)\varsigma\}}$, statement (viii) implies that $(p_c)^{j+1}[A]|_{[-\min\{\tau,j\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,j\varsigma\}}$. In both cases, we have shown that $(p_c)^{j+1}[A]|_{[-\min\{\tau,j\varsigma\},0]\times D_c}$ is precompact in $Y_{c,\min\{\tau,j\varsigma\}}$. By the principle of mathematical induction, the claim holds.

Finally, by definitions of $R_{c,k}$, p_c, q_c, and $r_{c,k}$, we easily see that $(R_{c,k})^j[A]\subseteq(p_c)^j[A]$ and $(r_{c,k})^j[A]\subseteq(q_c)^j[A]$ for all $k\in(0,1)$ and $j\in\mathbb{Z}_+$. This and statement (ix) produce statements (x) and (xi). This completes the proof. □

Lemma 4.3. *Suppose that $Q^+[\varphi]\geqslant Q[\varphi]\geqslant Q^-[\varphi]$ for all $\varphi\in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. Then, for any $k\in(0,1)$ and any real number c, $R_{c,k}^-\leqslant R_{c,k}\leqslant R_{c,k}^+$, $R_{c,k}|_{Y_c^r}$ and $R_{c,k}^\pm|_{Y_c^r}$ are continuous with given number $r>0$, and $R_{c,k}^\pm$ are order-preserving with the order induced by Y_c^+. Moreover, if $c\geqslant c_+^*$ is a rational number, then the following statements are true.*

- (i) *For any $k\in(0,1)$, there are $a_{c,k}^+, a_{c,k}^-\in Y_c^+$ with the following properties.*
  - (a) *$a_{c,k}^+\geqslant a_{c,k}^-$, $a_{c,k}^\pm(\cdot,\infty)=0$ and $a_{c,k}^\pm(\cdot,-\infty)=r^\pm$.*
  - (b) *$a_{c,k}^-(\cdot,s)$, $a_{c,k}^+(\cdot,s)$ are non-increasing in $s\in D_c$.*
  - (c) *$a_{c,k}^\pm=\lim_{n\to\infty}(R_{c,k}^\pm)^n[k\varphi]$ and hence $R_{c,k}[a_{c,k}^\pm]=a_{c,k}^\pm$.*
- (ii) *For any $k\in(0,1)$, there is $a_{c,k}\in C([-\tau,0]\times D_c,[0,r^+])\subseteq Y_c^+$ such that $R_{c,k}[a_{c,k}]=a_{c,k}$ and $a_{c,k}^-\leqslant a_{c,k}\leqslant a_{c,k}^+$ with the order induced by Y_c^+. Hence $a_{c,k}(\cdot,\infty)=0$ and $r^+\geqslant\limsup_{s\to-\infty}a_{c,k}(\cdot,s)\geqslant\liminf_{s\to-\infty}a_{c,k}(\cdot,s)\geqslant r^-$.*

Proof. By definitions of $R_{c,k}$ and $R_{c,k}^\pm$ and the monotonicity and continuity of Q and $Q^\pm$, we can easily check that, for any $k\in(0,1)$ and any real number c, $R_{c,k}^-\leqslant R_{c,k}\leqslant R_{c,k}^+$, $R_{c,k}|_{Y_c^r}$ and $R_{c,k}^\pm|_{Y_c^r}$ are continuous with given number $r>0$, and $R_{c,k}^\pm$ are order-preserving with the order induced by Y_c^+.

First we prove statement (i). Since c is a rational number, it follows that $D_c=\mathbb{R}$ when $\mathcal{H}=\mathbb{R}$ or D_c is discrete when $\mathcal{H}=\mathbb{Z}$. Then similar arguments as those in the proof of Theorem 4.2 in [15] give statement (i).

Next we prove (ii). Let $A = \{a \in Y_c^+ \colon a_{c,k}^- \leqslant a \leqslant a_{c,k}^+\}$ and $R \equiv R_{c,k}$. By the above discussions, we easily see that A is a convex and closed subset of Y_c such that $R(A) \subseteq A$. By Lemma 4.2(x), $R^n[A]$ is precompact in Y_c for any integer $n > 1 + \frac{\tau}{\varsigma}$.

Let $B = \bigcap_{n \geqslant 0} Cl(R^n[A])$. Then by the compactness of R, we know that B is the global attractor of $R|_A$ in the sense of Hale [10]. Let

$$\Pi = \left\{ K \subseteq A \colon \begin{array}{l} K \text{ is a convex and closed subset of } Y_c \\ \text{such that } R[K] \subseteq K \text{ and } B \subseteq K \end{array} \right\}.$$

Obviously, $A \in \Pi$. Let $\{K_\alpha\}_{\alpha \in I}$ be a totally ordered subset of Π under the usual set order. Then $K \triangleq \bigcap_{\alpha \in I} K_\alpha$ is a convex and closed subset of Y_c such that $B = R[B] \subseteq R[K] \subseteq K$, that is, $K \in \Pi$. By Zorn's lemma, there is a minimal element K^* of Π. Let $K^{**} = \overline{\mathrm{co}}(R[K^*])$. We can easily check that $B \subseteq R[K^*] \subseteq K^{**}$ and $R[K^{**}] \subseteq K^{**} \subseteq K^*$, that is, $K^{**} \in \Pi$. Thus, by the choice of K^*, we have $K^* = K^{**} = \overline{\mathrm{co}}(R[K^*])$.

We claim that K^* is compact. By Lemma 4.2(xi) and the fact that $K^* = \overline{\mathrm{co}}(R[K^*])$, we obtain that K^* is precompact in Y_c. Hence K^* is compact as K^* is closed. Note that $R|_{K^*} \colon K^* \to K^*$ is continuous and compact, where K^* is a nonempty, compact and convex subset of Y_c. Applying the Schauder fixed point theorem, we have the existence of $a_{c,k}$. Of course, $a_{c,k}^- \leqslant a_{c,k} \leqslant a_{c,k}^+$, $a_{c,k}(\cdot, \infty) = 0$, and $r^+ \geqslant \limsup_{s \to -\infty} a_{c,k}(\cdot, s) \geqslant \liminf_{s \to -\infty} a_{c,k}(\cdot, s) \geqslant r^-$. This completes the proof. □

Lemma 4.4. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. Assume that for given $c \geqslant c_+^*$ there exists a family of travelling waves $\{W_\alpha\}_{\alpha \in \mathcal{J}}$ of Q with the wave speed c such that $\sup\{W_\alpha(\theta, s) \colon (\theta, s) \in [-\tau, 0] \times D_c$ and $\alpha \in \mathcal{J}\} \leqslant r^+$, where $\mathcal{J}$ is a given index set. If there exist real numbers $\sigma > 0$ and $s_0 \in D_c$ such that $W_\alpha(\cdot, s) \geqslant \sigma$ for all $s \in D_c \cap (-\infty, s_0]$ and $\alpha \in \mathcal{J}$ and if* (GC)(b) *holds and Q satisfies* (UM), *then $\lim_{s \to -\infty} W_\alpha(\cdot, s) = r^*$ uniformly for all $\alpha \in \mathcal{J}$.*

Proof. Clearly, by (GC)(b) and Lemma 3.8, we have $c_+^* \geqslant c_-^* > 0$. Choose $c_0 \in (0, c_-^*)$. By (3.6) of Theorem 3.4, there exists $r_\sigma > 0$ such that if $u_0, v_0 \in C_+$ with $v_0 \geqslant u_0$ and $u_0(\cdot, x) \geqslant \sigma$ for all x on an interval of length $2r_\sigma$ then $\lim_{n \to \infty} \max\{|Q^n[\varphi](\theta, x) - r^*| \colon (\theta, x) \in \mathcal{A}_{n,c_0}$ and $u_0 \leqslant \varphi \leqslant v_0\} = 0$.

Suppose that the conclusion does not hold. Then there are a real number $W^* \in (0, r^+] \setminus \{r^*\}$ and sequences $\{\alpha_k\}_{k \in \mathbb{Z}_+} \subset \mathcal{J}$ and $\{(\theta_k, b_k)\}_{k \in \mathbb{Z}_+} \subset [-\tau, 0] \times D_c$ such that $\lim_{k \to \infty} b_k = -\infty$ and $\lim_{k \to \infty} W_{\alpha_k}(\theta_k, b_k) = W^*$. Without loss of generality, we may assume that there exist sequences $\{c_k\}_{k \in \mathbb{Z}_+} \subset [0, c) \cap D_c$ and $\{n_k\}_{k \in \mathbb{Z}_+} \subset \mathbb{Z}_+$ such that $b_k = c_k - n_k c$ and $n_k \to \infty$ as $k \to \infty$. Take $v_0 = r^+$ and $u_0 \in C_\sigma$ such that $u_0(\cdot, x) = \sigma$ for all $x \in [s_0 - c - 2r_\sigma - 1, s_0 - c - 1]$ and $u_0(\cdot, x) = 0$ for all $x \notin [s_0 - c - 2r_\sigma - 2, s_0 - c]$. Then $u_0 \leqslant W_{\alpha_k}(\cdot, \cdot + c_k)|_{[-\tau,0] \times \mathcal{H}} \leqslant v_0$ for all $k \in \mathbb{Z}_+$. This, combined with the above discussions, gives $\lim_{n \to \infty} \max\{|Q^n[W_{\alpha_k}(\cdot, \cdot + c_k)](\cdot, 0) - r^*| \colon k \in \mathbb{Z}_+\} = 0$. Thus there is a positive integer $L_1 > 0$ such that $|Q^n[W_{\alpha_k}(\cdot, \cdot + c_k)](\cdot, 0) - r^*| < \frac{|r^* - W^*|}{3}$ for all $k \in \mathbb{Z}_+$ and $n > L_1$. This, combined with (A1) and $c_k \in D_c$, gives $|W_{\alpha_k}(\theta_k, b_k) - r^*| = |W_{\alpha_k}(\theta_k, -n_k c + c_k) - r^*| = |Q^{n_k}[W_{\alpha_k}(\cdot, \cdot + c_k)](\theta_k, 0) - r^*| < \frac{|r^* - W^*|}{3}$ for all large $k \in \mathbb{Z}_+$. It follows that $|W^* - r^*| \leqslant \frac{|r^* - W^*|}{3}$, a contradiction. This completes the proof. □

Lemma 4.5. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. Assume that, for given $c \geqslant c_+^*$, there exists a travelling wave W of the map Q with the wave speed c and $\liminf_{s \to -\infty} W(\cdot, s) > 0$. If* (GC)(b) *holds and Q satisfies* (UM), *then $W(\cdot, -\infty) = r^*$ and $\liminf_{s \to \infty} W(\cdot, s) = 0$. Moreover, if c is a rational number, then $W(\cdot, \infty) = 0$.*

Proof. It follows from (GC)(b) and Lemma 3.8 that $c_+^* \geqslant c_-^* > 0$. Since $\liminf_{s \to -\infty} W(\cdot, s) > 0$, there exist $\sigma > 0$ and $s_0 \in D_c$ such that $W(\cdot, s) \geqslant \sigma$ for all $s \in D_c \cap (-\infty, s_0]$. This, together with Lemma 4.4, gives $W(\cdot, -\infty) = r^*$.

Now we claim that there exists $(\theta^*, p^*) \in [-\tau, 0] \times D_c$ such that $W(\theta^*, p^*) < \frac{r^-}{3}$. Otherwise, $W \geqslant \frac{r^-}{3}$. Let $a = \inf\{W(\theta, s) \colon (\theta, s) \in [-\tau, 0] \times D_c\}$ and $b = \sup\{W(\theta, s) \colon (\theta, s) \in [-\tau, 0] \times D_c\}$. Then $\frac{r^-}{3} \leqslant a < b \leqslant r^+$. By (UM), without loss of generality, we may assume that there is a positive integer $N(a, b)$

such that $I_{a,b} > a$, where $I_{a,b} \triangleq \inf\{Q^{N(a,b)}[\varphi](\theta,x)\colon \varphi \in C_{a,b}$ and $(\theta,x) \in [-\tau,0]\times\mathcal{H}\}$. Therefore, for any $m \in \mathbb{Z}$, $\inf\{Q^{N(a,b)}[W(\cdot,\cdot-mc)](\theta,x)\colon (\theta,x)\in[-\tau,0]\times\mathcal{H}\} \geqslant I_{a,b} > a$. This, together with the fact that $Q^{N(a,b)}[W(\cdot,\cdot-mc)](\theta,x) = W(\theta,x-(N(a,b)+m)c)$ for all $m\in\mathbb{Z}$, shows that $W(\theta,x-mc) \geqslant I_{a,b} > a$ for all $m\in\mathbb{Z}$ and $(\theta,x)\in[-\tau,0]\times\mathcal{H}$. By the definition of D_c, we have $W(\cdot,s) \geqslant I_{a,b} > a$ for all $s \in D_c$. This contradicts with the choice of a and hence the claim is proved.

Next we claim that $\liminf_{s\to\infty} W(\cdot,s) = 0$. Otherwise, there exist $\varepsilon \in (0,r^-)$ and $s_1 > 0$ such that $W(s) > \varepsilon$ for all $s \in D_c \cap [s_1,\infty)$. Take a positive number $c_1 < c_-^*$. By Proposition 3.3, there exists $r_\varepsilon > 0$ such that if $\varphi(\cdot,x) \geqslant \varepsilon$ for all x on an interval of length $2r_\varepsilon$ then $\lim_{n\to\infty} \min_{(\theta,x)\in\mathcal{A}_{n,c_1}} (Q^-)^n[\varphi](\theta,x) = r^-$. Choose $\psi \in C_\varepsilon$ such that $\psi(\cdot,x) = \varepsilon$ for all $x \in [-r_\varepsilon, r_\varepsilon]$ and $\psi(\cdot,x) = 0$ for all $x \notin [-(1+r_\varepsilon), 1+r_\varepsilon]$. Then, by choices of r_ε and ψ, there exists $n_1 > 0$ such that $(Q^-)^n[\psi](\theta,x) > \frac{r^-}{3}$ for all $(\theta,x)\in\mathcal{A}_{n,c_1}$ and $n \geqslant n_1$. Take a positive integer $n_2 > \max\{n_1, \frac{1+s_1+r_\varepsilon-p^*}{c}\}$. Clearly, $W(\cdot,\cdot+p^*+n_2c)|_{[-\tau,0]\times\mathcal{H}} \geqslant \psi$. By (A1) and Proposition 2.3 in [15], we have $W(\cdot,\cdot-nc+p^*+n_2c)|_{[-\tau,0]\times\mathcal{H}} = Q^n[W(\cdot,\cdot+p^*+n_2c)] \geqslant (Q^-)^n[\psi]$ for all $n \in \mathbb{Z}\cap[n_2,\infty)$. It follows that $\frac{r^-}{3} < Q^{n_2}[W(\cdot,\cdot+p^*+n_2c)](\theta^*,0) = W(\theta^*,-n_2c+p^*+n_2c) = W(\theta^*,p^*) < \frac{r^-}{3}$, a contradiction.

Suppose that c is a rational number. We claim that $\lim_{s\to\infty} W(\cdot,s) = 0$. Otherwise, by Lemma 4.1(ii), there are $\psi^* \in C_+ \setminus \{0\}$ and a sequence $\{s_k\}_{k\in\mathbb{Z}_+}$ in D_c such that $\lim_{k\to\infty} s_k = \infty$ and $\lim_{k\to\infty} W(\cdot,\cdot+s_k)|_{[-\tau,0]\times\mathcal{H}} = \psi^*$. Take $c_1 \in (0,c_-^*)$. By Theorem 3.10(iii), we have $\lim_{n\to\infty} \max\{|Q^n[\psi^*](\theta,x) - r^*|\colon (\theta,x)\in\mathcal{A}_{n,c_1}\} = 0$. This, together with Corollary 3.6, implies that for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $|W(\theta,s_k-nc+x) - r^*| = |Q^n[W(\cdot,\cdot+s_k)](\theta,x) - r^*| < \varepsilon$ for all integers n, $k > K_\varepsilon$ and $(\theta,x)\in[-\tau,0]\times([-nc_1,nc_1]\cap\mathcal{H})$. Let $F_\varepsilon = \{s_k - nc + x\colon n,k \in (K_\varepsilon,\infty)\cap\mathbb{Z}$ and $x \in [-nc_1,nc_1]\cap\mathcal{H}\}$. Since $\lim_{k\to\infty} s_k = \infty$ and c is a rational number, it follows from Lemma 4.2(i) that $F_\varepsilon = D_c$. Hence $|W(\theta,s) - r^*| < \varepsilon$ for all $(\theta,x)\in[-\tau,0]\times D_c$. Since ε is arbitrary, we get $W = r^*$, a contradiction. This completes the proof. □

Lemma 4.6. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. Assume that for any rational number $c \geqslant c_+^*$ there exists a travelling wave W_c of the map Q with the wave speed c such that $\sup\{W_c(\theta,s)\colon (\theta,s)\in[-\tau,0]\times D_c\} \leqslant r^+$, $W_c(\cdot,-\infty) = r^*$ and $\liminf_{s\to\infty} W_c(\cdot,s) = 0$. If* (GC)(b) *holds, then for any $c \geqslant c_+^*$ there exists a travelling wave W_c of the map Q with the wave speed c with $\liminf_{s\to-\infty} W_c(\cdot,s) > 0$. Moreover, if Q satisfies* (UM), *then $W_c(\cdot,-\infty) = r^*$ and $\liminf_{s\to\infty} W_c(\cdot,s) = 0$.*

Proof. Without loss of generality, we may assume that $c \geqslant c_+^*$ is an irrational number. Then there is a sequence of rational numbers $\{c_k\}_{k\in\mathbb{Z}_+}$ with $c_k \geqslant c_+^*$ and $c_k \to c$ as $k\to\infty$. By the properties of W_{c_k}, we may assume without loss of generality that $W_{c_k}(\cdot,s) \geqslant \frac{r^*}{3}$ for all $s \in D_{c_k}\cap(-\infty,-1]$ and $W_{c_k}(\theta_k,0) < \frac{r^*}{3}$ for some $\theta_k \in [-\tau,0]$.

By (A3*) and Lemma 4.1(ii), we know that $\{W_{c_k}(\cdot,\cdot-mc_k)|_{[-\tau,0]\times\mathcal{H}}\colon k\in\mathbb{Z}_+$ and $m\in\mathbb{Z}\}$ is precompact in C. Thus, by a usual diagonal argument, there exists a sequence $\{k_l\}_{l\in\mathbb{Z}_+}$ in $\mathbb{Z}_+$ such that $k_l\to\infty$, $\theta_{k_l}\to\theta^*\in[-\tau,0]$ and $W_{c_{k_l}}(\cdot,\cdot-mc_{k_l})|_{[-\tau,0]\times\mathcal{H}} \to U_{m,c} \in C_{r^+}$ as $l\to\infty$, where $m\in\mathbb{Z}$.

Define $W_c\colon[-\tau,0]\times D_c\to\mathbb{R}$ by $W_c(\theta,s) = U_{m,c}(\theta,s+mc)$ for all $s\in\mathcal{H}-mc$. Since $D_c = \bigcup_{m\in\mathbb{Z}}(\mathcal{H}-mc)$ and $(\mathcal{H}-mc)\cap(\mathcal{H}-nc) = \emptyset$ whenever $m\neq n$, W_c is well-defined. Moreover, $W_c(\theta^*,0) \leqslant \frac{r^*}{3}$, $W_c(\cdot,s) \geqslant \frac{r^*}{3}$ for all $s\in D_c\cap(-\infty,-1]$, and $Q^n[W_c(\cdot,\cdot-mc)](\theta,x) = W_c(\theta,x-(n+m)c)$ for all $n\in\mathbb{Z}_+$ and $m\in\mathbb{Z}$. This, together with Lemma 4.5, yields the other conclusions. Therefore, the proof is complete. □

Theorem 4.7. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi\in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. If* (GC)(b) *holds and Q satisfies* (UM), *then the following statements are true.*

(i) *If $c < c_-^*$, then Q has no travelling wave W with $\liminf_{s\to-\infty} W(\cdot,s) > 0$.*
(ii) *If $c \geqslant c_+^*$, then Q has a travelling wave W with $W \leqslant r^+$, $W(\cdot,-\infty) = r^*$ and $\liminf_{s\to\infty} W(\cdot,s) = 0$.*

Proof. To prove (i), by way of contradiction, assume that Q has a travelling wave $W(\theta, x-nc)$ such that $0 < \liminf_{s\to-\infty} W(\cdot, s) \triangleq w^*$. Since $c < c_-^*$, we can choose a real number $c_0 \in (c, c_-^*)$. Then, by Theorem 3.4, $\lim_{n\to\infty} \max\{|Q^n[W(\cdot,\cdot+d)](\theta, x) - r^*|: d \in [-1, 0] \cap D_c$ and $(\theta, x) \in \mathcal{A}_{n,c_0}\} = 0$. Since W is not constant, there exists $(\theta^*, s^*) \in [-\tau, 0] \times D_c$ such that $W(\theta^*, s^*) \neq r^*$. Take $x_n \in \mathcal{H} \cap [s^* + nc, s^* + nc + 1]$ and $d_n = s^* + nc - x_n$ for all $n \in \mathbb{Z}_+$. Then $d_n \in [-1, 0]$ and $|x_n| \leqslant nc_0$ for all large n. From the above discussions, we have $\lim_{n\to\infty} \max\{|Q^n[W(\cdot,\cdot+d_n)](\cdot, x_n) - r^*|: n \in \mathbb{Z}_+\} = 0$. Hence $W(\theta^*, s^*) = \lim_{n\to\infty} W(\theta^*, x_n + d_n - nc) = \lim_{n\to\infty} Q^n[W(\cdot,\cdot+d_n)](\theta^*, x_n) = r^*$, a contradiction. This shows statement (i).

For (ii), by Lemma 4.6, it suffices to consider the case where c is a rational number. In this case, $D_c = \mathbb{R}$ or D_c is discrete. By Lemma 4.3(ii), for any $k \in (0, 1)$, there is $a_{c,k} \in Y_c^{r^+}$ such that $R_{c,k}[a_{c,k}] = a_{c,k}$, $a_{c,k}^- \leqslant a_{c,k} \leqslant a_{c,k}^+$, $a_{c,k}(\cdot, \infty) = 0$ and $\liminf_{s\to-\infty} a_{c,k}(\cdot, s) \geqslant r^-$, where $a_{c,k}^{\pm}$ are defined as in Lemma 4.3(ii).

Let $A = \{a_{c,k}: k \in (0, 1)\}$. Clearly, $A \subseteq (\bigcup_{d\in D_c} T_d[A]) \subseteq (p_c)^n[Y_c^{r^+}]$ for any positive integer n, where p_c is defined in Lemma 4.2(ix). Then it follows from Lemma 4.2(ix) that A and $\bigcup_{d\in D_c} T_d[A]$ are precompact in Y_c.

For any $k \in (0, 1)$, let $p_k = \sup\{p \in D_c: a_{c,k}(\cdot, s) \geqslant \frac{r^-}{3}$ for all $s \in (-\infty, p] \cap D_c\}$. Since $a_{c,k}(\cdot, \infty) = 0$ and $\liminf_{s\to-\infty} a_{c,k}(\cdot, s) \geqslant r^-$, it is easy to see that $p_k \in D_c$, $a_{c,k}(\cdot, s + p_k) \geqslant \frac{r^-}{3}$ for all $s \in D_c \cap (-\infty, 0]$, and there exist $\theta^k \in [-\tau, 0]$ and $p^k \in (p_k, 1 + p_k] \cap D_c$ such that $a_{c,k}(\theta^k, p^k) < \frac{r^-}{3}$. Clearly, $\{a_{c,k}(\cdot, \cdot + p_k) \in Y_c: k \in (0, 1)\}$ is precompact in Y_c according to the compactness of $\bigcup_{d\in D_c} T_d[A]$. Then there is a sequence $\{k_l\}$ in $(0, 1)$ such that $k_l \to 0$, $\theta^{k_l} \to \theta^* \in [-\tau, 0]$, $p^{k_l} - p_{k_l} \to p^* \in [0, 1] \cap D_c$ and $a_{c,k_l}(\cdot, \cdot + p_{k_l}) \to W \in Y_c^{r^+}$ as $l \to \infty$. Therefore $W(\theta^*, p^*) = \lim_{l\to\infty} a_{c,k_l}(\theta^{k_l}, p^{k_l}) \leqslant \frac{r^-}{3}$ and $W(\cdot, s) = \lim_{l\to\infty} a_{c,k_l}(\cdot, s + p_{k_l}) \geqslant \frac{r^-}{3}$ for all $s \in D_c \cap (-\infty, 0]$. Then, by definitions of R_{c,k_l}, we know that $Q[W(\cdot, \cdot - nc)](\cdot, x) = W(\cdot, x - (1 + n)c)$ for all $n \in \mathbb{Z}$ and $x \in \mathcal{H}$. Note that $\frac{r^-}{3} \geqslant W(\theta^*, p^*)$ and $W(\cdot, s) \geqslant \frac{r^-}{3}$ for all $s \in D_c \cap (-\infty, 0]$. It follows from Lemma 4.5 that $W(\cdot, -\infty) = r^*$ and $\liminf_{s\to\infty} W(\cdot, s) = 0$. This completes the proof. □

The following theorem follows from Theorem 3.9(i) and Theorem 4.7.

Theorem 4.8. *Suppose that $Q^+[\varphi] \geqslant Q[\varphi] \geqslant Q^-[\varphi]$ for all $\varphi \in C_+$. Define spreading speeds $c_\pm^*$ with $Q^\pm$ similarly as c^* in Section 2. Assume that* (GC) *holds and Q satisfies* (UM)*. Let c^* be the spreading speed of Q as in Theorem* 3.9(i)*. Then the following statements are true.*

(i) *If $c < c^*$, then Q has no travelling wave W with $\liminf_{s\to-\infty} W(\cdot, s) > 0$.*
(ii) *If $c \geqslant c^*$, then Q has a travelling wave W with $W \leqslant r^+$, $W(\cdot, -\infty) = r^*$ and $\liminf_{s\to\infty} W(\cdot, s) = 0$.*

We also say that c^* in Theorem 4.8 is the minimal wave speed of Q.

In the rest of this section, we consider travelling waves for the continuous-time semiflow $\{Q_t\}_{t=0}^\infty$ on C_+ in the sense of Section 3 (see the paragraph before Theorem 3.11). We say that W is a travelling wave of $\{Q_t\}_{t=0}^\infty$ if $W: [-\tau, 0] \times \mathbb{R} \to \mathbb{R}_+$ is a bounded and nonconstant continuous function and $Q_t[W](\theta, x) = W(\theta, x - tc)$ for all $(\theta, x) \in [-\tau, 0] \times \mathcal{H}$ and $t \in \mathbb{R}_+$; moreover, we say that W connects r^* to 0 if $W(\cdot, -\infty) = r^*$ and $W(\cdot, \infty) = 0$. We emphasize that our method for the existence of travelling waves are quite different from those in [4–7,11,20,22,24,29,30,36].

Theorem 4.9. *Suppose that there are three positive numbers $r^*, r^\pm$ such that for any $t > 0$, there are three maps $Q_t^\pm, \mathcal{M}_t: C_+ \to C_+$ such that $Q \triangleq Q_t$, $Q^\pm \triangleq Q_t^\pm$ and $\mathcal{M} \triangleq \mathcal{M}_t$ satisfy all conditions in Theorem 4.8. Let c^* be the minimal wave speed of Q_1. Then the following statements are true.*

(i) *If $c < c^*$, then $\{Q_t\}_{t\in\mathbb{R}_+}$ has no travelling wave W with $\liminf_{s\to-\infty} W(\cdot, s) > 0$.*
(ii) *If $c \geqslant c^*$, then $\{Q_t\}_{t\in\mathbb{R}_+}$ has a travelling wave W connecting r^* to* 0*.*

Proof. (i) Suppose that $c < c^*$. By way of contradiction, assume that there exists a travelling wave W of $\{Q_t\}_{t\in\mathbb{R}_+}$ with $\liminf_{s\to-\infty} W(\cdot, s) > 0$. Thus $W|_{[-\tau,0]\times D_c}$ is a travelling wave of Q_1 with

$\liminf_{s\to-\infty} W|_{[-\tau,0]\times D_c}(\cdot,s)>0$. But, by applying Theorem 4.8(i) to Q_1, we can get a contradiction. This proves (i).

(ii) Suppose that $c\geqslant c^*$. Given $t\in\mathcal{J}\triangleq\{\frac{1}{c2^k}\colon k\in\mathbb{Z}_+\}$, by applying Theorem 4.8(ii) to Q_t, we know that there exists a travelling wave $W_t:[\tau,0]\times D_{tc}\to[0,r^+]$ of Q_t with the wave speed tc such that $W_t(\cdot,-\infty)=r^*$ and $\liminf_{s\to\infty} W_t(\cdot,s)=0$, where $D_{ct}=\{x-mtc\colon x\in\mathcal{H},\ m\in\mathbb{Z}\}$.

In the following, without loss of generality, we may assume that $W_t(\cdot,s)\geqslant\frac{r^*}{3}$ for all $(t,s)\in\mathcal{J}\times(D_{tc}\cap(-\infty,-1])$ and $W_t(\theta_t,0)<\frac{r^*}{3}$ for some $\theta_t\in[-\tau,0]$.

Let $\mathcal{D}=\{\frac{l}{2^k}\colon k,l\in\mathbb{Z}_+$ with $l\in[0,2^k-1]\}$ and $\mathcal{K}=\bigcup_{d\in\mathcal{D}}(\mathcal{H}-d)$, where $\mathcal{H}-d=\{x-d\colon x\in\mathcal{H}\}$. Given $d\in\mathcal{D}$, there exist k_0 and $l_0\in\mathbb{Z}_+$ with $l_0\in[0,2^{k_0}-1]$ such that $d=\frac{l_0}{2^{k_0}}$. Since Q_t satisfies (A3*) for any $t>0$, we may choose $l\in\mathbb{Z}_+\cap(c\tau-d,\infty)$ such that $Q_{\frac{d+l}{c}}[C_{r^+}]$ is precompact in C. Then, for any $k\in\mathbb{Z}_+\cap[k_0,\infty)$, it follows from (A1) and $\frac{d+l}{c}/\frac{1}{c2^k}=(d+l)2^k\in\mathbb{Z}_+$ that $W_{\frac{1}{c2^k}}(\cdot,\cdot-d)=T_{-l}[W_{\frac{1}{c2^k}}(\cdot,\cdot-(d+l))]=T_{-l}[Q_{\frac{d+l}{c}}[W_{\frac{1}{c2^k}}]]=Q_{\frac{d+l}{c}}[W_{\frac{1}{c2^k}}(\cdot,\cdot+l)]\in Q_{\frac{d+l}{c}}[C_{r^+}]$. We thus obtain that $\{W_{\frac{1}{c2^k}}(\cdot,\cdot-d)|_{[-\tau,0]\times\mathcal{H}}\colon k\in\mathbb{Z}_+\}$ is precompact in C for any $d\in\mathcal{D}$. Since $\mathcal{D}$ is a countable set, a usual diagonal argument shows that there are sequences $\{k_i\}_{i\in\mathbb{Z}_+}$ in $\mathbb{Z}_+$ and $\{U_d\}_{d\in\mathcal{D}}$ in C_{r^+} such that $k_i\to\infty$, $\theta_{\frac{1}{c2^{k_i}}}\to\theta^*$ and $W_{\frac{1}{c2^{k_i}}}(\cdot,\cdot-d)|_{[-\tau,0]\times\mathcal{H}}\to U_d$ as $i\to\infty$. Then by choices of W_t and θ_t, we know that $U_0(\theta^*,0)\leqslant\frac{r^*}{3}$ and $U_0(\cdot,s)\geqslant\frac{r^*}{3}$ for all $s\in\mathcal{H}\cap(-\infty,-1]$.

Define $U:[-\tau,0]\times\mathcal{K}\to\mathbb{R}$ such that $U(\theta,x)=U_d(\theta,x+d)$ for any $x\in\mathcal{H}-d$ with $d\in\mathcal{D}$, which is well-defined since $(\mathcal{H}-d)\cap(\mathcal{H}-\tilde{d})=\emptyset$ for all $d\neq\tilde{d}\in\mathcal{D}$. Clearly, $Q_{\frac{n}{c2^k}}[U](\theta,x)=U(\theta,x-\frac{n}{2^k})$ for all $x\in\mathcal{H}$ and $n,k\in\mathbb{Z}_+$ since $Q_{\frac{n}{c2^k}}[W_{\frac{1}{c2^{k_i}}}](\theta,x)=W_{\frac{1}{c2^{k_i}}}(\theta,x-\frac{n}{2^k})$ for all $k_i\geqslant k$. In other words, $Q_{\frac{s}{c}}[U](\theta,x)=U(\theta,x-s)$ for all $x\in\mathcal{H}$ and $s\in\mathcal{K}\cap[0,\infty)$.

For any $x\in\mathcal{H}$, let $V_x(\theta,s)=Q_{\frac{x-s}{c}}[U](\theta,x)$ for all $(\theta,s)\in[-\tau,0]\times(-\infty,x)$. By the previous discussions, we know that $V_x(\theta,s)=U(\theta,s)$ for all $(\theta,s)\in[-\tau,0]\times(\mathcal{K}\cap(-\infty,x))$. Again, for given $x\in\mathcal{H}$, the definition of V_x implies that $V_x|_{[-\tau,0]\times(-\infty,x)}$ is continuous. Since $\mathcal{K}$ is dense in $\mathbb{R}$, we can easily see that the definition of V_x is independent of $x\in\mathcal{H}$. Let us define $W:[-\tau,0]\times\mathbb{R}\to\mathbb{R}$ such that $W(\theta,s)=V_x(\theta,s)$ for any $(\theta,s)\in[-\tau,0]\times\mathbb{R}$ with $s<x\in\mathcal{H}$. Then W is well-defined. It follows easily that W is continuous and $W(\theta,s)=U(\theta,s)$ for all $(\theta,s)\in[-\tau,0]\times\mathcal{K}$. Thus $W(\theta^*,0)\leqslant\frac{r^*}{3}$, $W(\cdot,s)\geqslant\frac{r^*}{3}$ for all $s\in\mathcal{H}\cap(-\infty,-1]$ and $Q_{\frac{s}{c}}[W](\theta,x)=W(\theta,x-s)$ for all $(\theta,x)\in[-\tau,0]\times\mathcal{H}$ and $s\in\mathcal{K}\cap[0,\infty)$. These, together with the fact that $\mathcal{K}$ is dense in $\mathbb{R}$, yield that W is bounded and non-constant, and $Q_t[W](\theta,x)=W(\theta,x-ct)$ for all $(\theta,x)\in[-\tau,0]\times\mathcal{H}$ and $t\in\mathbb{R}_+$. By Theorem 3.11(ii) and $W>0$, we have $W(\cdot,-\infty)=\lim_{s\to-\infty}W(\cdot,s)=\lim_{s\to-\infty}Q_{\frac{-s}{c}}[W](\cdot,0)=r^*$.

We next prove $W(\cdot,\infty)=0$. Otherwise, by the compactness of Q_t, there exists a sequence $\{s_k\}_{k\in\mathbb{Z}_+}$ in $\mathbb{R}_+$ such that $s_k\to\infty$ and $W(\cdot,\cdot+s_k)|_{[-\tau,0]\times\mathcal{H}}\to\psi^*\in C_+\setminus\{0\}$ as $k\to\infty$.

We claim that there is a positive integer K_0 such that $\lim_{t\to\infty}\max\{\|Q_t[W(\cdot,s_k)]-r^*\|\colon k\in[K_0,\infty)\cap\mathbb{Z}\}=0$. Indeed, take $c_1=\frac{c^*}{2}$. By applying Theorem 3.9(iii) to Q_1, we obtain that $\lim_{n\to-\infty}\max\{\|Q_n[\psi^*](\cdot,x)-r^*\|\colon x\in[-nc_1,nc_1]\}=r^*$. Again by Corollary 3.6, there is a positive integer K_0 such that

$$\lim_{n\to\infty}\sup\bigl\{\bigl|Q_n\bigl[W(\cdot,\cdot+s_k)\bigr](\theta,x)-r^*\bigr|\colon(\theta,x)\in\mathcal{A}_{n,c_1}\text{ and }k\in[K_0,\infty)\cap\mathbb{Z}\bigr\}=0.\tag{4.1}$$

Note that $Q|_{C_{r^+}\times[0,1]}$ is continuous at r^* uniformly for $t\in[0,1]$ due to the continuity of Q. In particular, for any $\varepsilon>0$, there exist $\delta=\delta(\varepsilon)>0$ and $d=d(\varepsilon)>0$ such that if $u\in C_{r^+}$ with $\|u(\cdot,x)-r^*\|<\delta$ for all $x\in[-d,d]\cap\mathcal{H}$ then $\|Q_t[u](\cdot,0)-r^*\|<\varepsilon$ for all $t\in[0,1]$.

On the other hand, it follows from (4.1) that there is an integer $n_1>0$ such that $\|Q_n[W(\cdot,\cdot+s_k)](\cdot,x)-r^*\|<\delta$ for all $x\in[-nc_1,nc_1]\cap\mathcal{H}$, $k\in[K_0,\infty)\cap\mathbb{Z}$ and $n\in(n_1,\infty)\cap\mathbb{Z}$. Let $n_2=\max\{n_1,1+\frac{d}{c_1}\}$. Then $\|Q_n[W(\cdot,\cdot+s_k)](\cdot,x)-r^*\|<\delta$ for all $x\in[-d,d]\cap\mathcal{H}$, $k\in[K_0,\infty)\cap\mathbb{Z}$ and $n\in(n_2,\infty)\cap\mathbb{Z}$. Therefore, by choices of δ and ε, we have $\|Q_{t+n}[W(\cdot,\cdot+s_k)](\cdot,0)-r^*\|<\varepsilon$ for all $t\in[0,1]$, $k\in[K_0,\infty)\cap\mathbb{Z}$ and $n\in(n_2,\infty)\cap\mathbb{Z}$. In other words, $\|Q_t[W(\cdot,\cdot+s_k)](\cdot,0)-r^*\|<\varepsilon$ for all $k\in[K_0,\infty)\cap\mathbb{Z}$ and $t\in(n_2,\infty)$, that is, $\|W(\cdot,s_k-tc)-r^*\|<\varepsilon$ for all $k\in[K_0,\infty)\cap\mathbb{Z}$ and

$t \in (n_2, \infty)$. Let $F_\varepsilon = \{s_k - tc\colon\ k \in [K_0, \infty) \cap \mathbb{Z}$ and $t \in (n_2, \infty)\}$. Then $\|W(\cdot, s) - r^*\| < \varepsilon$ for all $s \in F_\varepsilon$ and $F_\varepsilon = \bigcup_{k=K_0}^{\infty} (-\infty, s_k - n_2 c) = \mathbb{R}$ due to $\lim_{k\to\infty} s_k = \infty$. Since ε is arbitrary, we get $W(\cdot, s) = r^*$ for all $s \in \mathbb{R}$, a contradiction to $W(\theta^*, 0) \leqslant \frac{r^*}{3}$. This completes the proof. $\square$

5. Applications

In this section, we apply the results obtained in Sections 3 and 4 to a delayed local/nonlocal reaction diffusion equation and a delayed nonlocal lattice differential system. To unify our discussions, we shall consider a class of integral equations, which include several local and nonlocal equations as special cases.

For a given interval $I \subseteq \mathbb{R}$, let $I + [-\tau, 0] = \{t + \theta\colon\ t \in I$ and $\theta \in [-\tau, 0]\}$. For $u \in BC((I + [-\tau, 0]) \times \mathcal{H}, \mathbb{R})$ and $t \in I$, we define $u_t(\cdot,\cdot) \in C$ by $u_t(\theta, x) = u(t + \theta, x)$ for all $\theta \in [-\tau, 0]$ and $x \in \mathcal{H}$.

Consider the following integral equation with the given initial function,

$$\begin{cases} u(t,\cdot) = T(t)\big(\varphi(0,\cdot)\big) + \mu \displaystyle\int_0^t T(t-s)\big(K\big[f\big(u(s-\tau,\cdot)\big)\big]\big)\,\mathrm{d}s, & t > 0, \\ u_0 = \varphi \in C_+, \end{cases} \tag{5.1}$$

where $\tau, \mu > 0$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $f(0) = 0$.

Assume that $T(\cdot)(\cdot) : \mathbb{R}_+ \times X \to X$ satisfies the following conditions:

(T1) $T(t) : X \to X$ is a linear operator such that $T|_{\mathbb{R}_+ \times X_r} : \mathbb{R}_+ \times X_r \to X$ is continuous, where $t \in \mathbb{R}_+$, $r \in (0, \infty)$, $X_r \triangleq \{\phi \in X_+\colon\ \phi \leqslant r\}$ and $T|_{\mathbb{R}_+ \times X_r}(t, \phi) = T(t)(\phi)$;
(T2) $T(t)(X_+) \subseteq X_+^o$ for all $t > 0$, and $(T(t)(a))(x) = ae^{-\mu t}$ for all $t \in \mathbb{R}_+$, $a \in \mathbb{R}$ and $x \in \mathcal{H}$;
(T3) $T(0) = \mathrm{Id}_X$, $T(t)T(s)(\phi) = T(t+s)(\phi)$ for all $t, s \in \mathbb{R}_+$ and $\phi \in X$;
(T4) $T(t)[\mathcal{R}[\phi]] = \mathcal{R}[T(t)[\phi]]$ and $T_y[T(t)[\phi]] = T(t)[T_y[\phi]]$ for all $\phi \in X$ and $y \in \mathcal{H}$;
(T5) $T(t)[X_r]$ is precompact in X for all $t, r \in (0, \infty)$.

The operator $K : X \to X$ is a linear operator with the following conditions:

(K1) $K|_{X_r} : X_r \to X$ is continuous, where $r \in (0, \infty)$;
(K2) $K(1) = 1$, $K(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$;
(K3) $K[\mathcal{R}[\phi]] = \mathcal{R}[K[\phi]]$ and $T_y[K[\phi]] = K[T_y[\phi]]$ for all $\phi \in X_+$ and $y \in \mathcal{H}$.

Here $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$.

In the coming concrete applications, it is easy to verify the above conditions on $T(t)$ and K.

By a simple application of the step-by-step argument, it is easy to see that, for $\varphi \in C_+$, the solution of (5.1) is unique and exists on $\mathbb{R}_+$. Such a solution is denoted by $u^\varphi(t, x; f)$. Moreover, it follows from (5.1), (T1)–(T2) and (K1)–(K2) that $(u^\varphi(\cdot,\cdot; f))_t \in C_+$ for all $(t, \varphi) \in \mathbb{R}_+ \times C_+$. Define $Q : \mathbb{R}_+ \times C_+ \to C_+$ by $Q[t, \varphi](\theta, x) = u^\varphi(t + \theta, x; f)$ for all $(t, \varphi) \in \mathbb{R}_+ \times C_+$ and $(\theta, x) \in [-\tau, 0] \times \mathcal{H}$. Let $Q_t \triangleq Q[t, \cdot]$ for all $t \in \mathbb{R}_+$.

To move forward, we impose the following conditions on f.

(H1) f is a continuously differentiable function on some right-neighborhood of 0;
(H2) $f'(0) > 1$ and $f(u) \leqslant f'(0)u$ for all $u \in [0, \infty)$;
(H3) f has a unique positive fixed point u^*.

It follows from (H1)–(H3) that $f(x) > x$ for all $x \in (0, u^*)$ and $f(x) < x$ for all $x \in (u^*, \infty)$.

Lemma 5.1. *The following statements are true.*

(i) *The solution map of (5.1) induces a continuous semiflow $\{Q_t\}_{t \geqslant 0}$ on C_+ in the sense of Section 3 (see the paragraph before Theorem 3.11).*

(ii) *If* $f(a)=a$ *for some* $a\in\mathbb{R}_+$, *then* $Q_t[a]=a$ *for all* $t\in\mathbb{R}_+$.
(iii) *If* $f([a,b])\subseteq[a,b]\subseteq\mathbb{R}_+$, *then* $Q_t[C_{a,b}]\subseteq C_{a,b}$ *for all* $t\in\mathbb{R}_+$.
(iv) Q_t *satisfies* (A1) *for all* $t\in\mathbb{R}_+$.
(v) *If* f *is non-decreasing on* $\mathbb{R}_+$, *then* Q_t *is monotone on* C_+.
(vi) *If* (H1)–(H3) *hold and if* f *is non-decreasing on* $\mathbb{R}_+$, *then for any* $t>\tau$ *we have* $Q_t[\alpha]\gg\alpha$ *for all* $\alpha\in(0,u^*)$ *and* $Q_t[\alpha]\ll\alpha$ *for all* $\alpha\in(u^*,\infty)$.

Proof. (i) Clearly, $Q_0=\mathrm{Id}|_{C_+}$. Since $T(t)$ is a semigroup and the solution of (5.1) is unique, a standard argument gives $Q_{t_1}\circ Q_{t_2}=Q_{t_1+t_2}$ for all $t_1,t_2\in\mathbb{R}_+$. Indeed, for any given $t_1\in\mathbb{R}_+$ and $\varphi\in C_+$, let $g(t)=u^{Q_{t_1}[\varphi]}(t,\cdot;f)$ and $h(t)=u^{\varphi}(t+t_1,\cdot;f)$ for all $t\in\mathbb{R}_+$. By definitions of g and h, we have $g_0=h_0=Q_{t_1}[\varphi]$. It follows from (5.1) and (T3) that

$$\begin{aligned}
h(t)&=T(t+t_1)\big(\varphi(0,\cdot)\big)(x)+\mu\int_0^{t+t_1}T(t+t_1-s)\big(K\big[f\big(h(s-t_1-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s\\
&=T(t+t_1)\big(\varphi(0,\cdot)\big)(x)+\mu\int_{t_1}^{t+t_1}T(t+t_1-s)\big(K\big[f\big(h(s-t_1-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s\\
&\quad+\mu\int_0^{t_1}T(t+t_1-s)\big(K\big[f\big(h(s-t_1-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s\\
&=T(t)\left[T(t_1)\big(\varphi(0,\cdot)\big)(x)+\mu\int_0^{t_1}T(t_1-s)\big(K\big[f\big(u^{\varphi}(s-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s\right]\\
&\quad+\mu\int_0^{t}T(t-s)\big(K\big[f\big(h(s-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s\\
&=T(t)\big(h(0)\big)+\mu\int_0^{t}T(t-s)\big(K\big[f\big(h(s-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s.
\end{aligned}$$

This shows that h satisfies (5.1) with the initial value h_0. According to the uniqueness of solution of (5.1) and the definition of g, we have $h(t)=g(t)$ for all $t\in\mathbb{R}_+$. By the arbitrariness of t and φ, we have $Q_{t_1}\circ Q_{t_2}=Q_{t_1+t_2}$ for all $t_1,t_2\in\mathbb{R}_+$.

Furthermore, by (T1), (K1) and (5.1) and by the similar proof of Theorem 2.7(i) in [37], we easily see that for any $r>0$, $Q|_{\mathbb{R}_+\times C_r}:\mathbb{R}_+\times C_r\to C_+$ is continuous. So, $\{Q_t\}_{t\geqslant 0}$ is a continuous semiflow on C_+ in the sense of Section 3.

(ii) In view of $f(a)=a$, we easily check that $u(t,\cdot)\equiv a$ satisfies (5.1). This proves (ii).

(iii) Suppose that $f([a,b])\subseteq[a,b]\subseteq\mathbb{R}_+$. By (5.1), (T1)–(T2) and (K1)–(K2), we know that for all $(t,x,\varphi)\in[0,\tau]\times\mathcal{H}\times C_{a,b}$,

$$\begin{aligned}
u^{\varphi}(t,x;f)&=T(t)\big(\varphi(0,\cdot)\big)(x)+\mu\int_0^{t}T(t-s)\big(K\big[f\big(\varphi(s-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s\\
&\geqslant ae^{-\mu t}+\int_0^{t}\mu T(t-s)\big(K[a]\big)(x)\,\mathrm{d}s
\end{aligned}$$

$$= ae^{-\mu t} + \mu a \int_0^t e^{-\mu(t-s)}\,\mathrm{d}s$$
$$= a.$$

Similarly, $u^{\varphi}(t, x; f) \leqslant b$ for all $(t, x, \varphi) \in [0, \tau] \times \mathcal{H} \times C_{a,b}$. In summary, $Q_t[C_{a,b}] \subseteq C_{a,b}$ for all $t \in [0, \tau]$. This, combined with the semigroup property of the solution map Q_t, implies that $Q_t[C_{a,b}] \subseteq C_{a,b}$ for all $t \in \mathbb{R}_+$.

(iv) follows from (5.1), (K3) and (T4).

(v) follows from (5.1), (K2), (T2) and the monotonicity of f.

(vi) Suppose that $t > \tau$ and f is non-decreasing on $\mathbb{R}_+$. We only consider the case where $\alpha \in (0, u^*)$ since the other case can be dealt with similarity. Note that $f([\alpha, u^*]) \subseteq [\alpha, u^*]$ due to the monotonicity of f. Thus, by (iii), $Q_t[C_{\alpha,u^*}] \subseteq C_{\alpha,u^*}$ for all $t \in \mathbb{R}_+$. Since $f(u) > \alpha$ for all $u \in [\alpha, u^*]$ due to (H1)–(H3), it follows from (5.1), (T2) and (K2) that for all $(t, x, \varphi) \in (0, \infty) \times \mathcal{H} \times C_{\alpha,u^*}$,

$$u^{\varphi}(t, x; f) = T(t)\big(\varphi(0, \cdot)\big)(x) + \mu \int_0^t T(t-s)\big(K\big[f\big(u^{\varphi}(s-\tau, \cdot; f)\big)\big]\big)(x)\,\mathrm{d}s$$
$$> T(t)\big(\varphi(0, \cdot)\big)(x) + \int_0^t \mu T(t-s)\big(K[\alpha]\big)(x)\,\mathrm{d}s$$
$$\geqslant \alpha e^{-\mu t} + \mu\alpha \int_0^t e^{-\mu(t-s)}\,\mathrm{d}s$$
$$= \alpha.$$

This implies that $u^{\varphi}(t, x; f) > \alpha$ for all $(t, x, \varphi) \in (0, \infty) \times \mathcal{H} \times C_{\alpha,u^*}$, and in particular, for any $t > \tau$, we have $Q_t[\alpha] \gg \alpha$.

This completes the proof. $\square$

By (T1)–(T3), (T5) and by using an argument similar to the proof of Theorem 2.1.8 in [35] with some necessary modifications, we may show the following compact result.

Lemma 5.2. *The following statements are true.*

(i) $Q_t[C_r](0, \cdot)$ *is precompact in* X *for all* $t, r \in (0, \infty)$.
(ii) $Q_t[C_r]$ *is precompact in* C *for all* $r \in (0, \infty)$ *and* $t \in (\tau, \infty)$.
(iii) *If* $t \in (0, \tau]$ *and* $D \subseteq C_r$ *with* $r > 0$ *and* $D(0, \cdot)$ *being a precompact subset of* X, *then* $S[D]$ *is precompact in* C, *where the operator*

$$S[u](\theta, x) := \begin{cases} u(0, x) & \text{if } -\tau \leqslant \theta < -t, \\ Q_t[u](\theta, x) & \text{if } -t \leqslant \theta \leqslant 0. \end{cases}$$

(iv) Q_t *satisfies* (A3*) *for all* $t \in (0, \infty)$.

Proof. (i) Fix $t, r \in (0, \infty)$. We prove that $Q_t[C_r](0, \cdot)$ is precompact in X. Let $r' = \max\{r, \max f([0, u^*])\}$. Then $r' \geqslant \max\{u^*, r\}$ and $f([0, r']) \subseteq [0, r']$. By Lemma 5.1(iii), we have $Q_{t'}[C_{r'}] \subseteq C_{r'}$ for all $t' \in \mathbb{R}_+$. It suffices to prove that $Q_t[C_{r'}](0, \cdot)$ is precompact in X. By (5.1), (K2) and (T2), we easily see that $Q_t[C_{r'}](0, \cdot) \subseteq T(t)(X_{r'}) + D_t$, where $D_t = \{\int_0^t T(t-s)g(s)\,\mathrm{d}s\colon\ g \in C([0, t], X_{\mu r'})\}$. According to (T5), it is enough to show that D_t is precompact in X. By $D_t \subseteq X_{r'}$ and the completeness of $X_{r'}$, we only need to prove that D_t is totally bounded. Indeed, by the proof of Lemma 1.6 in [35], we know that D_t^ε is precompact in X for all $\varepsilon > 0$, where $D_t^\varepsilon = \{\int_\varepsilon^t T(t-s)g(s)\,\mathrm{d}s\colon\ g \in C([0,t], X_{\mu r'})\}$. Thus, for any $\varepsilon > 0$, there is a finite subset $F \subseteq X_{r'}$ such that $D_t^\varepsilon \subseteq \{\phi \in X_{r'}\colon\ \|\phi - \eta\| \leqslant \varepsilon$ for some $\eta \in F\}$. Since $\int_0^\varepsilon T(t-s)g(s)\,\mathrm{d}s \leqslant \mu r'\varepsilon$ for all $g \in C([0,t], X_{\mu r'})$, it follows from definitions of D_t and D_t^ε that $D_t \subseteq \{\phi \in X_{r'}\colon\ \|\phi - \eta\| \leqslant \varepsilon + \mu r'\varepsilon$ for some $\eta \in F\}$. By the arbitrariness of ε and the finiteness of F, we see that D_t is totally bounded in $X_{r'}$. This proves (i).

To finish the proof of (ii) and (iii), we first show the following claim:

Claim. *If $t \in (0,\tau]$, $M \geqslant \max f([0,u^*])$ and $J \subseteq C_M$ with $J(0,\cdot)$ being precompact in X, then*

$$\prod = \bigl\{g_\varphi \in C\bigl([0,t], X\bigr)\colon\ \varphi \in J \text{ and } g_\varphi\bigl(t'\bigr) = Q_{t'}[\varphi](0,\cdot) \text{ for all } t' \in [0,t]\bigr\}$$

is equicontinuous.

By the compactness of $J(0,\cdot)$ and the continuity of T due to (T1), we get that $T\colon\ [0,t] \times J(0,\cdot) \to X$ is uniformly continuous and hence $\{T(\cdot)(J(0,\cdot))\}$ is equicontinuous. From (5.1), it suffices to prove that $\coprod := \{h_\varphi \in C([0,t], X)\colon\ \varphi \in J$ and $h_\varphi(t') = Q_{t'}[\varphi](0,\cdot) - T(t')(\varphi(0,\cdot))$ for all $t' \in [0,t]\}$ is equicontinuous. In fact, for any $\varepsilon > 0$, $\varphi \in J$, and $\tilde t, \tilde{\tilde t} \in [0,t]$ with $\tilde t \leqslant \tilde{\tilde t}$, we have

$$\begin{aligned}
&\bigl\|h_\varphi(\tilde{\tilde t}) - h_\varphi(\tilde t)\bigr\| \\
&\quad = \left\| \int_0^{\tilde{\tilde t}} T(\tilde{\tilde t} - s)\bigl(K\bigl[f\bigl(u^\varphi(s-\tau,\cdot;f)\bigr)\bigr]\bigr)\,\mathrm{d}s - \int_0^{\tilde t} T(\tilde t - s)\bigl(K\bigl[f\bigl(u^\varphi(s-\tau,\cdot;f)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| \\
&\quad = \left\| \int_0^{\tilde{\tilde t}} T(\tilde{\tilde t} - s)\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s - \int_0^{\tilde t} T(\tilde t - s)\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| \\
&\quad \leqslant \left\| \int_{\tilde t}^{\tilde{\tilde t}} T(\tilde{\tilde t} - s)\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| + \left\| \int_0^{\tilde t} \bigl[T(\tilde{\tilde t} - s) - T(\tilde t - s)\bigr]\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| \\
&\quad \leqslant M|\tilde{\tilde t} - \tilde t| + \left\| \int_0^{\tilde t} \bigl[T(\tilde{\tilde t} - s) - T(\tilde t - s)\bigr]\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| \\
&\quad \leqslant M|\tilde{\tilde t} - \tilde t| + \left\| \int_{\max\{0,\tilde t-\varepsilon\}}^{\tilde t} \bigl[T(\tilde{\tilde t} - s) - T(\tilde t - s)\bigr]\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| \\
&\qquad + \left\| \int_0^{\max\{0,\tilde t-\varepsilon\}} \bigl[T(\tilde{\tilde t} - s) - T(\tilde t - s)\bigr]\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\| \\
&\quad \leqslant M|\tilde{\tilde t} - \tilde t| + 2M\varepsilon + \left\| \int_0^{\max\{0,\tilde t-\varepsilon\}} \bigl[T(\tilde{\tilde t} - s) - T(\tilde t - s)\bigr]\bigl(K\bigl[f\bigl(\varphi(s-\tau)\bigr)\bigr]\bigr)\,\mathrm{d}s \right\|.
\end{aligned}$$

Thus, if $\tilde t \leqslant \varepsilon$, then $\|h_\varphi(\tilde{\tilde t}) - h_\varphi(\tilde t)\| \leqslant M|\tilde{\tilde t} - \tilde t| + 2M\varepsilon$. If $\tilde t > \varepsilon$, then

$$
\begin{aligned}
&\left\|h_\varphi(\tilde{\tilde{t}}) - h_\varphi(\tilde{t})\right\| \\
&\quad \leqslant M|\tilde{\tilde{t}} - \tilde{t}| + 2M\varepsilon + \left\| \int_0^{\tilde{t}-\varepsilon} \left[T(\tilde{\tilde{t}} - s) - T(\tilde{t} - s)\right]\left(K\left[f\left(\varphi(s-\tau)\right)\right]\right) \mathrm{d}s \right\| \\
&\quad \leqslant M|\tilde{\tilde{t}} - \tilde{t}| + 2M\varepsilon + \left\| \int_0^{\tilde{t}-\varepsilon} \left[T\left(\tilde{\tilde{t}} - s - \frac{\varepsilon}{2}\right) - T\left(\tilde{t} - s - \frac{\varepsilon}{2}\right)\right]\left(T\left(\frac{\varepsilon}{2}\right)\left(K\left[f\left(\varphi(s-\tau)\right)\right]\right)\right) \mathrm{d}s \right\|.
\end{aligned}
$$

Hence, when $\tilde{t} > \varepsilon$, we have

$$
\left\|h_\varphi(\tilde{\tilde{t}}) - h_\varphi(\tilde{t})\right\| \leqslant M|\tilde{\tilde{t}} - \tilde{t}| + 2M\varepsilon + |\tilde{t} - \varepsilon| H_{\tilde{t},\tilde{\tilde{t}}},
$$

where $H_{\tilde{t},\tilde{\tilde{t}}} = \sup\{\|[T(\tilde{\tilde{t}} - s - \frac{\varepsilon}{2}) - T(\tilde{t} - s - \frac{\varepsilon}{2})](\phi)\|\colon\ \phi \in T(\frac{\varepsilon}{2})(X_M)$ and $s \in [0, \tilde{t} - \varepsilon]\}$. By the compactness of $T(\frac{\varepsilon}{2})(X_M)$ due to (T5) and the continuity of T due to (T1), we get that $T : [0, t] \times T(\frac{\varepsilon}{2})(X_M) \to X$ is uniformly continuous, and hence $H_{\tilde{t},\tilde{\tilde{t}}} \to 0$ as $|\tilde{t} - \tilde{\tilde{t}}| \to 0$. This proves the Claim.

(ii) We next prove (ii). Fix $r \in (0, \infty)$ and $t > \tau$. Without loss of generality, we may assume that $r \geqslant \max f([0, u^*])$. By (i), we know that $Q_{t-\tau}[C_r](0, \cdot)$ is precompact in X. This, combined with claim, implies that $Q_t[C_r](\theta, \cdot) = Q_\tau[Q_{t-\tau}[C_r]](\theta, \cdot)$ is equicontinuous in $\theta \in [-\tau, 0]$. Again, by (i), we obtain that, for any $\theta \in [-\tau, 0]$, $Q_t[C_r](\theta, \cdot)$ is precompact in X. Conclusively, by Arzelà–Ascoli theorem, $Q_t[C_r]$ is precompact in C. This completes the proof of (ii).

(iii) follows from definitions of Q_t and S, the compactness of $D(0, \cdot)$, and an argument similar to the proof of (ii).

(iv) Finally, if $t > \tau$, then (A3*)(a) holds by (ii). If $t \in (0, \tau]$, then by the definition of Q_t and (iii), we get that (A3*)(b) holds. So, (A3*) holds.

This completes the proof. □

In the following, we always assume that (H1)–(H3) hold.

To obtain a convergence result, we will need the following assumption (stronger than (H3)).

(H4) f^2 has a unique positive fixed point u^*.

Obviously, (H4) implies (H3). In [38], Yi and Zou showed that assumption (H4) plays a key role on the delay independent global stability of a positive equilibrium for a class of delayed reaction diffusion equations in a bounded domain with the homogeneous Neumann boundary condition by establishing the relation between the globally stable dynamics of the map f and the delayed reaction diffusion equations.

We provide the following equivalent statement of (H4), which is very useful in verifying (UM).

(H5) For any interval $[a, b] \subseteq (0, \infty)$ with $a < b$, there exist $a', b' \in (0, \infty)$ such that $[a, b] \subseteq [a', b']$, $f([a', b']) \subseteq [a', b']$, and either $a < \min\{f(u)\colon\ u \in [a', b']\}$ or $b > \max\{f(u)\colon\ u \in [a', b']\}$.

Lemma 5.3. *The assumption* (H4) *is equivalent with the assumption* (H5).

Proof. Suppose that (H5) holds. We shall show that (H4) holds. Otherwise, there exists $a \in (0, \infty) \setminus \{u^*\}$ such that $f^2(a) = a$. Without loss of generality, we may assume that $a < f(a) \triangleq b$. Thus, $f([a, b]) \supseteq [a, b]$, which yields a contradiction to (H5).

Now suppose that (H4) holds. Let $[a,b]\subseteq(0,\infty)$ with $a<b$. By (H1)–(H3), there exist $M>\epsilon>0$ such that $[a,b]\subseteq[\epsilon,M]$ and $f([\epsilon,M])\subseteq[\epsilon,M]$. Assumption (H4) implies (see Proposition 2.1 in [38]), that

$$\lim_{n\to\infty}\operatorname{dist}\bigl(f^n([\epsilon,M]),u^*\bigr)=0.$$

Thus, $n^*\triangleq\sup\{n\geqslant 1:\ [a,b]\subseteq f^n([\epsilon,M])\}<\infty$. Clearly, there exist $a',b'\in(0,\infty)$ such that $[a',b']=f^{n^*}([\epsilon,M])$. By the choice of n^*, $[a,b]\subseteq[a',b']$, $f([a',b'])\subseteq[a',b']$ and $[a,b]\setminus f([a',b'])\neq\emptyset$. Hence, $a<\min\{f(u):\ u\in[a',b']\}$ or $b>\max\{f(u):\ u\in[a',b']\}$. This proves (H5). $\square$

Lemma 5.4. *Assume that* (H4) *holds. If $s>r>0$, then there are three numbers $N(r,s)\in(0,\infty)$, $I(r,s)\in(r,\infty)$ and $S(r,s)\in(0,s)$ such that either $Q_t[C_{r,s}]\geqslant I(r,s)$ for all $t\geqslant N(r,s)$ or $Q_t[C_{r,s}]\leqslant S(r,s)$ for all $t\geqslant N(r,s)$. In particular, Q_t satisfies assumption* (UM) *for all $t>0$.*

Proof. Suppose that $s>r>0$. By Lemma 5.3 and (H5), there exist $r',s'\in(0,\infty)$ such that $[r,s]\cup f([r',s'])\subseteq[r',s']$ and either $r<\min\{f(u):\ u\in[r',s']\}$ or $s>\max\{f(u):\ u\in[r',s']\}$. Without loss of generality, we may assume that $r<i_{r',s'}\triangleq\min\{f(u):\ u\in[r',s']\}$. It follows from Lemma 5.1(iii) that $Q_t[C_{r',s'}]\subseteq C_{r',s'}$ for all $t\in\mathbb{R}_+$, that is, $u^\varphi(t,x;f)\in[r',s']$ for all $(t,x,\varphi)\in\mathbb{R}_+\times\mathcal{H}\times C_{r',s'}$. By (5.1), (T2) and (K2), we know that for all $(t,x,\varphi)\in(0,\infty)\times\mathcal{H}\times C_{r',s'}$,

$$\begin{aligned}
u^\varphi(t,x;f)&=T(t)\bigl(\varphi(0,\cdot)\bigr)(x)+\mu\int_0^t T(t-s)\bigl(K\bigl[f\bigl(u^\varphi(s-\tau,\cdot;f)\bigr)\bigr]\bigr)(x)\,\mathrm{d}s\\
&\geqslant r'e^{-\mu t}+\int_0^t \mu T(t-s)\bigl(K[i_{r',s'}]\bigr)(x)\,\mathrm{d}s\\
&=r'e^{-\mu t}+i_{r',s'}\mu\int_0^t e^{-\mu(t-s)}\,\mathrm{d}s\\
&=i_{r',s'}+\bigl(r'-i_{r',s'}\bigr)e^{-\mu t}.
\end{aligned}$$

Thus, there is a positive number $N(r,s)$ such that $u^\varphi(t,x;f)\geqslant i_{r',s'}+(r'-i_{r',s'})e^{-\mu N(r,s)}\geqslant\frac{r+i_{r',s'}}{2}>r$ for all $(t,x,\varphi)\in[N(r,s),\infty)\times\mathcal{H}\times C_{r',s'}$. In view of $C_{r,s}\subseteq C_{r',s'}$, we know that $u^\varphi(t,x;f)\geqslant\frac{r+i_{r',s'}}{2}>r$ for all $(t,x,\varphi)\in[N(r,s),\infty)\times\mathcal{H}\times C_{r,s}$. Therefore, $Q_t[C_{r,s}]\geqslant\frac{r+i_{r',s'}}{2}>r$ for all $t\geqslant N(r,s)$. This proves the first conclusion in this lemma.

Now, we can easily see that Q_t satisfies assumption (UM) for all $t>0$. $\square$

For any $M\geqslant\max f([0,u^*])$, define f_M and $f_M^\pm$ respectively by

$$f_M(x)=\begin{cases} f(x), & x\in[0,M],\\ f(M), & x>M,\end{cases}\tag{5.2}$$

and

$$f_M^+(x)=\max f_M\bigl([0,x]\bigr),\qquad f_M^-(x)=\begin{cases}\min f_M([x,M]), & x\in[0,M],\\ f(M), & x>M,\end{cases}\tag{5.3}$$

$$f^L(x)=f'(0)x,\quad x\in\mathbb{R}_+.\tag{5.4}$$

According to the above definitions of f_M, $f_M^{\pm}$ and f^L, we easily obtain the following results.

Lemma 5.5. *Let $M \geqslant \max f([0, u^*])$ and let f_M, $f_M^{\pm}$ and f^L be defined as in* (5.2), (5.3) *and* (5.4)*, respectively. Then the following statements are true.*

(i) *$f_M^-(x) \leqslant f_M(x) \leqslant f_M^+(x) \leqslant f^L(x)$ for all $x \in \mathbb{R}_+$.*
(ii) *f_M^+ and f_M^- are non-decreasing and continuous on $\mathbb{R}_+$.*
(iii) *There exists $\delta > 0$ such that $f_M^{\pm}(x) = f(x)$ for all $x \in [0, \delta]$.*
(iv) *There exist positive numbers $u_{\pm}^*$ such that $f_M^{\pm}(u_{\pm}^*) = u_{\pm}^*$ and $0 < u_-^* \leqslant u^* \leqslant u_+^* \leqslant M$.*
(v) *f_M and $f_M^{\pm}$ satisfy assumptions* (H1)–(H3)*; moreover, if f satisfies* (H4)*, then f_M also satisfies assumption* (H4).
(vi) *For any $\varepsilon \in (0, 1)$ there is $\delta \in (0, u_-^*)$ such that $f_M^-(x) \geqslant (1-\varepsilon) f^L(x)$ for any $x \in [0, \delta]$.*

Define

$$Q_t[\varphi; M](\theta, x) = u^{\varphi}(t+\theta, x; f_M)$$

and

$$Q_t^{\pm}[\varphi; M](\theta, x) = u^{\varphi}\big(t+\theta, x; f_M^{\pm}\big),$$

where $(t, \varphi) \in \mathbb{R}_+ \times C_+$ and $(\theta, x) \in [-\tau, 0] \times \mathcal{H}$.

Define $\mathcal{M} : \mathbb{R}_+ \times C \to C$ by $\mathcal{M}(t, \varphi; f^L)(\theta, x) = u^{\varphi}(t+\theta, x; f^L)$ for all $(t, \varphi) \in \mathbb{R}_+ \times C$ and $(\theta, x) \in [-\tau, 0] \times \mathcal{H}$. For any $t \in \mathbb{R}_+$, we denote $\mathcal{M}(t, \cdot; f^L)$ by $\mathcal{M}_t$.

In view of Lemma 5.1(i), we know that $\{Q_t[\cdot; M]\}_{t\in\mathbb{R}_+}$, $\{Q_t^{\pm}[\cdot; M]\}_{t\in\mathbb{R}_+}$ and $\{\mathcal{M}_t\}_{t\in\mathbb{R}_+}$ are all continuous-time semiflows on C_+ in the sense of Section 3 (see the paragraph before Theorem 3.11) for any $M \geqslant \max f([0, u^*])$.

Lemma 5.6. *Let $M \geqslant \max f([0, u^*])$. Then the following statements are true.*

(i) *$Q_t^-[\cdot; M] \leqslant Q_t[\cdot; M] \leqslant Q_t^+[\cdot; M] \leqslant \mathcal{M}_t$ for all $t \in \mathbb{R}_+$.*
(ii) *$Q_t^{\pm}[\cdot; M]$ are monotone in C_+ and satisfy assumptions* (A1), (A2) *and* (A4) *with r^* being replaced by $u_{\pm}^*$, respectively, where u_-^* and u_+^* are defined in Lemma* 5.5(iv).
(iii) *$Q_t[\cdot; M]$ and $Q_t^{\pm}[\cdot; M]$ satisfy* (A3*) *for all $t > 0$.*
(iv) *$Q_t[C_M] \subseteq C_M$ and $Q_t[C_M; M] \subseteq C_M$ for all $t \in \mathbb{R}_+$ and $Q_t[\varphi; M] = Q_t[\varphi] \triangleq (u^{\varphi}(\cdot, \cdot; f))_t$ for all $\varphi \in C_M$ and $t \in \mathbb{R}_+$.*
(v) *$Q_t[\cdot; M]$ satisfies assumptions* (A1), (A2), (A4)(a) *for all $t \in \mathbb{R}_+$ with r^* being replaced by u^*.*
(vi) *If* (H4) *holds, then $Q_t[\cdot; M]$ satisfies the assumption* (UM) *for all $t \in (0, \infty)$.*
(vii) *For any $t \in \mathbb{R}_+$, $\mathcal{M}_t$ is a positive linear operator such that*

$$\big[f'(0) + \big(1 - f'(0)\big)e^{-\mu\tau}\big]^{\frac{t}{2\tau}-1} \leqslant \mathcal{M}_t[1] \leqslant \big[f'(0) + \big(1 - f'(0)\big)e^{-\mu\tau}\big]^{\frac{t}{\tau}+1}.$$

(viii) *For any $t \in (0, \infty)$, $\mathcal{M}_t$ satisfies assumptions* (AL1), (AL2) *and* (AL3).
(ix) *For any $t \in (0, \infty)$, $\mathcal{M}_t$, $Q_t^{\pm}[\cdot; M]$ satisfy the assumption* (GC).

Proof. (i) follows from Lemma 5.5(i) and (5.1) with f being replaced by f_M, $f_M^{\pm}$, f^L, respectively.

(ii) follows from Lemma 5.1 and Lemma 5.5(ii).

(iii) follows from Lemma 5.2.

(iv) By Lemma 5.1(iii), $f([0, M]) \subseteq [0, M]$ and $f_M([0, M]) \subseteq [0, M]$, hence we have $Q_t[C_M] \subseteq C_M$ and $Q_t[C_M; M] \subseteq C_M$ for all $t \in \mathbb{R}_+$. This, combined with (5.1) and the fact that $f_M|_{[0,M]} \equiv f|_{[0,M]}$, implies that $Q_t[\varphi; M] = Q_t[\varphi] \triangleq (u^{\varphi}(\cdot, \cdot; f))_t$ for all $\varphi \in C_M$ and $t \in \mathbb{R}_+$.

(v) follows from Lemma 5.1(i), (ii) and (iv).

(vi) follows from Lemma 5.4.

(vii) Clearly, by (5.1), (K2) and (T2), we can check that $\mathcal{M}_t[C_+] \subseteq C_+$ for all $t \in \mathbb{R}_+$. Thus, $\mathcal{M}_t$ is a positive linear operator.

To prove the inequalities in statement (vii), by (5.1), we have $\mathcal{M}_s[a](0,\cdot) = a[f'(0) + (1 - f'(0))e^{-\mu s}]$ for all $(s,a) \in [0,\tau] \times (0,\infty)$. It follows that $1 \leqslant \mathcal{M}_s[1](0,\cdot) \leqslant f'(0) + (1 - f'(0))e^{-\mu s}$ for all $s \in [0,\tau]$.

Now, we prove the second inequality in statement (vii). Fix $t \in \mathbb{R}_+$. Clearly, there are an integer $l \geqslant 0$ and $\tilde{t} \in [0,\tau)$ such that $t = \tilde{t} + l\tau$. In view of the semigroup properties of $\{\mathcal{M}_s\}_{s\in\mathbb{R}_+}$, we may obtain that $1 \leqslant \mathcal{M}_t[1] = \mathcal{M}_{l\tau+\tilde{t}}[1] = (\mathcal{M}_\tau)^l[\mathcal{M}_{\tilde{t}}[1]] \leqslant [f'(0) + (1 - f'(0))e^{-\mu\tau}]^{l+1} \leqslant [f'(0) + (1 - f'(0))e^{-\mu\tau}]^{\frac{t}{\tau}+1}$. This gives the second inequality.

Next, we prove the first inequality in statement (vii). It follows from (5.1) and the semigroup properties of $\{\mathcal{M}_s\}_{s\in\mathbb{R}_+}$ that for any $s \in [\tau, 2\tau]$ we have

$$\begin{aligned}
\mathcal{M}_s[1](0,\cdot) &= \mathcal{M}_{s-\tau}\big[\mathcal{M}_\tau[1]\big](0,\cdot) \\
&= T(s-\tau)\big(\mathcal{M}_\tau[1](0,\cdot)\big) + \mu \int_0^{s-\tau} T(s-\tau-\lambda)\big(K\big[f^L\big(\mathcal{M}_\tau[1](\lambda-\tau,\cdot)\big)\big]\big)\,\mathrm{d}\lambda \\
&\geqslant \big[f'(0) + \big(1 - f'(0)\big)e^{-\mu\tau}\big]e^{-\mu(s-\tau)} + \mu \int_0^{s-\tau} T(s-\tau-\lambda)\big(K\big[f'(0)\big]\big)\,\mathrm{d}\lambda \\
&\geqslant \big[f'(0) + \big(1 - f'(0)\big)e^{-\mu\tau}\big]e^{-\mu(s-\tau)} + f'(0)\big[1 - e^{-\mu(s-\tau)}\big] \\
&= f'(0) + \big(1 - f'(0)\big)e^{-\mu s} \\
&\geqslant f'(0) + \big(1 - f'(0)\big)e^{-\mu\tau}.
\end{aligned}$$

Again, fix $t \in \mathbb{R}_+$. Clearly, there are an integer $l \geqslant 0$ and $\tilde{t} \in [0,2\tau)$ such that $t = \tilde{t} + 2l\tau$. In view of the semigroup properties of $\{\mathcal{M}_s\}_{s\in\mathbb{R}_+}$, we may obtain that $\mathcal{M}_t[1] = \mathcal{M}_{2l\tau+\tilde{t}}[1] = (\mathcal{M}_{2\tau})^l[\mathcal{M}_{\tilde{t}}[1]] \geqslant (\mathcal{M}_{2\tau})^l[1] \geqslant [f'(0) + (1 - f'(0))e^{-\mu\tau}]^l \geqslant [f'(0) + (1 - f'(0))e^{-\mu\tau}]^{\frac{t}{2\tau}-1}$. This shows the first inequality in statement (vii) and completes the proof of (vii).

(viii) Suppose that $t > 0$. By Lemma 5.1(i) and (iv), and Lemma 5.2(iv), we easily see that $\mathcal{M}_t$ satisfies assumptions (A1), (A2) and (A3), and hence (AL1) holds.

We claim that $\mathcal{M}_{t'}[C_+ \setminus \{0\}] \subseteq C_+^o$ for all $t' \geqslant 2\tau$. Indeed, for any $\varphi \in C_+ \setminus \{0\}$, there is $\theta^* \in [-\tau, 0)$ such that $\varphi(\theta^*,\cdot) > 0$. Thus by (K2), $K[f^L(\varphi(\theta^*,\cdot))] > 0$. It follows from (5.1), (K2), (T2) and the fact that $f^L(x) = f'(0)x > 0$ for all $x \in (0,\infty)$ that, for all $(t', x, \varphi) \in [\tau,\infty) \times \mathcal{H} \times (C_+ \setminus \{0\})$, we have

$$\begin{aligned}
u^\varphi\big(t', x; f^L\big) &= T\big(t'\big)\big(\varphi(0,\cdot)\big)(x) + \mu \int_0^{t'} T\big(t'-s\big)\big(K\big[f^L\big(u^\varphi\big(s-\tau,\cdot; f^L\big)\big)\big]\big)(x)\,\mathrm{d}s \\
&\geqslant \mu \int_0^{t'} T\big(t'-s\big)\big(K\big[f^L\big(u^\varphi\big(s-\tau,\cdot; f^L\big)\big)\big]\big)(x)\,\mathrm{d}s \\
&\geqslant \mu \int_0^{\tau} T\big(t'-s\big)\big(K\big[f^L\big(\varphi(s-\tau,\cdot)\big)\big]\big)(x)\,\mathrm{d}s \\
&> 0.
\end{aligned}$$

This proves the above claim. Then (AL3) follows immediately.

In the following, for any $s \in \mathbb{R}_+$, let $\mathbf{M}_s \triangleq \mathcal{M}_s|_Y$. Then $\mathbf{M}_s[Y] \subseteq Y$ for all $s \in \mathbb{R}_+$ due to (A1). By the above claim, $(\mathcal{M}_t)^N[C_+ \setminus \{0\}] = \mathcal{M}_{Nt}[C_+ \setminus \{0\}] \subseteq C_+^o$ for all integers $N \geqslant \frac{2\tau}{t}$. Thus, for any integer $N \geqslant \frac{2\tau}{t}$, we have $\mathbf{M}_{Nt}[Y_+] \subseteq \mathrm{Int}(Y_+)$. Lemma 5.2(ii) shows that $\mathcal{M}_s[C_r]$ is precompact in C for all $s > \tau$ and $r > 0$. In particular, $\mathbf{M}_{Nt} : Y \to Y$ is compact for any integer $N > \frac{\tau}{t}$. Therefore, $\mathcal{M}_t$ satisfies (AL2)(a).

By (AL2)(a) and Lemma 3.1 in [15], the spectral radius λ_0 of $\mathbf{M}_t$ is the simple eigenvalue of $\mathbf{M}_t$ such that there is a strongly positive eigenvector associated with λ_0 and the modulus of any other eigenvalue is less than λ_0. It suffices to prove $\lambda_0 > 1$. Indeed, $\lambda_0 = \lim_{N\to\infty} \|(\mathbf{M}_t)^N\|^{\frac{1}{N}} = \lim_{N\to\infty} \|\mathbf{M}_{Nt}\|^{\frac{1}{N}} \geqslant \limsup_{N\to\infty} \|\mathbf{M}_{Nt}[1]\|^{\frac{1}{N}} = \limsup_{N\to\infty} \|\mathcal{M}_{Nt}[1]\|^{\frac{1}{N}}$. On the other hand, by the first inequity of (vii), we have

$$\lambda_0 \geqslant \limsup_{N\to\infty} \big\|\mathcal{M}_{Nt}[1]\big\|^{\frac{1}{N}} \geqslant \limsup_{N\to\infty} \big[f'(0) + \big(1 - f'(0)\big)e^{-\mu\tau}\big]^{\frac{t}{2\tau} - \frac{1}{N}} > 1.$$

This proves (AL2)(b), and thus (viii) holds.

(ix) Fix $t > 0$. The statement (i) gives (GC)(a). It suffices to prove (GC)(b). Indeed, it follows from Lemma 5.5(vi) that for any $\varepsilon \in (0,1)$ there is $\delta_1 \in (0, u_-^*)$ such that $f_M^-(x) \geqslant (1-\varepsilon)^{\frac{1}{1+\frac{t}{\tau}}} f^L(x)$ for any $x \in [0, \delta_1]$. Let $\alpha = (1-\varepsilon)^{\frac{1}{1+\frac{t}{\tau}}}$ and $\delta = \delta(t,\varepsilon) \triangleq \delta_1[f'(0) + (1 - f'(0))e^{-\mu\tau}]^{-1-\frac{t}{\tau}}$. This, combined with the second inequality of (vii) and the monotonicity of $\mathcal{M}$, implies that $\mathcal{M}_s[C_\delta] \subseteq C_{\delta_1}$ for all $s \in [0,t]$. Again, since $Q_s^-[\cdot; M] \leqslant \mathcal{M}_s$ for all $s \in \mathbb{R}_+$ due to (i), we know that $Q_s^-[C_\delta; M] \subseteq C_{\delta_1}$ for all $s \in [0,t]$. It follows from (5.1) and the choices of δ_1, α that for all $\varphi \in C_{\delta_1}$ and $t' \in [0,\tau]$

$$\begin{aligned}
Q_{t'}^-[\varphi; M](0,\cdot) &= T\big(t'\big)\big(\varphi(0,\cdot)\big) + \mu \int_0^{t'} T\big(t'-s\big)\big(K\big[f_M^-\big(\varphi(s-\tau,\cdot)\big)\big]\big)\,\mathrm{d}s \\
&\geqslant T\big(t'\big)\big(\varphi(0,\cdot)\big) + \mu \int_0^{t'} T\big(t'-s\big)\big(K\big[\alpha f^L\big(\varphi(s-\tau,\cdot)\big)\big]\big)\,\mathrm{d}s \\
&\geqslant \alpha\Bigg[T\big(t'\big)\big(\varphi(0,\cdot)\big) + \mu \int_0^{t'} T\big(t'-s\big)\big(K\big[f^L\big(\varphi(s-\tau,\cdot)\big)\big]\big)\,\mathrm{d}s\Bigg] \\
&= \alpha\mathcal{M}_{t'}[\varphi](0,\cdot).
\end{aligned}$$

In other words, $Q_{t'}^-[\varphi; M] \geqslant \alpha\mathcal{M}_{t'}[\varphi]$ for all $\varphi \in C_{\delta_1}$ and $t' \in [0,\tau]$.

Note that there are an integer $l \geqslant 0$ and $\tilde{t} \in [0,\tau)$ such that $t = \tilde{t} + l\tau$. Thus, we have $Q_{\tilde{t}+k\tau}^-[C_\delta; M] \subseteq C_{\delta_1}$ for all $k \in \mathbb{Z} \cap [0,l]$. By the semigroup properties of $\{Q_s^-\}_{s\in\mathbb{R}_+}$ and $\{\mathcal{M}_s\}_{s\in\mathbb{R}_+}$, we can see that, for any $\varphi \in C_\delta$,

$$\begin{aligned}
Q_t^-[\varphi; M] = Q_{\tilde{t}+l\tau}^-[\varphi; M] &= Q_{\tilde{t}}^-\big[Q_{l\tau}^-[\varphi; M]; M\big] \\
&\geqslant \alpha\mathcal{M}_{\tilde{t}}\big[Q_{l\tau}^-[\varphi; M]\big] \\
&\geqslant \alpha^{1+l}\mathcal{M}_t[\varphi].
\end{aligned}$$

By the choice of α, we get $Q_t^-[\varphi; M] \geqslant \alpha^{\frac{t}{\tau}+1}\mathcal{M}_t[\varphi] = (1-\varepsilon)\mathcal{M}_t[\varphi]$ for all $\varphi \in C_\delta$ and hence (GC)(b) holds. This completes the proof. $\square$

By Lemma 5.6, we easily see that for any given $t > 0$, $Q_t[\cdot; M]$, $Q_t^{\pm}[\cdot; M]$ and $\mathcal{M}_t$ with $M \geqslant \max f([0, u^*])$ satisfy all the conditions in Theorems 3.11 and 4.9.

With the help of Lemma 5.6(iv), the fact that for any $M \geqslant \max f([0, u^*])$, $Q_t[C_M; M] \subseteq C_M$, $Q_t[C_M] \subseteq C_M$ and $Q_t[\varphi; M] = Q_t[\varphi]$ for all $t \in \mathbb{R}_+$ and $\varphi \in C_M$, the following result can be easily deduced from Theorems 3.11 and 4.9.

Theorem 5.7. *Assume that* (H4) *holds. For system* (5.1)*, there exists $c^* > 0$ such that the following statements are true.*

(i) *For any $c > c^*$, if $\varphi \in C_{u^*}$ has compact support with $\varphi \ll u^*$ then*

$$\lim_{t\to\infty} \max\bigl\{u^{\varphi}(t, x; f): |x| \geqslant tc\bigr\} = 0.$$

(ii) *For any $c < c^*$ and $\varphi \in C_+ \setminus \{0\}$,*

$$\lim_{t\to\infty} \max\bigl\{\bigl|u^{\varphi}(t, x; f) - u^*\bigr|: |x| \leqslant tc\bigr\} = u^*.$$

(iii) *For any $c < c^*$,* (5.1) *has no travelling wave U such that $U(\cdot, -\infty) = u^*$.*
(iv) *For any $c \geqslant c^*$,* (5.1) *has a travelling wave solution U connecting u^* to* 0.

To study spreading speeds and travelling waves for a class of nonmonotone discrete-time integrodifference equation models, Hsu and Zhao [11] introduced the following much stronger nonlinearity assumption than (H4).

(HZ) f has the property (P1*) that $\frac{f(u)}{u}$ is strictly decreasing for $u \in (0, u_+^*]$, and the property (P2*) that for any $v, w \in (0, u_+^*]$ satisfying $v \leqslant u^* \leqslant w$, $v \geqslant f(w)$ and $w \leqslant f(v)$, we have $v = w$.

Assumption (HZ) has also been used to study the convergence and travelling waves for non-monotone integral equations [6] and non-monotone time-delayed lattice equations [5].

The following result tells us that property (P2*) in (HZ) implies (H4).

Lemma 5.8. *If property* (P2*) *in* (HZ) *holds, then* (H4) *holds.*

Proof. Suppose that (H4) does not hold. Then there are $a \in (0, \infty) \setminus \{u^*\}$ such that $a = f^2(a)$. Without loss of generality, we may assume that $a < f(a)$. Since $f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u$ for all $u \in (u^*, \infty)$, we easily see that $a < u^* < f(a) \leqslant u_+^*$. Let $v = a$ and $w = f(a)$. Then $f(v) = w$ and $f(w) = v$. By (P2*), we get $a = v = w = u^*$, a contradiction. This proves (H4). □

In the following applications, we only check the above assumptions (T1)–(T5), (K1)–(K3) and (H1)–(H4).

5.1. A delayed nonlocal reaction diffusion equation

Consider the following delayed reaction diffusion equation,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \Delta u(t, x) - \mu u(t, x) + \mu \displaystyle\int_{\mathbb{R}} f\bigl(u(t - \tau, y)\bigr)k(x - y)\,\mathrm{d}y, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(\theta, x) = \varphi(\theta, x), \quad (\theta, x) \in [-1, 0] \times \mathbb{R}, \end{cases} \tag{5.5}$$

where $\mu > 0$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (H1)–(H3) stated at the beginning of this section. We always assume that either $k(x) = \delta(x)$, or $k : \mathbb{R} \to [0, \infty)$ is continuous with $\int_{\mathbb{R}} k(y)\,\mathrm{d}y = 1$ and $k(x) = k(-x)$ for all $x \in \mathbb{R}$. The initial data φ belongs to C_+. Here $\mathcal{H} = \mathbb{R}$.

When $k(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$, (5.5) is the model derived by So et al. [27] to describe the growth of a single species matured population. On the other hand, when $k(x) = \delta(x)$, (5.5) reduces to the following delayed local reaction diffusion equation,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu f\big(u(t-\tau,x)\big), & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(\theta,x) = \varphi(\theta,x), \quad (\theta,x) \in [-1,0] \times \mathbb{R}. \end{cases}$$

Define

$$\big(T(t)(\phi)\big)(x) = \int_{\mathbb{R}} \frac{e^{-\mu t}}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right)\phi(y)\,\mathrm{d}y \tag{5.6}$$

and $T(0)(\phi) = \phi$ where $\phi \in X$, $t \in (0,\infty)$ and $x \in \mathbb{R}$.

We shall consider the following integral equation with the given initial function,

$$\begin{cases} u(t,\cdot) = T(t)\big(\varphi(0,\cdot)\big) + \displaystyle\int_0^t \mu T(t-s)\big(K\big[f\big(u(s-\tau,\cdot)\big)\big]\big)\,\mathrm{d}s, & t \geqslant 0, \\ u_0 = \varphi \in C_+, \end{cases} \tag{5.7}$$

where $T(t)$ is defined by (5.6) and $K : X \to X$ is defined by

$$K(\phi)(x) = \int_{\mathbb{R}} \phi(-\tau,y)k(x-y)\,\mathrm{d}y \quad \text{for } x \in \mathbb{R} \text{ and } \phi \in X_+. \tag{5.8}$$

By a simple application of the step-by-step argument, it is easy to see that, for $\varphi \in C_+$, the solution of (5.7) is unique and exists on $\mathbb{R}_+$, denoted by $u^{\varphi}(t,x)$; moreover, $(u^{\varphi}(\cdot,\cdot))_t \in C_+$ for all $t \in \mathbb{R}_+$.

Note that $\{T(t)|_{X^u}\}_{t\geqslant 0}$ is an analytic semigroup on X^u generated by the X^u-realization $\Delta_{X^u} - \mu\,\mathrm{Id}$ of $\Delta - \mu\,\mathrm{Id}$ (see, for example, Daners and Medina [3]), where $X^u = BUC(\mathbb{R},\mathbb{R})$ as the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the usual supremum norm $\|\cdot\|_{sup}$. Moreover, if f is a continuously differentiable function, then Corollary 2.2.5 in [35] and Remark 2.9 in [37] imply that a solution $u^{\varphi}(t,x)$ of (5.7) is also a classical solution of (5.5) for all $t > 3\tau$.

Lemma 5.9. *Let $T(t)$ and K be defined as in (5.6) and (5.8), respectively. Then the following results are true.*

(i) *$T(\cdot)$ satisfies assumptions* (T1)–(T5).
(ii) *K satisfies assumptions* (K1)–(K3).

Proof. (i) By (5.6) and by a simple computation, we easily see that $T(t)$ is a linear operator satisfying (T2)–(T3). By a similar proof as that of Lemma 2.4 in [37], we know that, for any $r > 0$, $T : \mathbb{R}_+ \times X_r \to X$ is continuous, where $X_r = \{\phi \in X_+ :\ \phi \leqslant r\}$ and $T(t,\phi) = T(t)(\phi)$ for all $(t,\phi) \in \mathbb{R}_+ \times X_r$. Thus, (T1) holds.

We shall prove that $T(t)[X_r]$ is precompact in X for all $t, r \in (0,\infty)$. Fix $t, r \in (0,\infty)$. By (T2), the Arzelà–Ascoli theorem and the compact and open topology of X, it suffices to show that, for any bounded and closed interval $I \equiv [a,b] \subseteq \mathbb{R}$, the set $A \equiv \{T(t)[\phi]|_I :\ \phi \in X_r\}$ is a family of equicontinuous functions in $C(I,\mathbb{R})$. In fact, for any $\varepsilon > 0$, there exists $K = K(\varepsilon) > 0$ such that $\int_{|y|\geqslant K} l(t,y)\,\mathrm{d}y < \frac{\varepsilon}{6r+1}$, where $l(t,x) = \frac{e^{-\mu t}}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$ for all $x \in \mathbb{R}$. Let $K^* = K + \max\{|a|,|b|,0\}$ and $I^* = [-K^*, K^*]$. Then there exists $\delta = \delta(\varepsilon, I) > 0$ such that $|l(t,z) - l(t,\tilde{z})| < \frac{\varepsilon}{1+3rK^*}$ for $z, \tilde{z} \in [-2K^*, 2K^*]$ satisfying $|z - \tilde{z}| < \delta$. It follows from the definition of $T(t)$ that, for any $\phi \in X_r$, $x, \tilde{x} \in I$ with $|x - \tilde{x}| < \delta$, we have

$$
\begin{aligned}
&\bigl|T(t)(\phi)(x)-T(t)(\phi)(\tilde{x})\bigr| \\
&\quad=\biggl|\int\limits_{\mathbb{R}}\phi(y)\bigl(l(t,x-y)-l(t,\tilde{x}-y)\bigr)\,\mathrm{d}y\biggr| \\
&\quad\leqslant\int\limits_{\mathbb{R}}\bigl|\phi(y)\bigr|\cdot\bigl|l(t,x-y)-l(t,\tilde{x}-y)\bigr|\,\mathrm{d}y \\
&\quad\leqslant\int\limits_{y\in[-K^*,K^*]}\bigl|\phi(y)\bigr|\cdot\bigl|l(t,x-y)-l(t,\tilde{x}-y)\bigr|\,\mathrm{d}y \\
&\qquad+\int\limits_{y\notin[-K^*,K^*]}\bigl|\phi(y)\bigr|\cdot\bigl|l(t,x-y)-l(t,\tilde{x}-y)\bigr|\,\mathrm{d}y \\
&\quad\leqslant\int\limits_{y\in[-K^*,K^*]}\bigl|\phi(y)\bigr|\cdot\bigl|l(t,x-y)-l(t,\tilde{x}-y)\bigr|\,\mathrm{d}y \\
&\qquad+\int\limits_{y\notin[-K,K]}\bigl|\phi(x+y)\bigr|l(t,-y)\,\mathrm{d}y+\int\limits_{y\notin[-K,K]}\bigl|\phi(\tilde{x}+y)\bigr|l(t,-y)\,\mathrm{d}y \\
&\quad\leqslant 2rK^*\frac{\varepsilon}{1+3rK^*}+2r\frac{\varepsilon}{6r+1} \\
&\quad<\varepsilon,
\end{aligned}
$$

that is, $A\equiv\{T(t)[\phi]|_I\colon\ \phi\in X_r\}$ is a family of equicontinuous functions.

(ii) By (5.8) and by a simple computation, we easily see that K is a linear operator satisfying (K2) and (K3). By a similar proof as that of Lemma 2.3 in [37], we know that for any $r>0$, $K: X_r\to X$ is continuous. Thus, (K1) holds. $\square$

The following result can be easily deduced from Theorem 5.7 with $\mathcal{H}=\mathbb{R}$ and Lemma 5.9.

Theorem 5.10. *Assume that* (H4) *holds. For system* (5.5)*, there exists* $c^*>0$ *such that the following statements are true.*

(i) *For any* $c>c^*$*, if* $\varphi\in C_{u^*}$ *has compact support with* $\varphi\ll u^*$ *then*

$$\lim_{t\to\infty}\max\bigl\{u^{\varphi}(t,x)\colon\ |x|\geqslant tc\bigr\}=0.$$

(ii) *For any* $c<c^*$ *and* $\varphi\in C_+\setminus\{0\}$*,*

$$\lim_{t\to\infty}\max\bigl\{\bigl|u^{\varphi}(t,x)-u^*\bigr|\colon\ |x|\leqslant tc\bigr\}=u^*.$$

(iii) *For any* $c<c^*$*,* (5.5) *has no travelling wave* U *such that* $U(\cdot,-\infty)=u^*$*.*
(iv) *For any* $c\geqslant c^*$*,* (5.5) *has a travelling wave solution* U *connecting* u^* *to* 0*.*

We remark that Fang and Zhao [6] studied a wide class of models including (5.5) as a special case. In particular, by using the classical upper and lower solution method, they addressed the existence of travelling waves under the strong nonlinear condition (HZ).

5.2. A delayed nonlocal lattice differential system

Consider

$$\begin{cases} \dfrac{du_j(t)}{dt} = D\Delta u_j(t) - du_j(t) + \dfrac{\mu}{2\pi}\displaystyle\sum_{k=-\infty}^{\infty} \beta_\alpha(j-k)b\big(u_k(t-\tau)\big), \quad j \in \mathbb{Z}, \\ u_j(\theta) = \varphi(\theta, j), \quad (\theta, j) \in [-\tau, 0] \times \mathbb{Z}, \end{cases} \tag{5.9}$$

where $\Delta u_j(t) = u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)$, $\beta_\alpha(l) = 2e^{-2\alpha}\int_0^\pi \cos(ly)e^{2\alpha\cos y}\,\mathrm{d}y$ and D, d, μ are all positive real numbers. System (5.9) was derived in [34] to model the growth of a single species matured population distributed over a patchy environment characterized by the lattice $\mathbb{Z}$. By Lemma 2.1 in [34], we know that $\beta_\alpha(l) = \beta_\alpha(|l|) \geqslant 0$ for all $l \in \mathbb{Z}$ and $\frac{1}{2\pi}\sum_{l=-\infty}^{\infty} \beta_\alpha(l) = 1$. Moreover, assume that the continuous function $f \equiv \frac{\mu b}{d} : \mathbb{R}_+ \ni u \mapsto \frac{\mu b(u)}{d} \in \mathbb{R}_+$ satisfies assumptions (H1)–(H4). By replacing b by f, we rewrite system (5.9) as follows,

$$\begin{cases} \dfrac{\partial u(t,j)}{\partial t} = D\Delta u(t,j) - du(t,j) + \dfrac{d}{2\pi}\displaystyle\sum_{k=-\infty}^{\infty} \beta_\alpha(j-k)f\big(u(t-\tau,k)\big), \quad j \in \mathbb{Z}, \\ u(\theta, j) = \varphi(\theta, j), \quad (\theta, j) \in [-\tau, 0] \times \mathbb{Z}, \end{cases} \tag{5.10}$$

where the initial data φ belongs to C_+. Here $\mathcal{H} = \mathbb{Z}$.

Define

$$\big(T(t)(\phi)\big)(j) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\left[\int_{-\pi}^{\pi} e^{i(j-k)y + D(2\cos y - 2) - d}\,\mathrm{d}y\right]\phi(k) \tag{5.11}$$

and $T(0)(\phi) = \phi$ where $\phi \in X$, $t \in (0, \infty)$ and $x \in \mathbb{R}$. By (5.11) and by simple computations, we may verify that $T(t)$ is a linear operator satisfying (T1)–(T5).

We shall consider the following integral equation with the given initial function,

$$\begin{cases} u(t,\cdot) = T(t)\big(\varphi(0,\cdot)\big) + \displaystyle\int_0^t \mu T(t-s)\big(K\big[f\big(u(s-\tau,\cdot)\big)\big]\big)\,\mathrm{d}s, \quad t \geqslant 0, \\ u_0 = \varphi \in C_+, \end{cases} \tag{5.12}$$

where $T(t)$ is defined by (5.11) and $K : X \to X$ is defined by

$$K(\phi)(j) = \frac{d}{2\pi}\sum_{k=-\infty}^{\infty} \beta_\alpha(j-k)\phi(k) \quad \text{for } x \in \mathbb{R} \text{ and } \phi \in X_+. \tag{5.13}$$

Obviously, K is a positive linear operator satisfying (K1)–(K3).

It is easy to see that for $\varphi \in C_+$ the solution of (5.10) is unique and exists on $\mathbb{R}_+$, denoted by $u^\varphi(t, j)$; moreover, $(u^\varphi)_t \in C_+$ for all $t \in \mathbb{R}_+$.

Therefore, applying Theorem 5.7 with $\mathcal{H} = \mathbb{Z}$, we can get the following result.

Theorem 5.11. *For system (5.10), there exists $c^* > 0$ such that the following statements are true.*

(i) *Let $u^\varphi(t, j)$ be a solution of (5.10) with $0 \leqslant \varphi(t, j)$ for all $t \in [-\tau, 0]$ and $j \in \mathbb{Z}$. Assume that $\varphi \ll u^*$, and $\varphi(t, j) = 0$ for all $t \in [-\tau, 0]$ and j outside a bounded interval. Then $\lim_{t\to\infty, |j|\geqslant tc} u^\varphi(t, j) = 0$ for any $c > c^*$.*

(ii) *Let $u^{\varphi}(t,j)$ be a solution of* (5.10) *with $0 \leqslant \varphi(t,j)$ for all $t \in [-\tau,0]$ and $j \in \mathbb{Z}$. If $\varphi(t,\cdot) \not\equiv 0$ for some $t \in [-\tau,0]$, then $\lim_{t\to\infty,\,|j|\leqslant tc} u^{\varphi}(t,j) = u^*$ for any $c < c^*$.*
(iii) *For any $c < c^*$,* (5.10) *has no travelling wave U connecting u^* to* 0.
(iv) *For any $c \geqslant c^*$,* (5.10) *has a travelling wave solution U connecting u^* to* 0.

5.3. Two concrete nonlinearities

In this subsection, we apply Theorem 5.10 and Theorem 5.11 respectively to system (5.5) and system (5.10) with two particular nonlinearities.

First, we consider the Ricker function $f(u) = que^{-pu}$, where $q > 1$ and $p > 0$. This is a widely used birth function (for fish population dynamics and for blowfly population, see, for example, [8,9,17,18, 23,25,26]). If $q \in (1, e^2]$, then by applying Theorems 5.10 and 5.11 and making use of the proof of Theorem 4.1 and Remark 4.3 in [38] we can obtain the following result.

Corollary 5.12. *Let $f(u) = que^{-pu}$ with $q > 1$ and $p > 0$. If $q \in (1, e^2]$ then both Theorems* 5.10 *and* 5.11 *hold.*

We should point out that Corollary 5.12 is not new and has been obtained by using the upper and lower solution method and monotone semiflows theory in [15]. See Theorems 3.3, 3.4, 4.1 and 4.2 in [6] for the nonlocal delayed reaction diffusion equation, and see [5] for the nonlocal delayed lattice differential equation.

Next, we assume the nonlinear function f to be the Mackey–Glass hematopoiesis function $f:\mathbb{R}_+ \to \mathbb{R}_+$, which is defined by $f(u) = \frac{pu}{q+u^m}$ for all $u \in \mathbb{R}_+$. This function was originally used by Mackey and Glass [21] to model the blood cell production in an ordinary differential equation model. Since then modified models have been studied by many researchers. One of the topics on these models is the stability of a positive equilibrium, accounting for a long term stable blood concentration level. See, for example, [12,28] and the references therein. Applying Theorems 5.10 and 5.11 and taking advantage of the proof of Theorem 4.2 and Remark 4.3 in [38], we have the following result.

Corollary 5.13. *Let $f(u) = \frac{pu}{q+u^m}$ with $p > q > 0$ and $m > 0$. If $m \leqslant \max\{2, \frac{2p}{p-q}\}$, then both Theorems* 5.10 *and* 5.11 *hold.*

To the best of our knowledge, Corollary 5.13 is new.

Clearly, function f in Corollaries 5.12 and 5.13 possess the property (P1*) in assumption (HZ).

To conclude this paper, we point out that property (P1*) in (HZ) is always assumed to get the asymptotic behavior in [5,6,11,39]. But our results do not need this property (P1*) in (HZ). Indeed, we assume that f satisfies assumptions (H1)–(H4). Take $u_1 = \frac{u^*}{f'(0)}$. Then $u_1 < u^*$ and $\max f([0,u_1]) \leqslant u^*$. Moreover, take $a, b, c \in (0, u_1)$ with $a < b < c$. Let $k = \frac{1}{2} + \frac{1}{2}\min\{\frac{f(a)}{a}, \frac{f(c)}{c}\}$. In view of $f(u) > u$ for all $u \in (0, u^*)$, we have $k \in (1, \min\{\frac{f(a)}{a}, \frac{f(c)}{c}\})$. Define $\tilde{f}:\mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{f}(u) = \begin{cases} f(u), & u \notin [a,c], \\ f(a) + \frac{kb-f(a)}{b-a}(u-a), & u \in [a,b], \\ f(c) + \frac{f(c)-kb}{c-b}(u-c), & u \in [b,c]. \end{cases}$$

Obviously, $\tilde{f}$ satisfies assumptions (H1)–(H4). But $\frac{\tilde{f}(u)}{u}$ is **not** decreasing on $[0,u^*]$ due to $\frac{\tilde{f}(b)}{b} = k < \frac{f(c)}{c} = \frac{\tilde{f}(c)}{c}$ and hence $\tilde{f}$ does **not** satisfy the property (P1*) in (HZ) which is used in [5,6,11,39]. Therefore, Theorems 5.10 and 5.11 can be applied to the above $\tilde{f}$ but the results in [5,6] **cannot**.

Corollary 5.14. *Assume that f satisfies assumptions* (H1)–(H4). *If $\tilde{f}$ is defined as above then both Theorems* 5.10 *and* 5.11 *hold with f being replaced by $\tilde{f}$.*

Acknowledgments

Research of Yi was supported partially by the National Natural Science Foundation of PR China (Grant No. 11171098) and the Hunan Provincial NSF (Grant No. 11JJ1001). Research of Chen was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and by the Ontario Early Researcher Award (ERA) Program. Research of Wu was supported partially by the Canada Research Chairs (CRC) Program, by Mprime, and by NSERC.

References

[1] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: Partial Differential Equations and Related Topics, in: Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5–49.
[2] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population dynamics, Adv. Math. 30 (1978) 33–76.
[3] D. Daners, P. Koch Medina, Abstract Evolution Equations, Periodic Problems and Applications, Pitman Res. Notes Math. Ser., vol. 279, Longman Scientific & Technical, Harlow, 1992.
[4] O. Diekmann, Thresholds and travelling waves for the geographical spread of infection, J. Math. Biol. 6 (1978) 109–130.
[5] J. Fang, J. Wei, X.-Q. Zhao, Spreading speeds and travelling waves for non-monotone time-delayed lattice equations, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 466 (2010) 1919–1934.
[6] J. Fang, X.-Q. Zhao, Existence and uniqueness of traveling waves for non-monotone integral equations with applications, J. Differential Equations 248 (2010) 2129–2226.
[7] T. Faria, W. Huang, J. Wu, Travelling waves for delayed reaction–diffusion equations with global response, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2006) 229–261.
[8] W. Gurney, S. Blythe, R. Nisbet, Nicholson's blowflies revisited, Nature 287 (1980) 17–21.
[9] I. Győri, S. Trofimchuk, Global attractivity in $x'(t) = -\delta x(t) + pf(x(t-\tau))$, Dynam. Systems Appl. 8 (1999) 197–210.
[10] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, American Mathematical Society, Providence, RI, 1988.
[11] S.-B. Hsu, X.-Q. Zhao, Spreading speeds and traveling waves for nonmonotone integrodifference equations, SIAM J. Math. Anal. 40 (2008) 776–789.
[12] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, London, 1993.
[13] B. Li, H.F. Weinberger, M.A. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, Math. Biosci. 196 (2005) 82–98.
[14] X. Liang, Y. Yi, X.-Q. Zhao, Spreading speeds and traveling waves for periodic evolution systems, J. Differential Equations 231 (2006) 57–77.
[15] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, Comm. Pure Appl. Math. 60 (2007) 1–40; Comm. Pure Appl. Math. 61 (2008) 137–138 (Erratum).
[16] X. Liang, X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Funct. Anal. 259 (2010) 857–903.
[17] E. Liz, Four theorems and one conjecture on the global asymptotic stability of delay differential equations, in: The First 60 Years of Nonlinear Analysis of Jean Mawhin, World Sci. Publ., River Edge, NJ, 2004, pp. 117–129.
[18] E. Liz, G. Röst, On the global attractivity of delay differential equations with unimodel feedback, Discrete Contin. Dyn. Syst. 24 (2009) 1215–1224.
[19] R. Lui, Biological growth and spread modeled by systems of recursions. I. Mathematical theory, Math. Biosci. 93 (1989) 269–295.
[20] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, J. Differential Equations 237 (2007) 259–277.
[21] M. Mackey, L. Glass, Oscillation and chaos in physiological control systems, Science 197 (1977) 287–289.
[22] M. Mei, J.W.-H. So, M.Y. Li, S.S.P. Shen, Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 579–594.
[23] A. Nicholson, An outline of the dynamics of animal populations, Aust. J. Zoology 2 (1954) 9–65.
[24] C. Ou, J. Wu, Persistence of wavefronts in delayed nonlocal reaction–diffusion equations, J. Differential Equations 235 (2007) 219–261.
[25] G. Röst, J. Wu, Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 463 (2007) 2655–2669.
[26] H.L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Math. Surveys Monogr., vol. 41, American Mathematical Society, Providence, RI, 1995.
[27] J.W.-H. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 457 (2001) 1841–1853.
[28] X.H. Tang, X. Zou, Stability of scalar delay differential equations with dominant delayed terms, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 951–968.
[29] H.R. Thieme, Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread, J. Math. Biol. 8 (1979) 173–187.
[30] H.R. Thieme, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction–diffusion models, J. Differential Equations 195 (2003) 430–470.

[31] A.I. Volpert, V.A. Volpert, V.A. Volpert, Traveling Wave Solutions of Parabolic Systems, Transl. Math. Monogr., vol. 140, American Mathematical Society, Providence, RI, 1994, translated from the Russian manuscript by James F. Heyda.
[32] H.F. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal. 13 (1982) 353–396.
[33] H.F. Weinberger, M.A. Lewis, B. Li, Analysis of linear determinacy for spread in cooperative models, J. Math. Biol. 45 (2002) 183–218.
[34] P. Weng, H. Huang, J. Wu, Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction, IMA J. Appl. Math. 68 (2003) 409–439.
[35] J. Wu, Theory and Applications of Partial Functional-Differential Equations, Appl. Math. Sci., vol. 119, Springer-Verlag, New York, 1996.
[36] J. Wu, X. Zou, Traveling wave fronts of reaction–diffusion systems with delay, J. Dynam. Differential Equations 13 (2001) 651–687.
[37] T. Yi, Y. Chen, J. Wu, The global asymptotic behavior of nonlocal delay reaction diffusion equation with unbounded domain, Z. Angew. Math. Phys. 63 (2012) 793–812.
[38] T. Yi, X. Zou, Map dynamics versus dynamics of associated delay reaction–diffusion equations with a Neumann condition, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 466 (2010) 2955–2973.
[39] X.-Q. Zhao, Global attractivity in a class of nonmonotone reaction–diffusion equations with time delay, Can. Appl. Math. Q. 17 (2009) 271–281.