

ESTIMATES OF PERIODS AND GLOBAL CONTINUA OF PERIODIC SOLUTIONS FOR STATE-DEPENDENT DELAY EQUATIONS*

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Abstract. We study the global Hopf bifurcation of periodic solutions for one-parameter systems of state-dependent delay differential equations, and specifically we obtain a priori estimates of the periods in terms of certain values of the state-dependent delay along continua of periodic solutions in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. We present an example of three-dimensional state-dependent delay differential equations to illustrate the general results.

Key words. state-dependent delay, Hopf bifurcation, analyticity, global continuation, slowly oscillating periodic solution

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1. Introduction. Hopf bifurcation is a phenomenon in nonlinear dynamical systems in which an equilibrium loses its stability as a system parameter passes a critical value, giving rise to small amplitude periodic solutions branching from this equilibrium. Classic Hopf bifurcation theorems can guarantee the existence of such bifurcated periodic solutions only when the bifurcation parameter is close to the critical value and are thus often referred to as local Hopf bifurcation theorems. Global Hopf bifurcation theorems seek conditions under which the bifurcated periodic solutions persist for *larger* or even *full* range of the bifurcation parameter values. There have been extensive and intensive studies on global Hopf bifurcations for various systems. The well-known Alexander–Yorke theorem [1] gives the global Hopf bifurcation for ordinary differential equations, using techniques from algebraic topology. Their result was refined and extended by, among many others, Chow and Mallet-Paret [7], Chow, Mallet-Paret, and Yorke [8], Mallet-Paret and Yorke [25], Alligood and Yorke [2], Fiedler [11], and Kielhöfer [20]. Many important global Hopf bifurcation theories for infinite dimensional dynamical systems have also been developed by, e.g., Ize [18] for abstract nonlinear evolution equations, Fiedler [10, 12, 13] for parabolic partial differential equations and Volterra integral equations, Nussbaum [28], Wu [31, 32], Krawcewicz, Wu, and Xia [21], Baptistini and Táboas [5], and Guo and Huang [14] for functional differential equations with constant delays and Mallet-Paret and Nussbaum [24] for some state-dependent delay differential equations.

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There is an increasing interest in state-dependent delay differential equations (SDDDEs). This is because more and more models in the form of SDDDEs arising from various practical fields have been proposed or derived; on the other hand, the fundamental theory for this type of equation has not been completely established. The recent survey [15] collects some SDDDE models, offers a large number (almost a complete up-to-date list) of references, and summarizes the recent progress of research on this type of equation.

There are many challenging mathematical problems for SDDDEs, among them the Hopf bifurcation problem, especially the global Hopf bifurcation problem. Eichmann [9] proved a local Hopf bifurcation result for SDDDEs. Hu and Wu [16] made an attempt by considering a very general class of SDDDEs and established an abstract theory on global Hopf bifurcations. Making use of this theory, Hu and Wu [17] obtained conditions for a class of SDDDEs that guarantee a global continuation of the bifurcated periodic solution with *small periods* in the sense explained later (hence referred to as *rapidly* oscillating periodic solutions). This paper is a continuation of [17], aiming to develop a framework and tools for the study of global continuation of *slowly* oscillating periodic solutions arising from Hopf bifurcation, for the same type of SDDDEs as studied in [17]. To this end, some a priori estimates for the periods of the bifurcated periodic solutions are inevitable, and a general approach is developed to obtain these a priori estimates in section 3, after introducing the same setup as in [17] and the required notation, terminology, and preliminaries in section 2. In section 4, we apply the obtained results to a neural network model consisting of two neurons. By verifying the conditions in the previous sections as well as in [16], we show that this model system exhibits Hopf bifurcations and global continua of both slowly and rapidly oscillating periodic solutions for an unbounded range of the bifurcation parameter.

2. Preliminaries. Consider the following parametrized differential equations with a state-dependent delay:

$$(2.1) \quad \begin{cases} \dot{x}(t) = f(x(t), x(t - \tau(t)), \sigma), \\ \dot{\tau}(t) = g(x(t), \tau(t), \sigma), \end{cases}$$

where $x(t) \in \mathbb{R}^N$, $\tau(t) \in \mathbb{R}^+ = [0, +\infty)$, and $\sigma \in \mathbb{R}$. This type of systems with a delay governed by an ordinary differential equation was formulated as an appropriate model for fish dynamics, and the existence of periodic solutions was considered by Arino, Hbid, and Bravo de la Parra [4], Arino, Hadeler, and Hbid [3], and Magal and Arino [23].

2.1. Notation and terminology. In what follows, we denote by $C(\mathbb{R}; \mathbb{R}^N)$ the normed space of bounded continuous functions from \mathbb{R} to \mathbb{R}^N equipped with the usual supremum norm $\|x\| = \sup_{t \in \mathbb{R}} |x(t)|$ for $x \in C(\mathbb{R}; \mathbb{R}^N)$, where $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^N . Denote by $C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ the subspace of $C(\mathbb{R}; \mathbb{R}^N)$ consisting of 2π -periodic functions. Denote by \mathbb{N} the set of all positive integers. For convenience, we now summarize the local and global Hopf bifurcation theory developed in [16]. Assume the following:

- (S1) The maps $f: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \rightarrow f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ and $g: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ni (\gamma_1, \gamma_2, \sigma) \rightarrow g(\gamma_1, \gamma_2, \sigma) \in \mathbb{R}$ are C^2 (twice continuously differentiable).
- (S2) There exist $L > 0$ and $M_g > 0$ such that $-M_g \leq g(\gamma_1, \gamma_2, \sigma) < \frac{L}{L+1}$ for all $\gamma_1 \in \mathbb{R}^N$, $\gamma_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

The period normalization $(x, \tau)(t) = (y, z)(2\pi t/p)$ by the period $p > 0$ of a periodic solution (x, τ) transforms (2.1) into

$$(2.2) \quad \begin{cases} \dot{y}(t) = \frac{p}{2\pi} f(y(t), y(t - \frac{2\pi}{p} z(t)), \sigma), \\ \dot{z}(t) = \frac{p}{2\pi} g(y(t), z(t), \sigma). \end{cases}$$

In what follows, we often talk about a solution (x, τ, σ) of (2.1) in the sense that (x, τ) is a solution of (2.1) with the parameter σ , and similarly a solution (y, z, σ) of (2.2). Then a solution (x, τ, σ) of (2.1) is p -periodic if and only if (y, z, σ) is a 2π -periodic solution of (2.2). In what follows, we will say that (x, τ, σ, p) is a p -periodic solution of (2.1) and (y, z, σ, p) is a 2π -periodic solution of (2.2). The spaces $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ and $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$, where the p -periodic solutions of (2.1) and the 2π -periodic solutions of (2.2) live, respectively, are called Fuller spaces.

In what follows, we will not distinguish a constant map defined in a certain interval from the constant value of the map. A stationary solution of (2.1) is a solution that is a constant map defined for all t . Therefore, the stationary solutions of (2.1) are obtained by solving the system $f(x, x, \sigma) = 0$ and $g(x, \tau, \sigma) = 0$. We assume throughout this paper that the stationary solution of (2.1) at given σ is $\xi(\sigma) = (x_\sigma, \tau_\sigma)$, where the mapping $\xi : \mathbb{R} \ni \sigma \mapsto (x_\sigma, \tau_\sigma) \in \mathbb{R}^{N+1}$ is continuous. For a stationary solution $(x_{\sigma_0}, \tau_{\sigma_0})$ of (2.1) at σ_0 , we say that $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ is a Hopf bifurcation point if there exists a sequence $\{(x_k, \tau_k, \sigma_k, T_k)\}_{k=1}^{+\infty} \subseteq C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ and $T_0 > 0$ such that $(x_k, \tau_k) \rightarrow (x_{\sigma_0}, \tau_{\sigma_0})$ (with respect to the respective supremum-norms), and $(\sigma_k, T_k) \rightarrow (\sigma_0, T_0)$ as $k \rightarrow \infty$, and (x_k, τ_k, σ_k) is a nonconstant T_k -periodic solution of (2.1).

Freezing the state-dependent delay of the term $y(t - 2\pi/pz(t))$ in (2.2) at $2\pi\tau_\sigma/p$ and then linearizing the resulting nonlinear system at the stationary point (x_σ, τ_σ) , we obtain the following inhomogeneous linear system:

$$(2.3) \quad \begin{pmatrix} \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \frac{p}{2\pi} \begin{bmatrix} a_1(\sigma) & 0 \\ b_1(\sigma) & b_2(\sigma) \end{bmatrix} \begin{pmatrix} y(t) - x_\sigma \\ z(t) - \tau_\sigma \end{pmatrix} + \frac{p}{2\pi} \begin{bmatrix} a_2(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y(t - \frac{2\pi}{p}\tau_\sigma) - x_\sigma \\ z(t - \frac{2\pi}{p}\tau_\sigma) - \tau_\sigma \end{pmatrix},$$

where

$$a_i(\sigma) = \frac{\partial}{\partial \theta_i} f(\theta_1, \theta_2, \sigma) \Big|_{\theta_1=x_\sigma, \theta_2=x_\sigma},$$

$$b_i(\sigma) = \frac{\partial}{\partial \gamma_i} g(\gamma_1, \gamma_2, \sigma) \Big|_{\gamma_1=x_\sigma, \gamma_2=\tau_\sigma},$$

for $i = 1, 2$.

Let

$$(2.4) \quad \det_{\mathbb{C}} \Delta_{(x_\sigma, \tau_\sigma, \sigma)}(\lambda) = 0$$

be the characteristic equation of the linear system corresponding to the formal linearization (2.3). See [16] for details. A solution $(x^*, \tau^*, \sigma^*) = (x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)$ is said to be a center of (2.1) if (2.4) with $\sigma = \sigma^*$ has a pair of purely imaginary roots $\pm i\beta^*$ with $\beta^* > 0$. In this case $p^* = 2\pi/\beta^*$ is called the *virtual period* associated with the center (x^*, τ^*, σ^*) . We say that (x^*, τ^*, σ^*) is an *isolated center* if it is the only center in some neighborhood of (x^*, τ^*, σ^*) in $\mathbb{R}^{N+1} \times \mathbb{R}$, that is, $\det_{\mathbb{C}} \Delta_{(x^*, \tau^*, \sigma^*)}(i\beta^*) = 0$

and for $\delta > 0$ sufficiently small,

$$(2.5) \quad \det \Delta_{(x_\sigma, \tau_\sigma, \sigma)}(i\beta) \neq 0$$

for every $(\sigma, \beta) \in ((\sigma^* - \delta, \sigma^*) \cup (\sigma^*, \sigma^* + \delta)) \times (0, +\infty)$ and $\det \Delta_{(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)}(i\beta) \neq 0$ for every $\beta \in (\beta^* - \delta, \beta^*) \cup (\beta^*, \beta^* + \delta)$.

We can then choose constants $b = b(\sigma^*, \beta^*) > 0$ and $c = c(\sigma^*, \beta^*) > 0$ such that the closure of $\Omega := (0, b) \times (\beta^* - c, \beta^* + c) \subseteq \mathbb{R}^2 \cong \mathbb{C}$ contains no other zero of $\det_{\mathbb{C}} \Delta_{(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)}(\lambda) = 0$. Then we can define the numbers

$$\gamma_{\pm}(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*, \beta^*) = \deg(\det_{\mathbb{C}} \Delta_{(x_{\sigma^* \pm \delta}, \tau_{\sigma^* \pm \delta}, \sigma^* \pm \delta)}(\cdot), \Omega),$$

where $\deg(\Delta_{(x_{\sigma^* \pm \delta}, \tau_{\sigma^* \pm \delta}, \sigma^* \pm \delta)}(\cdot), \Omega)$ is the usual Brouwer degree of the mapping $\det_{\mathbb{C}} \Delta_{(x_{\sigma^* \pm \delta}, \tau_{\sigma^* \pm \delta}, \sigma^* \pm \delta)}(\lambda)$ defined on Ω . The *crossing number* of (x^*, τ^*, σ^*) is defined as

$$(2.6) \quad \gamma(x^*, \tau^*, \sigma^*, \beta^*) = \gamma_{-}(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*) - \gamma_{+}(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*).$$

To state the local Hopf bifurcation theorem, we assume the following:

(S3) There exists σ_0 so that $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ is a center of system (2.1),

$$\left(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \right) f(\theta_1, \theta_2, \sigma) |_{\sigma=\sigma_0, \theta_1=\theta_2=x_{\sigma_0}}$$

is nonsingular (determinant is nonzero), and

$$\frac{\partial}{\partial \gamma_2} g(\gamma_1, \gamma_2, \sigma) |_{\sigma=\sigma_0, \gamma_1=x_{\sigma_0}, \gamma_2=\tau_{\sigma_0}} \neq 0.$$

THEOREM 2.1 (see [16]). *Assume (S1)–(S3) hold. Let $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ be an isolated center of system (2.1). If the crossing number $\gamma(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0) \neq 0$, then there exists a bifurcation of nonconstant periodic solutions of (2.1) near $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$. More precisely, there exists a sequence $\{(x_n, \tau_n, \sigma_n, \beta_n)\}$ such that $\sigma_n \rightarrow \sigma_0$, $\beta_n \rightarrow \beta_0$ and $\|(x_n, \tau_n) - (x_{\sigma_0}, \tau_{\sigma_0})\| \rightarrow 0$ as $n \rightarrow \infty$, where*

$$(x_n, \tau_n, \sigma_n) \in C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}$$

is a nonconstant $2\pi/\beta_n$ -periodic solution of system (2.1).

Let \mathcal{S}_0 be the closure of the set of all nonconstant periodic solutions of system (2.1) in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. $\gamma(x^*, \tau^*, \sigma^*, \beta^*) \neq 0$ implies that (x^*, τ^*, σ^*) is a Hopf bifurcation point, namely, there exists a connected component $C(x^*, \tau^*, \sigma^*, p^*)$ of \mathcal{S}_0 which contains $(x^*, \tau^*, \sigma^*, p^*)$.

Remark 2.2. Let $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*) \subseteq C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ be a nonconstant periodic solution. Note that p may not be the minimal period of the solution (x, τ, σ, p) . We say (x, τ, σ, p) is not p_0 -periodic, $p_0 > 0$, if p_0 is not a period of (x, τ, σ) . It is clear that if (x, τ, σ, p) is not p_0 -periodic, then $kp \neq p_0$ for every $k \in \mathbb{N}$.

To state the global Hopf bifurcation theory developed in [16], we further assume the following:

(S4) There exist constants $L_f > 0$, $L_g > 0$ such that

$$\begin{aligned} |f(\theta_1, \theta_2, \sigma) - f(\bar{\theta}_1, \bar{\theta}_2, \sigma)| &\leq L_f(|\theta_1 - \bar{\theta}_1| + |\theta_2 - \bar{\theta}_2|), \\ |g(\gamma_1, \gamma_2, \sigma) - g(\bar{\gamma}_1, \bar{\gamma}_2, \sigma)| &\leq L_g(|\gamma_1 - \bar{\gamma}_1| + |\gamma_2 - \bar{\gamma}_2|) \end{aligned}$$

for every $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2, \gamma_1, \bar{\gamma}_1 \in \mathbb{R}^N$, $\gamma_2, \bar{\gamma}_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

We can now state the global Hopf bifurcation theorem.

THEOREM 2.3 (see [16]). *Suppose that system (2.1) satisfies (S1)–(S4). Let M be the set of constant solutions of the system (2.2) and let \mathcal{S} denote the closure of the set of all nonconstant solutions of (2.2) in the Fuller space $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Assume that all the centers of (2.1) are isolated. If $(u_0, \sigma_0, p_0) = (x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, p_0) \in M$ is a bifurcation point, then either the connected component $C(u_0, \sigma_0, p_0)$ of the center (u_0, σ_0, p_0) in \mathcal{S} is unbounded or*

$$C(u_0, \sigma_0, p_0) \cap M = \{(u_0, \sigma_0, p_0), (u_1, \sigma_1, p_1), \dots, (u_q, \sigma_q, p_q)\},$$

where $p_i \in \mathbb{R}_+$, $(u_i, \sigma_i, p_i) = (x_{\sigma_i}, \tau_{\sigma_i}, \sigma_i, p_i) \in M$ for $i = 0, 1, 2, \dots, q$. Moreover, in the latter case, the crossing numbers $\gamma(u_i, \sigma_i, 2\pi/p_i)$ satisfy

$$(2.7) \quad \sum_{i=0}^q \epsilon_i \gamma(u_i, \sigma_i, 2\pi/p_i) = 0,$$

where $\epsilon_i = \text{sgn det} \begin{bmatrix} a_1(\sigma_i) + a_2(\sigma_i) & 0 \\ b_1(\sigma_i) & b_2(\sigma_i) \end{bmatrix}$.

2.2. Framework for global continuation of periodic solutions. We now outline the strategies to be carried out in the following sections for a priori estimates of periods of the solutions in a continuum of periodic solutions and thereafter to obtain global continuation of periodic solutions.

DEFINITION 2.4. *Let \mathcal{C} be a connected subset of the solution set of (2.1) in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. We say $(x, \tau, \sigma, p) \in C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ is a p -periodic solution of system (2.1) if (x, τ, σ) is a p -periodic solution of (2.1). We call \mathcal{C} a continuum of slowly oscillating periodic solutions if for every $(x, \tau, \sigma, p) \in \mathcal{C}$, there exists $t_0 \in \mathbb{R}$ so that $p > \tau(t_0) > 0$. Similarly, we call \mathcal{C} a continuum of rapidly oscillating periodic solutions if for every $(x, \tau, \sigma, p) \in \mathcal{C}$, there exists $t_0 \in \mathbb{R}$ so that $0 < p < \tau(t_0)$.*

Remark 2.5. The definition of slowly (rapidly, respectively) oscillating solutions is different from the familiar definition for scalar equations, where slow (rapid, respectively) oscillation means that successive zeros are spaced at distances larger (smaller, respectively) than the delay at the zero solution. See, for example, Kaplan and Yorke [19] for more details.

DEFINITION 2.6. *Let $C(x^*, \tau^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (2.1), bifurcated from $(x^*, \tau^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Let $I \subseteq \mathbb{R}$ be an interval, $m_0 \in \mathbb{N} \cup \{0\}$, and U be a subset in $C(x^*, \tau^*, \sigma^*, p^*)$. We call $I \times U \times \{m_0\}$ a delay-period disparity set if every solution $(x, \tau, \sigma, p) \in U$ satisfies $m_0\tau(t) \neq mp$ for every $t \in I$ and $m \in \mathbb{N}$. We call $I \times U \times \{m_0\}$ a delay-period disparity set at (\bar{t}, \bar{v}, m_0) if $(\bar{t}, \bar{v}, m_0) \in I \times U \times \{m_0\}$ and $I \times U \times \{m_0\}$ is a delay-period disparity set. Delay-period disparity sets associated with the Fuller space $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ are defined analogously.*

We note that the period normalization of a solution (x, τ, σ, p) does not change its norm in the respective Fuller spaces. Theorem 2.3 shows that a connected component of the closure of all the nonconstant periodic solutions of (2.1), bifurcated from $(x^*, \tau^*, \sigma^*, p^*)$, namely, $C(x^*, \tau^*, \sigma^*, p^*)$, either has finitely many bifurcation points with the sum of S^1 -equivariant degrees (the summation in (2.7); see [22] for more details) being zero or $C(x^*, \tau^*, \sigma^*, p^*)$ is unbounded in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Therefore, if global persistence of periodic solutions when the parameter is far from the local Hopf bifurcation value σ^* is desired, we should (1) verify that the sum of S^1 -equivariant degrees of the bifurcation points along $C(x^*, \tau^*, \sigma^*, p^*)$ is nonzero,

which implies that $C(x^*, \tau^*, \sigma^*, p^*)$ is unbounded, and (2) find conditions to ensure that the projection of $C(x^*, \tau^*, \sigma^*, p^*)$ on the σ -parameter space \mathbb{R} is unbounded. Assuming the nontriviality of the sum of the S^1 -degrees at the bifurcation points along $C(x^*, \tau^*, \sigma^*, p^*)$, it is sufficient to seek a continuous function $M : \mathbb{R} \ni \sigma \rightarrow M(\sigma) > 0$ such that for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$ we have

$$(2.8) \quad \|(x, \tau)\| + p \leq M(\sigma).$$

Then we can conclude from (2.8) that the projection of $C(x^*, \tau^*, \sigma^*, p^*)$ on the σ -parameter space \mathbb{R} is unbounded, for otherwise, by (2.8), $C(x^*, \tau^*, \sigma^*, p^*)$ is bounded in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$, which is a contradiction.

To achieve (2.8), we first give some sufficient geometric conditions ensuring the uniform boundedness of all possible periodic solutions (x, τ, σ) of (2.1), that is, we find a continuous function $M_1 : \mathbb{R} \ni \sigma \rightarrow M_1(\sigma) > 0$ such that for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$ we have

$$(2.9) \quad \|(x, \tau)\| \leq M_1(\sigma).$$

Furthermore, we seek a continuous function $M_2 : \mathbb{R} \ni \sigma \rightarrow M_2(\sigma) > 0$ such that for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$ we have

$$(2.10) \quad |p| \leq M_2(\sigma).$$

Seeking the uniform bound $M_2(\sigma)$ in (2.10) has been a major challenge and is the main focus of this paper and the earlier work in [17]. Earlier techniques for bounds of periods of periodic solutions of differential equations (see [17] for a short summary and the references therein) turn out to be not applicable for (2.1) due to the nature of the state-dependent delay.

In [17] we developed an approach to obtain a uniform upper bound for periods of the solutions in a continuum $C(x^*, \tau^*, \sigma^*, p^*)$ of rapidly oscillating periodic solutions where the virtual period p^* of the bifurcation point $(x^*, \tau^*, \sigma^*, p^*)$ satisfies

$$0 < p^* < \tau^* \text{ and } mp^* \neq \tau^* \text{ for every } m \in \mathbb{N}.$$

The approach can be outlined as follows. For each solution $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$, we find a period-delay disparity set $I \times (U \cap C(x^*, \tau^*, \sigma^*, p^*)) \times \{1\}$. More specifically, for each periodic solution $(x_0, \tau_0, \sigma_0, p_0)$, we show that it satisfies $\tau_0(t_0) \neq mp_0$ for some $t_0 \in \mathbb{R}$ and for all $m \in \mathbb{N}$. Then we find an open interval $I \ni t_0$ and a small open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ so that every $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $t \in I$ and $m \in \mathbb{N}$. We then develop a procedure to extend the period-delay disparity set along the continuum $C(x^*, \tau^*, \sigma^*, p)$ and a global estimate of the period values is then achieved.

However, extending this approach for continua of slowly oscillating periodic solutions is highly nontrivial for the following reasons. On the one hand, the period-delay disparity set in [17] was obtained through the self-mapping $l : [t_0, t_0 + \tau(t_0)] \ni t \rightarrow t - \tau(t) + \tau(t_0) \in [t_0, t_0 + \tau(t_0)]$, where $\dot{\tau}(t) < 1$ for all $t \in \mathbb{R}$. This mapping is essential in [17] for us to obtain the inequality $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I \subseteq \mathbb{R}$, along a continuum of rapidly oscillating periodic solutions in the Fuller space. However, for solutions (x, τ, σ, p) in a continuum of slowly oscillating periodic solutions, it is hard, if not impossible, to find an appropriate self-mapping that can lead to the inequality $m_0\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I$ with given $m_0 \in \mathbb{N}, m_0 \geq 1$. On

the other hand, the construction of the period-delay disparity set in [17] depends on the negative (or positive) feedback assumption on f . To be more specific, we assume that $xf(x, x, \sigma)$ is negative (or positive) if $f(x, x, \sigma) \neq 0$. In [17], we studied the global continua of rapidly oscillating periodic solutions for the state-dependent delay differential equations

$$(2.11) \quad \begin{cases} \dot{x}(t) = -\mu x(t) + \sigma b(x(t - \tau(t))), \\ \dot{\tau}(t) = 1 - h(x(t)) \cdot (1 + \tanh \tau(t)) \end{cases}$$

with $\tanh(\tau) = (e^{2\tau} - 1)/(e^{2\tau} + 1)$ and $\mu > 0$, where $b, h : \mathbb{R} \rightarrow \mathbb{R}$ are C^2 functions and $xb(x) < 0$ for $x \neq 0$. If the bifurcation parameter σ is positive, then $f(x) = -\mu x + \sigma b(x)$ satisfies that $xf(x)$ is negative for $x \neq 0$. If σ is negative, then $xf(x)$ is in general neither always negative nor always positive for $x \neq 0$. This means that if a continuum of nonconstant periodic solutions of (2.11) is located in the Fuller space where $\sigma < 0$, we cannot use the approach in [17] to obtain uniform lower and upper bounds of periods along the continuum.

These difficulties necessitate a new approach to find upper and lower bounds of periods for continua of periodic solutions of (2.1), in particular, for continua of slowly periodic solutions. To this end, we investigate the continuum of periodic solutions, bifurcated from $(x^*, \tau^*, \sigma^*, p^*)$, where the virtual period p^* satisfies

$$j_0\tau^* < p^* < k_0\tau^* \text{ for some } k_0, j_0 \in \mathbb{N} \cup \{0\}, k_0 > j_0 \geq 0$$

and

$$mp^* \neq m_0\tau^* \text{ for every } m \in \mathbb{N} \text{ and } m_0 \in \{k_0, j_0\}.$$

We then find sufficient conditions so that for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$, there exist $t_1, t_2 \in \mathbb{R}$ such that

$$(2.12) \quad j_0\tau(t_1) < p < k_0\tau(t_2).$$

Then by the uniform boundedness of the delay τ , the uniform boundedness of the p -component of $C(x^*, \tau^*, \sigma^*, p^*)$ follows from (2.12). Moreover, $C(x^*, \tau^*, \sigma^*, p^*)$ is a continuum of slowly oscillating periodic solutions if $j_0 \geq 1$ and is a continuum of rapidly oscillating periodic solutions if $j_0 = 0, k_0 = 1$.

We organize the rest of the paper as follows. In section 3 we find for each periodic solution a period-delay disparity set. Then we construct a monotonically increasing sequence of connected subsets $\{A_n\}_{n=1}^{+\infty}$ of $C(y^*, z^*, \sigma^*, p^*)$ which, combined with the uniform boundedness of the solutions (x, τ) , provides a priori estimates of the periods in terms of certain values of the state-dependent delay for continua of periodic solutions of (2.1) in the Fuller space. In the last section, we present a detailed case study to illustrate the general results.

3. A priori estimates for periods of periodic solutions in a connected component. We will need the following assumptions to construct the period-delay disparity sets.

- (S5) There exist $k_0, j_0 \in \mathbb{N} \cup \{0\}, k_0 > j_0 \geq 0$, so that $j_0\tau^* < p^* < k_0\tau^*$ and for every Hopf bifurcation point $(\bar{x}, \bar{\tau}, \bar{\sigma}, \bar{p}) \in C(x^*, \tau^*, \sigma^*, p^*)$, we have $m_0\bar{\tau} \neq m\bar{p}$ for all $m \in \mathbb{N}$, where $m_0 \in \{k_0, j_0\}$.

- (S6) There exists a continuous function $l : \mathbb{R}^N \times \mathbb{R} \ni (x, \sigma) \rightarrow l(x, \sigma) \in \mathbb{R}$ such that for every $(x, \sigma) \in \mathbb{R}^N \times \mathbb{R}$, $\tau = l(x, \sigma)$ is the unique solution for τ of $g(x, \tau, \sigma) = 0$ and the partial derivative $\partial_x l(x, \sigma)$ exists for all $x \in \mathbb{R}^N$ and $\sigma \in \mathbb{R}$.
- (S7) Let m_0 be as in (S5) with $m_0 \neq 0$ and $\sigma \in \mathbb{R}$. Let c be a constant in $(0, \tau_{\max}]$, where $0 < \tau_{\max} \leq +\infty$ is an upper bound of τ for every periodic solution (x, τ) of system (2.1) at $\sigma \in \mathbb{R}$. Let $\eta = [x_0, x_1, \dots, x_i, \dots, x_{m_0-1}]^T$ with $x_i \in \mathbb{R}^N$, $i = 0, 1, \dots, m_0 - 1$ and define

$$F(\eta, \sigma) = [f(x_0, x_1, \sigma), f(x_1, x_2, \sigma), \dots, f(x_{m_0-1}, x_0, \sigma)]^T \in \mathbb{R}^{m_0 N}.$$

Then the cyclic system of ordinary differential equations

$$(3.1) \quad \dot{\eta}(t) = F(\eta(t), \sigma)$$

has no nonconstant periodic solution which satisfies both of the following equations:

(i)

$$(3.2) \quad \begin{aligned} L(\eta(t), \sigma) &:= [l(x_0(t), \sigma), l(x_1(t), \sigma), \dots, l(x_{m_0-1}(t), \sigma)]^T \\ &= c[1, 1, \dots, 1]^T \end{aligned}$$

for all $t \in \mathbb{R}$;

(ii)

$$(3.3) \quad \begin{aligned} H(\eta(t), \sigma) &:= [\partial_x l(x_0(t), \sigma) f(x_0(t), x_1(t), \sigma), \partial_x l(x_1(t), \sigma) \\ &\quad f(x_1(t), x_2(t), \sigma), \dots, \partial_x l(x_{m_0-1}(t), \sigma) \\ &\quad f(x_{m_0-1}(t), x_0(t), \sigma)]^T \\ &= [0, 0, \dots, 0]^T \end{aligned}$$

for all $t \in \mathbb{R}$, where the product in $\partial_x l(\cdot) f(\cdot)$ is the standard inner product on \mathbb{R}^N .

- (S8) Every periodic solution (x, τ, σ) of (2.1) satisfies that $\tau(t) > 0$ for all $t \in \mathbb{R}$.

With future applications in mind, we consider here the existence of a delay-period disparity set associated with periodic solutions of system (2.1) along $C(x^*, \tau^*, \sigma^*, p^*)$ with greater generality than is immediately necessary for our work here.

LEMMA 3.1. *Let m_0 be as in (S5). If a solution $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $m_0 \tau_0(t_0) \neq m p_0$ for some $t_0 \in \mathbb{R}$ and for all $m \in \mathbb{N}$, then there exist an open neighborhood $I \ni t_0$ and an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that every solution $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $m_0 \tau_0(t) \neq m p$ for all $m \in \mathbb{N}$ and $t \in I$.*

Proof. The proof is similar to that of Lemma 2 in [17]. We omit the details here. \square

A global version with $I = \mathbb{R}$ in Lemma 3.1 is the following corollary.

COROLLARY 3.2. *Let m_0 be as in (S5). If $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $m_0 \tau_0(t) \neq m p_0$ for all $t \in \mathbb{R}$ and for all $m \in \mathbb{N}$, then there exists an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that every solution $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $m_0 \tau(t) \neq m p$ for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$.*

Proof. By way of contradiction, if not, then there exists a sequence

$$\{(x_k, \tau_k, \sigma_k, p_k, t_k)\}_{k=1}^{+\infty} \subset C(x^*, \tau^*, \sigma^*, p^*) \times \mathbb{R}$$

so that $m_0\tau_k(t_k) = n_k p_k$ with some $n_k \in \mathbb{N}$ and

$$\lim_{k \rightarrow +\infty} \|(x_k, \tau_k, \sigma_k, p_k) - (x_0, \tau_0, \sigma_0, p_0)\|_{C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2} = 0.$$

We note that the sequence of periods $\{p_k\}_{k=1}^{+\infty}$ is convergent. Without loss of generality, we assume that $\{t_k\}_{k=1}^{+\infty}$ is contained in a bounded interval in \mathbb{R} and $t_k \rightarrow t_0$ for some t_0 as $k \rightarrow +\infty$. Then we have $\lim_{k \rightarrow +\infty} n_k = \lim_{k \rightarrow +\infty} m_0\tau_k(t_k)/p_k = m_0\tau_0(t_0)/p_0$. If $m_0\tau_0(t_0)/p_0 \notin \mathbb{N}$, then it is a contradiction. Otherwise, there exists $n_0 \in \mathbb{N}$ so that $m_0\tau_0(t_0) = n_0 p_0$. This is also a contradiction. \square

THEOREM 3.3. *Assume that system (2.1) satisfies (S5)–(S8). Then for every nonconstant periodic solution $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$, there exist an open interval I and an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ such that every solution*

$$(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$$

satisfies that $m_0\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and for all $t \in I$.

Proof. The conclusion is trivial if $m_0 = 0$. We assume $m_0 \neq 0$ in the remaining part of the proof. Let $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ be a nonconstant periodic solution. We claim that there exists $t_0 \in \mathbb{R}$ such that $(x_0, \tau_0, \sigma_0, p_0)$ is not $m_0\tau_0(t_0)$ -periodic.

Suppose the claim is not true. Then, for every $t \in \mathbb{R}$, $(x_0, \tau_0, \sigma_0, p_0)$ is $m_0\tau_0(t)$ -periodic. It follows that τ_0 must be a constant function. Otherwise, by (S8) there exists a closed interval $[p_0, p_1]$ with $p_1 > p_0 > 0$ in the range of τ and for every $p \in [p_0, p_1]$, $m_0 p$ is a period of $(x_0, \tau_0, \sigma_0, p_0)$. Therefore, $(x_0, \tau_0, \sigma_0, p_0)$ has an arbitrary period. Hence $(x_0, \tau_0, \sigma_0, p_0)$ is a constant solution, which is a contradiction to the assumption that $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ is a nonconstant periodic solution.

Now we assume that $\tau_0(t) = c_0$ for every $t \in \mathbb{R}$, where $c_0 > 0$ is a constant. Then $(x_0, \tau_0, \sigma_0, p_0)$ is $m_0 c_0$ -periodic and x_0 is a nonconstant periodic function. Let $x_i(t) = x(t - ic_0)$, $i = 0, 1, \dots, m_0 - 1$. Then we have the following system of ordinary differential equations:

$$(3.4) \quad \begin{cases} \dot{x}_0(t) = f(x_0(t), x_1(t), \sigma_0), \\ \dots \dots \\ \dot{x}_i(t) = f(x_i(t), x_{i+1}(t), \sigma_0), \\ \dots \dots \\ \dot{x}_{m_0-1}(t) = f(x_{m_0-1}(t), x_0(t), \sigma_0). \end{cases}$$

Let $\eta(t) = [x_0(t), x_1(t), \dots, x_{m_0-1}(t)]^T$. Then by (S7) and (3.4) we have

$$(3.5) \quad \dot{\eta}(t) = F(\eta(t), \sigma_0) \text{ for every } t \in \mathbb{R}.$$

On the other hand, by (S6) and system (2.1), $\tau_0(t) = c_0$ for every $t \in \mathbb{R}$ implies that $l(x_0(t), \sigma_0) = c_0$ for every $t \in \mathbb{R}$. We notice that the ranges of x_i , $i \in \{0, 1, \dots, m_0 - 1\}$, and x_0 are the same. Then we have $l(x_i(t), \sigma_0) = c_0$ for every $t \in \mathbb{R}$ and hence by (3.4) we have $\partial_x l(x_i(t), \sigma_0) f(x_i(t), x_{i+1}(t), \sigma_0) = 0$ for every $t \in \mathbb{R}$ and $i = 0, 1, \dots, m_0 - 1$, where we identify m_0 with 0 for the index i of x_i . Then we have both

$$(3.6) \quad L(\eta(t), \sigma_0) = c_0 [1, 1, \dots, 1]^T \text{ for all } t \in \mathbb{R}$$

and

$$(3.7) \quad H(\eta(t), \sigma_0) = [0, 0, \dots, 0]^T \text{ for all } t \in \mathbb{R}.$$

But (3.5), (3.6), and (3.7) contradict (S7). Then the claim is proved and there exists $t_0 \in \mathbb{R}$ such that $(x_0, \tau_0, \sigma_0, p_0)$ is not $m_0\tau_0(t_0)$ -periodic. Therefore, we have $m_0\tau_0(t_0) \neq mp_0$ for all $m \in \mathbb{N}$. By Lemma 3.1, there exist an open interval I and an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ such that every solution

$$(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$$

satisfies that $m_0\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and for all $t \in I$. \square

Remark 3.4. If $N = 1$, we can replace assumptions (S6)–(S7) in Theorem 3.3 by

$$(S6') \quad \frac{\partial g}{\partial x}(x, \tau, \sigma) \neq 0 \text{ if } g(x, \tau, \sigma) = 0.$$

Indeed, by (S6'), we can conclude that x_0 is a constant function from the fact that $\tau_0(t) = c_0$ and $g(x_0(t), c_0, \sigma_0) = 0$ for every $t \in \mathbb{R}$.

Now we are in position to show the existence of a delay-period disparity set associated with every element, including bifurcation points, along $C(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. We have the next theorem.

THEOREM 3.5. *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant solutions of system (2.2), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that system (2.1) satisfies (S5)–(S8). Then for every $(y_0, z_0, \sigma_0, p_0) \in C(y^*, z^*, \sigma^*, p^*)$, there exist an open interval I and an open neighborhood $U \ni (y_0, z_0, \sigma_0, p_0)$ such that $m_0z(t) \neq mp$ for every solution $(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$ and for every $m \in \mathbb{N}$ and $t \in I$.*

Proof. Note that $p > 0$ for every solution (y, z, σ, p) in $C(y^*, z^*, \sigma^*, p^*)$. It is easy to show that the mapping

$$(3.8) \quad \begin{aligned} \iota : C(y^*, z^*, \sigma^*, p^*) &\rightarrow C(x^*, \tau^*, \sigma^*, p^*) \\ (y(\cdot), z(\cdot), \sigma, p) &\rightarrow \left(y\left(\frac{2\pi}{p}\cdot\right), z\left(\frac{2\pi}{p}\cdot\right), \sigma, p \right) \end{aligned}$$

is continuous, where $C(x^*, \tau^*, \sigma^*, p^*) \subseteq C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. (See Theorem 4 in [17] for a similar proof.) Therefore, $C(x^*, \tau^*, \sigma^*, p^*)$ is a connected component of periodic solutions of (2.1).

We note that ι is a homeomorphism and for every solution $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$, there exists a corresponding $(y_0, z_0, \sigma_0, p_0) \in C(y^*, z^*, \sigma^*, p^*)$ so that $(x_0, \tau_0, \sigma_0, p_0) = \iota(y_0, z_0, \sigma_0, p_0)$.

If $(x_0, \tau_0, \sigma_0, p_0)$ is a nonconstant periodic solution, then by Theorem 3.3, there exist an open interval I' and an open neighborhood $U' \ni (x_0, \tau_0, \sigma_0, p_0)$ such that $m_0z(t) \neq mp$ for every solution $(y, z, \sigma, p) \in U' \cap C(y^*, z^*, \sigma^*, p^*)$ and for every $m \in \mathbb{N}$ and $t \in I'$.

If $(x_0, \tau_0, \sigma_0, p_0)$ is a constant periodic solution, then it is a Hopf bifurcation point because $C(x^*, \tau^*, \sigma^*, p^*)$ which contains $(x_0, \tau_0, \sigma_0, p_0)$ is a connected component of the closure of all the nonconstant periodic solutions. By (S5), we have $m_0\tau_0 \neq mp_0$ for every $m \in \mathbb{N}$. Then by Lemma 3.1, for every $t_0 \in \mathbb{R}$, there exist an open interval $I' \ni t_0$ and an open neighborhood $U' \ni (x_0, \tau_0, \sigma_0, p_0)$ such that $m_0\tau(t) \neq mp$ for every solution $(x, \tau, \sigma, p) \in U' \cap C(x^*, \tau^*, \sigma^*, p^*)$ and for every $m \in \mathbb{N}$ and $t \in I'$.

Therefore, for every $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$, there exist an open interval $I' \ni t_0$ and an open neighborhood $U' \ni (x_0, \tau_0, \sigma_0, p_0)$ such that $m_0\tau(t) \neq mp$ for every solution $(x, \tau, \sigma, p) \in U' \cap C(x^*, \tau^*, \sigma^*, p^*)$ and for every $m \in \mathbb{N}$ and $t \in I'$.

Since ι is continuous, we can choose an open set $U \subseteq C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ small enough so that $(y_0, z_0, \sigma_0, p_0) \in U \subseteq \iota^{-1}(U')$ and the open set

$$I := \bigcap_{\{p:(y, z, \sigma, p) \in U\}} \frac{p}{2\pi} \cdot I'$$

is nonempty. Then it follows by the definition of ι that $m_0z(t) \neq mp$ for every $(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$, $m \in \mathbb{N}$ and $t \in I$. \square

In the following we want to use the period-delay disparity set to construct uniform lower and upper bounds for periods of the solutions in a continuum of periodic solutions and show that (2.10) is valid provided that (2.9) holds.

LEMMA 3.6 (the generalized intermediate value theorem [26]). *Let $f : X \rightarrow Y$ be a continuous map from a connected space X to a linearly ordered set Y with the order topology. If $a, b \in X$ and $y \in Y$ lies between $f(a)$ and $f(b)$, then there exists $x \in X$ such that $f(x) = y$.*

Now we are able to state our main results in this section.

THEOREM 3.7. *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (2.2), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that (2.1) satisfies (S5)–(S8) and all the periodic solutions are real analytic. Then for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, there exist $t_1, t_2 \in \mathbb{R}$ so that $j_0z(t_1) < p < k_0z(t_2)$.*

Proof. We only prove that for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, $p < k_0z(t_2)$ for some $t_2 \in \mathbb{R}$. The proof of $p > j_0z(t_1)$ for some $t_1 \in \mathbb{R}$ is similar.

By Corollary 3.2 and (S5), there exists an open set U^* in $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that $\mathbb{R} \times (U^* \cap C(y^*, z^*, \sigma^*, p^*)) \times \{k_0\}$ is a delay-period disparity set with $(y^*, z^*, \sigma^*, p^*) \in U^*$.

Let $A^* \ni (y^*, z^*, \sigma^*, p^*)$ be a connected component of $(U^* \cap C(y^*, z^*, \sigma^*, p^*))$. Then, $\mathbb{R} \times A^*$ is connected in $\mathbb{R} \times C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Define $S : \mathbb{R} \times C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$S(t, y, z, \sigma, p) = p - k_0z(t).$$

By (S5), we have $p^* < k_0z^*$ and hence $S(t, y^*, z^*, \sigma^*, p^*) = p^* - k_0z^* < 0$. Note that S is continuous. By Lemma 3.6, we have

$$(3.9) \quad S(t, y, z, \sigma, p) = p - k_0z(t) < 0$$

for every $(t, y, z, \sigma, p) \in \mathbb{R} \times A^*$, for otherwise there exists $(t_0, y_0, z_0, \sigma_0, p_0) \in \mathbb{R} \times A^*$ such that $p_0 = k_0z_0(t_0)$ which contradicts the fact that $\mathbb{R} \times A^* \times \{k_0\}$ is a subset of the delay-period disparity set $\mathbb{R} \times (U^* \cap C(y^*, z^*, \sigma^*, p^*))$.

Now we show that there exists a sequence of connected subsets of $C(y^*, z^*, \sigma^*, p^*)$, denoted by $\{A_n\}_{n=1}^{n_0}$, $n_0 \in \mathbb{N}$ or $n_0 = +\infty$, which satisfies that

- (i) $A^* \subseteq A_1 \subset A_2 \subset \dots \subset A_{n_0}$ and $\cup_{n=1}^{n_0} A_n = C(y^*, z^*, \sigma^*, p^*)$;
- (ii) for every $(y, z, \sigma, p) \in A_n$ with $n \in \{1, 2, \dots, n_0\}$, $p < k_0z(t_2)$ for some $t_2 \in \mathbb{R}$.

Let $A_1 := A^*$ and $I_1 = \mathbb{R}$. If $A_1 = C(y^*, z^*, \sigma^*, p^*)$, then we are done by setting $A_n = C(y^*, z^*, \sigma^*, p^*)$ and $I_n = \mathbb{R}$ for all $n \in \mathbb{N}$. If not, since the only sets both closed and open in the connected topological space $C(y^*, z^*, \sigma^*, p^*)$ are the empty set and the connected component $C(y^*, z^*, \sigma^*, p^*)$ itself, $A_1 \ni (y^*, z^*, \sigma^*, p^*)$ is not both closed and open. Then the boundary of A_1 in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$ is nonempty. That is,

$$(3.10) \quad \partial A_1 \neq \emptyset.$$

Let $\bar{v} = (\bar{y}, \bar{z}, \bar{\sigma}, \bar{p}) \in \partial A_1$. By (3.9) and by the continuity of S , we have $S(t, \bar{v}) \leq 0$ for all $t \in I_1$.

Claim. There exists $\bar{t} \in I_1$ so that $m\bar{p} \neq k_0\bar{z}(\bar{t})$ for all $m \in \mathbb{N}$.

Proof of the claim. Suppose the claim is not true. Then for every $t \in I_1$ there exists some $m \in \mathbb{N}$ so that $m\bar{p} = k_0\bar{z}(t)$. Then by the continuity of \bar{z} there exist a subinterval \bar{I} of I_1 and $\bar{m} \in \mathbb{N}$ so that $\bar{m}\bar{p} = k_0\bar{z}(t)$ for all $t \in \bar{I}$. Then \bar{z} is constant on \bar{I} . Then \bar{z} is constant on \mathbb{R} since \bar{z} is analytic.

Let $\bar{z}(t) = c_0$, where $c_0 > 0$ is a constant. Then we have $c_0 = \frac{\bar{m}\bar{p}}{k_0}$. Let $x_0(t) = \bar{y}(\frac{2\pi t}{\bar{p}})$, $t \in \mathbb{R}$. Then $x_0(t)$ is $\bar{m}\bar{p}$ -periodic and so is $x_i(t) = x_0(t - ic_0)$ for every $i = 1, 2, \dots, k_0$. Then $(x_0(t), x_1(t), \dots, x_{k_0-1}(t))$, $t \in \mathbb{R}$, satisfies the cyclic ordinary differential equation (3.4) with m_0 replaced by k_0 . Then by the proof of Theorem 3.3, we know that this is a contradiction with (S7). This completes the proof of the claim.

Then by the claim and by Lemma 3.1, there exists a delay-period disparity set $I' \times U' \times \{k_0\} \ni (\bar{t}, \bar{v}, k_0)$, and by Lemma 3.6, we have

$$(3.11) \quad S(t, v) = p - k_0z(t) < 0$$

for all $(t, v) \in I' \times U'$. Let $I_{\bar{v}} = I'$ and $A_{\bar{v}} \ni \bar{v}$ be the connected component of $U' \cap C(y^*, z^*, \sigma^*, p^*)$. Then it is clear that $A_1 \cup A_{\bar{v}}$ is connected. Note that $\bar{p} < k_0\bar{z}(\bar{t})$. Then by (3.11) we have

$$(3.12) \quad S(t, y, z, \sigma, p) = p - k_0z(t) < 0 \text{ for every } (t, y, z, \sigma, p) \in I_{\bar{v}} \times A_{\bar{v}}.$$

Therefore, for every $\bar{v} \in \partial A_1$, we can always find $A_{\bar{v}}$ and $I_{\bar{v}}$ satisfying (3.12). Then we define

$$A_2 = A_1 \cup \bigcup_{\bar{v} \in \partial A_1} A_{\bar{v}}.$$

It follows from (3.9) and (3.12) that for every $(y, z, \sigma, p) \in A_2$, $p < k_0z(t)$ for some $t \in \mathbb{R}$. Note that for every $\bar{v} \in \partial A_1$, $A_1 \cup A_{\bar{v}}$ is connected. Therefore, A_2 is connected.

Note that the existence of A_2 only depends on the fact that $\partial A_1 \neq \emptyset$, in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$. Beginning with $n = 1$, we can always recursively construct a connected subset of $C(y^*, z^*, \sigma^*, p^*)$ for each $n \geq 1$, $n \in \mathbb{N}$, with $\partial A_n \neq \emptyset$,

$$(3.13) \quad A_{n+1} = A_n \cup \bigcup_{\bar{v} \in \partial A_n} A_{\bar{v}}$$

satisfying that for every $(y, z, \sigma, p) \in A_n$,

$$(3.14) \quad p < k_0z(t) \text{ for some } t \in \mathbb{R}.$$

If the construction in (3.13) stops at some $n_0 \in \mathbb{N}$ with $\partial A_{n_0} = \emptyset$, then $A_{n_0} = C(y^*, z^*, \sigma^*, p^*)$ and we are done. If not, we have $n_0 = +\infty$ and we obtain a sequence of sets $\{A_n\}_{n=1}^{+\infty}$ which is a totally ordered family of sets with respect to the set inclusion relation \subseteq . Note that $\bigcup_{n=1}^{+\infty} A_n$ is an upper bound of $\{A_n\}_{n=1}^{+\infty}$. Then by Zorn's lemma, there exists a maximal element A_∞ for the sequence $\{A_n\}_{n=1}^{+\infty}$.

Now we show that $\partial A_\infty = \emptyset$ in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$. Suppose not; then for every $\bar{v} \in \partial A_\infty$ there exists a delay-period disparity set $I_\infty \times (U_\infty \times C(y^*, z^*, \sigma^*, p^*)) \times \{k_0\}$ with $\bar{v} \in U_\infty$ so that

$$\bar{p} - k_0\bar{z}(t) < 0.$$

Let $A_{\bar{v}}$ be the connected component of $U_\infty \cap C(y^*, z^*, \sigma^*, p^*)$. We distinguish two cases.

Case 1. $A_{\bar{v}} \setminus A_{\infty} = \emptyset$ for all $\bar{v} \in \partial A_{\infty}$. Then A_{∞} is a connected component of $C(y^*, z^*, \sigma^*, p^*)$. Recall that $C(y^*, z^*, \sigma^*, p^*)$ itself is a connected component of the closure of all the nonconstant periodic solutions of system (2.2). So we have $A_{\infty} = C(y^*, z^*, \sigma^*, p^*)$. That is, $\partial A_{\infty} = \emptyset$. This is a contradiction.

Case 2. $A_{\bar{v}} \setminus A_{\infty} \neq \emptyset$ for some $\bar{v} \in \partial A_{\infty}$. Then there exists $A'_{\infty} = A_{\infty} \cup_{\bar{v} \in \partial A_{\infty}} A_{\bar{v}}$ satisfying that for every $(y, z, \sigma, p) \in A'_{\infty}$, $p < k_0 z(t)$ for some $t \in \mathbb{R}$ and A_{∞} is a proper subset of A'_{∞} , where A'_{∞} is a member of the sequence of sets $\{A_n\}_{n=1}^{+\infty}$. This contradicts the maximality of A_{∞} .

The contradictions imply that $\partial A_{\infty} = \emptyset$, and hence $A_{\infty} = C(y^*, z^*, \sigma^*, p^*)$. Therefore, (3.14) holds for all $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$. This completes the proof that for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, we have $p < k_0 z(t)$ for all $t \in \mathbb{R}$. Similarly, we can prove that for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, $j_0 z(t) < p$ for all $t \in \mathbb{R}$. This completes the proof. \square

COROLLARY 3.8. *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (2.2), bifurcated at $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that (2.1) satisfies (S5)–(S8) and all the periodic solutions are real analytic. If there exists a continuous function $M_1 : \mathbb{R} \ni \sigma \rightarrow M_1(\sigma) > 0$ such that for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$ we have*

$$(3.15) \quad \|z\| \leq M_1(\sigma),$$

then for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, we have $p < k_0 M_1(\sigma)$.

Proof. By Theorem 3.7, we have for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$ that $j_0 z(t) < p < k_0 z(t)$ for all $t \in \mathbb{R}$. Then, by (3.15), we have $p < k_0 M_1(\sigma)$. \square

4. An example. In this section we study the global continua of periodic solutions for the following state-dependent delay differential equations,

$$(4.1) \quad \begin{cases} \dot{x}_1(t) = -\mu x_1(t) + \sigma b(x_2(t - \tau(t))), \\ \dot{x}_2(t) = -\mu x_2(t) + \sigma b(x_1(t - \tau(t))), \\ \dot{\tau}(t) = 1 - h(x(t)) \cdot (1 + \tanh \tau(t)), \end{cases}$$

where $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$, $\tau(t) \in \mathbb{R}$, $\tanh(\tau) = (e^{2\tau} - 1)/(e^{2\tau} + 1)$, and $\mu > 0$ is a constant. Equation (4.1) describes a neural network with two neurons where the time delay for information transmission from one neuron to another is state-dependent. Analogous models with constant delay for neural networks with two neurons have been widely studied in the literature. See, for example, Baptistini and Táboas [5], Chen and Wu [6], Ruan and Wei [29], Faria [30], and the references therein.

We make the following assumptions:

- (α_1) $b : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuously differentiable functions with $b'(0) = -1$.
- (α_2) There exist $h_0 < h_1$ in $(1/2, 1)$ such that $h_1 > h(x) > h_0$ for all $x \in \mathbb{R}^2$.
- (α_3) b is decreasing on \mathbb{R} and the map $\mathbb{R} \ni y \rightarrow yb(y) \in \mathbb{R}$ is injective.
- (α_4) $yb(y) < 0$ for $y \neq 0$, and there exists a continuous function $M : \mathbb{R} \ni \sigma \rightarrow M(\sigma) \in (0, +\infty)$ so that

$$\frac{b(y)}{y} > -\frac{\mu}{2|\sigma|}$$

for $|y| \geq M(\sigma)$.

- (α_5) There exists $M_0 > 0$ such that $|b'(x)| < M_0$ and $|h'(x)| < M_0$ for every $x \in \mathbb{R}$.

4.1. Uniform bound of periodic solutions of (4.1). We use the following lemma, which was proved in [17], to obtain the uniform boundedness of the range of periodic solutions (x, τ) of (4.1) with $\sigma \in \mathbb{R}$. Here \cdot denotes the standard inner product on \mathbb{R}^N and G^c denotes the complement of G .

LEMMA 4.1 (see [17]). *Consider system (2.1) at $\sigma \in \mathbb{R}$. Suppose that $G_1 \subset \mathbb{R}^N$ and $G_2 \subset \mathbb{R}$ are bounded, balanced, convex, and absorbing open subsets which define the Minkowski functionals $p_{G_1}(x)$ and $p_{G_2}(\tau)$. Let $G = G_1 \times G_2$ and*

$$F_{\max}(x, \sigma) = \max_{\{\tilde{x}: p_{G_1}(\tilde{x}) \leq p_{G_1}(x)\}} x \cdot f(x, \tilde{x}, \sigma),$$

$$F_{\min}(x, \sigma) = \min_{\{\tilde{x}: p_{G_1}(\tilde{x}) \leq p_{G_1}(x)\}} x \cdot f(x, \tilde{x}, \sigma).$$

Then the range of all periodic solutions of (2.1) is contained in G if either of the following conditions (H1) or (H2) holds:

(H1) $F_{\max}(x, \sigma) < 0$ for every $x \in G_1^c$ and $\tau \cdot g(x, \tau) < 0$ for every $\tau \in G_2^c$, $x \in \mathbb{R}^N$.

(H2) $F_{\min}(x, \sigma) > 0$ for every $x \in G_1^c$ and $\tau \cdot g(x, \tau) > 0$ for every $\tau \in G_2^c$, $x \in \mathbb{R}^N$.

LEMMA 4.2. *Assume (α_1) – (α_4) hold. Then the range of every periodic solution (x_1, x_2, τ) of (4.1) with $\sigma \in \mathbb{R}$ is contained in*

$$\Omega_1 = (-M(\sigma), M(\sigma)) \times (-M(\sigma), M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2}\right).$$

Proof. If $\sigma = 0$, the only periodic solution of (4.1) is $(0, 0, -\frac{\ln(2h(0)-1)}{2})$. By (α_2) and by (α_3) , we have $0 < -\frac{\ln(2h(0)-1)}{2} < -\frac{\ln(2h_0-1)}{2}$ and $0 \in (-M(0), M(0))$. It follows that

$$(4.2) \quad \left(0, -\frac{\ln(2h(0)-1)}{2}\right) \in (-M(0), M(0)) \times \left(0, -\frac{\ln(2h_0-1)}{2}\right).$$

Now we assume $\sigma \neq 0$. By (α_4) , there exists a continuous function $M : \mathbb{R} \ni \sigma \rightarrow M(\sigma) \in (0, +\infty)$ so that for every $|y| > M(\sigma)$,

$$\frac{b(y)}{y} > -\frac{\mu}{2|\sigma|}.$$

Let $G_1 = (-M(\sigma), M(\sigma)) \times (-M(\sigma), M(\sigma)) \subset \mathbb{R}^2$ and $G_1^c \subset \mathbb{R}^2$ be the complementary set of G_1 . Then the Minkowski functional $p_{G_1} : \mathbb{R}^2 \ni x \rightarrow p_{G_1}(x) \in \mathbb{R}$ determined by G_1 is $p_{G_1}(x) = \max\{\frac{|x_1|}{M(\sigma)}, \frac{|x_2|}{M(\sigma)}\}$. We first show that

$$(4.3) \quad F_{\max}(x, \sigma) = \max_{\{\tilde{x}: p_{G_1}(\tilde{x}) \leq p_{G_1}(x)\}} [x_1, x_2] \cdot [-\mu x_1 + \sigma b(\tilde{x}_2), -\mu x_2 + \sigma b(\tilde{x}_1)]^T < 0$$

for every $x \in G_1^c$, where $x = [x_1, x_2] \in \mathbb{R}^2$ and $\tilde{x} = [\tilde{x}_1, \tilde{x}_2] \in \mathbb{R}^2$. We distinguish the following four cases.

Case 1. $x_1 \geq \max\{M(\sigma), |x_2|\}$. Then we have

$$(4.4) \quad F_{\max}(x, \sigma) = \max_{\{\tilde{x}: \max\{|\tilde{x}_1|, |\tilde{x}_2|\} \leq x_1\}} -\mu x_1^2 + \sigma x_1 b(\tilde{x}_2) - \mu x_2^2 + \sigma x_2 b(\tilde{x}_1).$$

If $x_2 \geq 0$, then by (α_3) and (4.4), we have

$$F_{\max}(x, \sigma) = \begin{cases} -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) & \text{if } \sigma > 0, \\ -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(x_1) & \text{if } \sigma < 0. \end{cases}$$

By (α_4) and by the assumption $x_1 \geq \max\{M(\sigma), |x_2|\}$, we have for $\sigma > 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_1 b(-x_1) \\
 &= -\mu x_1^2 + 2\sigma x_1 b(-x_1) - \mu x_2^2 \\
 &= -2\sigma x_1^2 \left(\frac{\mu}{2\sigma} + \frac{b(-x_1)}{-x_1} \right) - \mu x_2^2 \\
 (4.5) \qquad &< 0,
 \end{aligned}$$

and for $\sigma < 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_1 b(x_1) \\
 &= -\mu x_1^2 + 2\sigma x_1 b(x_1) - \mu x_2^2 \\
 &= -2\sigma x_1^2 \left(\frac{\mu}{2\sigma} - \frac{b(x_1)}{x_1} \right) - \mu x_2^2 \\
 (4.6) \qquad &< 0.
 \end{aligned}$$

If $x_2 < 0$, then by (α_3) and (4.4), we have

$$F_{\max}(x, \sigma) = \begin{cases} -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(x_1) & \text{if } \sigma > 0, \\ -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) & \text{if } \sigma < 0. \end{cases}$$

By (α_4) and the assumption $x_1 \geq \max\{M, |x_2|\}$, we have for $\sigma > 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 - \sigma x_1 b(x_1) \\
 &= -\sigma x_1^2 \left(\frac{\mu}{\sigma} + \frac{b(-x_1)}{-x_1} + \frac{b(x_1)}{x_1} \right) - \mu x_2^2 \\
 (4.7) \qquad &< 0,
 \end{aligned}$$

and for $\sigma < 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 - \sigma x_1 b(-x_1) \\
 &= -\sigma x_1^2 \left(\frac{\mu}{\sigma} - \frac{b(x_1)}{x_1} - \frac{b(-x_1)}{-x_1} \right) - \mu x_2^2 \\
 (4.8) \qquad &< 0.
 \end{aligned}$$

Then inequalities (4.5), (4.6), (4.7), and (4.8) show that if $x_1 \geq \max\{M(\sigma), |x_2|\}$ and $\sigma \neq 0$, then $F_{\max}(x, \sigma) < 0$.

Case 2. $-x_1 \geq \max\{M, |x_2|\}$. Then we have

$$(4.9) \quad F_{\max}(x, \sigma) = \max_{\{\tilde{x}: \max\{|\tilde{x}_1|, |\tilde{x}_2|\} \leq -x_1\}} -\mu x_1^2 + \sigma x_1 b(\tilde{x}_2) - \mu x_2^2 + \sigma x_2 b(\tilde{x}_1).$$

If $x_2 > 0$, then by (α_3) and (4.9), we have

$$F_{\max}(x, \sigma) = \begin{cases} -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(x_1) & \text{if } \sigma > 0, \\ -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) & \text{if } \sigma < 0. \end{cases}$$

By (α_4) and the assumption $-x_1 \geq \max\{M, |x_2|\}$, we have for $\sigma > 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 - \sigma x_1 b(x_1) \\
 &= -\sigma x_1^2 \left(\frac{\mu}{\sigma} + \frac{b(-x_1)}{-x_1} + \frac{b(x_1)}{x_1} \right) - \mu x_2^2 \\
 (4.10) \qquad &< 0,
 \end{aligned}$$

and for $\sigma < 0$

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 - \sigma x_1 b(-x_1) \\
 &= -\sigma x_1^2 \left(\frac{\mu}{\sigma} - \frac{b(x_1)}{x_1} - \frac{b(-x_1)}{-x_1} \right) - \mu x_2^2 \\
 (4.11) \qquad &< 0.
 \end{aligned}$$

If $x_2 < 0$, then by (α_3) and (4.9), we have

$$F_{\max}(x, \sigma) = \begin{cases} -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) & \text{if } \sigma > 0, \\ -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(x_1) & \text{if } \sigma < 0. \end{cases}$$

By (α_4) and the assumption $-x_1 \geq \max\{M, |x_2|\}$, we have for $\sigma > 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_2 b(-x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(-x_1) - \mu x_2^2 + \sigma x_1 b(-x_1) \\
 &= -\mu x_1^2 + 2\sigma x_1 b(-x_1) - \mu x_2^2 \\
 &= -2\sigma x_1^2 \left(\frac{\mu}{2\sigma} + \frac{b(-x_1)}{-x_1} \right) - \mu x_2^2 \\
 (4.12) \qquad &< 0,
 \end{aligned}$$

and for $\sigma < 0$,

$$\begin{aligned}
 F_{\max}(x, \sigma) &= -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_2 b(x_1) \\
 &\leq -\mu x_1^2 + \sigma x_1 b(x_1) - \mu x_2^2 + \sigma x_1 b(x_1) \\
 &= -\mu x_1^2 + 2\sigma x_1 b(x_1) - \mu x_2^2 \\
 &= -2\sigma x_1^2 \left(\frac{\mu}{2\sigma} - \frac{b(x_1)}{x_1} \right) - \mu x_2^2 \\
 (4.13) \qquad &< 0.
 \end{aligned}$$

Then inequalities (4.10), (4.11), (4.12), and (4.13) show that if $-x_1 \geq \max\{M(\sigma), |x_2|\}$, then $F_{\max}(x, \sigma) < 0$. Then by Case 1 and Case 2 we have proved that if $\sigma \neq 0$ and $|x_1| \geq \max\{M(\sigma), |x_2|\}$, then $F_{\max}(x, \sigma) < 0$. By the symmetry between x_1 and x_2 in the first two equations of (4.1) we can similarly show that $F_{\max}(x, \sigma) < 0$ in each of the cases that $x_2 \geq \max\{M(\sigma), |x_1|\}$ and $-x_2 \geq \max\{M(\sigma), |x_1|\}$ with $\sigma \neq 0$.

It follows that $F_{\max}(x, \sigma) < 0$ for every $x \in G_1^c$. This completes the proof of (4.3).

It remains to show that the τ -coordinates are bounded so that all periodic solutions of (4.1) are bounded. We introduce the following change of variable:

$$(4.14) \quad z(t) = \tau(t) + \frac{\ln(2h_0 - 1)}{4}.$$

Then system (4.1) is transformed to

$$(4.15) \quad \begin{cases} \dot{x}_1(t) = -\mu x_1(t) + \sigma b \left(x_2 \left(t - z(t) + \frac{\ln(2h_0 - 1)}{4} \right) \right), \\ \dot{x}_2(t) = -\mu x_2(t) + \sigma b \left(x_1 \left(t - z(t) + \frac{\ln(2h_0 - 1)}{4} \right) \right), \\ \dot{z}(t) = 1 - h(x(t)) \left(1 + \tanh \left(z(t) - \frac{1}{4} \ln(2h_0 - 1) \right) \right). \end{cases}$$

By (α_2) and the monotonicity of $\tanh \tau$, we have, for every $z \leq \frac{\ln(2h_0 - 1)}{4} < 0$,

$$(4.16) \quad \begin{aligned} & z \cdot \left(1 - h(x) \left(1 + \tanh \left(z - \frac{1}{4} \ln(2h_0 - 1) \right) \right) \right) \\ & < z \cdot (1 - h(x) (1 + \tanh(0))) \\ & < 0. \end{aligned}$$

Similarly, by (α_2) and the monotonicity of $\tanh \tau$, for every $z \geq -\frac{\ln(2h_0 - 1)}{4} > 0$, we have

$$(4.17) \quad \begin{aligned} & z \cdot \left(1 - h(x) \left(1 + \tanh \left(z - \frac{1}{4} \ln(2h_0 - 1) \right) \right) \right) \\ & < z \cdot \left(1 - h(x) \left(1 + \tanh \left(-\frac{1}{2} \ln(2h_0 - 1) \right) \right) \right) \\ & = z \cdot \left(1 - h(x) \left(1 + \frac{1 - h_0}{h_0} \right) \right) \\ & < 0. \end{aligned}$$

Then, by (4.16) and (4.17), we have, for every $z \notin \left(\frac{\ln(2h_0 - 1)}{4}, -\frac{\ln(2h_0 - 1)}{4} \right)$,

$$(4.18) \quad z \cdot \left(1 - h(x) \left(1 + \tanh \left(z - \frac{1}{4} \ln(2h_0 - 1) \right) \right) \right) < 0.$$

Thus it follows from Lemma 4.1, (4.3), and (4.18) that the range of every periodic solution (x, z) of (4.15) with $\sigma \in \mathbb{R}$ is contained in $(-M(\sigma), M(\sigma)) \times \left(\frac{\ln(2h_0 - 1)}{4}, -\frac{\ln(2h_0 - 1)}{4} \right)$. Then, by (4.2) and by (4.14), the range of all the periodic solutions (x, τ) of (4.1) is contained in

$$\Omega_1 = (-M(\sigma), M(\sigma)) \times (-M(\sigma), M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2} \right).$$

This completes the proof. \square

4.2. Global bifurcation of periodic solutions. Now we consider the global Hopf bifurcation problem of system (4.1) under assumptions (α_1) – (α_5) . By (α_3) and (α_4) , $(x, \tau) = (0, \tau^*)$ is the only stationary solution of (4.1), where $\tau^* = -\frac{1}{2} \ln(2h(0) - 1) > 0$. Freezing the state-dependent delay $\tau(t)$ at τ^* for the term $x(t - \tau(t))$ of (4.1) and linearizing the resulting system with constant delay at the stationary solution $(0, \tau^*)$, we obtain the following formal linearization of system (4.1):

$$(4.19) \quad \begin{cases} \dot{X}(t) = -\mu X(t) - AX(t - \tau^*), \\ \dot{T}(t) = -\rho X(t) - qT(t), \end{cases}$$

where

$$(4.20) \quad A = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad \rho = \frac{\nabla h(0)}{h(0)}, \quad q = 2 - \frac{1}{h(0)} > 0.$$

In the following we regard σ as the bifurcation parameter. We obtain the following characteristic equation corresponding to (4.19):

$$(4.21) \quad \det[(\lambda + \mu)I + e^{-\tau^* \lambda} A](\lambda + q) = 0.$$

Note that the zero of $\lambda + q = 0$ is $-q$, which is real, and Hopf bifurcation points are related to zeros of the first factor

$$\det[(\lambda + \mu)I + e^{-\tau^* \lambda} A] = (\lambda + \mu - \sigma e^{-\lambda \tau^*})(\lambda + \mu + \sigma e^{-\lambda \tau^*}).$$

To locate local Hopf bifurcation points we let $\lambda = i\beta$, $\beta > 0$, in (4.21) and express the resulting equation in terms of its real and imaginary parts. We have

$$(4.22) \quad \begin{cases} \beta = -\sigma \sin(\tau^* \beta), \\ \mu = \sigma \cos(\tau^* \beta), \end{cases}$$

or

$$(4.23) \quad \begin{cases} \beta = \sigma \sin(\tau^* \beta), \\ \mu = -\sigma \cos(\tau^* \beta). \end{cases}$$

We summarize relevant information about (4.21) in the following.

LEMMA 4.3. *We have the following conclusions:*

- (i) *All the positive solutions of (4.22) and (4.23) can be represented by an infinite sequence $\{\beta_n\}_{n=1}^{+\infty}$ which satisfies $0 < \beta_1 < \beta_2 < \dots < \beta_n < \dots$, $\lim_{n \rightarrow +\infty} \beta_n = +\infty$, and*

$$\beta_n \in \left(\frac{(2n-1)\pi}{2\tau^*}, \frac{2n\pi}{2\tau^*} \right) \text{ for } n \geq 1.$$

- (ii) *$\pm i\beta_n$ are characteristic values of the stationary solution $(0, \tau^*, \sigma_n)$, where*

$$\sigma_n = \pm \sqrt{\beta_n^2 + \mu^2}.$$

If $\sigma \neq \sigma_n$, then the stationary solution $(0, \tau^, \sigma)$ has no purely imaginary characteristic value.*

(iii) Let $\lambda_n(\sigma) = u_n(\sigma) + iv_n(\sigma)$ be the root of (4.21) for σ close to σ_n such that $u_n(\sigma_n) + iv_n(\sigma_n) = i\beta_n$. Then

$$u'_n(\sigma)\Big|_{\sigma=\sigma_n} = \frac{\sigma_n(\mu + \tau^* \sigma_n^2)}{(\mu + (\mu^2 - 2\beta_n^2)\tau^*)^2 + (1 + 2\mu\tau^*)^2\beta_n^2}.$$

Proof. We note that $\mu > 0$ and hence $\cos(\tau^*\beta) \neq 0$. Then the solutions of (4.22) and (4.23) are also solutions of the following equations:

$$(4.24) \quad \begin{cases} \tan \tau^* \beta = -\frac{\beta}{\mu}, \\ \beta^2 = \sigma^2 - \mu^2. \end{cases}$$

Conversely, by assuming $\beta/\sigma = -\sin \theta$ and $\mu/\sigma = -\cos \theta$ in the second equation of (4.24), we have $\tan \theta = -\tan \tau^* \beta$ and hence $\theta = k\pi - \tau^* \beta$ for $k \in \mathbb{Z}$. Depending on whether k is even or odd, the set of solutions of (4.24) can be categorized into two classes which solve (4.22) and (4.23), respectively. That is, the solutions of (4.24) are also solutions of (4.22) and (4.23). Therefore, solving (4.22) and (4.23) for $\beta > 0$ is equivalent to solving (4.24).

To solve (4.24), we note that the function $z = \tan \tau^* \beta$ is a strictly increasing 1-to-1 mapping from the open interval $(\frac{(2j-1)\pi}{2\tau^*}, \frac{2j\pi}{2\tau^*})$ to $(-\infty, 0)$ with $\tan(\tau^* \frac{2j\pi}{2\tau^*}) = 0$ and $\lim_{\theta \rightarrow (\tau^* \frac{(2j-1)\pi}{2\tau^*})^+} \tan(\theta) = -\infty$ for every $j \geq 1$. Then, $z = \tan \tau^* \beta$ has a unique intersection with the straight line $z = -\beta/\mu$, $\mu > 0$, in the strip area $(\frac{(2n-1)\pi}{2\tau^*}, \frac{2n\pi}{2\tau^*}) \times (-\infty, 0)$ on the (β, z) -plane. That is, (4.24) has a unique solution $\beta_n \in (\frac{(2n-1)\pi}{2\tau^*}, \frac{2n\pi}{2\tau^*})$ for every $n \geq 1$, $n \in \mathbb{N}$. This completes the proof of (i).

The conclusion (ii) follows from (i) and from the second equation of (4.24).

To prove (iii), let $F(\lambda, \sigma) = (\lambda + \mu + \sigma e^{-\tau^* \lambda})(\lambda + \mu - \sigma e^{-\tau^* \lambda})$. Then we have

$$\begin{aligned} \frac{\partial F}{\partial \lambda}\Big|_{\lambda=i\beta_n, \sigma=\sigma_n} &= 2(i\beta_n + \mu + \sigma_n^2 \tau^* e^{-2\tau^* \cdot i\beta_n}) \\ &= 2(\mu + \sigma_n^2 \tau^* \cos 2\tau^* \beta_n + i(\beta_n - \sigma_n^2 \tau^* \sin 2\tau^* \beta_n)). \end{aligned}$$

We want to show that $\frac{\partial F}{\partial \lambda}\Big|_{\lambda=i\beta_n, \sigma=\sigma_n} \neq 0$. Assume the contrary. Then we have $\beta_n = \sigma_n^2 \tau^* \sin(2\tau^* \beta_n)$ and $\mu = -\sigma_n^2 \tau^* \cos(2\tau^* \beta_n)$. By (i), we have $\beta_n \neq 0$. Then $\beta_n = 2\sigma_n^2 \tau^* \sin(\tau^* \beta_n) \cos \tau^* \beta_n \neq 0$ implies that $\cos \tau^* \beta_n \neq 0$. It follows that $\beta_n/\mu = -\tan(2\tau^* \beta_n) = -2 \tan(\tau^* \beta_n)/(1 + \tan^2(\tau^* \beta_n))$. By (4.24), we have $\tan(\tau^* \beta_n) = 0$ or $\tan^2(\tau^* \beta_n) = 1$. If $\tan(\tau^* \beta_n) = 0$, then by (4.24), $\beta_n = 0$. This is a contradiction to (i). If $\tan^2(\tau^* \beta_n) = 1$, then by (4.24) we have $\beta_n = \pm\mu$ and hence $\sigma_n = 0$. This contradicts (ii). Therefore, we have $\frac{\partial F}{\partial \lambda}\Big|_{\lambda=i\beta_n, \sigma=\sigma_n} \neq 0$.

By the implicit function theorem, there exists a differentiable function $\sigma \rightarrow \lambda_n(\sigma) = u_n(\sigma) + iv_n(\sigma)$ which is a root of (4.21) for σ close to σ_n with $u_n(\sigma_n) + iv_n(\sigma_n) = i\beta_n$. Note that $\lambda_n(\sigma) \rightarrow i\beta_n \neq q$ as $\sigma \rightarrow \sigma_n$. We substitute λ by $\lambda_n(\sigma) = u_n(\sigma) + iv_n(\sigma)$ into (4.21) and obtain

$$(u_n(\sigma) + iv_n(\sigma) + \mu)^2 - \sigma^2 e^{-2\tau^*(u_n(\sigma) + iv_n(\sigma))} = 0.$$

Differentiating both sides of the above equation with respect to σ and then substituting $\sigma = \sigma_n$, we have

$$(4.25) \quad \begin{cases} (\mu + \tau^* \sigma^2 \cos(2\tau^* \beta_n))u'_n(\sigma_n) - (\beta_n - \tau^* \sigma_n^2 \sin(2\tau^* \beta_n))v'_n(\sigma_n) = \sigma_n \cos(2\tau^* \beta_n), \\ (\beta_n - \tau^* \sigma_n^2 \sin(2\tau^* \beta_n))u'_n(\sigma_n) + (\mu + \tau^* \sigma^2 \cos(2\tau^* \beta_n))v'_n(\sigma_n) = -\sigma_n \sin(2\tau^* \beta_n). \end{cases}$$

By (4.22) and by (4.23), we have

$$(4.26) \quad \begin{cases} \cos(2\tau^* \beta_n) = 1 - 2\beta_n^2/\sigma_n^2, \\ \sin(2\tau^* \beta_n) = -2\mu\beta_n/\sigma_n^2. \end{cases}$$

Note that we have $\mu > 0$, $\tau^* > 0$, and $\beta_n > 0$ for every $n \geq 1$. We combine (4.25) with (4.22), (4.23), and (4.26) to obtain

$$u'_n(\sigma) \Big|_{\sigma=\sigma_n} = \frac{\sigma_n(\mu + \tau^* \sigma_n^2)}{(\mu + (\mu^2 - 2\beta_n^2)\tau^*)^2 + (1 + 2\mu\tau^*)^2 \beta_n^2}.$$

This completes the proof. \square

As a preparation for describing the global continuation of periodic solutions of (4.1), we now consider assumption (S7) so that we can use Theorems 3.7 and 3.8 to obtain lower and upper bounds of periods in a connected component of the closure of all the nonconstant periodic solutions of (4.1).

For notational convenience, let

$$(4.27) \quad B(x) = \begin{bmatrix} b(x_2) \\ b(x_1) \end{bmatrix},$$

$$(4.28) \quad s(x) = -\frac{1}{2} \ln(2h(x) - 1),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. We now assume the following:

(α_6) For every $x = (x_1, x_2) \in \mathbb{R}^2$, $\frac{\partial}{\partial x_1} h(x) \cdot \frac{\partial}{\partial x_2} h(x) \neq 0$.

(α_7) For every $c > 0$ and $\sigma \in \mathbb{R}$, the system of algebraic equations

$$(4.29) \quad \begin{cases} s(x) = c, \\ s(\bar{x}) = c, \\ -\mu x + \sigma B(\bar{x}) = 0 \end{cases}$$

has at most one solution for $(x, \bar{x}) = (x_1, x_2, \bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ with $x_1 \neq \bar{x}_2$. As a preparation for our main results in this section we have the next lemma.

LEMMA 4.4. Assume (α_1 – α_4) and (α_6)–(α_7) hold. Let $\eta = [y_0, y_1, \dots, y_{m-1}]^T \in \mathbb{R}^{2m}$ for every $m \in \mathbb{N}$, where $y_i \in \mathbb{R}^2$, $i = 0, 1, \dots, m-1$. Let

$$(4.30) \quad F(\eta, \sigma) = [-\mu y_0 + \sigma B(y_1), -\mu y_1 + \sigma B(y_2), \dots, -\mu y_{m-1} + \sigma B(y_0)]^T \in \mathbb{R}^{2m};$$

$$(4.31) \quad L(\eta, \sigma) = [s(y_0), s(y_1), \dots, s(y_{m-1})]^T \in \mathbb{R}^m.$$

Then for every $c > 0$, there is no nonconstant periodic solution of $\dot{\eta}(t) = F(\eta(t), \sigma)$ which satisfies the constraint

$$(4.32) \quad L(\eta(t), \sigma) = c[1, 1, \dots, 1]^T$$

for every $t \in \mathbb{R}$.

Proof. By way of contradiction, we suppose that there exist $c_0 > 0$ and a nonconstant periodic solution of $\dot{\eta}(t) = F(\eta(t), \sigma)$ which satisfies the constraint $L(\eta, \sigma) = c_0[1, 1, \dots, 1]^T$, where $T_0 > 0$ is the minimal period of η .

We first show that for every $i \in \{0, 1, \dots, m-1\}$ and $j \in \{1, 2\}$, the coordinate function $y_{i,j}$ of η is a nonconstant function. Otherwise, there exist $i' \in \{0, 1, \dots, m-1\}$ and $j' \in \{1, 2\}$ so that $y_{i',j'}$ is a constant function. Without loss of generality

we assume $y_{i',1}$ is a constant function. Since $s(y_{i'})(t) = c$ for all $t \in \mathbb{R}$, $h(y_{i'})$ is a constant function. Then we have

$$\frac{\partial h}{\partial x_2}(y_{i',1}(t), y_{i',2}(t)) \cdot \dot{y}_{i',2}(t) = 0 \text{ for all } t \in \mathbb{R},$$

where $\frac{\partial h}{\partial x_2}$ denotes the partial derivative of h with respect to the second argument. By (α_6) , we have $\dot{y}_{i',2}(t) = 0$ for all $t \in \mathbb{R}$ and hence $y_{i',2}$ is also a constant function. By the cyclicity of the system $\dot{\eta}(t) = F(\eta, \sigma)$ with respect to $y_i, i \in \{0, 1, 2, \dots, m-1\}$, η is a constant solution of the system $\dot{\eta}(t) = F(\eta, \sigma)$. This a contradiction. So $y_{i,1}$ and $y_{i,2}$ are nonconstant periodic functions.

In the following we identify $i = m$ with $i = 0$ for the first index i of $y_{i,j}$ and identify $j = 3$ with $j = 1$ for the second index j . We claim that there exist $i_0 \in \{0, 1, 2, \dots, m-1\}$ and $j_0 \in \{1, 2\}$ so that if y_{i_0,j_0} assumes the global maximum or minimum at $t_0 \in \mathbb{R}$, then $y_{i_0+1,j_0+1}(t_0) \neq y_{i_0,j_0}(t_0)$. Suppose not. Then for every $i \in \{0, 1, 2, \dots, m-1\}$ and $j \in \{1, 2\}$, if $y_{i,j}$ assumes the global maximum or minimum at $t \in \mathbb{R}$, then $y_{i+1,j+1}(t) = y_{i,j}(t)$. By (4.27) and (4.30), we have

$$\dot{y}_{i,j}(t) = -\mu y_{i,j}(t) + \sigma b(y_{i,j}(t)) = 0.$$

Then, by (α_4) we have $y_{i,j}(t) = 0$. Then the global maximum and minimum of $y_{i,j}$ is 0. It follows that $y_{i,j}$ is a constant function. This is a contradiction and the claim is proved.

Now we choose $i_0 \in \{0, 1, 2, \dots, m-1\}$, $j_0 \in \{1, 2\}$, and $t^* \in \mathbb{R}$ so that y_{i_0,j_0} assumes its global maximum at t^* . Then $\dot{y}_{i_0,j_0}(t^*) = 0$. Since $h(y_{i_0})(t)$ is a constant for all $t \in \mathbb{R}$, we have

$$\frac{\partial h}{\partial x_1}(y_{i_0,1}, y_{i_0,2})\dot{y}_{i_0,1}(t) + \frac{\partial h}{\partial x_2}(y_{i_0,1}, y_{i_0,2})\dot{y}_{i_0,2}(t) = 0 \text{ for all } t \in \mathbb{R},$$

where $\frac{\partial h}{\partial x_1}$ denotes the partial derivative of h with respect to the first argument. Then by (α_7) , we have

$$(4.33) \quad \dot{y}_{i_0,1}(t^*) = \dot{y}_{i_0,2}(t^*) = 0.$$

By the definition of $L(\eta, \sigma)$, we have

$$(4.34) \quad s(y_i(t)) = s(y_{i,1}(t), y_{i,2}(t)) = c_0$$

for all $t \in \mathbb{R}$ and $i \in \{0, 1, 2, \dots, m-1\}$. Therefore, by (4.33) and (4.34), we know that $(y_{i_0}(t^*), y_{i_0+1}(t^*)) = (y_{i_0,1}(t^*), y_{i_0,2}(t^*), y_{i_0+1,1}(t^*), y_{i_0+1,2}(t^*))$ is a solution of the following algebraic equation of $(x, \bar{x}) \in \mathbb{R}^2 \times \mathbb{R}^2$:

$$(4.35) \quad \begin{cases} s(x) = c_0, \\ s(\bar{x}) = c_0, \\ -\mu x + \sigma B(\bar{x}) = 0, \end{cases}$$

where $y_{i_0,1}(t^*) \neq y_{i_0+1,2}(t^*)$. Similarly, we choose $\bar{t}^* \in \mathbb{R}$, where $y_{i_0,1}$ assumes its global minimum on \mathbb{R} . Then $(y_{i_0}(\bar{t}^*), y_{i_0+1}(\bar{t}^*)) = (y_{i_0,1}(\bar{t}^*), y_{i_0,2}(\bar{t}^*), y_{i_0+1,1}(\bar{t}^*), y_{i_0+1,2}(\bar{t}^*))$

is also a solution of (4.35) with $y_{i_0,1}(\bar{t}^*) \neq y_{i_0+1,2}(\bar{t}^*)$. By (α_7) , we know that (4.35) has at most one solution with $x_1 \neq \bar{x}_2$. If (4.35) does not have a solution, then it is a contradiction to the fact that $(y_{i_0}(t^*), y_{i_0+1}(t^*))$ and $(y_{i_0}(\bar{t}^*), y_{i_0+1}(\bar{t}^*))$ are solutions. If (4.35) has a unique solution, then $y_{i_0}(t^*) = y_{i_0}(\bar{t}^*)$, which means that the global maximum and global minimum of y_{i_0,j_0} are equal. Therefore, y_{i_0,j_0} is a constant function and so is y_{i_0} . This is a contradiction. \square

We discussed in [17] the global continuation (with respect to the parameter) of rapidly oscillating periodic solutions of system (2.1) where f is a negative or positive feedback, namely, $xf(x, x, \sigma) < 0$ or $xf(x, x, \sigma) > 0$ for all $x \in \mathbb{R}^N$ and $\sigma \in \mathbb{R}$. As we pointed out in section 1, the approach developed in section 3 for lower and upper bounds of periods along a continuum of periodic solutions in the Fuller space enables us to obtain global continuation of both slowly oscillating and rapidly oscillating periodic solutions of system (2.1) without the assumption that $xf(x, x, \sigma)$ is negative or positive definite.

In the following, we consider system (4.1) and illustrate the general results we obtained in the previous sections. By (i) of Lemma 4.3, we know that every possible virtual period $p_n = 2\pi/\beta_n$, $n \in \mathbb{N}$, satisfies that $4\tau^*/(2n) < p_n < 4\tau^*/(2n-1)$. Then we can conjecture that there exists a connected component $C(0, \tau^*, \sigma_n, p_n)$ of slowly oscillating periodic solutions bifurcated from $(0, \tau^*, \sigma_n, p_n)$ for $n \in \{1, 2\}$ and there exists a connected component $C(0, \tau^*, \sigma_n, p_n)$ of rapidly oscillating periodic solutions bifurcated from $(0, \tau^*, \sigma_n, p_n)$ for every $n \geq 3, n \in \mathbb{N}$.

We verify these conjectures and obtain global continuation with respect to the parameter $\sigma \in \mathbb{R}$ for $C(0, \tau^*, \sigma_n, p_n)$, $n \in \mathbb{N}$, in the following main result.

THEOREM 4.5. *Assume $(\alpha_1-\alpha_7)$ hold and all the periodic solutions of system (4.1) are real analytic. Let $\beta_n \in (\frac{(2n-1)\pi}{2\tau^*}, \frac{2n\pi}{2\tau^*})$, $n \geq 1$, be given in (i) of Lemma 4.3. Let $\sigma_n = \pm\sqrt{\mu^2 + \beta_n^2}$ for $n \in \mathbb{N}$. Then the following apply:*

- (i) *For $n \in \{1, 2\}$, there exists an unbounded continuum $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ of slowly oscillating periodic solutions of system (4.1). For every $n \geq 3, n \in \mathbb{N}$, there exists an unbounded continuum $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ of rapidly oscillating periodic solutions of system (4.1).*
- (ii) *For every $n \in \mathbb{N}$, let Σ be the projection of $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ onto the parameter space \mathbb{R} . Then Σ is unbounded with $\Sigma \subseteq (0, +\infty)$ if $\sigma_n > 0$ and $\Sigma \subseteq (-\infty, 0)$ if $\sigma_n < 0$.*
- (iii) *For every $n_0 \in \{1, 2\}$ and $n \geq 3, n \in \mathbb{N}$, $(0, \tau^*, \sigma_{n_0}, \frac{2\pi}{\beta_{n_0}}) \notin C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ and $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}) \notin C(0, \tau^*, \sigma_{n_0}, \frac{2\pi}{\beta_{n_0}})$.*

Proof. We first prove two claims.

Claim 1. For every $n \geq 1, n \in \mathbb{N}$, there exists a connected component $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ of the closure of all the nonconstant periodic solutions of system (4.1) in the Fuller space.

We prove Claim 1 by applying Theorem 2.1. Note that $\sigma_n = \pm\sqrt{\mu^2 + \beta_n^2}$ and by (i) and (ii) of Lemma 4.3, we have

$$(4.36) \quad \beta_n \in \left(\frac{(2n-1)\pi}{2\tau^*}, \frac{2n\pi}{2\tau^*} \right).$$

System (4.19) has infinitely many isolated centers $(0, \tau^*, \sigma_n)$. Except at these isolated centers, there is no purely imaginary characteristic value of (4.19) with $\sigma \in \mathbb{R}$.

By Lemma 4.3, we know that if $u_n(\sigma) + iv_n(\sigma)$ is the characteristic value of (4.19) such that $u_n(\sigma_n) + iv_n(\sigma_n) = i\beta_n$, then we have

$$(4.37) \quad \begin{aligned} \frac{d}{d\sigma}u_n(\sigma)\Big|_{\sigma=\sigma_n} &= u'_n(\sigma)\Big|_{\sigma=\sigma_n} \\ &= \frac{\sigma_n(\mu + \tau^*\sigma_n^2)}{(\mu + (\mu^2 - 2\beta_n^2)\tau^*)^2 + (1 + 2\mu\tau^*)^2\beta_n^2}. \end{aligned}$$

We note from (2.6) that the crossing number $\gamma(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ counts the difference, when σ varies from σ_n^- to σ_n^+ , of the number of nonreal characteristic values with positive real parts in a small neighborhood of $i\beta_n$ in the complex plane. Then the nontriviality of the crossing number implies the appearance of purely imaginary characteristic values.

By (4.37), the crossing number of the isolated center $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ in the Fuller space $C(\mathbb{R}; \mathbb{R}^2) \times \mathbb{R}^2$ satisfies

$$(4.38) \quad \gamma\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right) = -\text{sgn}(\sigma_n) \text{ for every } n \in \mathbb{N}.$$

Also, it is clear that (α_2) , (α_5) , and (α_6) imply (S1), (S2), and (S4). Let us check (S3). Noting that $\sigma_n = \pm\sqrt{\mu^2 + \beta_n^2}$ and $\beta_n > 0$, we obtain that

$$(4.39) \quad \left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right)(-\mu\theta_1 + \sigma B(\theta_2))\Big|_{\sigma=\sigma_n, \theta_1=\theta_2=0} = \begin{bmatrix} -\mu & \sigma_n \\ \sigma_n & -\mu \end{bmatrix}$$

is nonsingular for all $n \in \mathbb{N}$, where the map $\mathbb{R}^2 \times \mathbb{R}^2 \ni (\theta_1, \theta_2) \rightarrow (-\mu\theta_1 + \sigma B(\theta_2)) \in \mathbb{R}^2$ is given by the right-hand side of the first two equations of (4.1). Also, it follows from $\tau^* = -\frac{\ln(2h(0)-1)}{2}$ that

$$(4.40) \quad \frac{\partial}{\partial\gamma_2}(1 - h(\gamma_1))(1 + \tanh(\gamma_2))\Big|_{\sigma=\sigma_n, \gamma_1=0, \gamma_2=\tau^*} = -h(0) \cdot \frac{4e^{2\tau^*}}{(e^{2\tau^*} + 1)^2} < 0.$$

Therefore, condition (S3) is satisfied by system (4.19). Then by Theorem 2.1, we know that $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ is a Hopf bifurcation point of (4.1) and Claim 1 is proved.

Claim 2. For every $n \geq 1, n \in \mathbb{N}$, the connected component $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ is unbounded in the Fuller space $C(\mathbb{R}; \mathbb{R}^2) \times \mathbb{R}^2$.

Recall that the period normalization of periodic solutions does not change its norm in the Fuller space $C(\mathbb{R}; \mathbb{R}^2) \times \mathbb{R}^2$. So we transform the connected component $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ into a connected component, denoted by $\mathcal{C}(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$, in $C(\mathbb{R}/2\pi; \mathbb{R}^2) \times \mathbb{R}^2$ by period normalization and prove Claim 2 by means of Theorem 2.3. We note from Lemma 4.3 that all the centers of (4.1) are isolated. Now suppose that there exists $n_0 \in \mathbb{N}$ so that $\mathcal{C}(0, \tau^*, \sigma_{n_0}, 2\pi/\beta_{n_0})$ is bounded in the Fuller space. Then by Theorem 2.3, there are finitely many, namely, $q + 1$, bifurcation points $\{(0, \tau^*, \sigma_{n_j}, 2\pi/\beta_{n_j})\}_{j=0}^q$ in $\mathcal{C}(0, \tau^*, \sigma_{n_0}, 2\pi/\beta_{n_0})$ and

$$(4.41) \quad \sum_{j=0}^q \epsilon_{n_j} \gamma(0, \tau^*, \sigma_{n_j}, 2\pi/\beta_{n_j}) = 0,$$

where ϵ_{n_j} is the value of

$$\text{sgn det} \begin{pmatrix} \left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right)(-\mu\theta_1 + \sigma B(\theta_2)), & 0 \\ \frac{\partial}{\partial\gamma_1}[(1 - h(\gamma_1))(1 + \tanh(\gamma_2))], & \frac{\partial}{\partial\gamma_2}[(1 - h(\gamma_1))(1 + \tanh(\gamma_2))] \end{pmatrix}$$

evaluated at $(\theta_1, \theta_2, \sigma) = (0, 0, \sigma_{n_j})$ and $(\gamma_1, \gamma_2, \sigma) = (0, \tau^*, \sigma_{n_j})$. Then by (4.39) and (4.40) we have

$$(4.42) \quad \epsilon_{n_j} = 1 \text{ for all } j = 0, 1, 2, \dots, q.$$

If $\sigma = 0$, then system (4.1) becomes the system of ordinary differential equations

$$\begin{cases} \dot{x}_1(t) = -\mu x_1(t), \\ \dot{x}_2(t) = -\mu x_2(t), \\ \dot{\tau}(t) = 1 - h(x(t)) \cdot (1 + \tanh \tau(t)), \end{cases}$$

which clearly has no nonconstant periodic solution. Also, $(0, \tau^*, 0)$ is not a center of the linear system (4.19); then by Lemma 4.5 in [16], $(0, \tau^*, 0)$ is not a Hopf bifurcation point of (4.1). Therefore, every connected component $\mathcal{C}(0, \tau^*, \sigma_n, 2\pi/\beta_n)$, $n \in \mathbb{N}$, is located in the Fuller space where σ satisfies $\sigma \cdot \text{sgn}(\sigma_n) > 0$. In particular, $\sigma_{n_j} \cdot \text{sgn}(\sigma_{n_0}) > 0$ for $j = 0, 1, 2, \dots, q$. Then by (4.38) and (4.42) we have

$$\sum_{j=0}^q \epsilon_{n_j} \gamma(0, \tau^*, \sigma_{n_j}, 2\pi/\beta_{n_j}) = -q \text{sgn}(\sigma_{n_0}) \neq 0.$$

This is a contradiction to (4.41) and Claim 2 is proved.

Now we prove the conclusions (i)–(iii).

(i). We prove (i) by applying Theorem 3.7. By Claims 1 and 2, we know that for every $n \geq 1$, $n \in \mathbb{N}$, $C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$, is an unbounded and connected component of periodic solutions bifurcated from $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ in the Fuller space. Then by Theorem 3.7, it remains to verify conditions (S5)–(S8) and (S1')–(S2') so that we can identify continua of slowly and rapidly oscillating periodic solutions.

By (4.36) we know that the virtual period p_n , $n \geq 1$, of the bifurcation point $(0, \tau^*, \sigma_n, p_n)$ satisfies

$$(4.43) \quad \frac{2\tau^*}{n} < p_n < \frac{4\tau^*}{2n-1}.$$

For $n = 1$, it follows from (4.43) that $2\tau^* < p_1 < 4\tau^*$. Then there exist $k_0 = 4, j_0 = 2$ so that $j_0\tau^* < p_1 < k_0\tau^*$. Similarly, for $n = 2$, we have $\tau^* < p_2 < \frac{4}{3}\tau^*$. Then there exist $k_0 = 2, j_0 = 1$ so that $j_0\tau^* < p_2 < k_0\tau^*$. For $n \geq 3$, we have $0 < \frac{2}{n}\tau^* < p_n < \frac{4}{5}\tau^* < \tau^*$. Then there exist $k_0 = 1, j_0 = 0$ so that $j_0\tau^* < p_n < k_0\tau^*$. That is, for every $n \geq 1$, $n \in \mathbb{N}$, there exist $k_0, j_0 \in \mathbb{N} \cup \{0\}, k_0 > j_0$ so that $j_0\tau^* < p_n < k_0\tau^*$ and

$$(4.44) \quad (k_0, j_0) = \begin{cases} (4, 2) & \text{if } n = 1, \\ (2, 1) & \text{if } n = 2, \\ (1, 0) & \text{if } n \geq 3. \end{cases}$$

Also, we show that for every $m, n \in \mathbb{N}$, $m_0 \in \{0, 1, 2, 4\}$,

$$(4.45) \quad m_0\tau^* \neq mp_n.$$

It is clear that (4.45) is true if $m_0 = 0$. Suppose (4.45) is not true. Then there exist $\bar{m}, \bar{n} \in \mathbb{N}$, and $m_0 \in \{1, 2, 4\}$ so that $\bar{m}p_{\bar{n}} = m_0\tau^*$. Note that $\frac{(2\bar{n}-1)\pi}{2\tau^*} < \beta_{\bar{n}} < \frac{2\bar{n}\pi}{2\tau^*}$

and $p_n = \frac{2\pi}{\beta_n}$ for every $n \in \mathbb{N}$. It follows that

$$(4.46) \quad \frac{\bar{n}}{2} - \frac{1}{4} < \frac{\bar{m}}{m_0} < \frac{\bar{n}}{2}.$$

If $m_0 = 1$, then we have $\frac{\bar{n}}{2} - \frac{1}{4} < \bar{m} < \frac{\bar{n}}{2}$, which is impossible since $\bar{m} \in \mathbb{N}$. Similarly, it is clear that (4.46) does not hold if $m_0 = 2$ or $m_0 = 4$. The contradictions verify (S5).

Let $1 - h(x)(1 + \tanh(\tau)) = 0$; then we have $\tau = -\frac{1}{2} \ln(2h(x) - 1)$. By $(\alpha_1 - \alpha_2)$, the mapping $l : (x, \sigma) \rightarrow l(x, \sigma) = -\frac{1}{2} \ln(2h(x) - 1)$ is continuously differentiable. This verifies (S6).

By $(\alpha_1 - \alpha_4)$ and $(\alpha_6) - (\alpha_7)$ and Lemma 4.4, we know that (S7) is satisfied.

Moreover, by $(\alpha_1 - \alpha_4)$ and Lemma 4.2, the range of all periodic solutions (x, τ) of (4.1) with $\sigma \in \mathbb{R}$ is contained in

$$(4.47) \quad (-M(\sigma), M(\sigma)) \times (-M(\sigma), M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2}\right),$$

where $M : \mathbb{R} \ni \sigma \rightarrow M(\sigma) \in (0, +\infty)$ is a continuous function. Therefore, (S8) is satisfied.

Then by Theorem 3.7 and by (4.44), for every $n \in \mathbb{N}$ and every $(x, \tau, \sigma, p) \in C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$, we have

$$(4.48) \quad j_0\tau(t) < p < k_0\tau(t)$$

for all $t \in \mathbb{R}$, where (k_0, j_0) satisfies (4.44). Then (i) is proved.

(ii). Let Σ be the projection of $C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$, $n \geq 1, \mathbb{N}$, on the parameter space \mathbb{R} of σ . By the proof of Claim 2, we know that $\Sigma \subseteq (0, +\infty)$ if $\sigma_n > 0$ and $\Sigma \subseteq (-\infty, 0)$ if $\sigma_n < 0$. By (4.47), we know that for every $\sigma \in \Sigma$, there exists a constant $M(\sigma) > 0$ so that

$$(4.49) \quad \|(x, \tau)\| \leq M(\sigma),$$

where (x, τ, σ, p) is the solution associated with σ in $C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$.

We know from (4.47) and (4.48) that for every $n \in \mathbb{N}$ and every $(x, \tau, \sigma, p) \in C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$, we have

$$(4.50) \quad 0 < p < -\frac{k_0 \ln(2h_0 - 1)}{2}.$$

Therefore the projection of $C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$ on the parameter space \mathbb{R} of p is bounded. If Σ is bounded, then by (i), the projection of $C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$ on the space $C(\mathbb{R}; \mathbb{R}^{N+1})$ of (x, τ) is unbounded in the supremum norm. But by the continuity of M on \mathbb{R} and by (4.49), the projection of $C(0, \tau^*, \sigma_n, 2\pi/\beta_n)$ on the space $C(\mathbb{R}; \mathbb{R}^{N+1})$ of (x, τ) is uniformly bounded with respect to $\sigma \in \Sigma$. This is a contradiction and (ii) is proved.

(iii). (iii) follows from (4.43), (4.44), and (4.48) and the proof is complete. \square

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