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MONOTONE TRAVELING WAVES FOR DELAYED LOTKA-VOLTERRA COMPETITION SYSTEMS

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Abstract. We consider a delayed reaction-diffusion Lotka-Volterra competition system which does not generate a monotone semiflow with respect to the standard ordering relation for competitive systems. We obtain a necessary and sufficient condition for the existence of traveling wave solutions connecting the extinction state to the coexistence state, and prove that such solutions are monotone and unique (up to translation).

1. **Introduction.**

The classical Lotka-Volterra ODE competition model

$$\begin{cases} u' = r_1 u(1 - a_1 u - b_1 v), \\ v' = r_2 v(1 - a_2 v - b_2 u), \end{cases} \qquad r_i, a_i, b_i > 0, i = 1, 2 \tag{1}$$

has the trivial equilibrium $E_0 := (0,0)$ and two boundary equilibria E_1, E_2. Under the assumption that equilibria are isolated, this Lotka-Volterra ODE competition system may exhibit three different types of global dynamics. If one of the boundary equilibria is a saddle and the other is a sink, then the system has no positive coexistence equilibrium. Otherwise, the system has a coexistence equilibrium E_* which can be either a saddle point or a sink depending on whether the boundary equilibria are both sinks or are both saddle points. Such a system generates a monotone dynamical system with respect to the standard ordering for competitive systems, the aforementioned classification of global dynamics is natural and can be obtained by applying the powerful monotone dynamical systems theory, see for example [32]. This flow monotonicity with respect to the standard ordering relation for competitive systems has also made it possible to establish the existence of traveling waves connecting equilibria and to analytically describe the range of wave speeds for the corresponding reaction-diffusion model

$$\begin{cases} u_t - d_1 \Delta u = r_1 u(1 - a_1 u - b_1 v), \\ v_t - d_2 \Delta v = r_2 v(1 - a_2 v - b_2 u). \end{cases} \tag{2}$$

Consequently, it is well-known that traveling waves in (2) can be either *monostable* or *bistable* and this classification is completely (linearly) determined by the stability of the involved equilibria connected by a traveling wave. For example, traveling

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waves connecting two boundary equilibria E_1, E_2 are monostable if one of these equilibria is a saddle and the other is a sink, and the traveling waves connecting E_1, E_2 are bistable if both of the equilibria are sinks. General results about monostable and bistable traveling waves for reaction-diffusion equations admitting comparison principles can be found in the monograph [36], and relevant studies for monotone (i.e., order-preserving) dynamical systems can be found in [38, 26, 39, 20, 25]. There are substantial recent developments on the linear or nonlinear determinacy of minimal wave speeds for (2), see [23, 14, 13] and references therein.

Incorporating time delay into interspecific competition does not alter the aforementioned results for the corresponding ODE models since the order-preserving property remains valid. However, incorporating time delay in the intraspecific competition in model (1) may alter these results considerably. Even for a single species population with delayed intraspecific competition (self-limitation) such as the delayed Fisher-KPP equation

$$u_t = \Delta u + u(t)[1 - u(t - \tau)] \tag{3}$$

with τ being a given positive number, the generated semiflow (either the kinetic ordinary delayed differential equation or the PDE analogue) is no longer order-preserving with respect to any closed cone of phase space when τ is large. This is obvious since a Hopf bifurcation of stable periodic solutions may occur. Similar difficulties arise for the nonlocal Fisher-KPP equation

$$u_t = \Delta u + u[1 - u * k_\sigma], \quad k_\sigma(x) = \sigma^{-1} k(x/\sigma), \tag{4}$$

where k is a smooth function with $\int_{\mathbb{R}} k(x)dx = 1$. For model equation (3), there are some partial answers for the global dynamics and the existence of traveling waves. In particular, when the delay is small, Smith and Thieme [34, 35] obtained a general result that shows that the model equation without diffusion generates a monotone semiflow under an exponential ordering, and as such most solutions converge to equilibria and the stability of equilibria is essentially the same as that for the ordinary differential equation model. Additional results on the global dynamics in various delayed competition systems can be found in, for example, [16, 17, 18, 7, 9].

When both time delay and diffusion are incorporated, we generally obtain a time-delayed system with nonlocal interaction since populations in a specific spatial location at time $t - \tau$ will distribute over all spatial locations at time t due to diffusion (see, e.g., [3]). Friesecke [10] proved that most solutions converge to equilibria in some delayed reaction-diffusion equations similar to (3) in bounded domains subject to Neumann or Dirichlet boundary conditions when delay is sufficiently small. There seems to be no further study to verify whether solutions of such delayed reaction-diffusion equations can generate monotone semiflows even with small delay, except the work of [40] that introduced some PDE analogue of exponential ordering similar to that introduced by [34, 35] for delayed ODEs. For general classes of reaction-diffusion equations including (3) in unbounded domains, monotone traveling waves for non-quasi monotone delayed/nonlocal reaction-diffusion equation were established when delay is sufficiently small by Wu and Zou [41] (using the idea of exponential ordering), and similarly for (4) when the nonlocal response is sufficiently narrow by [12] (using a geometric singular perturbation method). These techniques have been further refined and used in [6, 30, 31, 8] to establish the existence of traveling waves when 1). the delay is small; or 2). the nonlocal interaction is narrow; or 3) the wave speed is large. See [21, 22] for another approach based on the cross-iteration scheme. Recently, Kwong and Ou [19] and Gomez and Trofimchuk [11] independently found the critical value of delay for the existence of monotone traveling waves of the delayed Fisher-KPP equation (and the work [11] also obtained the uniqueness and asymptotic behavior of such wave solutions). Moreover, Fang and Zhao [5] established the critical value of the rate of nonlocal interactions for the existence of monotone traveling waves and proved the uniqueness of such solutions to the nonlocal Fisher-KPP equation, for which Berestycki et al. [2] proved that traveling waves exist for all rates of nonlocal interactions and Nadin et al. [29] numerically showed that there is such a critical value for the existence of monotone traveling waves. Finally, we should mention the work [15, 24, 4, 37] for locally quasi-monotone systems, and the work [28, 33] for quasi-monotone delayed systems.

The purpose of this paper is to establish a *necessary and sufficient condition* for the existence of monotone traveling waves for the reaction-diffusion competition models with delayed intraspecific competition. In particular, we calculate precisely the minimal wave speed and provide the sharpest lower bound of the delay for which monotone traveling waves exist. We describe our results and develop our technical details for the following time-delayed model

$$\begin{cases} u_t - d_1 u_{xx} = r_1 u[1 - b_1 u(t-\tau, x) - a_1 v], \\ v_t - d_2 v_{xx} = r_2 v[1 - b_2 v(t-\tau, x) - a_2 u]. \end{cases} \tag{5}$$

For the simplicity of notations, we scale system (5) as follows:

$$\begin{cases} u_t - u_{xx} = u[1 - u(t-\tau, x) - a_1 v], \\ v_t - d v_{xx} = r v[1 - v(t-\tau, x) - a_2 u]. \end{cases} \tag{6}$$

Under the hypothesis that there exists a unique coexistence equilibrium E_*, that is, $(a_1 - 1)(a_2 - 1) > 0$, we will prove the following results on solutions connecting the trivial equilibrium $E_0 := (0, 0)$ to the positive equilibrium $E_* := (\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$ for system (6):

(I) There exist monotone traveling waves with speed c if and only if $c \geq c_{\min} := \max\{2, 2\sqrt{dr}\}$ and $\tau \leq \tau(c)$;

(II) $\tau(c)$ is strictly decreasing in c to the positive limit $\tau(\infty)$, which is the critical value for the existence of monotone heteroclinic orbits of the kinetic system;

(III) Such a monotone solution is unique up to translation.

In the above results, $\tau(c)$ and $\tau(\infty)$ are both implicitly determined by some eigenvalue problems. More precisely, $c \geq c_{\min}$ is necessary and sufficient for the existence of a non-negative eigenvector associated with a positive eigenvalue for the linearized problem of the wave profile equation at equilibrium E_0. $\tau \leq \tau(c)$ and $\tau \leq \tau(\infty)$ are the necessary and sufficient conditions for the existence of a non-positive eigenvector associated with a negative eigenvalue for the linearized problem of the wave profile equation and the kinetic system at equilibrium E_*, respectively. As such, we have the full linear determinacy of the considered model.

The rest of this paper is organized as follows. Section 2 is devoted to the study of the two aforementioned eigenvalue problems. Section 3 describes the properties of all possible monotone wave profiles and reveals their exact convergence rate to E_*. These preliminary results are then used in Sections 4 and 5 to derive our main results. We conclude the paper with a relevant system, for which we describe the existence of monotone traveling waves and describe their properties in the critical case. This resolves an unsolved problem in [11, 5].

2. **Eigenvalue problem.** This section is devoted to the study of the eigenvalue problems for the wave profile equation, which is obtained by substituting $u(t,x) = U(x+ct)$ and $v(t,x) = V(x+ct)$ in (6). Here (U,V) is called the wave profile, $\xi := x+ct$ the wave coordinate and c the speed. For the sake of convenience, we still use u, v, x instead of U, V, ξ, respectively. And hence, we have the following wave profile equation after the scaling $u(x) \to u(x/c)$ and $v(x) \to v(x/c)$:

$$\begin{cases} c^{-2}u''(x) - u'(x) + u(x)[1 - u(x-\tau) - a_1 v(x)] = 0, \\ c^{-2}dv''(x) - v'(x) + rv(x)[1 - v(x-\tau) - a_2 u(x)] = 0. \end{cases} \tag{7}$$

In what follows, we say $u \in \mathbb{R}^2$ is strongly positive if $u \gg 0$ in the sense that each component of u is positive.

We first study the eigenvalue problem at E_0. Linearizing (7) yields

$$\begin{cases} c^{-2}u''(x) - u'(x) + u(x) = 0, \\ c^{-2}dv''(x) - v'(x) + rv(x) = 0, \end{cases} \tag{8}$$

which implies that eigenvalue μ is governed by the following equation

$$(c^{-2}\mu^2 - \mu + 1)(c^{-2}d\mu^2 - \mu + r) = 0. \tag{9}$$

Direct computations show that there exists at least one positive eigenvalue if and only if $c \geq c_{\min} := \max\{2, 2\sqrt{dr}\}$. There are nonnegative eigenvectors associated with these eigenvalues. Moreover, for $c > c_{\min}$, there are two eigenvalues μ_1, μ_3 solving $c^{-2}\mu^2 - \mu + 1 = 0$ and two eigenvalues μ_2, μ_4 solving $c^{-2}d\mu^2 - \mu + r = 0$.

Next we analyze the eigenvalue problem at $E_* = (u^*, v^*)^T$. Linearizing (7) yields

$$\begin{cases} c^{-2}u''(x) - u'(x) - u^* u(x-\tau) - a_1 u^* v(x) = 0, \\ c^{-2}dv''(x) - v'(x) - rv^* v(x-\tau) - ra_2 v^* u(x) = 0. \end{cases} \tag{10}$$

Plugging $u(x) = m_1 e^{\lambda x}, v(x) = m_2 e^{\lambda x}$ into the above equation, we obtain the following eigenvalue problem:

$$\det A(\lambda,\tau) = 0, \quad A(\lambda,\tau)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0, \tag{11}$$

where

$$A(\lambda,\tau) := \begin{pmatrix} h_1(\lambda,\tau) & -a_1 u^* \\ -ra_2 v^* & h_2(\lambda,\tau) \end{pmatrix}$$

with

$$h_1(\lambda,\tau) = c^{-2}\lambda^2 - \lambda - u^* e^{-\lambda\tau} \quad \text{and} \quad h_2(\lambda,\tau) = c^{-2}d\lambda^2 - \lambda - rv^* e^{-\lambda\tau}.$$

About the distribution of negative eigenvalues and their associated eigenvectors, we have the following results:

Lemma 2.1. *Assume that $c \geq c_{\min}$, then we have the following statements.*

(i) There exists $\tau(c) > 0$ such that the (λ, y)-system

$$\begin{cases} \det A(\lambda,\tau) = 0, \\ A(\lambda,\tau)y = 0, \\ \lambda < 0, \ y \gg 0 \end{cases} \tag{12}$$

has a solution if and only if $\tau \leq \tau(c)$.

(ii) $\tau(c)$ is decreasing in c to the positive limit $\tau(\infty) > 0$.

(iii) Define

$$\lambda_1 := \max\{\lambda < 0 : \lambda \text{ solves } (12)\}, \tag{13}$$

and

$$D_c := \{\tau : \frac{d}{d\lambda} \det A(\lambda, \tau)|_{\lambda=\lambda_1} = 0\}. \tag{14}$$

The set D_c consists of finitely many points.

(iv) Fix $\tau \le \tau(c)$ with $\tau \notin D_c$. Then there exists $\epsilon_0 > 0$ such that for any $\beta \gg 0$ and $\epsilon \in (0, \epsilon_0)$, equation $A(\lambda_1 - \epsilon, \tau)y = \beta$ has a strongly positive solution.

Proof. (i) In order to define the value $\tau(c)$, we begin with the study of the possible location of a solution $\bar{\lambda}$ to (12). Equation $\det A(\bar{\lambda}, \tau) = 0$ requires that $h_i(\bar{\lambda}, \tau), i = 1, 2$ has the same sign which, together with $A(\bar{\lambda}, \tau)y = 0$ with $y \gg 0$, requires that $h_i(\bar{\lambda}, \tau) > 0, i = 1, 2$. This requirement implies that equation $h_i(\lambda, \tau) = 0$ has only two negative roots $\lambda_{i1} > \lambda_{i2}$ with $\bar{\lambda} \in (\lambda_{i2}, \lambda_{i1})$ and $h_i(\lambda, \tau) > 0$ for $\lambda \in (\lambda_{i2}, \lambda_{i1})$. Here we shall mention that $\bar{\lambda}, \lambda_{i1}, \lambda_{i2}$ all depend on τ. Now we can define the set

$$S(\tau) := \{\lambda : h_i(\lambda, \tau) > 0, i = 1, 2\}. \tag{15}$$

Then it is easy to see $S(\tau)$ is an interval having the form $(s_1(\tau), s_2(\tau))$ with $s_i(\tau)$ being the zeros of either $h_1(\lambda, \tau)$ or $h_2(\lambda, \tau)$. Note that $h_i(\lambda, \tau)$ is decreasing in τ. It then follows that $s_1(\tau)$ is increasing in τ while $s_2(\tau)$ is decreasing. These facts imply that quantity

$$\max_{\lambda \in S(\tau)} h_1(\lambda, \tau) h_2(\lambda, \tau) \tag{16}$$

is decreasing in τ. It is not difficult to see that $\lim_{\tau \to 0} S(\tau) = (-\infty, s_2(0))$, and hence, for small τ equation $\det A(\lambda, \tau) = 0$ always has a negative solution. On the other hand, $h_i(\lambda, \tau) < 0$ for all $\lambda < 0$ if τ is sufficiently large. Therefore, there exists a critical value $\tau(c)$ such that $\max_{\lambda \in S(\tau)} h_1(\lambda, \tau) h_2(\lambda, \tau) > r a_1 a_2 u^* v^*$ if and only if $\tau < \tau(c)$. This proves statement (i).

(ii) In order to study the dependence on c, we write $h_i(\lambda, \tau, c)$ instead of $h_i(\lambda, \tau)$. Note that $h_i(\lambda, \tau, c)$ is decreasing in c, so is $\max_{\lambda \in S(\tau)} h_1(\lambda, \tau, c) h_2(\lambda, \tau, c)$. It then follows from the analysis in the proof of statement (i) that $\tau(c)$ is decreasing in c and the limit $\tau(\infty)$ is the maximal value of τ such that system (12) with $c = +\infty$ has a solution.

(iii) Denote the solution set of (λ, τ)-system

$$\begin{cases} \det A(\lambda, \tau) = 0, \\ \frac{d}{d\lambda} \det A(\lambda, \tau) = 0 \end{cases} \tag{17}$$

by $\mathcal{K}$. Hence, there are finitely many elements in any bounded subset of $\mathcal{K}$ due to the analyticity of the functions $\det A(\lambda, \tau)$ and $\frac{d}{d\lambda} \det A(\lambda, \tau)$. Since $D_c \subset \{\tau : \exists \lambda, \, s.t. \, (\lambda, \tau) \in \mathcal{K}\}$, it then suffices to prove $\lambda_1 = \lambda_1(\tau)$ is bounded in τ. Note that $\lambda_1(\tau) > s_1(\tau)$, which is the left endpoint of interval $S(\tau)$ defined in (15). It follows that $\lambda_1(\tau)$ is bounded for large τ due to the monotonicity of $s_1(\tau)$. On the other hand, there exists some λ_0 such that $\det A(\lambda, \tau) > 0$ for all small τ and $\lambda \le \lambda_0$. Therefore, $\lambda_1(\tau)$ is bounded in τ.

(iv) From statement (i), we see that $\det A(\lambda_1 - \epsilon, \tau)$ is positive for small ϵ when $\tau < \tau(c)$ with $\tau \notin D_c$, so are $h_i(\lambda_1 - \epsilon, \tau), i = 1, 2$. Then for any $\beta \gg 0$, equation $A(\lambda_1 - \epsilon, \tau)y = \beta$ has a unique solution

$$y = \frac{1}{\det A(\lambda_1 - \epsilon, \tau)} \begin{pmatrix} h_2(\lambda_1 - \epsilon, \tau) & a_1 u^* \\ r a_2 v^* & h_1(\lambda_1 - \epsilon, \tau) \end{pmatrix} \beta \gg 0. \tag{18}$$

This completes the proof. □

3. **Properties of wave profiles.** In this section, we conduct some qualitative analysis for wave profiles. The results will be used in the proof of uniqueness and nonexistence of traveling waves. In what follows, we always assume that (u, v) is a given wave profile between E_0 and E_*.

Lemma 3.1. *Any non-constant wave profile between E_0 and E_* is strictly increasing and connecting E_0 to E_*.*

Proof. Assume, for the sake of contradiction and by translation invariance, that $u'(0) = 0$. Since u satisfies

$$c^{-2}u''(x) - u'(x) + u(x)[1 - u(x - \tau) - a_1 v(x)] = 0, \tag{19}$$

we see that

$$c^{-2}(u'(x)e^{c^2x})' = -u(x)[1 - u(x - \tau) - a_1 v(x)]e^{c^2x} \leq 0. \tag{20}$$

Hence, $u'(x)e^{c^2x}$ is nonincreasing, which together with $u'(0) = 0$ implies that

$$u'(x) \leq 0, \forall x \geq 0 \quad \text{and} \quad u'(x) \geq 0, \forall x \leq 0.$$

As u is bounded, $u(+\infty) \leq u(0)$ exists and $u'(+\infty) = 0$. Next, we claim $u(x) = u(0), \forall x \geq 0$. Otherwise, $u(+\infty) < u(0)$, which gives rise to the existence of $x_1 > 0$ such that

$$u''(x_1) = 0, \quad u'(x_1) < 0, \quad u(0) > u(x_1) > 0.$$

Choosing $x = x_1$ in (19), we obtain the contradiction

$$0 \geq -u'(x_1) + u(x_1)[1 - u^* - a_1 v^*] = -u'(x_1) > 0. \tag{21}$$

Since u is not a constant, we see that u is non-decreasing with $0 \leq u(-\infty) < u(0) = u(+\infty) \leq u^*$. Consequently, we can find $x_2 < 0$ such that $u(x_2) = u(0)$ and $u(x) < u(0), \forall x < x_2$. Choosing $x = x_2$ in (19), we obtain another contradiction

$$0 = 1 - u(x_2 - \tau) - a_1 v(x_2) > 1 - u(0) - a_1 v^* \geq 1 - u^* - a_1 v^* = 0. \tag{22}$$

Similarly, v is strictly increasing.

Now that non-constant u, v are increasing and bounded, we have $u'(\pm\infty) = 0, v'(\pm\infty) = 0$ and that $u(\pm\infty), v(\pm\infty)$ all exist. Taking $x \to +\infty$ along some appropriate sequence $x_n \to +\infty$ in (19), we see that both $(u(-\infty), v(-\infty))$ and $(u(+\infty), v(+\infty))$ are equilibria of (6). This implies that the wave profile connects E_0 to E_*. □

Next we show that $w_1(x) := u^* - u(x)$ and $w_2(x) := v^* - v(x)$ are exponentially decreasing as $x \to +\infty$.

Lemma 3.2. *There exist positive constants $B_{11}, B_{12}, \beta_{11}$ and β_{12} such that*

$$B_{11}e^{-\beta_{11}x} \leq w_1(x) \leq B_{12}e^{-\beta_{12}x}, \quad \forall x \geq 0. \tag{23}$$

Similar results hold for w_2.

Proof. It is easy to see that $w_1(-\infty) = u^*, w_1(+\infty) = 0$. We can rewrite (19) as the following integral equation for w_1:

$$\begin{aligned} w_1(x) &= \int_x^\infty m_c(x - y)[u^* - w(y)][w_1(y - \tau) + a_1 w_2(y)]dy \\ &= \int_{-\infty}^0 m_c(y)[u^* - w_1(x - y)][w_1(x - y - \tau) + a_1 w_2(x - y)]dy, \end{aligned} \tag{24}$$

where $m_c(x) = 1 - e^{c^2 x}$. For any $\epsilon > 0$, there exists $x' > 0$ such that $w_1(x) \leq \epsilon, \forall x \geq x'$. Since $w \geq 0$ by hypothesis, it follows that

$$\begin{aligned} w_1(x) &\geq (u^* - \epsilon) \int_{-\infty}^{0} m_c(y) w_1(x - y - \tau) \\ &\geq (u^* - \epsilon) \int_{-\tau/2}^{0} m_c(y) w_1(x - y - \tau) dy \\ &\geq M_1 w_1(x - \tau/2), \end{aligned} \tag{25}$$

where $M_1 := (u^* - \epsilon) \int_{-\tau/2}^{0} m_c(y) dy$.

On the other hand, we can choose $M_2 > 0$ such that $\int_{-M_2}^{0} m_c(y) dy > 1$. Then we have

$$w_1(x) \geq (u^* - \epsilon) \int_{-M_2}^{0} m_c(y) w_1(x - y - \tau) dy. \tag{26}$$

Consequently, by arguments similar to those in the proof of [5, Lemma 3.1] we can obtain

$$w_1(x) \leq M_3 w_1(x - M_4) \tag{27}$$

for some positive numbers M_3 and M_4. Combining (25) and (26), we reach the conclusion by employing arguments similar to those in the proof of [4, Lemma 3.1]. □

In view of Lemma 3.2, we may define the Laplace transform

$$\mathcal{L}_i(\lambda) = \int_{\mathbb{R}} w_i(x) e^{-\lambda x} dx, \quad \lambda < 0. \tag{28}$$

Owing to Lemma 3.2, $\mathcal{L}_i(\lambda)$ converges at least for $\lambda > -\beta_{i1}$ and diverges for $\lambda < -\beta_{i2}$. As w_i is positive, there exists a singular point $\lambda_i^* \in (-\beta_{i1}, -\beta_{i2})$ such that $\mathcal{L}_i(\lambda) < +\infty$ if $\lambda > \lambda_i^*$ and $\mathcal{L}_i(\lambda) = +\infty$ if $\lambda < \lambda_i^*$.

Based on Lemma 3.2, we further give an exact priori estimation of $w_i(x)$ as $x \to +\infty$.

Theorem 3.3. *There exists an eigenvector $(m_1, m_2)^T$ associated with eigenvalue λ_1 defined in (13) and a polynomial p of order k such that*

$$\lim_{x \to +\infty} \left(\frac{w_1(x)}{p(x) e^{\lambda_1 x}}, \frac{w_2(x)}{p(x) e^{\lambda_1 x}} \right) = (m_1, m_2),$$

where $k + 1$ is the multiplicity of eigenvalue λ_1.

Proof. Note that (w_1, w_2) satisfies the following system of differential equations

$$\begin{cases} c^{-2} w_1''(x) - w_1'(x) - w_1(x) = [u^* - w_1(x)][w_1(x - \tau) + a_1 w_2(x)] + w_1(x), \\ c^{-2} d w_2''(x) - w_2'(x) - r w_2(x) = r[v^* - w_2(x)][w_2(x - \tau) + a_2 w_1(x)] + r w_2(x). \end{cases} \tag{29}$$

Using the variation of constants formula for the second order ODE, we have the following integral equation

$$\begin{aligned} \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} &= \int_x^{+\infty} \begin{pmatrix} g_1(x - y)\{u^*[w_1(y - \tau) + a_1 w_2(y)] + w_1(y)\} \\ g_2(x - y)\{v^*[w_1(y - \tau) + a_2 w_1(y)] + w_2(y)\} \end{pmatrix} dy \\ &\quad - \int_x^{+\infty} \begin{pmatrix} g_1(x - y) w_1(y)[w_1(y - \tau) + a_1 w_2(y)] \\ g_2(x - y) w_2(y)[w_1(y - \tau) + a_2 w_1(y)] \end{pmatrix} dy, \end{aligned} \tag{30}$$

where g_1, g_2 are defined as in (39). Under the transform (28), equation (30) becomes

$$\begin{aligned}
&\mathcal{A}(\lambda,\tau)\begin{pmatrix}\mathcal{L}_1(\lambda)\\ \mathcal{L}_2(\lambda)\end{pmatrix}\\
= &-\int_{\mathbb{R}}\int_x^{+\infty} e^{-\lambda x}\begin{pmatrix} g_1(x-y)w_1(y)[w_1(y-\tau)+a_1w_2(y)]\\ g_2(x-y)w_2(y)[w_1(y-\tau)+a_2w_1(y)]\end{pmatrix} dydx\\
:= &\begin{pmatrix}G_1(\lambda)\\ G_2(\lambda)\end{pmatrix} \qquad (31)
\end{aligned}$$

with

$$\mathcal{A}(\lambda,\tau)=\begin{pmatrix}\frac{h_1(\lambda,\tau)}{c^{-2}\lambda^2-\lambda+1} & \frac{-a_1u^*}{c^{-2}\lambda^2-\lambda+1}\\ \frac{-ra_2v^*}{c^{-2}d\lambda^2-\lambda+r} & \frac{h_2(\lambda,\tau)}{c^{-2}d\lambda^2-\lambda+r}\end{pmatrix}. \qquad (32)$$

Clearly, $\mathcal{A}(\lambda,\tau)$ has the same eigenvalues and eigenvectors with $A(\lambda,\tau)$, which is defined in (11).

Note that if $\mathcal{L}_i(\lambda)$ converges for $\lambda>\bar{\lambda}$, then the right-hand side of (31) converges for $\lambda>2\bar{\lambda}$. Such difference in the abscissa of convergence gives rise to the conclusion that $\mathcal{L}_i(\lambda)$ has possible singularity only at the zeros of $\det A(\lambda,\tau)$. Further, $\mathcal{L}_1(\lambda)$ and $\mathcal{L}_2(\lambda)$ shares the singular point, say $\lambda^*<0$. Next, we employ the argument in the proof of [27, Proposition 6.1] (see Page 29-31), where in particular we choose $\tau=-\infty$. Hence we see (31) is the similar required version as equation (7.5) with $\psi=0$ in [27]. Then following the steps in the proof of [27, Proposition 6.1], we can choose $a=\lambda^*-\epsilon$ and $b=\lambda^*-\epsilon$ with $\epsilon>0$ small enough to reach that

$$\lim_{x\to+\infty}\left(\frac{w_1(x)}{p(x)e^{\lambda^*x}},\frac{w_2(x)}{p(x)e^{\lambda^*x}}\right)=(m_1,m_2),\quad \text{for some positive number } m_1,m_2, \qquad (33)$$

where p is a polynomial of order k and $k+1$ is the multiplicity of λ^* as the pole of

$$A^{-1}(\lambda,\tau)\begin{pmatrix}G_1(\lambda)\\ G_2(\lambda)\end{pmatrix}. \qquad (34)$$

Next we prove $\lambda^*=\lambda_1$ defined in (13) and that $(m_1,m_2)^T$ is an associated eigenvector with the assumption that the coefficient of x^k in p is the unit. Assume for the sake of contradiction that $\lambda^*<\lambda_1$. Then choosing $\lambda=\lambda_1$ in (31) reads

$$0\gg\mathcal{A}(\lambda,\tau)\begin{pmatrix}\mathcal{L}_1(\lambda)\\ \mathcal{L}_2(\lambda)\end{pmatrix}|_{\lambda=\lambda_1}=\begin{pmatrix}\frac{h_1(\lambda,\tau)}{c^{-2}\lambda^2-\lambda+1} & \frac{-a_1u^*}{c^{-2}\lambda^2-\lambda+1}\\ \frac{-ra_2v^*}{c^{-2}d\lambda^2-\lambda+r} & \frac{h_2(\lambda,\tau)}{c^{-2}d\lambda^2-\lambda+r}\end{pmatrix}\begin{pmatrix}\mathcal{L}_1(\lambda)\\ \mathcal{L}_2(\lambda)\end{pmatrix}|_{\lambda=\lambda_1}. \qquad (35)$$

However, two components should have different signs due to $h_i(\lambda_1,\tau)>0$ and $\det\mathcal{A}(\lambda_1,\tau)=0$, a contradiction. Assume again for the sake of contradiction that $\lambda^*>\lambda_1$, then $h_i(\lambda^*,\tau)<0$ due to the definition of λ_1. Dividing the term $x^ke^{\lambda^*x}$ in both sides of the integral equality (30) and taking $x\to+\infty$, we arrive at

$$\mathcal{A}(\lambda^*,\tau)\begin{pmatrix}m_1\\ m_2\end{pmatrix}=0, \qquad (36)$$

which means that $(m_1,m_2)^T$ is a positive eigenvector of λ^*, a contradiction with $h_i(\lambda^*,\tau)<0$. Thus, $\lambda^*=\lambda_1$ and $(m_1,m_2)^T$ is an associated eigenvector. □

Remark 1. From the proof of Theorem 3.3, we see that there is no traveling wave if λ_1, as defined in (13), does not exist. Therefore, monotone traveling wave with speed $c\geq c_{\min}$ does not exist when $\tau>\tau(c)$.

4. **Existence and nonexistence.** For the nonexistence, we have already seen from Remark 1 that no monotone traveling waves with speed $c \geq c_{\min}$ and $\tau > \tau(c)$, so it remains to show that no traveling wave exists if $c \in [0, c_{\min})$. This can be done using the same argument as in the proof of [2, Lemma 3.8], where the nonlocal Fisher-KPP equation was considered.

In what follows, we focus on the existence of monotone traveling waves with speed $c \geq c_{\min}$ and $\tau \leq \tau(c)$. We develop our proof in two steps. (i) For the case where $c > c_{\min}$ and $\tau \notin D_c$, we employ the results for eigenvalue problems to construct upper and lower fixed points for an appropriate monotone integral operator; (ii) For the case where $c = c_{\min}$ or $\tau \in D_c$, we employ some limiting arguments.

Define the differential operator $L : C^2(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by the left hand side of wave profile equation (7), that is,

$$L[\phi](x) = \begin{pmatrix} c^{-2}\phi_1''(x) - \phi_1'(x) + \phi_1(x)[1 - \phi_1(x-\tau) - a_1\phi_2(x)] \\ c^{-2}d\phi_2''(x) - \phi_2'(x) + r\phi_2(x)[1 - \phi_2(x-\tau) - a_2\phi_1(x)] \end{pmatrix}. \tag{37}$$

For fixed $c > c_{\min}$, we have four eigenvalues $\mu_i, i = 1 \cdots 4$, of the linearized problem at E_0. Define the integral operator $T : C(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by separating the linear and nonlinear parts of the wave profile equation and solving it with the variation of constants formula, that is,

$$T[\phi](x) = \begin{pmatrix} \int_x^{+\infty} g_1(x-y)\phi_1(y)[\phi_1(y-\tau) + a_1\phi_2(y)] \\ \int_x^{+\infty} g_2(x-y)\phi_2(y)[\phi_2(y-\tau) + a_1\phi_1(y)] \end{pmatrix} \tag{38}$$

with

$$g_1(y) = \frac{c^2}{\mu_3 - \mu_1}(e^{\mu_1 y} - e^{\mu_3 y}), \quad g_2(y) = \frac{rc^2 d^{-1}}{\mu_4 - \mu_2}(e^{\mu_2 y} - e^{\mu_4 y}). \tag{39}$$

Note that zeros of operator L and fixed points of T are the same, and both are wave profiles with speed $c > c_{\min}$.

Next we construct upper and lower solutions to $L[\phi] = 0$.

Let λ_1 be defined as in (13) and $(m_1, m_2)^T$ its associating eigenvector. Define function $\phi^- = (\phi_1^-, \phi_2^-)^T$ with

$$\phi_1^-(x) = \begin{cases} u^* - m_1 e^{\lambda_1 x}, & x \geq x_1^-, \\ l_1 e^{\mu_1 x}, & x < x_1^-, \end{cases} \quad \phi_2^-(x) = \begin{cases} v^* - m_2 e^{\lambda_1 x}, & x \geq x_2^-, \\ l_2 e^{\mu_2 x}, & x < x_2^-, \end{cases} \tag{40}$$

where l_1, x_1^- and l_2, x_2^- are uniquely determined, respectively, by

$$\begin{cases} u^* - m_1 e^{\lambda_1 x_1^-} = l_1 e^{\mu_1 x_1^-}, \\ -m_1\lambda_1 e^{\lambda_1 x_1^-} = l_1 \mu_1 e^{\mu_1 x_1^-} \end{cases} \quad \text{and} \quad \begin{cases} v^* - m_2 e^{\lambda_1 x_2^-} = l_2 e^{\mu_2 x_2^-}, \\ -m_2\lambda_1 e^{\lambda_1 x_2^-} = l_2 \mu_2 e^{\mu_2 x_2^-}. \end{cases} \tag{41}$$

Lemma 4.1. *$L[\phi^-](x) \leq 0$ for any $x \in \mathbb{R} \setminus \{x_1^-, x_2^-\}$.*

Proof. It suffices to show that the first component $(L[\phi^-])_1(x) \leq 0$ for $x \in \mathbb{R} \setminus \{x_1^-\}$ and the second component $(L[\phi^-])_2(x) \leq 0$ for $x \in \mathbb{R} \setminus \{x_2^-\}$. Note that for all $x \in \mathbb{R}$

$$\phi_1^-(x) \geq u^* - m_1 e^{\lambda_1 x} \quad \text{and} \quad \phi_2^-(x) \geq v^* - m_2 e^{\lambda_1 x} \tag{42}$$

due to $u^* - m_1 e^{\lambda_1 x} \le l_1 e^{\mu_1 x}$ and $v^* - m_2 e^{\lambda_1 x} \le l_2 e^{\mu_2 x}$. It then follows that for $x > x_1^-$,

$$\begin{aligned} & (L[\phi^-])_1(x) \\ = \;& -m_1 e^{\lambda_1 x}[c^{-2}\lambda_1^2 - \lambda_1] + [u^* - m_1 e^{\lambda_1 x}][1 - \phi_1^-(x) - a_1(v^* - m_2 e^{\lambda_1 x})] \\ \le \;& -m_1 e^{\lambda_1 x}[c^{-2}\lambda_1^2 - \lambda_1] + [u^* - m_1 e^{\lambda_1 x}][1 - (u^* - m_1 e^{\lambda_1 (x-\tau)}) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad -a_1(v^* - m_2 e^{\lambda_1 x})] \\ = \;& -m_1 e^{\lambda_1 x}[c^{-2}\lambda_1^2 - \lambda_1] + [u^* - m_1 e^{\lambda_1 x}][m_1 e^{\lambda_1 (x-\tau)} + a_1 m_2 e^{\lambda_1 x}] \\ = \;& -m_1 e^{\lambda_1 x} h_1(\lambda_1, \tau) + u^* a_1 m_2 e^{\lambda_1 x} - m_1 e^{\lambda_1 x}[m_1 e^{\lambda_1 (x-\tau)} + a_1 m_2 e^{\lambda_1 x}] \\ \le \;& -m_1 e^{\lambda_1 x} h_1(\lambda_1, \tau) + u^* a_1 m_2 e^{\lambda_1 x} \\ = \;& 0, \end{aligned} \tag{43}$$

where equalities $1 - u^* - a_1 v^* = 0$ and $A(\lambda_1, \tau)m = 0$ are used. For $x < x_1^-$, we have

$$(L[\phi^-])_1 \le l_1 e^{\mu_1 x}[c^{-2}\mu_1^2 - \mu_1 + 1] - r l_1 e^{\mu_1 x}[\phi_1^-(x-\tau) + a_2\phi_2^-(x)] \le 0 \tag{44}$$

due to $c^{-2}\mu_1^2 - \mu_1 + 1 = 0$. This proves $(L[\phi^-])_1(x) \le 0, \forall x \in \mathbb{R} \setminus \{x_1^-\}$.

Similarly, for $x > x_2^-$ we have

$$(L[\phi^-])_2(x) \le r v^* a_2 m_1 e^{\lambda_1 x} - m_2 e^{\lambda_1 x} h_2(\lambda_1, \tau) = 0 \tag{45}$$

and for $x < x_2^-$ we have

$$(L[\phi^-])_2(x) \le l_2 e^{\mu_2 x}[c^{-2} d\mu_2^2 - \mu_2 + r] - r l_2 e^{\mu_2 x}[\phi_2^-(x-\tau) + a_2\phi_1^-(x)] \le 0. \tag{46}$$

This proves $(L[\phi^-])_2(x) \le 0, \forall x \in \mathbb{R} \setminus \{x_2^-\}$. □

For any $\tau \notin D_c$ and small $\epsilon > 0$, we see from Lemma 2.1(iv) that there exists $p := (p_1, p_2)^T \gg 0$ such that

$$\begin{pmatrix} h_1(\lambda_1 - \epsilon) & -u^* a_1 \\ -r a_2 v^* & h_2(\lambda_1 - \epsilon) \end{pmatrix} p \gg 0. \tag{47}$$

Moreover, we fix the value of p_1/p_2. Then we define $\phi_p^+ := (\phi_{1,p}^+, \phi_{2,p}^+)^T$ with

$$\phi_{1,p}^+(x) = \begin{cases} u^* - m_1 e^{\lambda_1 x} + p_1 e^{(\lambda_1 - \epsilon)x}, & x \ge x_1^+, \\ \delta_1, & x < x_1^+ \end{cases} \tag{48}$$

and

$$\phi_{2,p}^+(x) = \begin{cases} v^* - m_2 e^{\lambda_1 x} + p_2 e^{(\lambda_1 - \epsilon)x}, & x \ge x_2^+, \\ \delta_2, & x < x_2^+, \end{cases} \tag{49}$$

where δ_1, x_1^+ and δ_2, x_2^+ are uniquely determined, respectively, by

$$\begin{cases} -m_1\lambda_1 e^{\lambda_1 x_1^+} + p_1(\lambda_1 - \epsilon) e^{(\lambda_1 - \epsilon)x_1^+} = 0, \\ \delta_1 = u^* - m_1 e^{\lambda_1 x_1^+} + p_1 e^{(\lambda_1 - \epsilon)x_1^+} \end{cases} \tag{50}$$

and

$$\begin{cases} -m_2\lambda_1 e^{\lambda_1 x_1^+} + p_2(\lambda_1 - \epsilon) e^{(\lambda_1 - \epsilon)x_2^+} = 0, \\ \delta_2 = v^* - m_2 e^{\lambda_1 x_2^+} + p_2 e^{(\lambda_1 - \epsilon)x_2^+}. \end{cases} \tag{51}$$

Clearly, $x_i^+ = \frac{1}{\epsilon} \ln \frac{p_i(\lambda_1 - \epsilon)}{m_i \lambda_1}$ and it is increasing in p_i to $+\infty$.

Lemma 4.2. *$L[\phi_p^+](x) \ge 0$ for any $x \in \mathbb{R} \setminus \{x_1^+, x_2^+\}$ if p is sufficiently large.*

Proof. It suffices to show that the first component $(L[\phi_p^+])_1(x) \geq 0$ for $x \in \mathbb{R} \setminus \{x_1^+\}$ and the second component $(L[\phi_p^+])_2(x) \geq 0$ for $x \in \mathbb{R} \setminus \{x_2^+\}$. Note that

$$\phi_{1,p}^+(x) \leq u^* - m_1 e^{\lambda_1 x} + p_1 e^{(\lambda_1-\epsilon)x}, \quad \phi_{2,p}^+(x) \leq v^* - m_2 e^{\lambda_1 x} + p_2 e^{(\lambda_1-\epsilon)x}, \ \forall x \in \mathbb{R}$$

due to $u^* - m_1 e^{\lambda_1 x} + p_1 e^{(\lambda_1-\epsilon)x} \geq \delta_1$ and $v^* - m_2 e^{\lambda_1 x} + p_2 e^{(\lambda_1-\epsilon)x} \geq \delta_2$, respectively, and

$$p_i e^{-\epsilon x} < p_i e^{-\epsilon x_i^+} = \frac{m_i \lambda_1}{\lambda_1 - \epsilon}, \quad \forall p_i > 0, x > x_i^+, \ i = 1, 2.$$

It then follows that for $x > x_1^+$,

$$\begin{aligned}
&(L[\phi_p^+])_1(x) \\
= \ & -m_1 e^{\lambda_1 x}[c^{-2}\lambda_1^2 - \lambda_1] + p_1 e^{(\lambda_1-\epsilon)x}[c^{-2}(\lambda_1-\epsilon)^2 - (\lambda_1-\epsilon)] \\
& +[u^* - m_1 e^{\lambda_1 x} + p_1 e^{(\lambda_1-\epsilon)x}][1 - \phi_{1,p}^+(x-\tau) - a_1\phi_{2,p}^+(x)] \\
\geq \ & -m_1 e^{\lambda_1 x}[c^{-2}\lambda_1^2 - \lambda_1] + p_1 e^{(\lambda_1-\epsilon)x}[c^{-2}(\lambda_1-\epsilon)^2 - (\lambda_1-\epsilon)] \\
& +[u^* - m_1 e^{\lambda_1 x} + p_1 e^{(\lambda_1-\epsilon)x}][1 - (u^* - m_1 e^{\lambda_1(x-\tau)} + p_1 e^{(\lambda_1-\epsilon)(x-\tau)}) \\
& -a_1(v^* - m_2 e^{\lambda_1 x} + p_2 e^{(\lambda_1-\epsilon)x})] \\
= \ & e^{\lambda_1 x}[-m_1 h_1(\lambda_1,\tau) + a_1 u^* m_2] + p_1 e^{(\lambda_1-\epsilon)x} h_1(\lambda_1-\epsilon) - u^* a_1 p_2 e^{(\lambda_1-\epsilon)x} \\
& +e^{2\lambda_1 x}[-m_1 + p_1 e^{-\epsilon x}][m_1 e^{-\lambda_1\tau} - p_1 e^{-(\lambda_1-\epsilon)\tau-\epsilon x} + a_1 m_2 - a_1 p_2 e^{-\epsilon x}],
\end{aligned}$$

and hence,

$$(L[\phi_p^+])_1(x) e^{(\lambda_1-\epsilon)x} \geq p_1 h_1(\lambda_1-\epsilon) - u^* a_1 p_2 - M e^{(\lambda_1+\epsilon)x} \tag{52}$$

because $A(\lambda_1,\tau)m = 0$ and $p_i e^{-\epsilon x}$ are uniformly bounded in p_i and $x > x_i^+$, where $M > 0$ is a constant depending on fixed parameters $m, \lambda_1, \epsilon, \tau$ and p_1/p_2. This, together with the fact that $A(\lambda_1-\epsilon,\tau)p \gg 0$, implies that $(L[\phi_p^+])_1(x) > 0$ for $x > x_1^+$ if p is sufficiently large. Since $\phi_{1,p}^+(x) < u^*$ and $\phi_{2,p}^+(x) < v^*$, we have

$$(L[\phi_p^+])_1(x) = \delta_1[1 - \phi_1^+(x-\tau) - a_1\phi_{2,p}^+(x)] > \delta_1[1 - u^* - a_1 v^*] = 0, \ \forall x < x_1^+.$$

Similarly, for $x > x_2^+$ we have

$$(L[\phi_p^+])_2(x) e^{(\lambda_1-\epsilon)x} \geq p_2 h_2(\lambda_1-\epsilon) - r a_2 v^* p_1 - M r e^{(\lambda_1+\epsilon)x}. \tag{53}$$

It then follows that $(L[\phi_p^+])_2(x) \geq 0, x > x_2$ if p is sufficiently large due to $A(\lambda_1 - \epsilon,\tau)p \gg 0$. For $x < x_2^+$, we have

$$(L[\phi_p^+])_2(x) = r\delta_2[1 - \phi_{2,p}^+(x-\tau) - a_2\phi_{1,p}^+(x)] > r\delta_2[1 - v^* - a_2 u^*] = 0.$$

Therefore, $L[\phi_p^+](x) \geq 0$ for $x \in \mathbb{R} \setminus \{x_1^+, x_2^+\}$ if p is sufficiently large. □

Now we are ready to state and prove the existence of monotone traveling waves.

Theorem 4.3. *For any $c \geq c_{\min}$ and $\tau \leq \tau(c)$, system (6) admits a monotone traveling wave connecting E_0 to E_*.*

Proof. We first consider the case where $c > c_{\min}$ and $\tau \notin D_c$. Let ϕ^- and ϕ_p^+ be defined as in (40) and (48)-(49), respectively. Then ϕ^- and ϕ_p^+ are C^1-functions on $\mathbb{R}$. In view of Lemmas 4.1 and 4.2, together with [11, Corollary 16], we see that ϕ^- and ϕ_p^+ (after necessary translations) are a pair of ordered lower and upper fixed points of the monotone operator T for sufficiently large p. Define the iteration scheme

$$\psi_0 = \phi^-, \psi_{n+1} = T[\psi_n], \ \forall n \geq 0.$$

We then obtain a sequence of functions $\{\psi_n\}$ with

$$\phi^- = v_0 \le \psi_1 \le \cdots \le \psi_n \le \cdots \le \phi_p^+.$$

It follows that the sequence ψ_n converges, as $n \to \infty$, to a continuous and nondecreasing function ψ pointwise on $\mathbb{R}$, and $\psi(\pm\infty)$ both exist. Clearly, ψ is a fixed point of T with $\phi^- \le \psi \le \phi_p^+$. Moreover, it is easy to see from (38) that $\psi(\pm\infty)$ satisfy the algebraic (x, y)-equation

$$\begin{cases} x = x(1 - x - a_1 y), \\ y = ry(1 - y - a_2 x), \end{cases} \tag{54}$$

which, together with the fact that $\psi(-\infty) \le \phi_p^+(-\infty) = (\delta_1, \delta_2)^T \ll (u^*, v^*)^T = \phi^-(+\infty) \le \psi(+\infty)$, implies that

$$\psi(-\infty) = (0, 0)^T \quad \text{and} \quad \psi(+\infty) = (u^*, v^*)^T.$$

In the case where $c = c_{\min}$ or $\tau \in D_c^1$, we employ a limiting argument. Without loss of generality, we only consider the case $c = c_{\min}$ and $\tau = \tau(c)$. Choose sequences $c_n > c_{\min}$ and $\tau_n < \tau(c_n)$ with $c_n \to 2, \tau_n \to \tau(c)$ and $\tau_n \notin D_{c_n}$. Then for each n, there is a monotone traveling wave ψ_n with speed c_n. By appropriate translations, we fix $(\psi_n)_1(0) = 1/2$ for all n. Since ψ_n is monotone in n, we see from Helly's theorem that there exists a subsequence of ψ_n converging to a monotone function ψ pointwise. By Lebesgue's dominated convergence theorem, it follows that ψ is a fixed point of T, and hence, is of C^2. Furthermore, because $(\psi_n)_1(0) = 1/2$, we see that ψ connects E_0 to E_*. $\square$

5. **Uniqueness.** By uniqueness of traveling waves, we mean that any two wave profiles with the same speed must be the same after appropriate translations. The main result of this section is as follows.

Theorem 5.1. *Traveling waves between constant solutions E_0 and E_* are unique up to translation.*

Proof. Assume, for the sake of contradiction, that there are two wave profiles $(u_i, v_i), i = 1, 2$ with speed c. From Theorem 3.3 we have

$$\lim_{x \to +\infty} \left(\frac{u^* - u_i(x)}{p(x)e^{\lambda_1 x}}, \frac{v^* - v_i(x)}{p(x)e^{\lambda_1 x}} \right) = (m_1^i, m_2^i) \tag{55}$$

with $(m_1^i, m_2^i)^T$ being eigenvectors associated with λ_1. Consequently,

$$m_1^i / m_2^i = a_1 u^* / h_1(\lambda_1, \tau), \quad \forall i = 1, 2.$$

Set $x_0 = \frac{1}{\lambda_1} \ln \frac{a_1 u^*}{h_1(\lambda_1, \tau)}$ and construct the comparison function (P, Q) as follows:

$$P(x) = |u_1(x) - u_2(x + x_0)|e^{-\lambda_1 x}, \quad Q(x) = |v_1(x) - v_2(x + x_0)|e^{-\lambda_1 x}. \tag{56}$$

Clearly, $P(-\infty) = 0 = Q(-\infty)$.

Using the fact $0 \le (u_i, v_i) \le (u^*, v^*)$ and triangular inequality, we obtain from the wave profile equation $\begin{pmatrix} u_i \\ v_i \end{pmatrix} = T \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ that

$$\begin{pmatrix} P(x) \\ Q(x) \end{pmatrix} \le \int_{-\infty}^{0} \begin{pmatrix} g_1(x-y)e^{-\lambda(x-y)}[P(y) + u^* e^{-\lambda\tau} P(y - \tau) + a_1 u^* Q(y)]dy \\ g_2(x-y)e^{-\lambda(x-y)}[Q(y) + v^* e^{-\lambda\tau} Q(y - \tau) + a_2 v^* P(y)]dy \end{pmatrix}. \tag{57}$$

Define

$$J_i(\lambda,y)=\begin{cases}0, & y\ge 0\\ g_i(y)e^{-\lambda y}, & y<0\end{cases}, \quad i=1,2 \tag{58}$$

and

$$J(\lambda,y)=\begin{pmatrix} J_1(\lambda,y)+e^{-\lambda\tau}J_1(\lambda,y-\tau) & a_1u^*J_1(\lambda,y)\\ a_2v^*J_2(\lambda,y) & J_2(\lambda,y)+e^{-\lambda\tau}J_2(\lambda,y-\tau)\end{pmatrix}. \tag{59}$$

Then (57) can be rewrite as

$$f(x):=\int_{\mathbb{R}} J(\lambda_1,y)\begin{pmatrix}P(x-y)\\ Q(x-y)\end{pmatrix}dy-\begin{pmatrix}P(x)\\ Q(x)\end{pmatrix}\ge 0. \tag{60}$$

Now we claim that $P(+\infty)=0=Q(+\infty)$. Indeed, direct computations show that $\int_{\mathbb{R}} J(\lambda,y)dy-Id=-\mathcal{A}(\lambda,\tau)$ which is defined in (32). Assume, for the sake of contradiction, that $P(x)$ and $Q(x)$ have limits greater than 0 (possibly $+\infty$) as x goes to $+\infty$. Then multiplying the matrix $\begin{pmatrix}\sigma_1P^{-1}(x) & 0\\ 0 & \sigma_2Q^{-1}(x)\end{pmatrix}$ in both sides of (60) and letting $x\to+\infty$, we obtain

$$0\le \lim_{x\to+\infty}\begin{pmatrix}\sigma_1P^{-1}(x) & 0\\ 0 & \sigma_2Q^{-1}(x)\end{pmatrix}f(x)=-\mathcal{A}(\lambda_1,\tau)\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}, \quad \forall\sigma_1,\sigma_2>0. \tag{61}$$

However, the components of the right hand side have different signs due to the property of $\mathcal{A}(\lambda_1,\tau)$, which leads to a contradiction.

Assume that x_1^* and x_2^* are the points at which P and Q attain the maximum P^* and Q^*, respectively. Consequently, from (57) we have

$$\begin{pmatrix}P^*\\ Q^*\end{pmatrix}\le\int_{-\infty}^0\begin{pmatrix}g_1(y)e^{-\lambda y}[(u^*+u^*e^{-\lambda\tau}+a_1v^*)P^*+a_1u^*Q^*]\\ g_2(y)e^{-\lambda y}[(v^*+v^*e^{-\lambda\tau}+a_2u^*)Q^*+a_2v^*P^*]\end{pmatrix}dy. \tag{62}$$

Note that

$$\int_{-\infty}^0 g_1(y)e^{-\lambda y}dy=\frac{1}{c^{-2}\lambda^2-\lambda+1},\quad \int_{-\infty}^0 g_2(y)e^{-\lambda y}dy=\frac{r}{c^{-2}d\lambda^2-\lambda+r} \tag{63}$$

due to the facts $\mu_1\mu_3=c^2,\mu_1+\mu_3=c^2$ and $\mu_2\mu_4=c^2d^{-1}r,\mu_2+\mu_4=c^2d^{-1}$. Therefore, we obtain the inequality

$$A(\lambda_1,\tau)\begin{pmatrix}P^*\\ Q^*\end{pmatrix}=\begin{pmatrix}h_1(\lambda_1,\tau) & -a_1u^*\\ -ra_2v^* & h_2(\lambda_1,\tau)\end{pmatrix}\begin{pmatrix}P^*\\ Q^*\end{pmatrix}\le 0. \tag{64}$$

As $h_i(\lambda_1,\tau)>0$ and $\det A(\lambda_1,\tau)=0$, we see that $A(\lambda_1,\tau)\begin{pmatrix}P^*\\ Q^*\end{pmatrix}=0$. This implies

$$P^*=P(x_1^*)=P(x_1^*-y)\quad\text{and}\quad Q^*=Q(x_2^*)=Q(x_2^*-y)\quad\forall y\in\mathbb{R}. \tag{65}$$

Thus, $P^*=Q^*=0$ due to $P(\pm\infty)=0=Q(\pm\infty)$, a contradiction. □

6. **Remarks.** Based on the proof and results in the previous sections, we also obtain the following result on the heteroclinic orbits of the kinetic system.

$$\begin{cases}u'=u[1-u(\cdot-\tau)-a_1v],\\ v'=rv[1-v(\cdot-\tau)-a_2u].\end{cases} \tag{66}$$

Theorem 6.1. *Equation (66) admits heteroclinic orbits between E_0 to E_* if and only if $\tau\le\tau(\infty)$, which is defined in Lemma 2.1(ii). Moreover, such orbits are unique.*

The idea of the proof is to pass $c \to \infty$ in Theorems 4.3 and 5.1, we omit the details here.

To conclude this paper, we address an unsolved question on the uniqueness of traveling waves for the delayed Fisher-KPP equation

$$u_t = u_{xx} + u(t,x)[1 - u(t-h,x)]. \tag{67}$$

It has been proved in [11, Theorem 4] that a monotone traveling wave exists if and only if delay h is less than some value h_1 and speed $c \in [0, c^*(h)]$, and such traveling wave is unique if $c \in [2, c^*(h))$. The following result fills the gap on the uniqueness when $c = c^*(h)$ by employing Lemma 7 in [1]. The similar gap in [5, Thorem 1.2] for the nonlocal Fisher-KPP equation can also be filled in the same way.

Theorem 6.2. *Let number h_1 be defined as in* [11, Theorem 4] *and $h \le h_1$. Then the traveling wave of* (67) *with speed $c = c^*(h)$ is unique.*

Proof. Assume, for the sake of contradiction, that ϕ_1, ϕ_2 are two traveling waves with speed $c^*(h)$. Then from [11, Theorem 6] we see that

$$\phi_i(x) = 1 - C_i x e^{\lambda^* x} + O(e^{(\lambda^*-\sigma)x}) \quad \text{for some } C_i > 0 \text{ and } \sigma > 0, \tag{68}$$

where $\lambda^* < 0$ is the root with multiplicity two to the equation $c^{-2}\lambda^2 - \lambda - e^{-\lambda h} = 0$ with $c = c^*(h)$. If $C_1 = C_2$, then one can prove the theorem (specially $\phi_1 = \phi_2$) in the same way as for the case where $c < c^*(h)$. So we assume that $C_1 \neq C_2$. Choose x_0 such that $C_1 = C_2 e^{\lambda^* x_0}$. Hence, the function

$$w(x) = (C_2 x_0 e^{\lambda^* x_0})^{-1} |\phi_1(x) - \phi_2(x + x_0)| e^{-\lambda^* x} \tag{69}$$

has limit one at plus infinity and limit zero at minus infinity.

Note that ϕ_i with $i = 1, 2$ satisfy the following integral equation

$$\phi_i(x) = \int_{-\infty}^{0} K(y)\phi_i(x-y)\phi_i(x-y-h)dy \tag{70}$$

with $K(y) = \frac{c^2}{\lambda-\mu}(e^{\mu y} - e^{\lambda y})$ with $0 < \lambda \le \mu$ being the roots to $c^{-2}z^2 - z + 1 = 0$, where $K(y)$ is understood as the limit $-4ye^{2y}$ when $c = 2$. Thus, by the triangular inequality, we have

$$\begin{aligned} w(x) \quad &\le \quad \int_{-\infty}^{0} K(y)e^{-\lambda^* y}[w(x-y-h)e^{-\lambda^* h} + w(x-y)]dy \\ &:= \quad \int_{-\infty}^{+\infty} N(y)w(x-y)dy, \end{aligned} \tag{71}$$

where

$$N(y) = \begin{cases} 0, & y > h, \\ K(y-h)e^{-\lambda^* y}, & y \in [0,h], \\ [K(y-h) + K(y)]e^{-\lambda^* y}, & y < 0. \end{cases}$$

Since $\int_{\mathbb{R}} N(y)dy = 1, \int_{\mathbb{R}} yN(y) = 0$ and $\int_{\mathbb{R}} |y|N(y)dy < \infty$, we can employ [1, Lemma 7] to conclude that such w can not exist. This completes the proof. ☐

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REFERENCES

[1] M. Aguerrea, C. Gomez and S. Trofimchuk, *On uniqueness of semi-wavefronts: DiekmannKaper theory of a nonlinear convolution equation re-visited*, Math. Ann., online press.

[2] H. Berestycki, G. Nadin, B. Perthame and L. Ryzhik, *The non-local Fisher-KPP equation: Travelling waves and steady states*, Nonlinearity, **22** (2009), 2813–2844.

[3] N. Britton, *Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model*, SIAM J. Appl. Math., **50** (1990), 1663–1688.

[4] J. Fang and X.-Q. Zhao, *Existence and uniqueness of traveling waves for non-monotone integral equations with applicaitons*, J. Diff. Eqs., **248** (2010), 2199–2226.

[5] J. Fang and X.-Q. Zhao, *Monotone wavefronts of the nonlocal Fisher-KPP equation*, Nonlinearity, **24** (2011), 3043–3054.

[6] T. Faria, W. Huang and J. Wu, *Travelling waves for delayed reaction-diffusion equations with global response*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **462** (2006), 229–261.

[7] T. Faria and J. Oliveira, *Local and global stability for Lotka-Volterra systems with distributed delays and instantaneous negative feedbacks*, J. Diff. Eqs., **244** (2008), 1049–1079.

[8] T. Faria and S. Trofimchuk, *Positive travelling fronts for reaction-diffusion systems with distributed delay*, Nonlinearity, **23** (2010), 2457–2481.

[9] T. Faria, *Stability and extinction for Lotka-Volterra systems with infinite delay*, J. Dynam. Diff. Eqs., **22** (2010), 299–324.

[10] G. Friesecke, *Convergence to equilibrium for delay-diffusion equations with small delay*, J. Dynam. Diff. Eqs., **5** (1993), 89–103.

[11] A. Gomez and S. Trofimchuk, *Monotone traveling wavefronts of the KPP-Fisher delayed equation*, J. Diff. Eqs., **250** (2011), 1767–1787.

[12] S. Gourley, *Travelling front solutions of a nonlocal Fisher equation*, J. Math. Biol., **41** (2000), 272–284.

[13] J.-S. Guo and X. Liang, *The minimal speed of traveling fronts for the Lotka-Volterra competition system*, J. Dynam. Diff. Eqs., **23** (2011), 353–363.

[14] W. Huang and M. Han, *Non-linear determinacy of minimum wave speed for a Lotka-Volterra competition model*, J. Diff. Eqs., **251** (2011), 1549–1561.

[15] S.-B. Hsu and X.-Q. Zhao, *Spreading speeds and traveling waves for nonmonotone integrodifference equations*, SIAM J. Math. Anal., **40** (2008), 776–789.

[16] Y. Kuang and H. Smith, *Convergence in Lotka-Volterra type diffusive delay systems without dominating instantaneous negative feedbacks*, J. Austral. Math. Soc. Ser. B, **34** (1993), 471–494.

[17] Y. Kuang and H. Smith, *Global stability for infinite delay Lotka-Volterra type systems*, J. Diff. Eqs., **103** (1993), 221–246.

[18] Y. Kuang, *Global stability in delay differential systems without dominating instantaneous negative feedbacks*, J. Diff. Eqs., **119** (1995), 503–532.

[19] M. K. Kwong and C. Ou, *Existence and nonexistence of monotone traveling waves for the delayed Fisher equation*, J. Diff. Eqs., **249** (2010), 728–745.

[20] M. Lewis, B. Li and H. Weinberger, *Spreading speed and linear determinacy for two-species competition models*, J. Math. Biol., **45** (2002), 219–233.

[21] W.-T. Li, G. Lin and S. Ruan, *Existence of traveling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems*, Nonlinearity, **19** (2006), 1253–1273.

[22] G. Lin, W.-T. Li, and M. Ma, *Traveling wave solutions in delayed reaction diffusion systems with applications to multi-species models*, Dis. Cont. Dyn. Syst. - Series B, **13** (2010), 393–414.

[23] B. Li, H. Weinberger and M. Lewis, *Spreading speeds as slowest wave speeds for cooperative systems*, Math. Biosci., **196** (2005), 82–98.

[24] B. Li, M. Lewis and H. Weinberger, *Existence of traveling waves for integral recursions with nonmonotone growth functions*, J. Math. Biol., **58** (2009), 323–338.

[25] X. Liang and X.-Q. Zhao, *Spreading speeds and traveling waves for abstract monostable evolution systems*, J. Funct. Anal., **259** (2010), 857–903.

[26] R. Lui, *Biological growth and spread modeled by systems of recursions. I. Mathematical theory*, Math. Biosci., **93** (1989), 269–295.

[27] J. Mallet-Paret, *The Fredholm alternative for functional-differential equations of mixed type*, J. Dynam. Diff. Eqs., **11** (1999), 1–47.

[28] R. H. Martin and H. L. Smith, *Abstract functional-differential equations and reaction-diffusion systems*, Trans. Amer. Math. Soc., **321** (1990), 1–44.

[29] G. Nadin, B. Perthame and M. Tang, *Can a traveling wave connect two unstable states? The case of the nonlocal Fisher equation*, C. R. Acad. Sci. Paris, **349** (2011), 553–557.

[30] C. Ou and J. Wu, *Traveling wavefronts in a delayed food-limited population model*, SIAM J. Math. Anal., **39** (2007), 103–125.

[31] C. Ou and J. Wu, *Persistence of wavefronts in delayed nonlocal reaction-diffusion equations*, J. Diff. Eqs., **235** (2007), 219–261.

[32] H. Smith, *Systems of ordinary differential equations which generate an order preserving flow. A survey of results*, SIAM Rev., **30** (1988), 87–113.

[33] H. Smith, "Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems," Math. Surveys and Monographs, **41**, American Mathematical Society, Providence, R.I., 1995.

[34] H. Smith and H. Thieme, *Monotone semiflows in scalar non-quasi-monotone functional-differential equations*, J. Math. Anal. Appl., **150** (1990), 289–306.

[35] H. Smith and H. Thieme, *Strongly order preserving semiflows generated by functional-differential equations*, J. Diff. Eqs., **93** (1991), 332–363.

[36] A. I. Volpert, V. A. Volpert and V. A. Volpert, "Traveling Wave Solutions of Parabolic Systems," Translation of Mathematical Monographs, **140**, Amer. math. Soc., Providence, RI, 1994.

[37] H. Wang, *Spreading speeds and traveling waves for non-cooperative reaction-diffusion systems*, J. Non. Sci., **21** (2011), 747–783.

[38] H. F. Weinberger, *Long-time behavior of a class of biological models*, SIAM J. Math. Anal., **13** (1982), 353–396.

[39] H. F. Weinberger, *On spreading speeds and traveling waves for growth and migration models in a periodic habitat*, J. Math. Biol., **45** (2002), 511–548.

[40] J. Wu and X.-Q. Zhao, *Diffusive monotonicity and threshold dynamics of delayed reaction diffusion equations*, J. Diff. Eqs., **186** (2002), 470–484.

[41] J. Wu and X. Zou, *Traveling wave fronts of reaction-diffusion systems with delay*, J. Dyn. Diff. Eqs., **13** (2001), 651–687.

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