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Global dynamics of delayed reaction–diffusion equations in unbounded domains

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Abstract. We consider a nonlocal delayed reaction–diffusion equation in an unbounded domain that includes some special cases arising from population dynamics. Due to the non-compactness of the spatial domain, the solution semiflow is not compact. We first show that, with respect to the compact open topology for the natural phase space, the solutions induce a compact and continuous semiflow Φ on a bounded and positively invariant set Y in $C_+ = C([-1, 0], X_+)$ that attracts every solution of the equation, where X_+ is the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $[0,\infty)$. Then, to overcome the difficulty in describing the global dynamics, we establish a priori estimate for nontrivial solutions after describing the delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator. The estimate enables us to show the permanence of the equation with respect to the compact open topology. With the help of the permanence, we can employ standard dynamical system theoretical arguments to establish the global attractivity of the nontrivial equilibrium. The main results are illustrated with the diffusive Nicholson's blowfly equation and the diffusive Mackey–Glass equation.

Mathematics Subject Classification. Primary 34K25, 35K57; Secondary 92D25.

Keywords. Compact open topology, Global attractivity, Nonlocal delayed reaction–diffusion equation, Permanence, Unbounded domain.

1. Introduction

Consider the following delayed reaction–diffusion equation

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x) = \mathrm{d}\Delta u(t,x) - \delta u(t,x) + \displaystyle\int_{\mathbb{R}} f(u(t-\tau,y))k_\alpha(x-y)\mathrm{d}y, & (t,x)\in(0,\infty)\times\mathbb{R},\\ u(\theta,x)=\varphi(\theta,x), & (\theta,x)\in[-\tau,0]\times\mathbb{R},\end{cases} \tag{1.1}$$

where $\mathbb{R} = (-\infty,\infty), \Delta$ is the Laplacian operator on $\mathbb{R}, \varphi : [-\tau, 0]\times\mathbb{R}\to\mathbb{R}$ is a bounded and continuous function, and $k_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{-\frac{x^2}{4\alpha}}$. Such an equation arises naturally from the interaction of intrinsic dynamics (birth and death) and the spatial diffusion in a structured population (Metz and Diekmann [24]). In particular, u in (1.1) can be regarded as the density of the matured individuals in a two-stage population (juvenile and adult, with a fixed maturation time τ), f is the birth rate nonlinearity, δ is the death rate, and the nonlocal integration $\int_{\mathbb{R}} f(u(t-\tau,y))k_\alpha(x-y)\mathrm{d}y$ is exactly the maturation rate of those juvenile individuals born at time τ ago. As juvenile individuals may also move around during the maturation period, the delayed nonlocal integration is required. The model derivation, detailed biological

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backgrounds, and historical accounts of the development can be found in the paper of So et al. [29] and in a recent survey by Gourley and Wu [11]. Note that if we let $\alpha \to 0^+$ in (1.1), then we obtain the following (local) delayed reaction–diffusion equation

$$\frac{\partial u}{\partial t}(t,x) = \mathrm{d}\Delta u(t,x) - \delta u(t,x) + f(u(t-\tau,x)).$$

In the case where the space is a bounded domain, the global dynamics of the semiflow generated by the model subject to either the Dirichlet or the Neumann boundary condition has been intensively and successfully studied (see, for example, [2,7,14,17,26,27,35–37]). The order-preserving property when f is monotone (or when there is an attractive interval where f is monotone) has been explored, in conjunction with the monotone dynamical systems theory, to establish various threshold dynamics. Roughly speaking, there exists a quantity depending on the parameters of the system such that if the quantity is less than 1 then the model has a unique equilibrium (the trivial equilibrium) and every nonnegative solution converges to this equilibrium, while if this quantity is larger than 1 then the model has an additional nonnegative equilibrium, which is globally attractive for all nonnegative and nontrivial solutions. Obtaining such a threshold result when f is non-monotone, even when the domain is bounded, is difficult (and of course, such a result may not hold as time lag may generate nonlinear oscillation through the Hopf bifurcation mechanism). Nevertheless, some progress has been made, at least for the diffusive Nicholson's equation (a special case where f takes a particular form, see Sect. 4) with homogeneous Neumann boundary condition [40] or Dirichlet boundary condition [38]. See also [41] for some relevant results for a more general nonlinearity f. Recently, by using the fluctuation method, Zhao [43] established the global attractivity of the positive steady state for a class of non-monotone time-delayed reaction–diffusion equations with local/nonlocal effect, subject to the Neumann boundary condition.

However, results about the global dynamics for the local/nonlocal delayed reaction–diffusion equation in an unbounded spatial domain are scarce, due to the non-compactness of the unbounded spatial domain and the difficulty in describing the dynamics of the solution semiflow near the trivial equilibrium. Until now, the existing results only focus on the global dynamics for the local/nonlocal delayed reaction–diffusion equation in an unbounded spatial domain when the solution semiflow essentially is monotone or the initial values enjoy compact supports [1,4,16]. Most studies have been devoted to the existence and other qualitative properties of traveling wave fronts. See [1,4–6,9–11,23,28,29,32–34] and the references therein.

This motivates us to investigate the global asymptotic behavior for (1.1). In fact, we shall study the following more general system,

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu \displaystyle\int_{\mathbb{R}} f(u(t-1,y))k(x-y)\mathrm{d}y, & (t,x)\in(0,\infty)\times\mathbb{R},\\ u(\theta,x) = \varphi(\theta,x), & (\theta,x)\in[-1,0]\times\mathbb{R},\end{cases} \tag{1.2}$$

where $\mu > 0, f : \mathbb{R}_+ = [0,\infty) \to \mathbb{R}_+$ is continuous with $f(0) = 0$, and $k : \mathbb{R} \to (0,\infty)$ is continuous with $\int_{\mathbb{R}} k(y)\mathrm{d}y = 1$. After rescaling, (1.1) is a special case of (1.2). We emphasize that here k is not necessarily even. The initial data φ belong to C_+, where $X = UBC(\mathbb{R},\mathbb{R})$ is the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $||\cdot||_X, X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}\}, C = C([-1,0],X)$ is the Banach space of continuous functions from $[-1,0]$ into X with the supremum norm $||\cdot||_C$, and $C_+ = C([-1,0],X_+) \subset C$.

We will consider the mild solution of system (1.2), which solves the following integral equation with the given initial function,

$$\begin{cases} u(t,\cdot) = \frac{1}{\mu}L^t(\varphi(0,\cdot)) + \int_0^t L^{t-s}(F(u_s))\mathrm{d}s, & t\geq 0,\\ u_0 = \varphi \in C_+,\end{cases} \tag{1.3}$$

where L^0 and L^t are, respectively, defined by (2.2) and (2.3), and $F : C_+ \to X_+$ is defined by

$$F(\varphi)(x) = \int_{\mathbb{R}} f(\varphi(-1, y))k(x - y)\mathrm{d}y \qquad \text{for } x \in \mathbb{R} \text{ and } \varphi \in C_+.$$

The remaining of this paper is organized as follows. In Sect. 2, we first show that there exists a bounded and positively invariant set Y in C_+ such that Y attracts every solution of (1.3). When Y is endowed with the compact open topology, we then prove that the solution map of (1.3) induces a compact and continuous semiflow Φ on Y. Due to the non-compactness of the unbounded spatial domain, when there is a nontrivial equilibrium, it is quite difficult to show that nontrivial solutions are expelled from a given neighborhood of the trivial equilibrium. In Sect. 3, to overcome this difficulty, through describing the delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator, we establish a priori estimate for nontrivial solutions. This estimate enables us first to show the permanence of the equation with respect to the compact open topology. Then, we employ standard dynamical system theoretical arguments to obtain the global attractivity of the nontrivial equilibrium of (1.3). We conclude the paper with the applications of our general results to the nonlocal delayed reaction–diffusion Nicholson's blowfly equation and the nonlocal delayed reaction–diffusion Mackey–Glass equation.

2. Preliminary results

Define $X = BUC(\mathbb{R}, \mathbb{R})$ as the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the usual supremum norm $\|\cdot\|_X$. Let $X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$, $X_+^\circ = \{\phi \in X : \phi(x) > 0 \text{ for all } x \in \mathbb{R}\}$. It follows that X_+ is a closed cone in X. Note that $X_+^\circ \neq \mathrm{Int}(X_+)$, for example, $f(x) = \mathrm{e}^{-|x|} \in X_+^\circ$ but $f \notin \mathrm{Int}(X_+)$. Let $C = C([-1, 0], X)$ be the Banach space of continuous functions from $[-1, 0]$ into X with the supremum norm $||\cdot||_C$, and let $C_+ = C([-1, 0], X_+)$ and $C_+^\circ = C([-1, 0], X_+^\circ)$. Clearly, C_+ is a closed cone of C but $C_+^\circ \neq \mathrm{Int}(C_+)$.

For convenience, we shall identify an element $\varphi \in C$ as a function from $[-1, 0] \times \mathbb{R}$ into $\mathbb{R}$. For $a \in \mathbb{R}, \hat{a} \in X$ is defined as $\hat{a}(x) = a$ for all $x \in \mathbb{R}$, and $\hat{\hat{a}} \in C$ is defined as $\hat{\hat{a}}(\theta) = \hat{a}$ for all $\theta \in [-1, 0]$.

For any $\phi, \psi \in X$, we write $\phi \geq_X \psi$ if $\phi - \psi \in X_+, \phi >_X \psi$ if $\phi \geq \psi$ and $\phi \neq \psi$, and $\phi \gg_X \psi$ if $\phi - \psi \in X_+^\circ$. Similarly, for any $\xi, \eta \in C$, we write $\xi \geq_C \eta$ if $\xi - \eta \in C_+, \xi >_C \eta$ if $\xi \geq_C \eta$ and $\xi \neq \eta, \xi \gg_C \eta$ if $\xi - \eta \in C_+^\circ$. For simplicity of notations, when there is no confusion about the spaces, for $a \in \mathbb{R}$, we write $\mathbf{a} \triangleq \hat{a}$ or $\mathbf{a} \triangleq \hat{\hat{a}}$; also, we just write $\geq, >, \gg$, and $||\cdot||$ for $\geq_*, >_*, \gg_*$, and $||\cdot||_*$, respectively, where $*$ stands for X or C.

For a real interval I, let $I + [-1, 0] = \{t + \theta : t \in I \text{ and } \theta \in [-1, 0]\}$. For $u : (I + [-1, 0]) \times \mathbb{R} \to \mathbb{R}$ and $t \in I$, we define $u_t(\cdot, \cdot) \in C$ by $u_t(\theta, x) = u(t + \theta, x)$ for all $\theta \in [-1, 0]$ and $x \in \mathbb{R}$.

For $x \in \mathbb{R}$ and $t > 0$, let

$$l(x) = \frac{\sqrt{\mu}}{2} \exp\left(-\sqrt{\mu x^2}\right) \qquad \text{and} \qquad l(t, x) = \frac{\mu \mathrm{e}^{-\mu t}}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

Define L, L^0, and $L^t : X \to X (t > 0)$, respectively, by

$$(L(\phi))(x) = \int_{\mathbb{R}} l(x - y)\phi(y)\mathrm{d}y, \tag{2.1}$$

$$(L^0(\phi))(x) = \mu\phi(x), \tag{2.2}$$

and

$$(L^t(\phi))(x) = \int_{\mathbb{R}} l(t, x - y)\phi(y)\mathrm{d}y, \tag{2.3}$$

where $\phi \in X$ and $x \in \mathbb{R}$. Clearly, $\{\frac{1}{\mu} L^t\}_{t \geq 0}$ is an analytic strongly continuous semigroup on X generated by the X-realization $\Delta_X - \mu Id$ of $\Delta - \mu Id$ (see, for example, Daners and Medina [3]).

Define $F : C_+ \to X_+$ by

$$F(\varphi)(x) = \int_{\mathbb{R}} f(\varphi(-1,y))k(x-y)\mathrm{d}y \qquad \text{for all } x \in \mathbb{R} \text{ and } \varphi \in C_+.$$

Associated with (1.2) is the following integral equation with the given initial function

$$\begin{cases} u(t,\cdot) = \frac{1}{\mu} L^t \varphi(0,\cdot) + \displaystyle\int_0^t L^{t-s} F(u_s)\mathrm{d}s, & t \geq 0, \\ u_0 = \varphi \in C_+. \end{cases} \tag{2.4}$$

For a given $\varphi \in C_+$, let $u^\varphi(t,x)$ represents a solution of (2.4), that is, a mild solution of (1.2) in the sense of Martin and Smith [21,22].

Throughout the remaining part of this paper, we make the following baseline assumptions for the nonlinearity f.

(H1). There exists $M > 0$ such that $f(x) \in (0, M]$ for all $x \in (0,\infty)$.
(H2). f is a continuously differentiable function on some right-neighborhood of 0.

By the step argument and the definition of F, it is easy to see that, for $\varphi \in C_+$, the solution of (2.4) is unique and exists on $\mathbb{R}_+$, denoted by u^φ; moreover, $(u^\varphi)_t \in C_+$ for all $t \in \mathbb{R}_+$. Thus, the solutions of (2.4) induce a continuous semiflow in C_+. Since the semigroup $\{\frac{1}{\mu}L^t\}_{t\geq 0}$ is analytic, by Corollary 2.2.5 [33], we know that a mild solution of (1.2) is also a classical solution of (1.2) for all $t > 1$ (for instance, see [21,22,31,33]). Therefore, as far as asymptotic behaviors are concerned, it is sufficient to consider only mild solutions.

Lemma 2.1. *Let L, L^0, and L^t be defined by (2.1)–(2.3), respectively. Then, the following statements are true.*

(i) $(L(\mathbf{1}))(x) = 1$ *for all* $x \in \mathbb{R}$.
(ii) $(L^t(\mathbf{a}))(x) = a\mu e^{-\mu t}$ *for all* $t \in \mathbb{R}_+, a \in \mathbb{R}$, *and* $x \in \mathbb{R}$.
(iii) $L^t(X_+) \subseteq X_+$ *for* $t \in \mathbb{R}_+$ *and* $L^t(X_+ \backslash \{\mathbf{0}\}) \subseteq X_+^\circ$ *for* $t > 0$.
(iv) $\int_{\mathbb{R}_+} l(t,x)\mathrm{d}t = l(x)$ *for all* $x \in \mathbb{R}$. *Hence,* $(L(\phi))(x) = \int_{\mathbb{R}_+} (L^t(\phi))(x)\mathrm{d}t$ *for all* $\phi \in X$ *and* $x \in \mathbb{R}$.

Proof. Obviously, statements (i–iii) follow directly from the explicit expressions of L and L^t.

Now, we show statement (iv). Letting $s = \sqrt{\mu t}$ and using the formula that $\int_0^\infty \mathrm{e}^{-(s^2+\frac{c^2}{s^2})}\mathrm{d}s = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-2c}$ (see Example 2 of Chapter 9.5 in [42]), we obtain that

$$\begin{aligned} \int_0^\infty \frac{\mu \mathrm{e}^{-\mu t}}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)\mathrm{d}t &= \sqrt{\frac{\mu}{\pi}} \int_0^\infty \exp\left(-\left(s^2 + \frac{\mu x^2}{4s^2}\right)\right)\mathrm{d}s \\ &= \sqrt{\frac{\mu}{\pi}} \times \frac{\sqrt{\pi}}{2}\mathrm{e}^{-2\sqrt{\frac{\mu x^2}{4}}} = l(x). \end{aligned}$$

This, combined with Fubini's theorem, implies that $\int_{\mathbb{R}_+}\int_{\mathbb{R}} l(t,x-y)\phi(y)\mathrm{d}y\mathrm{d}t = \int_{\mathbb{R}} l(x-y)\phi(y)\mathrm{d}y$ for any $\phi \in X$ and hence statement (iv) holds. □

Remark 2.2. If $t > 0$ and ϕ is a bounded and continuous function from $\mathbb{R}$ to $\mathbb{R}$, then $|(L^t(\phi))(x) - (L^t(\phi))(z)| \leq \frac{\mu||\phi||_X}{\sqrt{\pi t}}|x-z|$ for all $x, z \in \mathbb{R}$. Here, we have extended the operator L^t to the set of all bounded and continuous functions. Indeed, for any $x, z \in \mathbb{R}$,

$$
\begin{aligned}
&|(L^t(\phi))(x) - (L^t(\phi))(z)| \\
&\le \int_{\mathbb{R}} |l(t, x-y) - l(t, z-y)||\phi(y)|\mathrm{d}y \\
&\le ||\phi||_X \int_{\mathbb{R}} |l(t, x-y) - l(t, z-y)|\mathrm{d}y \\
&= \frac{\mu \mathrm{e}^{-\mu t}||\phi||_X}{\sqrt{4\pi t}} \int_{\mathbb{R}} \left| \exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(z-y)^2}{4t}\right) \right| \mathrm{d}y \\
&\le \frac{\mu \mathrm{e}^{-\mu t}||\phi||_X}{\sqrt{4\pi t}} \int_{\mathbb{R}} \int_0^1 \left| \frac{(x-z)(z-y+\theta(x-z))}{2t} \exp\left(-\frac{(z-y+\theta(x-z))^2}{4t}\right) \right| \mathrm{d}\theta \mathrm{d}y \\
&\le \frac{\mu ||\phi||_X}{\sqrt{4\pi t}} |x-z| \int_{\mathbb{R}} \frac{|y|}{2t} \exp\left(-\frac{y^2}{4t}\right) \mathrm{d}y \\
&= \frac{\mu ||\phi||_X}{\sqrt{\pi t}} |x-z|.
\end{aligned}
$$

However, due to the non-compactness of the spatial domain, it is generally difficult and inconvenient to describe the global asymptotic behaviors with respect to the supremum norm. To overcome this difficulty, we shall introduce a coarser topology such as the compact open topology. We now define some new norms $\|\cdot\|_{co}^X$ on X and $\|\cdot\|_{co}^C$ on C by $||\phi||_{co}^X \triangleq \sum_{n\in\mathbb{N}} 2^{-n} \sup\{|\phi(x)| : x \in [-n, n]\}$ for $\phi \in X$ and $||\varphi||_{co}^C = \sup\{||\varphi(\theta)||_{co}^X : \theta \in [-1, 0]\}$ for $\varphi \in C$, respectively, where $\mathbb{N} = \{1, 2, \ldots\}$. Again, for the simplicity of notation, when there is no confusion about the spaces involved, we just write $||\cdot||_{co}$ for one of the two norms we just defined. Moreover, we denote the normed vector spaces $(X, ||\cdot||_{co})$ and $(C, ||\cdot||_{co})$ by X_{co} and C_{co}, respectively.

Lemma 2.3. *Given $r > 0$, let $B_r = \{\phi \in X : ||\phi|| \le r\}$ and $d_r(\phi, \psi) = ||\phi - \psi||_{co}$ for ϕ, $\psi \in B_r$. Then, for any $\phi \in B_r$ and $\{\phi_n\}_{n\in\mathbb{N}} \subset B_r$, $\lim_{n\to\infty} d_r(\phi_n, \phi) = 0$ if and only if $\lim_{n\to\infty}(\sup\{|\phi_n(x) - \phi(x)| : x \in I\}) = 0$ for any bounded and closed interval $I \subseteq \mathbb{R}$.*

Lemma 2.4. *Define $K_1 : X \to X$ by*

$$K_1(\phi)(x) = \int_{\mathbb{R}} \phi(y) k(x-y) \mathrm{d}y \qquad \text{for all } x \in \mathbb{R} \text{ and } \phi \in X.$$

Then, the following statements are true:

(i) *$K_1(X_+) \subseteq X_+$ and $K_1(X_+ \backslash \{\mathbf{0}\}) \subseteq X_+^\circ$.*
(ii) *For any $r > 0$, $K_1(B_r) \subseteq B_r$ and $K_1|_{B_r} : B_r \to B_r$ is continuous, where B_r is equipped with the topology induced by d_r.*

Proof. (i) follows from the definition of K_1 and the fact that $k : \mathbb{R} \to (0, \infty)$.

Clearly, we know that $K_1(B_r) \subseteq B_r$. We next prove that $K_1|_{B_r}$ is continuous. Take $\{\phi_n\}_{n\in\mathbb{N}} \subset B_r$ and $\phi \in B_r$ such that $\lim_{n\to\infty} d_r(\phi_n, \phi) = 0$. By Lemma 2.3, we only need to show that $\lim_{n\to\infty}(\sup\{|K_1(\phi_n)(x) - K_1(\phi)(x)| : x \in I\}) = 0$ for any bounded and closed interval $I \equiv [a, b] \subseteq \mathbb{R}$. Indeed, for any $\varepsilon > 0$, there exists $T > 0$ such that $\int_{|y|\ge T} k(y)\mathrm{d}y < \frac{\varepsilon}{3r+1}$. Let $I^* = [a - T, b + T]$. By Lemma 2.3, there exists $n_0 > 1$ such that $|\phi_n(x) - \phi(x)| < \frac{\varepsilon}{3}$ for all $x \in I^*$ and $n \ge n_0$. It follows from the definition of K_1 that, for any $x \in I$ and $n \ge n_0$,

$$
\begin{aligned}
|K_1(\phi_n)(x) - K_1(\phi)(x)| &= |K_1(\phi_n - \phi)(x)| \\
&\le \int_{\mathbb{R}} |\phi_n(x+y) - \phi(x+y)| k(-y) \mathrm{d}y \\
&= \int_{y\in[-T,T]} |\phi_n(x+y) - \phi(x+y)| k(-y) \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{y\notin[-T,T]}|\phi_n(x+y)-\phi(x+y)|k(-y)\mathrm{d}y\\
&\leq \frac{\varepsilon}{3}+2r\cdot\frac{\varepsilon}{3r+1}\\
&<\varepsilon.
\end{aligned}
$$

This means that $K_1|_{B_r}$ is continuous and hence statement (ii) is proved. □

Lemma 2.5. *Assume that $L^t(\cdot)$ is defined by (2.3). Then, the following statements are true:*

(i) *For any $r>0$ and $t\in\mathbb{R}_+$, $L^t(B_r)\subseteq B_{\mu r}$.*

(ii) *Let $r>0$ and define $\mathcal{L}:\mathbb{R}_+\times B_r\to B_{\mu r}$ by $\mathcal{L}(t,\phi)=L^t(\phi)$ for all $(t,\phi)\in\mathbb{R}_+\times B_r$. Let B_r and $B_{\mu r}$ be equipped with the topologies induced by d_r and $d_{\mu r}$, respectively. Then, $\mathcal{L}$ is a continuous map.*

Proof. Obviously, statement (i) follows from the explicit expression of L^t.

To prove (ii), we first claim that, for any bounded closed interval I, $s>0$ and $\varepsilon>0$, there exists $\delta>0$ such that $|\mathcal{L}(t,\phi)(x)|<\varepsilon$ for all $\phi\in B_r$, $x\in I$, and $t\in(0,s]$ with $||\phi||_{co}<\delta$. Indeed, there exists $T>0$ such that $\int_{|y|\geq T}\mathrm{e}^{-y^2}\mathrm{d}y\leq\frac{\sqrt{\pi}}{3r\mu}\varepsilon$. Let $I\equiv[a,b]\subseteq\mathbb{R}$ and let $I^*=[a-\sqrt{4s}T,b+\sqrt{4s}T]$. Take $\delta>0$ such that $|\phi(x)|<\frac{\varepsilon}{3\mu}$ for all $x\in I^*$ and $\phi\in B_r$ with $||\phi||_{co}<\delta$. Then, we easily see that for any $\phi\in B_r$, $x\in I$, and $t\in[0,s]$ with $||\phi||_{co}<\delta$,

$$
\begin{aligned}
|\mathcal{L}(t,\phi)(x)|&=\left|\int_{\mathbb{R}}\phi(x-y)l(t,y)\mathrm{d}y\right|\\
&\leq\int_{y\in[-\sqrt{4t}T,\sqrt{4t}T]}|\phi(x-y)|l(t,y)\mathrm{d}y+\int_{y\notin[-\sqrt{4t}T,\sqrt{4t}T]}|\phi(x-y)|l(t,y)\mathrm{d}y\\
&=\frac{\mu\mathrm{e}^{-\mu t}}{\sqrt{\pi}}\left[\int_{z\in[-T,T]}|\phi(x-\sqrt{4t}z)|\mathrm{e}^{-z^2}\mathrm{d}z+\int_{z\notin[-T,T]}|\phi(x-\sqrt{4t}z)|\mathrm{e}^{-z^2}\mathrm{d}z\right]\\
&\leq\frac{\mu}{\sqrt{\pi}}\left[\frac{\varepsilon}{3\mu}\int_{z\in[-T,T]}\mathrm{e}^{-z^2}\mathrm{d}z+r\int_{z\notin[-T,T]}\mathrm{e}^{-z^2}\mathrm{d}z\right]\\
&<\varepsilon.
\end{aligned}
$$

This proves the claim.

Take $\{(t_n,\phi_n)\}_{n\in\mathbb{N}}\subset\mathbb{R}_+\times B_r$ and $(t,\phi)\in\mathbb{R}_+\times B_r$ such that $\lim_{n\to\infty}|t_n-t|=0$ and $\lim_{n\to\infty}d_r(\phi_n,\phi)=0$. Given a bounded closed interval I and a positive number ε, then, by the above claim, there exists a positive integer N_0 such that $|\mathcal{L}(t_n,\phi_n-\phi)(x)|<\frac{\varepsilon}{3}$ for all $x\in I$ and $n>N_0$. Because $\{\frac{1}{\mu}L^t\}_{t\geq0}$ is a strongly continuous semigroup on X, there exists a positive integer N_1 such that $|\mathcal{L}(t_n,\phi)(x)-\mathcal{L}(t,\phi)(x)|<\frac{\varepsilon}{3}$ for all $x\in I$ and $n>N_1$. It follows from the linearity of L^t that, for all $x\in I$ and $n>\max\{N_0,N_1\}$, we have

$$
\begin{aligned}
|\mathcal{L}(t_n,\phi_n)(x)-\mathcal{L}(t,\phi)(x)|&=|\mathcal{L}(t_n,\phi_n-\phi)(x)|+|\mathcal{L}(t_n,\phi)(x)-\mathcal{L}(t,\phi)(x)|\\
&\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\\
&<\varepsilon.
\end{aligned}
$$

So, $\mathcal{L}$ is continuous. □

Proposition 2.6. *The following statements are true:*

(i) *If $\varphi\in C_+\backslash\{\mathbf{0}\}$, then $u^\varphi(t,x)>0$ for all $(t,x)\in(0,\infty)\times\mathbb{R}$.*

(ii) *For any $\varphi\in C_+$, there exists $T=T(\varphi)>0$ such that $u^\varphi(t,x)\leq M+1$ for all $(t,x)\in[T,\infty)\times\mathbb{R}$.*

(iii) *If $\varphi\in C_+$ and $\varphi(\theta,x)\leq M+1$ for all $(\theta,x)\in[-1,0]\times\mathbb{R}$, then $u^\varphi(t,x)\in[0,M+1]$ for all $(t,x)\in[-1,\infty)\times\mathbb{R}$.*

Proof. (i) follows from (2.4) and Lemmas 2.1(iii) and 2.4(i).

Now, suppose $\varphi \in C_+$. Then, there exists $A = A(\varphi) > 0$ such that $||\varphi|| \leq A$. It follows from (2.4), (H1), Lemmas 2.1(ii–iii) and 2.4(i) that

$$\begin{aligned} u^\varphi(t,x) &= \frac{1}{\mu}L^t(\varphi(0,\cdot))(x) + \int_0^t L^{t-s}(K_1(f(u^\varphi(s-1,\cdot))))(x)\mathrm{d}s \\ &\leq \frac{1}{\mu}L^t(\mathbf{A})(x) + \int_0^t L^{t-s}(K_1(\mathbf{M}))(x)\mathrm{d}s \\ &= A\mathrm{e}^{-\mu t} + M(1-\mathrm{e}^{-\mu t}), \end{aligned}$$

which implies that there exists $T = T(\varphi) > 0$ such that $u^\varphi(t,x) \leq M+1$ for all $(t,x) \in [T,\infty)\times\mathbb{R}$. This proves (ii).

The proof of (iii) is similar to that of (ii) and hence is omitted. □

Let

$$W = \{\phi \in X_+ : \phi(x) \leq 1 + M \text{ for all } x \in \mathbb{R}\}$$

and

$$Y = \{\varphi \in C_+ : \varphi(\theta,x) \leq 1+M \text{ for all } (\theta,x) \in [-1,0]\times\mathbb{R}\}. \tag{2.5}$$

By Proposition 2.6, we know that Y is a positively invariant set of the solution semiflow and Y attracts every point in C_+ in the sense of Hale [13]. In the following, we shall study the asymptotic behavior of the solution semiflow in Y. Define $d_w : W \times W \to \mathbb{R}_+$, $d : Y\times Y \to \mathbb{R}_+$ and $\Phi : \mathbb{R}_+ \times Y \to Y$ as follows,

$$\begin{aligned} d_w(\phi,\psi) &= ||\phi-\psi||_{co} && \text{for all } (\phi,\psi) \in W\times W, \\ d(\xi,\eta) &= ||\xi-\eta||_{co} && \text{for all } (\xi,\eta)\in Y\times Y, \\ \Phi(t,\varphi) &= (u^\varphi)_t && \text{for all } (t,\varphi)\in\mathbb{R}_+\times Y. \end{aligned} \tag{2.6}$$

The following result can be easily shown.

Lemma 2.7. *Assume that Y is defined by (2.5). Then, the following results are true:*

- (i) *Y is a bounded subset of C_{co}.*
- (ii) *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset Y$ and $\varphi \in Y$. Then $\lim_{n\to\infty} d(\varphi_n,\varphi) = 0$ if and only if $\lim_{t\to\infty}(\sup\{|\varphi_n(\theta,x) - \varphi(\theta,x)| : (\theta,x)\in[-1,0]\times I\}) = 0$ for any bounded and closed interval $I \subseteq \mathbb{R}$.*
- (iii) *Let $A \subseteq Y \cap C^{1,1}([-1,0]\times\mathbb{R},\mathbb{R})$, where $C^{1,1}([-1,0]\times\mathbb{R},\mathbb{R})$ is the set of all continuously differentiable functions from $[-1,0]\times\mathbb{R}$ to $\mathbb{R}$. If there is $\kappa > 0$ such that $|\frac{\partial\varphi(\theta,x)}{\partial\theta}| + |\frac{\partial\varphi(\theta,x)}{\partial x}| \leq \kappa$ for all $(\theta,x)\in[-1,0]\times\mathbb{R}$ and $\varphi \in A$, then A is pre-compact in Y.*

In what follows, we always assume that the topologies of W and Y are induced by d_w and d, respectively.

Theorem 2.8. *Assume that Y and Φ are defined, respectively, by (2.5) and (2.6). Then, the following statements are true:*

- (i) *Φ is a continuous semiflow on Y.*
- (ii) *There exists $t_0 > 1$ such that $\Phi(t,\cdot)$ is compact with respect to the compact open topology for $t > t_0+1$.*

Proof. It follows easily from the definition of Φ that $\Phi(0,\varphi) = \varphi$ and $\Phi(t+s,\varphi) = \Phi(t,\Phi(s,\varphi))$ for all $\varphi \in Y$ and $s,t\in\mathbb{R}_+$, that is, Φ is a semigroup on Y.

Define $g : \mathbb{R}_+\times Y \to W$ by $g(t,\varphi) = \Phi(t,\varphi)(0,\cdot)$ for all $(t,\varphi)\in\mathbb{R}_+\times Y$. To prove the continuity of Φ, it suffices to prove the continuity of g. Clearly, by (2.4), Lemmas 2.3(i), 2.4(ii), 2.5(ii), and 2.7(ii), some standard arguments easily yield that $g(t,\varphi)$ is continuous in $(t,\varphi)\in[0,1]\times Y$. Then, this and the semigroup property of Φ imply the continuity of g and hence Φ is continuous. This proves (i).

Now, we prove (ii). Since the semigroup $\{\frac{1}{\mu}L^t\}_{t\geq 0}$ is analytic, by Corollary 2.2.5 [33], we know that a mild solution of (1.2) is also a classical solution of (1.2) for all $t > 1$. In particular, $u^\varphi(\cdot,\cdot) \in C^{1,2}((1,\infty)\times\mathbb{R},\mathbb{R})$, where $C^{1,2}((1,\infty)\times\mathbb{R})$ is the set of all functions from $(1,\infty)\times\mathbb{R}$ to $\mathbb{R}$, which are continuously differentiable with respect to the first variable and twice continuously differentiable with respect to the second variable. By applying the standard parabolic estimates (see the proof of Proposition 4.3 in [32]), there exists $t_0 > 1$ and $M^* > 0$ such that $|\frac{\partial u^\varphi(t,x)}{\partial t}|+|\frac{\partial u^\varphi(t,x)}{\partial x}| < M^*$ for all $(t,x) \in [t_0,\infty)\times\mathbb{R}$ and $\varphi \in Y$. Thus, by Lemma 2.7(iii), we know that $\Phi(t,\cdot)$ is compact with respect to the compact open topology for $t > t_0 + 1$. □

Remark 2.9. We point out that the same asymptotic behaviors of the integral equation (2.4) hold when the initial value φ is a bounded and continuous function from $[-1,0]\times\mathbb{R}$ to $\mathbb{R}_+$. By the step argument and the definition of F, it is easy to see that the solution $(u^\varphi)_t$ of the integral equation (2.4) is well defined for all $t \in \mathbb{R}_+$. We easily see that all the conclusions of Proposition 2.6 hold when the initial value φ is a bounded and continuous function from $[-1,0]\times\mathbb{R}$ to $\mathbb{R}_+$. We only show that $(u^\varphi)_t \in C_+$ for all $(t,\varphi) \in [2,\infty)\times C([-1,0]\times\mathbb{R},[0,1+M])$. Indeed, for any $(t,\varphi) \in [1,2]\times C([-1,0]\times\mathbb{R},[0,1+M])$, and $x, z \in \mathbb{R}$, it follows from (2.4) and Remark 2.2 that

$$\begin{aligned}
|u^\varphi(t,x) - u^\varphi(t,z)| &\leq \frac{1}{\mu}|L^t(\varphi(0,\cdot))(x) - L^t(\varphi(0,\cdot))(z)| \\
&\quad + \int_0^t |L^{t-s}(F(u_s^\varphi))(x) - L^{t-s}(F(u_s^\varphi))(z)|\mathrm{d}s \\
&\leq \frac{||\varphi(0,\cdot)||_X}{\sqrt{\pi t}}|x-z| + \int_0^t \frac{\mu||F(u_s^\varphi)||_X}{\sqrt{\pi(t-s)}}|x-z|\mathrm{d}s \\
&\leq \frac{M+1}{\sqrt{\pi t}}|x-z| + \int_0^t \frac{\mu M}{\sqrt{\pi(t-s)}}|x-z|\mathrm{d}s \\
&= \left(\frac{M+1}{\sqrt{\pi t}} + \frac{2\mu M\sqrt{t}}{\sqrt{\pi}}\right)|x-z| \\
&\leq \left(\frac{M+1}{\sqrt{\pi}} + \frac{2\sqrt{2}\mu M}{\sqrt{\pi}}\right)|x-z|,
\end{aligned}$$

which, combined with the semigroup property of Φ, yields the claim.

Definition 2.10. An element $\varphi \in Y$ is called an equilibrium of Φ if $\Phi(t,\varphi) = \varphi$ for all $t \in \mathbb{R}_+$. A subset $\mathcal{A}$ of Y is said to be positively invariant under Φ if $\Phi(t,\varphi) \in \mathcal{A}$ for every $\varphi \in \mathcal{A}$ and $t \in \mathbb{R}_+$.

We write $O(\varphi) = \{\Phi(t,\varphi) : t \in \mathbb{R}_+\}$ for the positive semi-orbit through the point φ. The ω-limit set of $O(\varphi)$ is defined by $\omega(\varphi) = \bigcap_{t\in\mathbb{R}_+}\overline{O(\Phi(t,\varphi))}$, where $\overline{O(\Phi(t,\varphi))}$ represents the closure of $O(\Phi(t,\varphi))$ with respect to the compact open topology.

Definition 2.11. We say that the Eq. (2.4) is permanent with respect to the compact open topology if there exists $0 < a < b$ such that

$$\lim_{t\to\infty}(\inf\{||(u^\varphi)_t - \psi||_{co} : \psi \in C_+ \text{ with } \mathbf{a} \leq \psi \leq \mathbf{b}\}) = 0 \text{ for all } \varphi \in C_+\backslash\{\mathbf{0}\}.$$

The following result follows from Proposition 2.6 (i–ii) and the above definition.

Corollary 2.12. *If there exists $a > 0$ such that $\xi \geq \mathbf{a}$ for any $\varphi \in Y\backslash\{\mathbf{0}\}$ and $\xi \in \omega(\varphi)$, then the Eq. (2.4) is permanent with respect to the compact open topology.*

Definition 2.13. Let $\mathbf{u}^*$ be an equilibrium and $\mathcal{A}$ be a positively invariant set of the semiflow Φ. We say that $\mathbf{u}^*$ is globally attractive in $\mathcal{A}$ if $\omega(\varphi) = \{\mathbf{u}^*\}$ for all $\varphi \in \mathcal{A}$.

Definition 2.14. We say that $\mathbf{0}$ is globally attractive in C_+ with respect to the usual supremum norm if $\lim_{t\to\infty} ||(u^{\varphi})_t||_C = 0$ for all $\varphi \in C_+$.

Definition 2.15. Let $\mathbf{u}^*$ be an equilibrium. We say that $\mathbf{u}^*$ is globally attractive in $C_+\backslash\{\mathbf{0}\}$ with respect to the compact open topology if $\lim_{t\to\infty} ||(u^{\varphi})_t - \mathbf{u}^*||_{co} = 0$ for all $\varphi \in C_+\backslash\{\mathbf{0}\}$.

In the sequel, we shall omit the term "with respect to the compact open topology" in Definition 2.15. The following result follows from Proposition 2.6 and the above definitions.

Corollary 2.16. *If $\mathbf{u}^*$ is a globally attractive equilibrium in $Y\backslash\{\mathbf{0}\}$, then $\mathbf{u}^*$ is a globally attractive equilibrium in $C_+\backslash\{\mathbf{0}\}$.*

It is not difficult to establish the following result. For the sake of completeness, the proof is provided.

Theorem 2.17. *If $f(x) < x$ for all $x > 0$, then $\mathbf{0}$ is a globally attractive equilibrium of (2.4) in C_+ with respect to the usual supremum norm.*

Proof. Suppose that $\varphi \in C_+$. By Proposition 2.6(ii–iii), without loss of generality, we may assume that $\varphi \in Y$. Let $u_+ = \limsup_{t\to\infty} ||u^{\varphi}(t,\cdot)||_X$. It suffices to show that $u_+ = 0$. By the way of contradiction, suppose $u_+ > 0$. It follows from the assumption that there exists $u_{++} > u_+$ such that $f_+ \triangleq \max f([0, u_{++}]) < u_+$. Then, by the definition of u_+, there exists $T^* > 0$ such that $u^{\varphi}(t,x) < u_{++}$ for all $(t,x) \in [T^*,\infty)\times\mathbb{R}$. For any $t \geq T^*+1$ and $x \in \mathbb{R}_+$, it follows from (2.4), the semigroup property of Φ and the choices of u_+, u_{++}, and T^* that

$$\begin{aligned}
u^{\varphi}(t,x) &= u^{(u^{\varphi})_{1+T^*}}(t-1-T^*,x)\\
&= \tfrac{1}{\mu}L^{t-1-T^*}(u^{\varphi}(1+T^*,\cdot))(x) + \int_0^{t-1-T^*} L^{t-1-T^*-s}(K_1(f(u^{\xi}(s+T^*,\cdot))))(x)\mathrm{d}s\\
&\leq \tfrac{1}{\mu}L^{t-1-T^*}(\mathbf{u}_{++})(x) + \int_0^{t-1-T^*} L^{t-1-T^*-s}K_1(\mathbf{f}_+)(x)\mathrm{d}s\\
&= \mathrm{e}^{-\mu(t-1-T^*)}u_{++} + f_+(1-\mathrm{e}^{-\mu(t-1-T^*)}).
\end{aligned}$$

This implies that $u_+ \leq f_+$, a contradiction to the choices of u_+ and f_+. Therefore, $u_+ = 0$ and the proof is complete. □

Clearly, it follows from the assumption of Theorem 2.17 that $f'(0) \leq 1$. If $f'(0) > 1$, then $\mathbf{0}$ is not a locally attractive equilibrium. In the coming section, we tackle the global dynamics of (2.4) when $f'(0) > 1$.

3. Permanence and global attractivity

In this section, we always assume that $f'(0) > 1$. In this case, to overcome the difficulty in showing that the trivial equilibrium expels nontrivial solutions due to the lack of compactness of the spatial domain, we establish a priori estimate for nontrivial solutions after describing the delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator. The estimate enables us to show the permanence of the system. Then, we obtain the global attractivity of the nontrivial equilibrium by employing standard dynamical system theoretical arguments.

For $t \geq 0$, we define K and $K(t,\cdot) : X \to X$, respectively, by

$$K(\phi)(x) = \int_{\mathbb{R}^2} \phi(y)k(z-y)l(x-z)\mathrm{d}y\mathrm{d}z$$

and

$$K(t,\phi)(x) = \int_{\mathbb{R}^2} \phi(y)k(z-y)\int_0^t l(t,x-z)\mathrm{d}t\mathrm{d}y\mathrm{d}z$$

for all $\phi \in X$ and $x \in \mathbb{R}$. Obviously, K and $K(t,\cdot)$ are linear on X. It is easy to see that all the operators L^0, L^t, K_1, K, and $K(t,\cdot)$ can be extended to the linear space of all measurable and bounded functions from $\mathbb{R}$ to $\mathbb{R}$ into itself. Moreover, the extended operators are order preserving in the sense of the pointwise order. In the sequel, the operators are the extended ones.

For given positive numbers δ and T, define the function $h_\delta^T : \mathbb{R} \to \mathbb{R}_+$ by $h_\delta^T(x) = \delta$ for all $x \in [-T, T]$ and $h_\delta^T(x) = 0$ for all $x \notin [-T, T]$. Define the functions $h_\pm : \mathbb{R} \to \mathbb{R}$ by $h_\pm(x) = 1$ for all $x \in I_\pm$ and $h_\pm(x) = 0$ for all $x \notin I_\pm$, where $I_- = (-\infty, 0]$ and $I_+ = \mathbb{R}_+$. Let $a_n^\pm = (K^n(h_\pm)(0))^{\frac{1}{n}}$ and $a_n = \min\{a_n^-, a_n^+\}$ for all $n \in \mathbb{N}$, where K^n represents the nth-composition of K.

Lemma 3.1. *The following statements are true:*

(i) $a_n^\pm \in (0, 1]$ *for all* $n \in \mathbb{N}$.
(ii) $a_{mn}^\pm \geq a_m^\pm$ *for all* m, $n \in \mathbb{N}$.
(iii) $\limsup_{n\to\infty} a_n^\pm \geq a_m^\pm$ *for all* $m \in \mathbb{N}$.
(iv) *For any* $n \in \mathbb{N}$ *and* $\delta > 0$, *there exists* $T_{n,\delta} > 0$ *such that* $K^n(h_1^T) \geq h^T_{(a_n)^n - \delta}$ *for all* $T \geq T_{n,\delta}$.

Proof. (i) follows directly from the definitions of $a_n^\pm$.

To prove (ii), let $D_\pm^n = \{\mathbf{y} = (y_1, z_1, y_2, z_2, \ldots, y_n, z_n) \in \mathbb{R}^{2n} : \sum_{i=1}^n (y_i + z_i) \in I_\pm\}$, where $n \in \mathbb{N}$. Define $g_n : \mathbb{R}^{2n} \to \mathbb{R}$ by $g_n(\mathbf{y}) = \prod_{i=1}^n (k_1(-y_i)l(-z_i))$ for all $\mathbf{y} \in \mathbb{R}^{2n}$. We claim that $K^n(h_\pm)(0) = \int_{D_\pm^n} g_n(\mathbf{y})\mathrm{d}\mathbf{y}$ for all $n \in \mathbb{N}$. Actually, by Fubini's Theorem, for any $x \in \mathbb{R}$ and any measurable and bounded function ζ from $\mathbb{R}$ to $\mathbb{R}$,

$$K^n(\zeta)(x) = \int_{\mathbb{R}^{2n}} \zeta(y_1)k(z_n - y_n)l(x - z_n) \prod_{i=1}^{n-1} (k(z_i - y_i)l(y_{i+1} - z_i)) \prod_{i=1}^{n} \mathrm{d}y_i \mathrm{d}z_i.$$

It follows from the linear transformations of variables that

$$K^n(\zeta)(x) = \int_{\mathbb{R}^{2n}} \zeta\left(x + \sum_{i=1}^n (y_i + z_i)\right) \prod_{i=1}^n (k(-y_i)l(-z_i)) \prod_{i=1}^n \mathrm{d}y_i \mathrm{d}z_i.$$

Then, $K^n(\zeta)(x) = \int_{\mathbb{R}^{2n}} \zeta(x + \sum_{i=1}^n (y_i + z_i))g_n(\mathbf{y})\mathrm{d}\mathbf{y}$. Letting $\zeta = h_\pm$, we have proved the claim.

For any $m, n \in \mathbb{N}$, the above claim, combined with the definitions of $g_m, g_{mn}, D_\pm^m$, and $D_\pm^{mn}$, gives us

$$\begin{aligned}(a_{mn}^\pm)^{mn} &= K^{mn}(h_\pm)(0) \\ &= \int_{D_\pm^{mn}} g_{mn}(\mathbf{y})\mathrm{d}\mathbf{y} \\ &\geq \left(\int_{D_\pm^m} g_m(\mathbf{y})\mathrm{d}\mathbf{y}\right)^n \\ &= (a_m^\pm)^{mn}.\end{aligned}$$

This gives $a_{mn}^\pm \geq a_m^\pm$ and hence (ii) is proved.

For any $m \in \mathbb{N}$, statement (ii) implies that $\{a_{nm}^\pm\}_{n\in\mathbb{N}}$ is a subsequence such that $a_{nm}^\pm \geq a_m^\pm$ for all $n \in \mathbb{N}$. It follows immediately that $\limsup_{n\to\infty} a_n^\pm \geq \limsup_{n\to\infty} a_{nm}^\pm \geq a_m^\pm$, that is, (iii) holds.

Finally, suppose that $n \in \mathbb{N}$ and $\delta > 0$. Let $I_-^T = [-T, 0]$, $I_+^T = [0, T]$, and $D_\pm^T = \{\mathbf{y} = (y_1, z_1, y_2, z_2, \ldots, y_n, z_n) \in \mathbb{R}^{2n} : \sum_{i=1}^n (y_i + z_i) \in I_\pm^T\}$ for all $T \in \mathbb{R}_+$. Define $\eta_\pm : \mathbb{R}_+ \to \mathbb{R}$ by $\eta_\pm(T) = \int_{D_\pm^T} g_n(\mathbf{y})\mathrm{d}\mathbf{y}$ for all $T \in \mathbb{R}_+$. Obviously, both η_- and η_+ are continuous and increasing on $\mathbb{R}_+$ with $\lim_{T\to\infty} \eta_\pm(T) = (a_n^\pm)^n$. Therefore, there exists $T_{n,\delta} > 0$ such that $\eta_\pm(T) \geq (a_n^\pm)^n - \delta$ for all $T \geq T_{n,\delta}$. Since $D_-^T \subseteq B(x, T) \triangleq \{(y_1, z_1, y_2, z_2, \ldots, y_n, z_n) \in \mathbb{R}^{2n} : \sum_{i=1}^n x + (y_i + z_i) \in [-T, T]\}$ for all $x \in [0, T]$, we know that if $T \geq T_{n,\delta}$, then $K^n(h_1^T)(x) = \int_{\mathbb{R}^{2n}} h_1^T(x + \sum_{i=1}^n (y_i + z_i))g_n(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{\mathbf{y}\in B(x,T)} g_n(\mathbf{y})\mathrm{d}\mathbf{y} \geq \eta_-(T) \geq (a_n^-)^n - \delta$ for all $x \in [0, T]$. Similarly, $K^n(h_1^T)(x) \geq \eta_+(T) \geq (a_n^+)^n - \delta$

for all $x \in [-T, 0]$. Thus, $K^n(h_1^T) \geq h^T_{(a_n)^n-\delta}$ for all $T \geq T_{n,\delta}$. This proves (iv) and hence the proof is complete. □

Lemma 3.2. *Suppose that* $n \in \mathbb{N}$. *Then,* $\lim_{t\to\infty}(\sup\{|(K(t,\cdot))^n(h_1^T)(x) - K^n(h_1^T)(x)| : x \in \mathbb{R}$ *and* $T \in \mathbb{R}_+\}) = 0$.

Proof. It follows easily from Lemma 2.1(iv) that $K^n(h_1^T)(x) - (K(t,\cdot))^n(h_1^T)(x) \geq 0$ for any $T \in \mathbb{R}_+$ and $x \in \mathbb{R}$. We claim that $K^n(h_1^T)(x) - (K(t,\cdot))^n(h_1^T)(x) \leq n\mathrm{e}^{-\mu t}$ for any $T \in \mathbb{R}_+$ and $x \in \mathbb{R}$. We prove the claim by the mathematical induction. First, suppose $n = 1$. Then, by Fubini's theorem and the fact that $\int_{\mathbb{R}_+} l(t,x)\mathrm{d}t = l(x)$ for all $x \in \mathbb{R}$, for any $T \in \mathbb{R}_+$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} K\left(h_1^T\right)(x) - K(t,\cdot)\left(h_1^T\right)(x) &= \int_t^\infty L^s K_1\left(h_1^T\right)(x)\mathrm{d}s \\ &\leq \int_t^\infty L^s K_1(\mathbf{1})(x)\mathrm{d}s \\ &= \int_t^\infty \mu \mathrm{e}^{-\mu t}\mathrm{d}s \\ &= \mathrm{e}^{-\mu t}, \end{aligned}$$

namely the claim holds for $n = 1$. Now, assume that the claim holds for n_0. Then, for any $T \in \mathbb{R}_+$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} &K^{n_0+1}\left(h_1^T\right)(x) - (K(t,\cdot))^{n_0+1}\left(h_1^T\right)(x) \\ &\quad = K^{n_0+1}(h_1^T)(x) - (K(K(t,\cdot))^{n_0})(h_1^T)(x) \\ &\qquad + (K(K(t,\cdot))^{n_0})\left(h_1^T\right)(x) - (K(t,\cdot))^{n_0+1}\left(h_1^T\right)(x) \\ &\quad = (K(K^{n_0} - (K(t,\cdot))^{n_0}))(h_1^T)(x) + ((K - K(t,\cdot))(K(t,\cdot))^{n_0})(h_1^T)(x) \\ &\quad = K\left((K^{n_0} - (K(t,\cdot))^{n_0})\left(h_1^T\right)\right)(x) + ((K - K(t,\cdot))\left((K(t,\cdot))^{n_0})(h_1^T)\right)(x) \\ &\quad \leq K(\mathbf{n_0}\mathrm{e}^{-\mu \mathbf{t}})(x) + ((K - K(t,\cdot))(\mathbf{1})(x) \\ &\quad = n_0\mathrm{e}^{-\mu t} + \mathrm{e}^{-\mu t} \\ &\quad = (n_0+1)\mathrm{e}^{-\mu t}, \end{aligned}$$

i.e., the claim holds for $n_0 + 1$. By the induction principle, the claim is proved. It follows that $K^n(h_1^T)(x) - (K(t,\cdot))^n(h_1^T)(x) \in [0, n\mathrm{e}^{-\mu t}]$ for all $T \in \mathbb{R}_+$ and $x \in \mathbb{R}$, which obviously implies that $\lim_{t\to\infty}(\sup\{|(K(t,\cdot))^n(h_1^T)(x) - K^n(h_1^T)(x)| : x \in \mathbb{R} \text{ and } T \in \mathbb{R}_+\}) = 0$. This completes the proof. □

Lemma 3.3. *For any* $n \in \mathbb{N}$ *and* $\delta > 0$, *there exists* $T_{n,\delta} > 0$ *and* $s_{n,\delta} > 0$ *such that* $K^n(h_1^T) \geq h^T_{(a_n)^n-\delta}$ *and* $(K(s,\cdot))^n(h_1^T) \geq h^T_{(a_n)^n-\delta}$ *for all* $T \geq T_{n,\delta}$ *and* $s \geq s_{n,\delta}$, *where* K^n *and* $(K(s,\cdot))^n$ *represent the* n*th-composition of* K *and* $K(s,\cdot)$, *respectively.*

Proof. By Lemma 3.1(iv), there exists $T_{n,\delta} > 0$ such that $K^n(h_1^T) \geq h^T_{\delta_1}$ for all $T \geq T_{n,\delta}$, where $\delta_1 = (a_n)^n - \frac{\delta}{3}$. On the other hand, Lemma 3.2 implies that there exists $s_{n,\delta} > 0$ such that $\sup\{|(K(s,\cdot))^n(h_1^T)(x) - K^n(h_1^T)(x)| : x \in \mathbb{R} \text{ and } T \in \mathbb{R}_+\} \leq \frac{\delta}{3}$ for all $s \geq s_{n,\delta}$. It follows that $K^n(h_1^T)(x) - K(s,\cdot))^n(h_1^T)(x) \in (0, \frac{\delta}{3}]$ for all $x \in \mathbb{R}$, $T \in \mathbb{R}_+$, and $s \geq s_{n,\delta}$. This, combined with the fact that $K^n(h_1^T) \geq h^T_{\delta_1}$ for all $T \geq T_{n,\delta}$, implies that $(K(s,\cdot))^n(h_1^T)(x) \geq K^n(h_1^T)(x) - \frac{\delta}{3} \geq \delta_1 - \frac{\delta}{3} > (a_n)^n - \delta$ for all $T \geq T_{n,\delta}, s \geq s_{n,\delta}$ and $x \in [-T, T]$. Thus, $(K(s,\cdot))^n(h_1^T) \geq h^T_{(a_n)^n-\delta}$ for all $T \geq T_{n,\delta}$ and $s \geq s_{n,\delta}$. This completes the proof. □

To continue our discussions, we make the following assumption which links together the nonlocal reaction and the diffusion.

(NRD). $f'(0) > \frac{1}{\limsup_{n\to\infty}(K^n(h_\pm)(0))^{\frac{1}{n}}}$, where K^n represents the nth-composition of K.

Lemma 3.4. *The assumption (NRD) holds if and only if $(f'(0))^{n^*}K^{n^*}(h_\pm)(0) > 1$ for some $n^* \in \mathbb{N}$.*

Proof. We only need to prove the sufficiency. Suppose that $(f'(0))^{n^*}K^{n^*}(h_\pm)(0) > 1$ for some $n^* \in \mathbb{N}$. It follows from Lemma 3.1(ii) that, for any $m \in \mathbb{N}$, $a^\pm_{mn^*} \geq a^\pm_{n^*} > \frac{1}{f'(0)}$. This gives $f'(0)a^\pm_{mn^*} \geq f'(0)a^\pm_{n^*} > 1$. It follows easily that $\limsup_{n\to\infty} f'(0)(K^n(h_\pm)(0))^{\frac{1}{n}} \geq f'(0)a^\pm_{n^*} > 1$, that is, assumption (NRD) holds. This completes the proof. □

The following result shows that the assumption (NRD) automatically holds for a wide range of kernel functions, including those derived from the structured population model in So et al. [29].

Lemma 3.5. *If $f'(0) > 1$ and $k(x) = k(-x)$ for all $x \in \mathbb{R}$, then assumption (NRD) holds.*

Proof. Since $f'(0) > 1$ and $k(x) = k(-x)$ for all $x \in \mathbb{R}$, we have $(a^+_n)^n + (a^-_n)^n = 1$ and $(a^+_n)^n = (a^-_n)^n$. It follows that $(a^\pm_n)^n = \frac{1}{2}$ for all $n \in \mathbb{N}$. Then, $\limsup_{n\to\infty}(K^n(h_\pm)(0))^{\frac{1}{n}} = \limsup_{n\to\infty} a^\pm_n = \lim_{n\to\infty}\sqrt[n]{\frac{1}{2}} = 1$. Immediately, we know that assumption (NRD) holds. This completes the proof. □

The following result gives a priori estimate for nontrivial solutions of (2.4), which plays a key role in the proof of the permanence and global attractivity of (2.4).

Proposition 3.6. *Suppose that $f'(0) > 1$ and assumption (NRD) holds. Then, there exists $\varepsilon_0 > 0, T_0 > 0$ and $T^* > 0$ such that for all $\varepsilon \in [0, \varepsilon_0], T \in [T_0, \infty)$, and a solution $u : [-1, \infty) \times \mathbb{R} \to [0, M+1]$ of (2.4) with $u(t, \cdot) \geq h^T_\varepsilon$ for all $t \in [-1, T^*]$, we have $u(t, \cdot) \geq h^T_\varepsilon$ for all $t \in [-1, \infty)$ and $u(t, \cdot) \gg h^T_\varepsilon$ for all $t \in (T^*, \infty)$.*

Proof. Lemma 3.4 implies that $(f'(0))^n(a_n)^n = (f'(0))^n K^n(h_\pm)(0) > 1$ for some $n \in \mathbb{N}$. Thus, there exists $\delta > 0$ and $\beta_1 \in (1, f'(0))$ such that $(\beta_1)^n((a_n)^n - \delta) > 1$.

By the choice of β_1 and assumption (H2), there exists a $\varrho \in (0, M+1)$ such that $f(u) \geq \beta_1 u$ for $u \in [0, \varrho]$. Let $\eta = \min\{f(u) : u \in [\varrho, M+1]\}$. Denote $\varepsilon_1 = \min\{\varrho, \eta/\beta_1\}$. Then, one can easily see that $f(u) \geq \beta_1 u$ for all $u \in [0, \varepsilon_1]$ and $f(u) \geq \beta_1\varepsilon_1$ for all $u \in [\varepsilon_1, M+1]$.

By applying Lemma 3.3, we know that there exists $T_{n,\delta} > 0$ and $s_{n,\delta} > 0$, such that $K^n(h^T_1) \geq h^T_{(a_n)^n-\delta}$ and $(K(s,\cdot))^n(h^T_1) \geq h^T_{(a_n)^n-\delta}$ for all $T \geq T_{n,\delta}$ and $s \geq s_{n,\delta}$.

Let $\varepsilon_0 = \frac{\varepsilon_1}{(\beta_1)^{n+1}}$, $T_0 = T_{n,\delta}$, $T_1 = s_{n,\delta}$ and $T^* = ns_{n,\delta} + n - 1$. Suppose that $\varepsilon \in [0, \varepsilon_0]$, $T \in [T_0, \infty)$, $u : [-1, \infty) \times \mathbb{R} \to [0, M+1]$ is a solution of (2.4) such that $u(t, \cdot) \geq h^T_\varepsilon$ for all $t \in [-1, T^*]$. Let $\varphi = u_0$. Then, $u(t, x) = u^\varphi(t, x) = \Phi(t+1, \varphi)(-1, x)$ for all $(t, x) \in [-1, \infty) \times \mathbb{R}$. Due to the choices of ε and β_1, one can easily obtain $\beta_1^j(K(t,\cdot))^j(h^T_\varepsilon) < \varepsilon_{\mathbf{1}}$ and $f(\beta_1^j(K(t,\cdot))^j(h^T_\varepsilon)) \geq \beta_1^{j+1}(K(t,\cdot))^j(h^T_\varepsilon)$ for all $t \geq 0$ and $j = 0, 1, \ldots, n$.

Now, we claim that if $j \in \{1, 2, \ldots, n\}$, then $u(t + jt^*, \cdot) \geq (\beta_1)^j(K(t^*, \cdot))^j(h^T_\varepsilon)$ for all $t, t^* \in \mathbb{R}_+$ such that $t \geq j-1$ and $t + jt^* \in [0, T^*+1]$. We use mathematical induction to prove the claim. First, for any $t, t^* \in \mathbb{R}_+$ such that $t + t^* \in [0, T^*+1]$, it follows from (2.4) and Fubini's theorem that

$$\begin{aligned}
u(t+t^*, \cdot) &= \Phi(t^*, \Phi(t, \varphi))(0, \cdot) \\
&= \frac{1}{\mu}L^{t^*}(u(t,\cdot)) + \int_0^{t^*} L^{t^*-s}(F(u_{s+t}))\mathrm{d}s \\
&= \frac{1}{\mu}L^{t^*}(u(t,\cdot)) + \int_0^{t^*} L^{t^*-s}(K_1(f(u(s+t-1,\cdot))))\mathrm{d}s \\
&\geq \beta_1 \int_0^{t^*} L^{t^*-s}(K_1(h^T_\varepsilon))\mathrm{d}s \\
&= \beta_1 K(t^*, h^T_\varepsilon),
\end{aligned}$$

that is, the claim holds for $j = 1$. Next, assume that the claim holds for $j_0 \in \{1, 2, \ldots, n-1\}$. Suppose that $t, t^* \in \mathbb{R}_+$ such that $t \geq j_0$ and $t + (j_0+1)t^* \in [0, T^*+1]$. Then, by the induction assumption, we have $u(s + t + j_0t^* - 1, \cdot) \geq (\beta_1)^{j_0}(K(t^*, \cdot))^{j_0}(h_\varepsilon^T)$ for all $s \in [0, t^*]$. It follows from (2.4) and Fubini's theorem that

$$\begin{aligned}
u(t + (j_0+1)t^*, \cdot) &= u(t^* + (t + j_0t^*), \cdot) \\
&= \Phi(t^*, \Phi(t + j_0t^*, \varphi))(0, \cdot) \\
&= \frac{1}{\mu} L^{t^*}(u(t + j_0t^*, \cdot)) + \int_0^{t^*} L^{t^*-s}(F(u_{s+t+j_0t^*}))\mathrm{d}s \\
&\geq \int_0^{t^*} L^{t^*-s}(F(u_{s+t+j_0t^*}))\mathrm{d}s \\
&= \int_0^{t^*} L^{t^*-s}(K_1(f(u(s + t + j_0t^* - 1, \cdot))))\mathrm{d}s \\
&\geq \int_0^{t^*} L^{t^*-s}\left(K_1((\beta_1)^{j_0+1}(K(t^*, \cdot))^{j_0}\left(h_\varepsilon^T\right))\right)\mathrm{d}s \\
&= (\beta_1)^{j_0+1}(K(t^*, \cdot))^{j_0+1}(h_\varepsilon^T),
\end{aligned}$$

which means that the claim holds for $j_0 + 1$. By the induction principle, we have proved the claim.

For any $t \in [T^*, T^*+1]$, applying the above claim with $j = n$ and $t^* = T_1$, we obtain that $t - nT_1 \geq n-1$ and

$$u(t, \cdot) = u((t - nT_1) + nT_1, \cdot) \geq (\beta_1)^n(K(T_1, \cdot))^n\left(h_\varepsilon^T\right) \geq (\beta_1)^n((a_n)^n - \delta)(h_\varepsilon^T),$$

where $(K(T_1, \cdot))^n$ represents the nth-composition of the map $K(T_1, \cdot)$. This, combined with the fact that $(\beta_1)^n((a_n)^n - \delta) > 1$, implies that $u(t, \cdot) \geq h_\varepsilon^T$ for all $t \in [-1, T^*+1]$ and $u(t, \cdot) \gg h_\varepsilon^T$ for all $t \in [T^*, T^*+1]$. Now, the results easily follow from Proposition 2.6 and the semigroup property of the semiflow Φ. □

Theorem 3.7. *Suppose that $f'(0) > 1$ and assumption (NRD) holds. If $\varphi \in Y \setminus \{\mathbf{0}\}$, then there exists $a > 0$ such that $\xi \geq \mathbf{a}$ for all $\xi \in \omega(\varphi)$.*

Proof. By Proposition 2.6(i), we may assume that $\varphi \in C_+^\circ$ and hence $u^\varphi(t, x) > 0$ for all $(t, x) \in [-1, \infty) \times \mathbb{R}$. Choose T_0, T^*, and ε_0 as in Proposition 3.6. Let $\varepsilon_1 = \inf\{u(t, x) : (t, x) \in [-1, T^*] \times [-T_0, T_0]\}$ and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$. Then, $\varepsilon_1 > 0$ and $\varepsilon > 0$. By Proposition 3.6 and the choices of T_0, T^*, and ε_0, we get $u^\varphi(t, \cdot) \geq h_\varepsilon^{T_0}$ for all $t \geq -1$. This, combined with the definition of $\omega(\varphi)$, implies $\xi \geq h_\varepsilon^{T_0}$ for all $\xi \in \omega(\varphi)$.

For any $\xi \in \omega(\varphi)$, let $a_\xi = \sup\{a \in \mathbb{R}_+ : \xi(\theta, x) \geq \varepsilon$ for all $(t, x) \in [-1, 0] \times [-a, 0]\}$ and $b_\xi = \sup\{b \in \mathbb{R}_+ : \xi(\theta, x) \geq \varepsilon$ for all $(t, x) \in [-1, 0] \times [0, b]\}$. Then, $a_\xi, b_\xi \geq T_0$. Let $I_\xi = [-a_\xi, b_\xi]$ if $a_\xi, b_\xi \in \mathbb{R}_+$; $I_\xi = [-a_\xi, \infty)$ if $a_\xi \in \mathbb{R}_+$ and $b_\xi = \infty$; $I_\xi = (-\infty, b_\xi]$ if $a_\xi = \infty$ and $b_\xi \in \mathbb{R}_+$; and otherwise $I_\xi = \mathbb{R}$. Denote $I = \bigcap_{\xi \in \omega(\varphi)} I_\xi$. Then, $I \supseteq [-T_0, T_0]$ and thus there exists $c_1, c_2 \geq T_0$ such that $I = [-c_1, c_2]$ or $[c_1, \infty)$ or $(-\infty, c_2]$ or $\mathbb{R}$.

We claim that $I = \mathbb{R}$. To prove the claim, we just show that $I = [-c_1, c_2]$ cannot hold as the proofs for the other two cases are similar. By the way of contradiction, suppose that $I = [-c_1, c_2]$. Without loss of generality, we may also assume that $c_1 \geq c_2$. Taking $\xi \in \omega(\varphi)$, we obtain by the invariance of $\omega(\varphi)$ that $u^\xi(t, \cdot) \geq h_\varepsilon^{c_2}$ for all $t \in [-1, T^*]$. Again, by Proposition 3.6 and the choices of T_0, T^*, and ε_0, we have $u^\xi(t, \cdot) \gg h_\varepsilon^{c_2}$ for all $(t, x) \in (T^*, \infty) \times \mathbb{R}$. In particular, there exists $T > c_2$ such that $u^\xi(t, \cdot) \gg h_\varepsilon^T$ for all $t \in [1 + T^*, 2 + 2T^*]$. On the other hand, by the definition of $\omega(\varphi)$, there exists a sequence $\{s_n\}_{n \in \mathbb{N}}$ such that $\lim_{n\to\infty} ||(u^\varphi)_{s_n} - \xi||_{co} = 0$. It follows that $\lim_{n\to\infty}(\sup\{|u^\varphi(s_n + t, x) - u^\xi(t, x)| : (t, x) \in [1 + T^*, 1 + 2T^*] \times [-T, T]\}) = 0$. Thus, there exists $n^* > 1$ such that $u^\varphi(s_{n^*} + t, \cdot) \geq h_\varepsilon^T$ for all $t \in [1 + T^*, 1 + 2T^*]$. It follows from Proposition 3.6 that $u^\varphi(s_{n^*} + t, \cdot) \geq h_\varepsilon^T$ for all $t \in [1 + T^*, \infty)$. This and the definition of $\omega(\varphi)$ produce $\xi \geq h_\varepsilon^T$ for all $\xi \in \omega(\varphi)$. Since $T > c_2$, we have $b_\xi \geq T > c_2$ for all

$\xi \in \omega(\varphi)$. Then, $c_2 = \inf\{b_\xi : \xi \in \omega(\varphi)\} \geq T > c_2$, a contradiction. This proves the claim, that is, $I = \mathbb{R}$. Hence, this claim and the choice of I imply that we can take $a = \varepsilon$ to complete the proof. □

The following remark and Corollary 2.12 combined tell us that that (2.4) is permanent with respect to the compact open topology.

Remark 3.8. Suppose that $f'(0) > 1$ and assumption (NRD) holds. Then, there exists $a^* > 0$ such that $a^* \leq a_\varphi \triangleq \inf\{\xi(\theta, x) : (\theta, x) \in [-1, 0] \times \mathbb{R}$ and $\xi \in \omega(\varphi)\}$ for all $\varphi \in Y \backslash \{\mathbf{0}\}$. Clearly, Theorem 3.7 implies that $a_\varphi \in (0, 1 + M]$ for all $\varphi \in Y \backslash \{\mathbf{0}\}$. By the assumptions (H1) and (H2), we choose $a^* > 0$ such that $f(x) > x$ for all $x \in (0, a^*]$ and $f(x) > a^*$ for all $x \in [a^*, 1+M]$. Take $\varphi \in Y \backslash \{\mathbf{0}\}$. We shall show $a^* \leq a_\varphi$; otherwise, $a^* > a_\varphi$. Let $a^{**} = \inf\{f(x) : x \in [a_\varphi, 1 + M]\}$. Then, $a^{**} > a_\varphi$. It follows from (2.4) that for any $(t, x) \in [0, \infty) \times \mathbb{R}$,

$$\begin{aligned} u^\xi(t, x) &= \frac{1}{\mu} L^t(\xi(0, \cdot))(x) + \int_0^t L^{t-s}(K_1(f(u^\xi(s - 1, \cdot))))(x)\mathrm{d}s \\ &\geq \mathrm{e}^{-\mu t} a_\varphi + \int_0^t \mu \mathrm{e}^{-\mu t} a^{**} \mathrm{d}s \\ &= a_\varphi + (1 - \mathrm{e}^{-\mu t})(a^{**} - a_\varphi). \end{aligned}$$

Thus, by the invariance of $\omega(\varphi)$, we have $\xi \geq a_\varphi + (1 - \mathrm{e}^{-\mu})(a^{**} - a_\varphi) > a_\varphi$ for all $\xi \in \omega(\varphi)$, a contradiction with the definition of a_φ. This shows that $a_\varphi \geq a^*$ for all $\varphi \in Y \backslash \{\mathbf{0}\}$.

Under assumption (H1), if $f'(0) > 1$, then f has a positive fixed point, which is also a fixed point of f^2. Below, to further study the global attractivity for (2.4), we formulate the following non-monotone assumption on the nonlinearity f.

(H3). f^2 has a unique positive fixed point u^*.

Lemma 3.9. *Suppose $f'(0) > 1$. Then, assumption* (H3) *holds if and only if, for any interval $[a, b] \subseteq (0, \infty)$ with $a < b$, either $a < \min\{f(u) : u \in [a, b]\}$ or $b > \max\{f(u) : u \in [a, b]\}$.*

Proof. By the remark just before assumption (H3), we know that f^2 has at least one positive fixed point, which is also a fixed point of f. Let us fix one of them, say $\hat{u}$.

We first prove the sufficiency by the way of contradiction. Suppose there exists a $\tilde{u} \in (0, \infty)$ such that $\tilde{u} \neq \hat{u}$ and $f^2(\tilde{u}) = \tilde{u}$. Let $[a, b] = [\min\{\hat{u}, \tilde{u}\}, \max\{\hat{u}, \tilde{u}\}]$ if $f(\tilde{u}) = \tilde{u}$ while $[a, b] = [\min\{\tilde{u}, f(\tilde{u})\}, \max\{\tilde{u}, f(\tilde{u})\}]$ if $f(\tilde{u}) \neq \tilde{u}$. Then, we can easily see that $[a, b] \subseteq f([a, b])$, a contradiction.

Now, we prove the necessity. Assume that (H3) holds. Then, f has a unique positive fixed point and it is also u^*. This, combined with assumption (H1) and the fact that $f'(0) > 1$, implies that $f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u$ for all $u \in (u^*, \infty)$. Let $[a, b] \subseteq (0, \infty)$. If $a \geq u^*$, then obviously $b > \max\{f(u) : u \in [a, b]\}$, while if $b \leq u^*$ then obviously $a < \min\{f(u) : u \in [a, b]\}$. Hence, without loss of generality, we assume that $u^* \in (a, b)$. By the way of contradiction, suppose that there exists $a^*, b^* \in [a, b]$ such that $f(a^*) \leq a$ and $f(b^*) \geq b$. Clearly, $a^* \in (u^*, b]$ and $b^* \in [a, u^*)$. Then, $a^* \in [u^*, b] \subseteq [u^*, f(b^*)] \subseteq f([b^*, u^*]) \subseteq f([a, u^*])$. It follows that there exists $a^{**} \in [a, u^*)$ such that $a^* = f(a^{**})$ and hence $f^2(a^{**}) = f(a^*) \leq a \leq a^{**}$. On the other hand, pick $v^* \in (0, a^{**})$ such that $f(v^*) < u^*$. Then, $f^2(v^*) > f(v^*) > v^*$. By the mean value theorem, there exists $u^{**} \in (v^*, a^{**}]$ such that $f^2(u^{**}) = u^{**}$, that is, u^{**} $(\neq u^*)$ is also a fixed point of f^2, a contradiction. This completes the proof. □

Theorem 3.10. *Suppose that $f'(0) > 1$ and assumptions (NRD) and* (H3) *hold. Then, $\mathbf{u}^*$ is a globally attractive equilibrium of* (2.4) *in $Y \backslash \{\mathbf{0}\}$ and hence it is also a globally attractive equilibrium in $C_+ \backslash \{\mathbf{0}\}$.*

Proof. By Corollary 2.16, it suffices to prove that $\mathbf{u}^*$ is a globally attractive equilibrium in $Y\backslash\{\mathbf{0}\}$. Suppose that $\varphi \in Y\backslash\{\mathbf{0}\}$. By Theorem 3.7 or Remark 3.8, there exists $a > 0$ such that $\xi \geq \mathbf{a}$ for all $\xi \in \omega(\varphi)$. Let $A = \{\xi(\theta, x) : \xi \in \omega(\varphi), \theta \in [-1, 0] \text{ and } x \in \mathbb{R}\}$, $u_+ = \sup A$, $u_- = \inf A$.

We claim that $u_+ = u_-$. Otherwise, suppose $u_+ \neq u_-$ and hence $u_+ > u_- \geq a > 0$. Then, it follows from (H3) and Lemma 3.9 that either $u_- < \min\{f(u) : u \in [u_-, u_+]\}$ or $u_+ > \max\{f(u) : u \in [u_-, u_+]\}$. Without loss of generality, we may assume that $u_- < \min\{f(u) : u \in [u_-, u_+]\}$. Let $u^- = \min\{f(u) : u \in [u_-, u_+]\}$. Then, $f(u) \geq u^- > u_-$ for all $u \in [u_-, u_+]$. Hence, for any $\xi \in \omega(\varphi)$, it follows from (2.4) that

$$\begin{aligned} u^\xi(t,x) &= \frac{1}{\mu} L^t(\xi(0,\cdot))(x) + \int_0^t L^{t-s}(K_1(f(u^\xi(s-1,\cdot))))(x)\mathrm{d}s \\ &\geq \frac{1}{\mu} L^t(\mathbf{u}_-)(x) + \int_0^t L^{t-s}(K_1(\mathbf{u}^-))(x)\mathrm{d}s \\ &= \mathrm{e}^{-\mu t} u_- + (1-\mathrm{e}^{-\mu t})u^- \\ &= u^- + \mathrm{e}^{-\mu t}(u_- - u^-), \end{aligned}$$

which implies that $u^\xi(t,x) \geq u^- + \mathrm{e}^{-\mu}(u_- - u^-) > u_-$ for all $\xi \in \omega(\varphi)$, $t \in [1, \infty)$, and $x \in \mathbb{R}$. This, combined with the invariance of $\omega(\varphi)$, shows that $\xi \geq \mathbf{u}^- + \mathrm{e}^{-\mu}(\mathbf{u}_- - \mathbf{u}^-)$ for all $\xi \in \omega(\xi)$, a contradiction to the choices of u^- and u_-. This proves the claim.

It follows from the claim that u_- is a fixed point of f. Since f has the only fixed point u^*, we have $\omega(\varphi) = \{\mathbf{u}^*\}$. This completes the proof. □

Under conditions similar to (H3), by applying a quite different method, Yi and Zou [41] studied the global stability of a class of delayed reaction–diffusion equations in bounded domains.

We now formulate a geometric condition on the nonlinearity f.

(H4). There is a $u^* > 0$ such that $f(u^*) = u^*$, and $|f(b) - f(u^*)| \leq |b - u^*|$ for all $b \geq 0$; and the equality $|f(b) - f(u^*)| = |b - u^*|$ holds for some $b \geq 0$ if and only if either $b = 0$ or $b = u^*$.

(H4) implies that f is a contraction map about the fixed point u^*.

Theorem 3.11. *Suppose that $f'(0) > 1$ and assumptions (NRD) and* (H4) *hold. Then,* $\mathbf{u}^*$ *is a globally attractive equilibrium of* (2.4) *in* $Y\backslash\{\mathbf{0}\}$ *and hence it is also a globally attractive equilibrium in* $C_+\backslash\{\mathbf{0}\}$.

Proof. To apply Theorem 3.10, it suffices to verify assumption (H3). Note that u^* is also a fixed point of f^2. Suppose that $u^{**} \in (0, \infty)$ such that $f^2(u^{**}) = u^{**}$. Then, by (H4), $|u^{**} - u^*| = |f^2(u^{**}) - f^2(u^*)| \leq |f(u^{**}) - f(u^*)| \leq |u^{**} - u^*|$, which gives $|f(u^{**}) - f(u^*)| = |u^{**} - u^*|$. This, combined with (H4), implies that $u^{**} = u^*$. Therefore, (H3) is verified. □

If f is continuously differentiable on $(0, \infty)$ and $f'(0) > 1$, then the conclusions of Theorem 3.11 remain true when (H4) is replaced with the following assumption which, as illustrated in the next section, is complementary to (H4).

(H5). f is continuously differentiable on $\mathbb{R}_+$ and has a unique critical point u^c and a unique fixed point u^* such that either $u^c \geq u^*$ or ($u^c < u^*$ and $f(f(u)) > u$ for all $u \in [u^c, u^*)$).

Theorem 3.12. *Suppose that $f'(0) > 1$ and assumptions (NRD) and (H5) hold. Then,* $\mathbf{u}^*$ *is a globally attractive equilibrium of* (2.4) *in* $Y\backslash\{\mathbf{0}\}$ *and hence it is also a globally attractive equilibrium in* $C_+\backslash\{\mathbf{0}\}$.

Proof. Again, the results follow from Theorem 3.10 after verifying (H3). To verify (H3), we suppose by the way of contradiction that there exists $u^{**} \in (0, \infty)\backslash\{u^*\}$ such that $f^2(u^{**}) = u^{**}$. We discuss case by case.

Case 1: $u^* \leq u^c$. In this case, $u^* > f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u$ for all $u \in (u^*, \infty)$. First, assume $u^{**} \in (0, u^*)$. Then $u^* > f(u^{**}) > u^{**}$. It follows that $u^{**} = f^2(u^{**}) > f(u^{**}) > u^{**}$, a contradiction. Next assume $u^{**} \in (u^*, \infty)$. If $f(u^{**}) \leq u^*$ then $u^{**} = f^2(u^{**}) \leq u^*$, a contradiction; while if $f(u^{**}) > u^*$ then $u^{**} = f^2(u^{**}) < f(u^{**}) < u^{**}$, a contradiction.

Case 2: $u^* > u^c$. In this case, $f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u^*$ for all $u \in (u^*, \infty)$. Clearly, $\min\{u^{**}, f(u^{**})\} < u^*$. Since $f(u^{**})$ is also a fixed point of f^2, without loss of generality, we may assume that $u^{**} < u^*$, and thus $f(u_{**}) > u^*$.

We claim that $u^{**} \in [u^c, u^*)$. Otherwise, $u^{**} < u^c$. Since $f(u^{**}) \in [u^*, f(u^c)] \subseteq f([u^c, u^*])$, there exists $u^{cc} \in [u^c, u^*)$ such that $f(u^{cc}) = f(u^{**})$. Then, $f^2(u^{cc}) = f^2(u^{**}) = u^{**} < u^c \le u^{cc}$, a contradiction to assumption (H5). This proves the claim.

If $u^{**} \in [u^c, u^*)$ then $u^{**} = f(f(u^{**})) > u^{**}$ by (H5), a contradiction; while if $f(u^{**}) \in [u^c, u^*)$, then $f(u^{**}) = f(f^2(u^{**})) = f(f(f(u^{**}))) > f(u^{**})$ by (H5), a contradiction. □

Remark 3.13. Theorems 2.17, 3.7, 3.10, 3.11, and 3.12 still hold if we replace (H1) with

(H1*). There exists a sequence $\{u_n\}_{n\ge1}$ such that $\lim_{n\to\infty} u_n = \infty$ and $f([0, u_n]) \subseteq [0, u_n]$.

We emphasize that assumption (NRD) is crucial for the above results to be true. If (NRD) does not hold then, for example, nonconstant steady-state solutions exist [39].

Finally, for simplicity of exposition, we only consider Eq. (1.1) on $\mathbb{R}$. We mention that our approach can be slightly modified to study the global asymptotic behavior for the higher-dimensional case.

4. Applications

In this section, we illustrate our main results with two important examples.

First, we consider the following diffusive Nicholson's blowfly equation with nonlocal effect

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + \beta \displaystyle\int_{\mathbb{R}} k(x-y)u(t-\tau,y)\mathrm{e}^{-au(t-\tau,y)}\mathrm{d}y, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) \ = \varphi(\theta,x) \qquad \text{for } (\theta,x) \in [-\tau,0]\times\mathbb{R}, \end{cases} \tag{4.1}$$

where $a, d, \beta, \delta, \tau \in (0,\infty)$, and $\varphi \in C_+$. System (4.1) arises from population biology. See [5,6,9,12,18,19,25–29,37,38,40,41,43] and the references therein. After scaling in (4.1), we obtain

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu \left[(\beta/\delta) \displaystyle\int_{\mathbb{R}} k(x-y)u(t-1,y)\mathrm{e}^{-u(t-1,y)}\mathrm{d}y\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) = \varphi(\theta,x), \qquad (\theta,x) \in [-1,0]\times\mathbb{R}, \end{cases} \tag{4.2}$$

where $\mu, \delta, \beta > 0$, and $k : \mathbb{R} \to (0,\infty)$ are continuous with $\int_{\mathbb{R}} k(y)\mathrm{d}y = 1$.

The following threshold dynamics of (4.2) follows from Theorems 2.17 and 3.11.

Theorem 4.1. *If $\beta/\delta \in (0, e^2]$, then the following statements are true:*

(i) *If $\beta/\delta \le 1$, then $\mathbf{0}$ is a globally attractive equilibrium of (4.2) in C_+ with respect to the usual supremum norm.*

(ii) *If $\beta/\delta > 1$ and assumption (NRD) holds, then $\ln(\beta/\delta)$ is a globally attractive equilibrium of (4.2) in $C_+\backslash\{\mathbf{0}\}$.*

Proof. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(u) = (\beta/\delta)u\mathrm{e}^{-u}$ for all $u \in \mathbb{R}_+$. If $\beta/\delta \le 1$, then $f'(0) = \beta/\delta \in (0,1]$ and $f(u) < u$ for all $u > 0$. By Theorem 2.17, we conclude that $\mathbf{0}$ is a globally attractive equilibrium in C_+ with respect to the usual supremum norm. This proves (i).

Now, suppose $\beta/\delta > 1$. Then, obviously, assumption (H4) follows from Lemma 2.3 of Yi and Zou [40]. The other conditions of Theorem 3.11 can be easily verified and hence (ii) follows from Theorem 3.11. This completes the proof. □

Note that for the case where the space is a bounded domain, the threshold dynamics of the diffusive delay Nicholson blowfly equation with the local/nonlocal effect and the Dirichlet/Neumann boundary condition has been obtained in [27,37,38,40,41,43]. As for the case of unbounded domain, to the best of our knowledge, existing results on the global asymptotic behavior of (4.1) are obtained only when the solution semiflow essentially is monotone or the initial values enjoy compact supports [1,4,16].

Next, we consider the following scalar diffusive Mackey–Glass equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu \displaystyle\int_{\mathbb{R}} k(x-y)\frac{pu(t-1,y)}{1+(u(t-1,y))^n}\mathrm{d}y, & (t,x)\in(0,\infty)\times\mathbb{R},\\ u(\theta,x)=\varphi(\theta,x) & \text{for } (\theta,x)\in[-1,0]\times\mathbb{R}, \end{cases} \tag{4.3}$$

where p, μ, and n are all positive constants and $k:\mathbb{R}\to(0,\infty)$ is continuous with $\int_{\mathbb{R}} k(y)\mathrm{d}y = 1$. By rescaling the diffusive version of the original model system proposed by Mackey and Glass [20] to model the blood cell production, we easily get (4.3). The non-diffusive version of (4.3) has been studied by many researchers. See [8,15,18,19,30] and the references therein. We mention that for the case where the space is a bounded domain, the threshold dynamics of the diffusive delay Mackey–Glass equation with the local/nonlocal effect and the Dirichlet/Neumann boundary condition has been obtained in [38,41,43]. However, for the case where the space is an unbounded domain, to the best of our knowledge, there exists no result comparable to Theorem 4.5 to be shown soon.

Lemma 4.2. *Assume that* $p>1$. *Let* $u^*=(p-1)^{\frac{1}{n}}$, $f(b)=\frac{pb}{1+b^n}$ *and* $h_\pm(b)=-u^*\pm|b-u^*|+f(b)$ *for all* $b\in\mathbb{R}_+$. *If* $0<n\le 2$, *then the following results are true:*

(i) $h_+(0)=h_+(u^*)=0$.
(ii) $h_+(b)>0$ *for all* $b>u^*$.
(iii) $h_+(b)>0$ *for all* $b\in(0,u^*)$.
(iv) $h_-(u^*)=0$.
(v) $h_-(b)<0$ *for all* $b>u^*$.
(vi) $h_-(b)<0$ *for all* $b\in(0,u^*)$.

Hence, f *satisfies assumption* (H4).

Proof. By the definitions of h_+ and h_-, we know that statements (i), (iii), (iv), and (v) hold. It suffices to prove statement (ii) since the proof of statement (vi) is similar.

Clearly,

$$h_+(b) = b-2u^*+f(b) = \frac{(1+p)b-2u^*b^n+b^{1+n}-2u^*}{1+b^n}.$$

Let $g(b)=(1+p)b-2u^*b^n+b^{1+n}-2u^*$ for all $b\in\mathbb{R}_+$. Then, for any $b\in\mathbb{R}_+$, we have $g'(b)=1+p-2nu^*b^{n-1}+(n+1)b^n$ and $g''(b)=-2n(n-1)u^*b^{n-2}+(n+1)nb^{n-1}$. We will finish the proof by distinguishing two cases.

Case 1: $n\in(0,1]$. If $b>u^*$, then $g'(b)=1+p+b^{n-1}((n+1)b-2nu^*)>1+p+b^{n-1}((n+1)-2n)u^*>0$. This and $g(u^*)=0$ imply $g(b)>0$, and hence $h_+(b)>0$ for all $b>u^*$.

Case 2: $n\in(1,2]$. In this case, $g'(u^*)=(2-n)p+n>0$ and $g''(b)=nb^{n-2}(-2(n-1)u^*+(n+1)b)\ge nb^{n-2}(-2(n-1)u^*+(n+1)u^*)=(3-n)nb^{n-2}u^*>0$ for all $b>u^*$. It follows that $g'(b)>0$ for all $b>u^*$. This, combined with $g(u^*)=0$, implies $g(b)>0$ for $b>u^*$. Again, we have $h_+(b)>0$ for all $b>u^*$.

This completes the proof. □

Lemma 4.3. *Let* $n>2$. *If* $p>\frac{n}{n-2}$, *then* $f(u)=\frac{pu}{1+u^n}$ *for* $u\in\mathbb{R}_+$ *does not satisfy* (H4).

Proof. Let $u^*=(p-1)^{\frac{1}{n}}$, the unique positive fixed point of f; $u^c=(\frac{1}{n-1})^{\frac{1}{n}}$, the unique nonnegative critical point. Moreover, $u^c<u^*$. Then, $f'(u)<0$ for $u>u^c$ as $f'(0)=p>0$. It follows that $f(u)>f(u^*)=u^*$ for $u\in[u^c,u^*)$. Let $g(u)=f(u)+u$ for $u\in\mathbb{R}_+$. Then, $g'(u^*)=\frac{n-(n-2)p}{p}<0$. This tells us that there exists $\hat{u} \in [u^c, u^*)$ such that $g(\hat{u}) = f(\hat{u}) + \hat{u} > g(u^*) = 2u^*$. Thus, $|f(\hat{u}) - u^*| = f(\hat{u}) - u^* > u^* - \hat{u} = |\hat{u} - u^*|$, which means that (H4) is not satisfied. This completes the proof. □

Does $f(u) = \frac{pu}{1+u^n}$ satisfy (H4) when $n > 2$ and $1 < p \leq \frac{n}{n-2}$? The answer may not always be affirmative. Indeed, let $n = 2 + 10^{-4}$ and $p = 10001$. Then, $1 < p \leq \frac{n}{n-2}$ and f have a unique positive fixed point $u^* = 10^{\frac{4}{n}}$. With $b = 10$, we have $|b - u^*| \approx 89.98 < 889.995 \approx |f(b) - u^*|$, which indicates that assumption (H4) does not hold. However, the following result tells us that, in this case, f satisfies (H5). Therefore, (H5) is complementary to (H4).

Lemma 4.4. *Assume that $p > 1$. Let $u^* = (p-1)^{\frac{1}{n}}$, $u^c = (\frac{1}{n-1})^{\frac{1}{n}}$, $f(u) = \frac{pu}{1+u^n}$ for all $u \in \mathbb{R}_+$. If $n > 2$ and $1 < p \leq \frac{n}{n-2}$, then f satisfies assumption (H5).*

Proof. Obviously, f is continuously differentiable on $\mathbb{R}_+$. Moreover, u^c and u^* are the unique critical point and unique fixed point of f, respectively. We distinguish two cases to finish the proof.

Case 1: $1 < p \leq \frac{n}{n-1}$. In this case, one can check that $u^c \geq u^*$ and hence (H5) holds.

Case 2: $\frac{n}{n-1} < p \leq \frac{n}{(n-2)}$. In this case, $u^c < u^*$. To prove (H5), it suffices to prove $f(f(u)) > u$ for all $u \in [u^c, u^*)$, which, after a simple computation, is equivalent to prove $(1 + u^n)^n - p^2(1 + u^n)^{n-1} + p^n(1 + u^n) - p^n < 0$ for all $u \in [u^c, u^*)$, that is, $h(y) < 0$ for all $y \in [\frac{n}{n-1}, p)$, where $h(y) = y^n - p^2y^{n-1} + p^ny - p^n$. Note that $h'(y) = ny^{n-1} - (n-1)p^2y^{n-2} + p^n$ and $h''(y) = (n-1)(n-2)y^{n-3}(\frac{n}{n-2}y - p^2)$. Then $h''(y) < 0$ if $0 < y < \frac{(n-2)p^2}{n}$ and $h''(y) > 0$ if $y > \frac{(n-2)p^2}{n}$. It follows that $h'(y) > h'(\frac{(n-2)p^2}{n}) = (1 - [\frac{n-2}{n}p]^{n-2})p^n \geq 0$ for $y \in \mathbb{R}_+ \backslash \{\frac{(n-2)p^2}{n}\}$. Therefore, $h(y) < h(p) = 0$ for $y \in [\frac{n}{n-1}, p)$. This completes the proof. □

From the proof of Lemma 4.4, we see that if $n > 2$ and $p > \frac{n}{n-2}$, then $h(p) = 0$ and $h'(p) < 0$. It follows that $h(y) > 0$ for y less than p but close enough to p. Therefore, f does not satisfy assumption (H5) in the case where $n > 2$ and $p > \frac{n}{n-2}$.

Theorem 4.5. *The following statements are true for (4.3).*

(i) *If $p \leq 1$, then $\mathbf{0}$ is a globally attractive equilibrium in C_+ for all $\mu > 0$ with respect to the usual supremum norm.*

(ii) *Assume that assumption (NRD) holds. If either ($p > 1$ and $0 < n \leq 2$) or ($1 < p \leq \frac{n}{n-2}$ and $n > 2$), then $(\mathbf{p-1})^{\frac{1}{\mathbf{n}}}$ is a globally attractive equilibrium in $C_+ \backslash \{\mathbf{0}\}$ for all $\mu > 0$.*

Proof. Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(u) = \frac{pu}{1+u^n}$ for all $u \in \mathbb{R}_+$.

Under the assumption in (i), one can easily see that $f(u) < u$ when $u > 0$. By Theorem 2.17, we conclude that $\mathbf{0}$ is a globally attractive equilibrium in C_+ with respect to the usual supremum norm.

Now, we prove (ii). First, suppose that $p > 1$ and $0 < n \leq 2$. Then, assumption (H4) holds by Lemma 4.2. It follows from Theorem 3.11 that $(\mathbf{p-1})^{\frac{1}{\mathbf{n}}}$ is a globally attractive equilibrium in $C_+ \backslash \{0\}$. Second, suppose $1 < p \leq \frac{n}{n-2}$ and $n > 2$. Then, by Lemma 4.4, assumption (H5) holds. Therefore, Theorem 3.12 implies that $(\mathbf{p-1})^{\frac{1}{\mathbf{n}}}$ is a globally attractive equilibrium in $C_+ \backslash \{0\}$. This completes the proof. □

References

1. Aronson, D.G., Weinberger, H.F.: Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: Partial Differential Equations and Related Topics (Program, Tulane University, New Orleans, 1974), Lecture Notes in Mathematis, vol. 446, pp. 5–49. Springer, Berlin (1975)
2. Cooke, K.L., Huang, W.: Dynamics and Global Stability for a Class of Population Models with Delay and Diffusion Effects, CDSNS, pp. 92–76. (1992)
3. Daners, D., Medina, P.K.: Abstract Evolution Equations, Periodic Problems and Applications. Pitman Research Notes in Mathematics Series, no. 279, Longman Scientific & Technical, Harlow (1992)

4. Fang, J., Zhao, X.: Existence and uniqueness of traveling waves for non-monotone integral equations with applications. J. Differ. Equ. **248**, 2199–2226 (2010)
5. Faria, T., Huang, W., Wu, J.: Travelling waves for delayed reaction–diffusion equations with global response. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **462**, 229–261 (2006)
6. Faria, T., Trofimchuk, S.: Nonmonotone travelling waves in a single species reaction–diffusion equation with delay. J. Differ. Equ. **228**, 357–376 (2006)
7. Friesecke, G.: Convergence to equilibrium for delay-diffusion equations with small delay. J. Dyn. Differ. Equ. **5**, 89–103 (1993)
8. Gopalsamy, K., Trofimchuk, S., Bantsur, N.: A note on global attractivity in models of hematopoiesis. Ukrainian Math. J. **50**, 3–12 (1998)
9. Gourley, S.A., Ruan, S.: Dynamics of the diffusive Nicholson's blowflies equation with distributed delay. Proc. Roy. Soc. Edinb. Sect. A **130**, 1275–1291 (2000)
10. Gourley, S., So, J.W.-H., Wu, J.: Non-locality of reaction–diffusion equations induced by delay: biological modeling and nonlinear dynamics. In: Anosov, D.V., Skubachevskii, A. (eds.) Contemporary Mathematics, Thematic Surveys. Kluwer Plenum 2004 (English version is published in Journal of Mathematical Sciences, vol. 124, pp. 5119–5153 (2004))
11. Gourley, S.A., Wu, J.: Delayed non-local diffusive systems in biological invasion and disease spread. In: Brunner, H., Zhao, X.-Q., Zou, X. (eds.) Nonlinear Dynamics and Evolution Equations, pp. 137–200. American Mathematical Society, Providence (2006)
12. Gurney, W.S.C., Blythe, S.P., Nisbet, R.M.: Nicholson's blowflies revisited. Nature **287**, 17–21 (1980)
13. Hale, J.: Asymptotic Behavior of Dissipative Systems. Mathematical Surveys and Monographs, vol. 20. American Mathematical Society, Providence (1988)
14. Huang, W.: Global dynamics for a reaction–diffusion equation with time delay. J. Differ. Equ. **143**, 293–326 (1998)
15. Kuang, Y.: Delay Differential Equations with Applications in Population Dynamics. Academic, London (1993)
16. Kyrychko, Y., Gourley, S.A., Bartuccelli, M.V.: Comparison and convergence to equilibrium in a nonlocal delayed reaction–diffusion model on an infinite domain. Discret Contin. Dyn. Syst. Ser. B **5**, 1015–1026 (2005)
17. Kyrychko, Y., Gourley, S.A., Bartuccelli, M.V.: Dynamics of a stage-structured population model on an isolated finite lattice. SIAM J. Math. Anal. **37**, 1688–1708 (2006)
18. Liz, E., Rost, G.: On the global attractivity of delay differential equations with unimodel feedback. Discret. Contin. Dyn. Syst. **24**, 1215–1224 (2009)
19. Liz, E., Rost, G.: Dichotomy results for delay differential equations with negative Schwarzian derivative. Nonlinear Anal. RWA, doi:10.1016/j.nonrwa.2009.02.030
20. Mackey, M.C., Glass, L.: Oscillation and chaos in physiological control systems. Science **197**, 287–289 (1977)
21. Martin, R.H. Jr., Smith, H.L.: Abstract functional-differential equations and reaction–diffusion systems. Trans. Am. Math. Soc. **321**, 1–44 (1990)
22. Martin, R.H. Jr., Smith, H.L.: Reaction–diffusion systems with time delays: monotonicity, invariance, comparison and convergence. J. Reine Angew. Math. **413**, 1–35 (1991)
23. Mei, M., So, J.W.-H., Li, M.Y., Shen, S.S.P.: Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion. Proc. Roy. Soc. Edinb. Sect. A **134**, 579–594 (2004)
24. Metz, J.A.J., Diekmann, O.: Age. In: Metz, J.A.J., Diekmann, O. (eds.) The Dynamics of Physiologically Structured Populations, Lecture Notes in Biomathematics, vol. 68, pp. 136–184. Springer, Berlin (1986)
25. Nicholson, A.J.: An outline of the dynamics of animal populations. Aust. J. Zool. **2**, 9–65 (1954)
26. Smith, H.L.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Mathematical Surveys and Monographs, vol. 41. American Mathematical Society, Providence (1995)
27. So, J.W.-H., Yang, Y.: Dirichlet problem for the diffusive Nicholson's blowflies equation. J. Differ. Equ. **150**, 317–348 (1998)
28. So, J.W.-H., Zou, X.: Traveling waves for the diffusive Nicholson's blowflies equation. Appl. Math. Comput. **122**, 385–392 (2001)
29. So, J.W.-H., Wu, J., Zou, X.: A reaction–diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457**, 1841–1853 (2001)
30. Tang, X.H., Zou, X.: Stability of scalar delay differential equations with dominant delayed terms. Proc. Roy. Soc. Edinb. A **133**, 951–968 (2003)
31. Travis, C.C., Webb, G.F.: Existence and stability for partial functional differential equations. Trans. Am. Math. Soc. **200**, 395–418 (1974)
32. Wang, Z., Li, W., Ruan, S.: Entire solutions in bistable reaction–diffusion equations with nonlocal delayed nonlinearity. Trans. Am. Math. Soc. **361**, 2047–2084 (2009)
33. Wu, J.: Theory and Applications of Partial Functional-Differential Equations, Applied Mathematical Sciences, vol. 119. Springer, New York (1996)
34. Wu, J., Zou, X.: Traveling wave fronts of reaction–diffusion systems with delay. J. Dyn. Differ. Equ. **13**, 651–687 (2001)

35. Wu, J., Zhao, X.-Q.: Diffusive monotonicity and threshold dynamics of delayed reaction diffusion equations. J. Differ. Equ. **186**, 470–484 (2002)
36. Yamada, Y.: Asymptotic behavior of solutions for semilinear Volterra diffusion equations. Nonlinear Anal. **21**, 227–239 (1993)
37. Yang, Y., So, J.W.-H.: Dynamics for the diffusive Nicholson's blowflies equation. Discret Contin. Dyn. Syst. **II**, 333–352 (1998)
38. Yi, T., Chen, Y., Wu, J.: Threshold dynamics of a delayed reaction diffusion equation subject to the Dirichlet condition. J. Biol. Dyn. **3**, 331–341 (2009)
39. Yi, T., Chen, Y., Wu, J.: Spreading speeds and travelling waves for non-monotone dynamical systems with applications, preprint
40. Yi, T., Zou, X.: Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: a non-monotone case. J. Differ. Equ. **245**, 3376–3388 (2008)
41. Yi, T., Zou, X.: Map dynamics versus dynamics of associated delay reaction–diffusion equations with a Neumann condition. Proc. R. Soc. A Math. Phys. Eng. Sci. **466**, 2955–2973 (2010)
42. Yosida, K.: Functional Analysis. Springer, Berlin (1980)
43. Zhao, X.: Global attractivity in a class of nonmonotone reaction–diffusion equations with time delay. Can. Appl. Math. Q. **17**, 271–281 (2009)

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