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Global Continua of Rapidly Oscillating Periodic Solutions of State-Dependent Delay Differential Equations

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Abstract We apply our recently developed global Hopf bifurcation theory to examine global continuation with respect to the parameter for periodic solutions of functional differential equations with state-dependent delay. We give sufficient geometric conditions to ensure the uniform boundedness of periodic solutions, obtain an upper bound of the period of non-constant periodic solutions in a connected component of Hopf bifurcation, and establish the existence of rapidly oscillating periodic solutions.

Keywords Differential equations · State-dependent delay · Hopf bifurcation · Global continuation · Upper bound of period

Mathematics Subject Classification (2000) 34K18 · 46A30

1 Introduction

In this paper, with the aid of the global Hopf bifurcation theory recently established in [7], we examine the global continuation of periodic solutions for the following system of delay differential equations with state-dependent delays:

Dedicated to the 80th birthday of Professor Jack K. Hale.

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$$\begin{cases} \dot{x}(t) = f(x(t),\ x(t-\tau(t)),\ \sigma), \\ \dot{\tau}(t) = g(x(t),\ \tau(t),\ \sigma), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N$, $\tau \in \mathbb{R}$ and $\sigma \in \mathbb{R}$.

For the sake of simplicity, we assume:

(S1) The maps $f\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ and $g\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ni (\gamma_1,\ \gamma_2,\ \sigma) \to g(\gamma_1,\ \gamma_2,\ \sigma) \in \mathbb{R}$ are C^2 (twice continuously differentiable).

(S2) There exist $L > 0$ and $M_g > 0$ such that $-M_g \le g(\gamma_1,\ \gamma_2,\ \sigma) < \frac{L}{L+1}$ for every $\gamma_1 \in \mathbb{R}^N$, $\gamma_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

The normalization $(x,\ \tau)(t) = (y,\ z)(2\pi t/p)$ by the period $p > 0$ of a periodic solution transfers (1.1) into

$$\begin{cases} \dot{y}(t) = \frac{p}{2\pi} f\left(y(t),\ y\left(t - \frac{2\pi}{p} z(t)\right),\ \sigma\right), \\ \dot{z}(t) = \frac{p}{2\pi} g(y(t),\ z(t),\ \sigma). \end{cases} \tag{1.2}$$

So a solution $(x,\ \tau)$ of (1.1) with the parameter σ is p-periodic if and only if $(y,\ z)$ is a 2π-periodic solution of (1.2) with the given σ and p. In what follows, we will say that $(x,\ \tau,\ \sigma,\ p)$ is a p-periodic solution of (1.1) and $(y,\ z,\ \sigma,\ p)$ is a 2π-periodic solution of (1.2). Sometimes we say $(x,\ \tau,\ \sigma)$ is a solution of (1.1).

The stationary solutions of (1.1) are given by solving the system $f(x, x, \sigma) = 0$ and $g(x, \tau, \sigma) = 0$. We assume throughout this paper that the stationary solution of (1.1) at given σ is given by (x_σ, τ_σ) and the mapping $\mathbb{R} \ni \sigma \mapsto (x_\sigma, \tau_\sigma) \in \mathbb{R}^{N+1}$ is continuous.

Freezing the state-dependent delay of the term $y\left(t - \frac{2\pi}{p} z(t)\right)$ in (1.2) at $2\pi\tau_\sigma/p$ and then linearizing the resulting nonlinear system, we obtain the formal linearization of (1.2) at the stationary point $(x_\sigma,\ \tau_\sigma)$ as follows:

$$\begin{aligned} \begin{pmatrix} \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} &= \frac{p}{2\pi} \begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} \begin{pmatrix} y(t) - x_\sigma \\ z(t) - \tau_\sigma \end{pmatrix} \\ &\quad + \frac{p}{2\pi} \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y(t - \frac{2\pi}{p}\tau_\sigma) - x_\sigma \\ z(t - \frac{2\pi}{p}\tau_\sigma) - \tau_\sigma \end{pmatrix}, \end{aligned} \tag{1.3}$$

where

$$\partial_i f(\sigma) = \frac{\partial}{\partial \theta_i} f(\theta_1,\ \theta_2,\ \sigma)\Big|_{\theta_1 = x_\sigma,\, \theta_2 = x_\sigma},$$

$$\partial_i g(\sigma) = \frac{\partial}{\partial \gamma_i} g(\gamma_1,\ \gamma_2,\ \sigma)\Big|_{\gamma_1 = x_\sigma,\, \gamma_2 = \tau_\sigma},$$

for $i = 1,\ 2$.

Let

$$\det\nolimits_{\mathbb{C}} \Delta_{(x_\sigma,\, \tau_\sigma,\, \sigma)}(\lambda) = 0 \tag{1.4}$$

be the characteristic equation of the linear system

$$\begin{pmatrix} \dot{Y}(t) \\ \dot{Z}(t) \end{pmatrix} = \frac{p}{2\pi} \begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} + \frac{p}{2\pi} \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} Y\left(t - \frac{2\pi}{p}\tau_\sigma\right) \\ Z\left(t - \frac{2\pi}{p}\tau_\sigma\right) \end{pmatrix} \tag{1.5}$$

corresponding to (1.3). The stationary solution $(x^*,\ \tau^*,\ \sigma^*) = (x_{\sigma^*},\ \tau_{\sigma^*},\ \sigma^*)$ is said to be a center of (1.1), if (1.4) with $\sigma = \sigma^*$ has a pair of purely imaginary roots $\pm i\beta^*$ with $\beta^* > 0$.

In this case, $p^* = 2\pi/\beta^*$ is called the *virtual period* associated with the center (x^*, τ^*, σ^*). We say that (x^*, τ^*, σ^*) is an *isolated center* if it is the only center in some neighborhood of (x^*, τ^*, σ^*) in $\mathbb{R}^{N+1} \times \mathbb{R}$, that is, $\det_{\mathbb{C}} \Delta_{(x^*, \tau^*, \sigma^*)}(i\beta^*) = 0$ and for $\delta > 0$ sufficiently small,

$$\det \Delta_{(x_\sigma, \tau_\sigma, \sigma)}(i\beta) \neq 0, \tag{1.6}$$

for any $(\sigma, \beta) \in ((\sigma^* - \delta, \sigma^*) \cup (\sigma^*, \sigma^* + \delta)) \times (0, +\infty)$ and $\det \Delta_{(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)}(i\beta) \neq 0$ for any $\beta \in (\beta^* - \delta, \beta^*) \cup (\beta^*, \beta^* + \delta)$.

We can then choose constants $b = b(\sigma^*, \beta^*) > 0$ and $c = c(\sigma^*, \beta^*) > 0$ such that the closure of $\Omega := (0, b) \times (\beta^* - c, \beta^* + c) \subseteq \mathbb{R}^2 \cong \mathbb{C}$ contains no other zero of $\det_{\mathbb{C}} \Delta_{(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)}(\lambda) = 0$. Then we can define the numbers

$$\gamma_\pm(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*, \beta^*) = \deg(\det_{\mathbb{C}} \Delta_{(x_{\sigma^*\pm\delta}, \tau_{\sigma^*\pm\delta}, \sigma^*\pm\delta)}(\cdot), \Omega),$$

where $\deg(\det_{\mathbb{C}} \Delta_{(x_{\sigma^*\pm\delta}, \tau_{\sigma^*\pm\delta}, \sigma^*\pm\delta)}(\cdot), \Omega)$ is the usual Brouwer degree of the determinant $\det_{\mathbb{C}} \Delta_{(x_{\sigma^*\pm\delta}, \tau_{\sigma^*\pm\delta}, \sigma^*\pm\delta)}(\cdot)$ on Ω. The *crossing number* of (x^*, τ^*, σ^*) is defined as

$$\gamma(x^*, \tau^*, \sigma^*, \beta^*) = \gamma_-(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*, \beta^*) - \gamma_+(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*, \beta^*). \tag{1.7}$$

To state the local and global Hopf bifurcation theory developed in [7], we further assume that

(S3) There exists σ_0 so that $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ is a center of the linearized system (1.3), $\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right) f(\theta_1, \theta_2, \sigma)|_{\sigma=\sigma_0, \theta_1=\theta_2=x_{\sigma_0}}$ is non-singular (determinant is nonzero) and

$$\frac{\partial}{\partial\gamma_2} g(\gamma_1, \gamma_2, \sigma)|_{\sigma=\sigma_0, \gamma_1=x_{\sigma_0}, \gamma_2=\tau_{\sigma_0}} \neq 0.$$

(S4) There exist constants $L_f > 0$ and $L_g > 0$ such that

$$|f(\theta_1, \theta_2, \sigma) - f\left(\overline{\theta}_1, \overline{\theta}_2, \sigma\right)| \leq L_f\left(|\theta_1 - \overline{\theta}_1| + |\theta_2 - \overline{\theta}_2|\right)$$
$$|g(\gamma_1, \gamma_2, \sigma) - g\left(\overline{\gamma}_1, \overline{\gamma}_2, \sigma\right)| \leq L_g(|\gamma_1 - \overline{\gamma}_1| + |\gamma_2 - \overline{\gamma}_2|)$$

for all $\theta_1, \theta_2, \overline{\theta}_1, \overline{\theta}_2, \gamma_1, \overline{\gamma}_1 \in \mathbb{R}^N, \gamma_2, \overline{\gamma}_2 \in \mathbb{R}, \sigma \in \mathbb{R}$.

In what follows, we will also write $u = (x, \tau)$ for the state variable. We now recall the local and global Hopf bifurcation theorems in [7].

Theorem 1 *Assume (S1)–(S3) hold. Let $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ be an isolated center of system (1.3) with the virtual period $2\pi/\beta_0$. If the crossing number defined by (1.7) satisfies $\gamma(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0) \neq 0$, then there exists a bifurcation of non-constant periodic solutions of (1.1) near $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$. More precisely, $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ is a Hopf bifurcation point, that is, there exists a sequence $\{(x_n, \tau_n, \sigma_n, \beta_n)\}$ such that $\sigma_n \to \sigma_0$, $\beta_n \to \beta_0$ as $n \to \infty$, and $\lim_{n\to\infty} \|x_n - x_{\sigma_0}\| = 0$, $\lim_{n\to\infty} \|\tau_n - \tau_{\sigma_0}\| = 0$, where*

$$(x_n, \tau_n, \sigma_n) \in C\left(\mathbb{R}; \mathbb{R}^{N+1}\right) \times \mathbb{R}$$

is a non-constant $2\pi/\beta_n$-periodic solution of system (1.1) and $\|\cdot\|$ is the supremum norm for the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^{N+1}$, and from $\mathbb{R}$ to $\mathbb{R}$, respectively.

Let $\mathcal{S}_0$ be the closure of the set of all the non-constant periodic solutions of system (1.1) in the Fuller space

$$C\left(\mathbb{R};\mathbb{R}^{N+1}\right)\times\mathbb{R}^2=\left\{(x,\ \tau,\ \sigma,\ p):(x,\ \tau)\in C\left(\mathbb{R};\mathbb{R}^{N+1}\right),\ (\sigma,\ p)\in\mathbb{R}^2\right\}.$$

Theorem 1 shows that $\gamma(x^*,\ \tau^*,\ \sigma^*,\ \beta^*)\neq 0$ implies that $(x^*,\ \tau^*,\ \sigma^*)$ is a Hopf bifurcation point, namely, there exists a connected component $C(x^*,\ \tau^*,\ \sigma^*,\ p^*)\subseteq\mathcal{S}_0$ of $\mathcal{S}_0$. By the period normalization $(x,\ \tau)(t)=(y,\ z)(2\pi t/p)$, we obtain a connected component $C(y^*,\ z^*,\ \sigma^*,\ p^*)$ in the Fuller space

$$C\left(\mathbb{R}/2\pi;\mathbb{R}^{N+1}\right)\times\mathbb{R}^2=\left\{(y,\ z,\ \sigma,\ p):(y,\ z)\in C\left(\mathbb{R}/2\pi;\mathbb{R}^{N+1}\right),\ (\sigma,\ p)\in\mathbb{R}^2\right\}.$$

The $(y,\ z)$-component of a 2π-periodic solution $(y,\ z,\ \sigma,\ p)$ of (1.2) gives an element in $C\left(\mathbb{R}/2\pi;\mathbb{R}^{N+1}\right)$ and in this sense, we say that $(y,\ z,\ \sigma,\ p)\in C\left(\mathbb{R}/2\pi;\mathbb{R}^{N+1}\right)\times\mathbb{R}^2$.

Theorem 2 *Suppose that system* (1.1) *satisfies* (*S1–S4*). *Let* $\mathscr{M}$ *be the set of constant solutions of the system* (1.2) *and* $\mathcal{S}$ *denote the closure of the set of all non-constant* 2π*-periodic solutions of* (1.2) *in the Fuller space* $C\left(\mathbb{R}/2\pi;\mathbb{R}^{N+1}\right)\times\mathbb{R}^2$. *Assume that all the centers of* (1.3) *are isolated and* $\mathscr{M}$ *is complete. If* $(u_0,\ \sigma_0,\ p_0)=(x_{\sigma_0},\ \tau_{\sigma_0},\ \sigma_0,\ p_0)\in\mathscr{M}$ *is a bifurcation point, then either the connected component* $C(u_0,\ \sigma_0,\ p_0)$ *of the center* $(u_0,\ \sigma_0,\ p_0)$ *in* $\mathcal{S}$ *is unbounded, or*

$$C(u_0,\ \sigma_0,\ p_0)\cap\mathscr{M}=\{(u_0,\ \sigma_0,\ p_0),\ (u_1,\ \sigma_1,\ p_1),\ldots,(u_q,\ \sigma_q,\ p_q)\},$$

where $p_i\in\mathbb{R}_+$, $(u_i,\ \sigma_i,\ p_i)=(x_{\sigma_i},\ \tau_{\sigma_i},\ \sigma_i,\ p_i)\in\mathscr{M}$ *for* $i=0,\ 1,\ 2,\ldots,\ q$. *Moreover, in the latter case, the crossing numbers* $\gamma(u_i,\ \sigma_i,\ 2\pi/p_i)$ *satisfy*

$$\sum_{i=0}^{q}\epsilon_i\gamma(u_i,\ \sigma_i,\ 2\pi/p_i)=0,$$

where $\epsilon_i=$ *sgn* $\det\begin{bmatrix}\partial_1 f(\sigma_i)+\partial_2 f(\sigma_i) & 0\\ \partial_1 g(\sigma_i) & \partial_2 g(\sigma_i)\end{bmatrix}$.

Definition 1 Let $\mathscr{C}$ be a connected component of the closure of all non-constant periodic solutions of (1.1) in the Fuller space $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$. We call $\mathscr{C}$ a continuum of slowly oscillating periodic solutions, if for every $(x,\ \tau,\ \sigma,\ p)\in\mathscr{C}$, there exists $t_0\in\mathbb{R}$ so that $p>\tau(t_0)>0$. Similarly, we call $\mathscr{C}$ a continuum of rapidly oscillating periodic solutions, if for every $(x,\ \tau,\ \sigma,\ p)\in\mathscr{C}$, there exists $t_0\in\mathbb{R}$ so that $0<p<\tau(t_0)$.

Note that the period normalization of a solution $(x,\ \tau,\ \sigma,\ p)$ does not change its norm in the Fuller space. Theorem 2 shows that for a given trivial solution $(x^*,\ \tau^*,\ \sigma^*)$ with the virtual period p^*, the connected component $C\ (x^*,\ \tau^*,\ \sigma^*,\ p^*)$ either has finitely many bifurcation points with the sum of S^1-equivariant degrees being zero or $C\ (x^*,\ \tau^*,\ \sigma^*,\ p^*)$ is unbounded in the Fuller space $C\left(\mathbb{R};\mathbb{R}^{N+1}\right)\times\mathbb{R}^2$. Therefore, if global persistence of periodic solutions when the parameter is far away from the local Hopf bifurcation value σ^* is desired, we should find conditions to ensure that the connected component $C\ (x^*,\ \tau^*,\ \sigma^*,\ p^*)$ of Hopf bifurcation is unbounded in the Fuller space $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$ and $C(x^*,\ \tau^*,\ \sigma^*,\ p^*)$ will not blow up to infinity at any given σ in the norm of the Fuller space $C(\mathbb{R};\ \mathbb{R}^{N+1})\times\mathbb{R}^2$. That is, there exists a continuous function $M:\mathbb{R}\ni\sigma\to M(\sigma)>0$ such that for every $(x,\tau,\sigma,\ p)\in C(x^*,\ \tau^*,\ \sigma^*,\ p^*)$ we have

$$\|(x,\tau,\ p)\|_{C(\mathbb{R};\,\mathbb{R}^{N+1})\times\mathbb{R}}\leq M(\sigma). \tag{1.8}$$

To achieve this goal, we shall give some sufficient geometric conditions ensuring the uniform boundedness of all possible periodic solutions (x, τ, σ) of (1.1), that is, we show that there exists a continuous function $M_1 : \mathbb{R} \ni \sigma \to M_1(\sigma) > 0$ such that for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$ we have

$$\|(x, \tau)\|_{C(\mathbb{R};\mathbb{R}^{N+1})} \leq M_1(\sigma). \tag{1.9}$$

Then we seek for a continuous function $M_2 : \mathbb{R} \ni \sigma \to M_2(\sigma) > 0$ such that for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$ we have

$$|p| \leq M_2(\sigma). \tag{1.10}$$

Remark 1 In Lemma 4.2 of [7] we have shown that if (S2) and (S4) are satisfied, then p has a positive lower bound. In the following, we assume that $p > 0$ for every $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$, or correspondingly $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$.

The main challenge we are encountering in this study is to seek a uniform upper bound of the periods of the periodic solutions in a connected Hopf bifurcation branch. For differential equations with constant delays, Chow and Mallet-Paret [1] and Wu [17] used the idea of excluding periodic solutions of period twice of the constant delay for some types of scalar delay differential equations. Earlier results on bounds of periods for periodic solutions of ordinary differential equations can be found in Diliberto [2], Fuller [3,4], Lau [9], Smith [14] and Schwartzman [15] and the references therein. Recent studies to rule out periodic solutions with large periods for delay differential equations with a constant delay have been linked to circulant matrices in the celebrated paper of Nussbaum [11]. Spectral analysis of circulant matrices have been used in Xia and Wu [18] and in Wei and Li [16] to rule out periodic solutions using either a Liapunov functional approach or a compound matrix technique. See also [12] for additional results on the range of periods of periodic solutions of scalar equations with a constant delay. However, none of the above mentioned results can be applied to obtain a uniform bound for the periods of non-constant periodic solutions of differential equations with a state-dependent delay.

In this paper, we develop a novel approach in obtaining a uniform upper bound of the periods of the periodic solutions in a connected component $C(x^*, \tau^*, \sigma^*, p^*)$. First, for each periodic solution $(x_0, \tau_0, \sigma_0, p_0)$ we exclude certain values of the period. More specifically, we show that this periodic solution satisfies $\tau_0(t_0) \neq mp_0$ for some t_0 and for all $m \in \mathbb{N}$. Then we find an open interval $I \ni t_0$ and a small open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ so that every $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $t \in I$ and $m \in \mathbb{N}$. We then develop a procedure to glue these local exclusions of period values together to obtain a global exclusion of the period values in the component $C(x^*, \tau^*, \sigma^*, p)$.

We organize the remaining part of the paper as follows. In Sect. 2, we construct a monotonically increasing sequence of connected subsets $\{A_n\}_{n=1}^{+\infty}$ of $C(y^*, z^*, \sigma^*, p^*)$ which, combined with the uniform boundedness of the solutions (x, τ) and the global exclusion of the period values, provides an upper bound of the period, p, for rapidly oscillating periodic solutions of (1.1). In Sect. 3 we follow the idea of Gustafson and Schmit [5] to obtain sufficient geometric conditions for the uniform boundedness of all the periodic solutions of (1.1), based on a Razumikhin-type technique. In Sect. 4, we present a detailed case study to illustrate our general results.

2 Uniform Bounds for Periods of Periodic Solutions in a Connected Component

We will need the following assumptions to derive some important properties of an interval map.

(S5) For every $(\sigma, \tau) \in \mathbb{R}^2$, $\frac{\partial g}{\partial \tau}(x_\sigma, \tau, \sigma) \neq 0$.
(S6) $\frac{\partial g}{\partial x}(x, \tau, \sigma) f(x, x, \sigma) \neq 0$ for $(x, \tau, \sigma) \in \mathbb{R}^{N+1} \times \mathbb{R}$ such that $x \neq x_\sigma$ and $g(x, \tau, \sigma) = 0$.

2.1 Properties of an Interval Map

In order, by way of contradiction, to exclude certain values of the period of the periodic solutions in a given connected component, we need some analytic properties of the following map

$$l(t) = t - \tau(t) + \tau(t_0),$$

where $t_0 \in \mathbb{R}$ is fixed. In this section, to avoid notational complications, we use superscripts to denote function compositions, e.g., $l^j(t)$ denotes the j-th composition of l evaluated at time t.

Lemma 1 *Suppose that* (1.1) *satisfies* (*S1, S2, S5, S6*) *and* (x, τ, σ_0) *is a non-constant periodic solution of* (1.1)*. If* (x, τ) *is* $\tau(t_0)$*-periodic and if* $\tau(t_0) \neq \tau_{\sigma_0}$*, then the function* $l(t) = t - \tau(t) + \tau(t_0)$ *defined on* $[t_0, t_0 + \tau(t_0)]$ *satisfies the following properties:*

(a). $l(t)$ *is a self-mapping on* $[t_0, t_0 + \tau(t_0)]$;
(b). $l(t)$ *has only finitely many fixed points* $\{t_i\}_{i=1}^n$ *in* $[t_0, t_0 + \tau(t_0)]$ *with* $t_i < t_{i+1}$ *for every* $i \in \{1, 2, \ldots, n-1\}$;
(c). *For every* $t \in (t_i, t_{i+1}) \subseteq [t_0, t_0 + \tau(t_0)]$,

$$\lim_{j\to+\infty} l^j(t) = \begin{cases} t_i, & if\ there\ exists\ \bar{t} \in [t_i, t_{i+1}] such\ that\ \bar{t} > l(\bar{t}), \\ t_{i+1}, & if\ there\ exists\ \bar{t} \in [t_i, t_{i+1}] such\ that\ \bar{t} < l(\bar{t}); \end{cases}$$

(d). *Let* $\{t_{i_k}\}_{k=1}^{k_0} \subseteq \{t_i\}_{i=1}^n$ *be all the fixed points such that* $\lim_{j\to+\infty} l^j(t) = t_{i_k}$ *for every* $t \in [t_{i_k}, t_{i_k+1})$*. Then for* $\delta > 0$ *small enough*

$$\lim_{j\to+\infty} \sup_{t\in[t_{i_k}, t_{i_k+1}-\delta]} |l^j(t) - t_{i_k}| = 0,$$

$$\lim_{j\to+\infty} \sup_{t\in[t_i+\delta, t_{i+1}],\, t_i\in\{t_1, t_2, \ldots, t_n\}\setminus\{t_{i_k}\}_{k=1}^{k_0}} |l^j(t) - t_{i+1}| = 0;$$

(e). *Let* $h(t) = t - \tau(t)$*, then* $l^j(t) = h^j(t) + j\tau(t_0)$ *for every* $t \in [t_0, t_0 + \tau(t_0)]$ *and* $j \in \mathbb{N}$;
(f). $h^j(t + \tau(t_0)) = h^j(t) + \tau(t_0)$ *for all* $t \in \mathbb{R}$ *and* $j \in \mathbb{N}$.

Proof (a). Note that by (S2), $\dot{l}(t) = 1 - \dot{\tau}(t) = 1 - g(x(t), \tau(t), \sigma) > 1/(L+1) > 0$ implies that $l(t)$ is strictly increasing on $[t_0, t_0 + \tau(t_0)]$. Also, $\tau(t)$ is $\tau(t_0)$-periodic implies that $l(t_0) = t_0$ and $l(t_0 + \tau(t_0)) = t_0 + \tau(t_0)$. Therefore, $l(t)$ is a self-mapping on $[t_0, t_0 + \tau(t_0)]$.

(b). Next, we show that $l(t)$ has only finitely many fixed points in $[t_0, t_0 + \tau(t_0)]$. That is, $l(t) = t - \tau(t) + \tau(t_0) = t \Leftrightarrow \tau(t) = \tau(t_0)$ has only finitely many solutions in $[t_0, t_0 + \tau(t_0)]$.

Indeed, suppose that $t_f \in [t_0, t_0 + \tau(t_0)]$ is a fixed point of $l(t)$, that is, $\tau(t_f) = \tau(t_0)$, we then have three possible cases:

Case 1. $\dot{\tau}(t_f) > 0$. By the continuity of $\dot{\tau}(t)$, there exists an open interval $I_f \ni t_f$ such that $\dot{\tau}(t) > 0$ for all $t \in I_f$. Then, by the strict monotonicity of $\tau(t)$ on I_f, t_f is an isolated fixed point of $l(t)$.

Case 2. $\dot{\tau}(t_f) < 0$. We can similarly show that t_f is an isolated fixed point of $l(t)$.

Case 3. $\dot{\tau}(t_f) = 0$. We have

$$g(x(t_f),\ \tau(t_f),\ \sigma_0) = g(x(t_f),\ \tau(t_0),\ \sigma_0) = 0. \tag{2.2}$$

By assumption (S5), we have $x(t_f) \neq x_{\sigma_0}$. By (1.1) and (S1), we have

$$\begin{aligned}\ddot{\tau}(t_f) &= \frac{\partial g}{\partial x}(x(t_f),\ \tau(t_f),\ \sigma_0)\,\dot{x}(t_f) + \frac{\partial g}{\partial \tau}(x(t_f),\ \tau(t_f),\ \sigma_0)\,\dot{\tau}(t_f)\\ &= \frac{\partial g}{\partial x}(x(t_f),\ \tau(t_0),\ \sigma_0)\,f(x(t_f),\ x(t_f),\ \sigma_0).\end{aligned} \tag{2.3}$$

Then by (2.2), (2.3) and assumption (S6), we have $\ddot{\tau}(t_f) \neq 0$. Without loss of generality, we assume $\ddot{\tau}(t_f) > 0$. By the continuity of $\ddot{\tau}(t)$ and the fact that $\dot{\tau}(t_f) = 0$, there exists $\delta > 0$ small enough so that $\dot{\tau}(t) < 0$ on $(t_f - \delta,\ t_f)$ and $\dot{\tau}(t) > 0$ on $(t_f,\ t_f + \delta)$. Therefore, t_f is the unique zero of $\tau(t) - \tau(t_0)$ in $(t_f - \delta,\ t_f + \delta)$, and hence is an isolated zero.

It follows from the isolatedness of t_f and the compactness of the interval $[t_0,\ t_0 + \tau(t_0)]$ that l has only finite number of fixed points in $[t_0,\ t_0 + \tau(t_0)]$.

(c) Since by (b) the number of fixed points of l in $[t_0,\ t_0 + \tau(t_0)]$ is finite, we can order all of them as

$$t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_n = t_0 + \tau(t_0).$$

Then for every $t \in [t_0,\ t_0 + \tau(t_0)] \setminus \{t_i\}_{i=0}^n$, there exists $i \in \{0,\ 1,\ \ldots n-1\}$ such that $t_i < t < t_{i+1}$. It is clear that $l(t) \neq t$ and that $l|_{[t_i,\, t_{i+1}]}$, $i \in \{0,\ 1,\ \ldots,\ n-1\}$, is a strictly increasing self-mapping on $[t_i,\ t_{i+1}]$ with t_i and t_{i+1} the only fixed points.

If there exists $\bar{t} \in [t_i,\ t_{i+1}]$ such that $\bar{t} > l(\bar{t})$, then $t > l(t)$ for every $t \in (t_i,\ t_{i+1})$. Otherwise, by the continuity of l there exists another fixed point of l in $(t_i,\ t_{i+1})$, which is impossible. Therefore, for every $t \in (t_i,\ t_{i+1})$ we have,

$$t_{i+1} > t > l(t) > l^2(t) > \cdots > l^j(t) > l^{j+1}(t) > \cdots > t_i.$$

Therefore, $\{l^j(t)\}_{j=1}^{\infty}$ is a strictly decreasing sequence with the lower bound t_i. Then there exists $t^* \in [t_i,\ t_{i+1})$ such that $\lim_{j\to+\infty} l^j(t) = t^*$, and hence $l(t^*) = t^*$. Since t_i is the only fixed point of l in $[t_i,\ t_{i+1})$, $t^* = t_i$.

Similarly, if there exists $\bar{t} \in [t_i,\ t_{i+1}]$ such that $\bar{t} < l(\bar{t})$, then $t < l(t)$ for every $t \in (t_i,\ t_{i+1})$. For, otherwise, by the continuity of l there exists another fixed point of l in $(t_i,\ t_{i+1})$, which is impossible. Therefore, for every $t \in (t_i,\ t_{i+1})$, we have

$$t_i < t < l(t) < l^2(t) < \cdots < l^j(t) < l^{j+1}(t) < \cdots < t_{i+1}.$$

That is, $\{l^j(t)\}_{j=1}^{\infty}$ is a strictly increasing sequence with the upper bound t_{i+1}. Then, there exists $t^* \in (t_i,\ t_{i+1}]$ such that $\lim_{j\to+\infty} l^j(t) = t^*$, and hence $l(t^*) = t^*$. Since t_{i+1} is the only fixed point of $l(t)$ in $(t_i,\ t_{i+1}]$, $t^* = t_{i+1}$.

(d) Note that $l|_{[t_i,\, t_{i+1}]}$ is a strictly increasing self-mapping on $[t_i,\ t_{i+1}]$, where $\{t_i\}_{i=1}^n$ is the set of fixed points of l. Therefore, l^j is also a strictly increasing self-mapping on $[t_i,\ t_{i+1}]$ for every $j \in \mathbb{N}$. Then, by (c), we have

$$\lim_{j\to+\infty}\sup_{t\in[t_{i_k},\, t_{i_k+1}-\delta]}|l^j(t)-t_{i_k}| = \lim_{j\to+\infty}|l^j(t_{i_k+1}-\delta)-t_{i_k}| = 0,$$

$$\lim_{j\to+\infty}\sup_{t\in[t_i+\delta,\, t_{i+1}],\, t_i\in\{t_1,t_2,\ldots,t_n\}\backslash\{t_{i_k}\}_{k=1}^{k_0}}|l^j(t)-t_{i+1}| = \lim_{j\to+\infty}|l^j(t_i+\delta)-t_{i+1}| = 0.$$

(e) We prove (e) by mathematical induction. It is clear by the definitions of $h(t)$ and $l(t)$ that $l(t) = h(t) + \tau(t_0)$. That is, $l^j(t) = h^j(t) + j\tau(t_0)$ holds for $j = 1$. Suppose, this holds for $j = k$. Then, for every $t \in [t_0,\ t_0 + \tau(t_0)]$ we have,

$$\begin{aligned} l^{k+1}(t) &= l(l^k(t)) \\ &= l(h^k(t) + k\tau(t_0)) \\ &= h(h^k(t) + k\tau(t_0)) + \tau(t_0) \\ &= h^k(t) + k\tau(t_0) - \tau(h^k(t) + k\tau(t_0)) + \tau(t_0) \\ &= h^k(t) - \tau(h^k(t)) + (k+1)\tau(t_0) \\ &= h^{k+1}(t) + (k+1)\tau(t_0). \end{aligned}$$

That is, $l^j(t) = h^j(t) + j\tau(t_0)$ holds for $j = k+1$. By mathematical induction, $l^j(t) = h^j(t) + j\tau(t_0)$ holds for every $j \in \mathbb{N}$ and $t \in [t_0,\ t_0 + \tau(t_0)]$.

(f) By the definition of $h(t)$ and the assumption that $\tau(t)$ is $\tau(t_0)$-periodic, it is clear that for every $t \in \mathbb{R}$,

$$\begin{aligned} h(t + \tau(t_0)) &= t + \tau(t_0) - \tau(t + \tau(t_0)) \\ &= t + \tau(t_0) - \tau(t)) \\ &= h(t) + \tau(t_0). \end{aligned}$$

That is, $h^j(t + \tau(t_0)) = h^j(t) + \tau(t_0)$ holds for $j = 1$. Suppose this holds for $j = k$. Then for every $t \in [t_0,\ t_0 + \tau(t_0)]$, we have

$$\begin{aligned} h^{k+1}(t + \tau(t_0)) &= h(h^k(t + \tau(t_0))) \\ &= h^k(t + \tau(t_0)) - \tau(h^k(t + \tau(t_0))) \\ &= h^k(t) + \tau(t_0) - \tau(h^k(t) + \tau(t_0)) \\ &= h^k(t) + \tau(t_0) - \tau(h^k(t)) \\ &= h^{k+1}(t) + \tau(t_0). \end{aligned}$$

That is, $h^j(t + \tau(t_0)) = h^j(t) + \tau(t_0)$ holds for $j = k+1$. By mathematical induction, $h^j(t + \tau(t_0)) = h^j(t) + \tau(t_0)$ holds for every $j \in \mathbb{N}$ and $t \in \mathbb{R}$. This completes the proof. □

2.2 Excluding Certain Periods: Locally

Recall that $C(x^*,\ \tau^*,\ \sigma^*,\ p^*)$ denotes the connected component of the closure of all the non-constant periodic solutions of system (1.1) bifurcated at $(x^*,\ \tau^*,\ \sigma^*,\ p^*)$ in the Fuller space $C(\mathbb{R};\ \mathbb{R}^{N+1}) \times \mathbb{R}^2$. In this subsection we exclude, for each periodic solution $(x_0, \tau_0, \sigma_0,\ p_0)$ certain values of the period. To be specific, we find an open interval I and a small open neighborhood $U \ni (x_0, \tau_0, \sigma_0,\ p_0)$ so that every $(x, \tau, \sigma,\ p) \in U \cap C(x^*,\ \tau^*,\ \sigma^*,\ p^*)$ satisfies $\tau(t) \neq mp$ for all $t \in I$ and $m \in \mathbb{N}$. In the next subsection, we will glue up these local exclusions to a global upper bound for the period along the rescaled (by period normalization) connected component $C(y^*, z^*, \sigma^*, p^*)$.

Now we consider the periods of the solutions in the neighborhood of a periodic solution which does not assume a certain period.

Lemma 2 *If a solution* $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ *satisfies* $\tau_0(t_0) \neq mp_0$ *for some* $t_0 \in \mathbb{R}$ *and for all* $m \in \mathbb{N}$*, then there exist an open neighborhood* $I \ni t_0$ *and an open neighborhood* $U \ni (x_0, \tau_0, \sigma_0, p_0)$ *in* $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ *such that every solution* $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ *satisfies* $\tau(t) \neq mp$ *for all* $m \in \mathbb{N}$ *and* $t \in I$.

Proof By way of contradiction, we suppose that for every open interval $I \ni t_0$ and every open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$, there exist $t \in I$, $m \in \mathbb{N}$ and a periodic solution $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ so that $\tau(t) = mp$. Then there exist sequences $\{(x_k, \tau_k, \sigma_k, p_k, t_k)\}_{k=1}^{+\infty} \subseteq U \cap C(x^*, \tau^*, \sigma^*, p^*)$ and $\{m_k : m_k \in \mathbb{N}\}_{k=1}^{+\infty}$ such that

$$\begin{cases} \tau_k(t_k) = m_k p_k, \\ \lim_{k\to+\infty} (x_k, \tau_k, \sigma_k, p_k, t_k) = (x_0, \tau_0, \sigma_0, p_0, t_0). \end{cases} \tag{2.4}$$

Without loss of generality, we assume $m_k \to m_0 \in \mathbb{N}$ as $k \to +\infty$ (otherwise we take a subsequence). Then it follows from (2.4) and (S2) that

$$m_0 = \lim_{k\to+\infty} m_k = \lim_{k\to+\infty} \frac{\tau_k(t_k)}{p_k} = \frac{\tau_0(t_0)}{p_0}. \tag{2.5}$$

Therefore, we have $\tau_0(t_0) = m_0 p_0$ which is a contradiction to the assumption. □

We note that for a non-constant periodic solution (x, τ, σ) of system (1.1), it is allowed that $\tau(t)$ assume its stationary value τ_σ, or even $\tau(t) = \tau_\sigma$ for all $t \in \mathbb{R}$. Ruling out these cases turns out to be crucial for us to exclude certain values of periods of the periodic solutions.

We first consider the periods of the periodic solutions in a neighborhood of a given non-constant periodic solution in the Fuller space, for which the delay τ-component is not equal to the corresponding stationary value at some time t.

We need the following condition:

(S7) (i) $f(0, 0, \sigma) = 0$ for all $\sigma \in \mathbb{R}$;
(ii) $xf(x, x, \sigma)$ is positive (or negative) if $f(x, x, \sigma) \neq 0$.

Theorem 3 *Suppose that system* (1.1) *satisfies* (*S5–S7*)*. Let* $(x_0, \tau_0, \sigma_0, p_0)$ *be a nonconstant periodic solution in* $C(x^*, \tau^*, \sigma^*, p^*)$*. If* $\tau_0(t_0) \neq \tau_{\sigma_0}$ *for some* t_0*, then there exist an open interval* I *and an open neighborhood* U *of* $(x_0, \tau_0, \sigma_0, p_0)$ *in* $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ *such that every solution* (x, τ, σ, p) *in* $U \cap C(x^*, \tau^*, \sigma^*, p^*)$ *satisfies* $\tau(t) \neq mp$ *for all* $m \in \mathbb{N}$ *and* $t \in I$.

Proof We first show that there exist an open neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$ and an open neighborhood I_0 of t_0 such that $\tau(t) \neq \tau_{\sigma_0}$ for any $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ and $t \in I_0$.

By way of contradiction, suppose for every neighborhood $\tilde{I}$ of t_0 and neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$, there exist $t \in \tilde{I}$ and a non-constant solution $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ such that $\tau(t) = \tau_{\sigma_0}$. Then there exist a sequence of periodic solutions $\{(x_k, \tau_k, \sigma_k, p_k)\}_{k=1}^{+\infty}$ and $\{t_k\}_{k=1}^{+\infty}$ such that

$$\begin{cases} \tau_k(t_k) = \tau_{\sigma_k}, \\ \lim_{k\to+\infty} (x_k, \tau_k, \sigma_k, p_k, t_k) = (x_0, \tau_0, \sigma_0, p_0, t_0). \end{cases} \tag{2.6}$$

Then it follows from (2.6) and (S2) that

$$\begin{aligned} |\tau_k(t_k) - \tau_0(t_0)| &\le |\tau_k(t_k) - \tau_k(t_0)| + |\tau_k(t_0) - \tau_0(t_0)| \\ &\le |t_k - t_0| + \sup_{t\in\mathbb{R}} \|\tau_k - \tau_0\| \\ &\to 0 \ \text{ as } k \to +\infty. \end{aligned} \tag{2.7}$$

Therefore, by (2.6) and (2.7) we have

$$\tau_0(t_0) = \lim_{k\to+\infty} \tau_k(t_k) = \lim_{k\to+\infty} \tau_{\sigma_k} = \tau_{\sigma_0}.$$

This is a contradiction to the assumption that $\tau_0(t_0) \neq \tau_{\sigma_0}$, and hence the claim is proved.

If $(x_0, \tau_0, \sigma_0, p_0)$ satisfies $\tau_0(t_0) \neq mp_0$ for all $m \in \mathbb{N}$, then the existence of I and U is followed from Lemma 2. Otherwise, $(x_0, \tau_0, \sigma_0, p_0)$ is $\tau_0(t_0)$-periodic. Let Γ_{σ_0} be the nonempty solution set of the equation $f(x, x, \sigma_0) = 0$ for $x \in \mathbb{R}^N$. Then by (S5), for every $x \in \Gamma_{\sigma_0}$, τ_{σ_0} is the unique solution of $g(x, \tau, \sigma_0) = 0$ for $\tau \in \mathbb{R}$. Now we distinguish two cases:

Case 1. $x_0(t_0) = x_{\sigma_0}$ for some $x_{\sigma_0} \in \Gamma_{\sigma_0}$. Since $\tau_0(t_0) \neq \tau_{\sigma_0}$, by system (1.1) and by (S5), we have

$$\begin{cases} \dot{x}_0(t_0) = f(x_{\sigma_0}, x_{\sigma_0}, \sigma_0) = 0, \\ \dot{\tau}_0(t_0) = g(x_{\sigma_0}, \tau_0(t_0), \sigma_0) \neq 0. \end{cases} \tag{2.8}$$

Without loss of generality, we suppose $\dot{\tau}_0(t) > 0$ for t in some open neighborhood of t_0. Then, by the continuity and local monotonicity of $\tau_0(t)$, there exists $\delta > 0$ small enough so that

$$0 < \tau_0(t) - \tau_0(t_0) < p_{\min}, \ t \in (t_0, t_0 + \delta),$$

where $p_{\min} > 0$ is the minimal period of (x_0, τ_0). Then, $\tau_0(t) \neq m\, p_{\min}$ for any $m \in \mathbb{N}$. Therefore, (x_0, τ_0) is not $\tau_0(t)$-periodic for all $t \in (t_0, t_0 + \delta)$. So we have $\tau_0(t) \neq mp_0$ for all $t \in (t_0, t_0 + \delta)$ and $m \in \mathbb{N}$.

By Lemma 2, for every $t^* \in (t_0, t_0 + \delta)$, there exist an open interval I of t^* and an open neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that every solution (x, τ, σ, p) in $U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I$.

Case 2. $x_0(t_0) \neq x_\sigma$ for every $x_\sigma \in \Gamma_{\sigma_0}$. By Lemma 1 (c), there are finitely many fixed points $\{t_i\}_{i=1}^n$ of $l(t) = t - \tau_0(t) + \tau_0(t_0)$ in $[t_0, t_0 + \tau_0(t_0)]$ which are in ascending order (we assume in the proof that all the sequences of the fixed points of l are in ascending order). And we can let the subsequence $\{t_{i_k}\}_{k=1}^{k_0} \subseteq \{t_i\}_{i=1}^n$ be all the fixed points such that $\lim_{j\to+\infty} l^j(t) = t_{i_k}$ for every $t \in [t_{i_k}, t_{i_k+1})$. Note that $\tau_0(t_i) = \tau_0(t_0)$ and $\tau_0(t_0) \neq \tau_{\sigma_0}$ implies that $\tau_0(t_i) \neq \tau_{\sigma_0}$ for all $i \in \{1, 2, \ldots, n\}$. If $x_0(t_{i_0}) = x_{\sigma_0}$ for some $i_0 \in \{1, 2, \ldots, n\}$ and for some $x_{\sigma_0} \in \Gamma_{\sigma_0}$, then the conclusion follows by Case 1 with t_0 replaced by t_{i_0}.

Now we exclude that $x_0(t_i) \neq x_\sigma$ for every $i \in \{0, 1, 2, \ldots, n\}$ and for every $x_\sigma \in \Gamma_{\sigma_0}$. Assume that the contrary is true. We want to obtain a contradiction under the assumption that (x_0, τ_0) is $\tau_0(t_0)$-periodic.

For $\delta > 0$ small enough, we consider the following compact subset I_δ of $[t_0, t_0 + \tau_0(t_0)]$:

$$I_\delta = \bigcup_{t_{i_k}\in\{t_{i_1}, t_{i_2}, \ldots, t_{i_{k_0}}\}} [t_{i_k}, t_{i_k+1} - \delta] \bigcup \bigcup_{t_i\in\{t_1, t_2, \ldots, t_n\}\setminus\{t_{i_k}\}_{k=1}^{k_0}} [t_i + \delta, t_i].$$

Note that, for each interval $[t_i, t_{i+1}]$, only one of the endpoints is the limit of $\lim_{j\to+\infty} l^j(t)$ for every $t \in (t_i, t_{i+1})$. Note also that when δ goes to zero, I_δ goes to $[t_0, t_0 + \tau_0(t_0)]$ in the sense of Lebesgue measure.

Now for $\delta > 0$ small enough we introduce the following piecewise constant function $\chi(t)$ on the compact subset I_δ of $[t_0, t_0 + \tau_0(t_0)]$:

$$\chi(t) = \begin{cases} t_{i_k}, & \text{if } t \in [t_{i_k}, t_{i_k+1} - \delta],\ t_{i_k} \in \{t_{i_k}\}_{k=1}^{k_0}, \\ t_{i+1}, & \text{if } t \in [t_i + \delta, t_{i+1}], t_i \in \{t_1, t_2, \dots, t_n\} \setminus \{t_{i_k}\}_{k=1}^{k_0}. \end{cases}$$

Since the number of intervals with the end points being the fixed points of $l(t)$ is finite, it is clear from Lemma 1 (d) that

$$\lim_{j\to+\infty} \sup_{t\in I_\delta} |l^j(t) - \chi(t)| = 0. \tag{2.10}$$

Note that $(x(t), \tau(t))$ is a periodic solution of system (1.1). There exists $\widetilde{M} > 0$ such that $|\dot{x}(t)| \le \widetilde{M}$ for every $t \in [t_0, t_0 + \tau(t_0)]$. Let I_i with $i \in \{1, 2, \dots, n\}$ be the sub-interval of I_δ which is either $[t_{i-1}, t_i - \delta]$ or $[t_{i-1} + \delta, t_i]$. Then we have $\chi(t) = t_{i-1}$ or $\chi(t) = t_i$ for $t \in I_i$ and hence we have

$$x_0(\chi(t)) = x_0(t_{i-1}) \text{ or } x_0(\chi(t)) = x_0(t_i) \text{ for every } t \in I_i. \tag{2.11}$$

Since $x_0(t_i) \ne x_\sigma$ for every $i \in \{0, 1, 2, \dots, n\}$ and for every $x_\sigma \in \Gamma_{\sigma_0}$, by (2.11), we have

$$x_0(\chi(t)) \notin \Gamma_{\sigma_0} \text{ for every } t \in I_\delta. \tag{2.12}$$

By (2.10), for every $\epsilon > 0$, there exists $N_0 > 0$ large enough so that

$$\sup_{t\in I_\delta} |l^j(t) - \chi(t)| \le \epsilon, \quad \text{for every } j > N_0. \tag{2.13}$$

Let $(x_j(t), \tau_j(t)) = (x_0(h^j(t)), \tau_0(h^j(t))$ for $j = 0, 1, 2, \dots,$ where we define $h^0(t) = t$. Then by Lemma 1 (e), we have $(x_j(t), \tau_j(t)) = (x_0(l^j(t)), \tau_0(l^j(t))$. Note that I_δ is composed of finitely many sub-intervals. By applying the Integral Mean Value Theorem on each sub-interval of I_δ and by (2.13), we have for every $j > N_0$ that

$$\sup_{t\in I_\delta} |x_0(l^j(t)) - x_0(\chi(t))| \le \sup_{t\in I_\delta} |\dot{x}_0(t)| \sup_{t\in I_\delta} |l^j(t) - \chi(t)| \le \widetilde{M}\epsilon. \tag{2.14}$$

Differentiating $x_j(t)$ for $j = 1, 2, \dots$, we can obtain from system (1.1) that

$$\dot{x}_j(t) = \prod_{m=0}^{j-1} (1 - g(x_m(t), \tau_m(t)), \sigma_0) f(x_j(t), x_{j+1}(t), \sigma_0). \tag{2.15}$$

As $g(x, \tau, \sigma) < 1$, we have

$$\prod_{m=0}^{j-1} (1 - g(x_m(t), \tau_m(t)), \sigma_0) > 0,\ t \in \mathbb{R}. \tag{2.16}$$

Also by (ii) of (S7), $xf(x, x, \sigma_0) > 0$ as long as $x \notin \Gamma_{\sigma_0}$. Then by (2.12) we have

$$x_0(\chi(t)) f(x_0(\chi(t)), x_0(\chi(t)), \sigma_0) > 0 \tag{2.17}$$

for every $t \in I_\delta$. By (2.14), (2.17) and by the continuity of f, it follows that there exists $N_1 > N_0$ so that

$$x_j(t) f(x_j(t), x_{j+1}(t), \sigma_0) > 0 \text{ for } j > N_1 \text{ and } t \in I_\delta. \tag{2.18}$$

Therefore, for every $t \in I_\delta$ and $j > N_1$, by (2.15), (2.16) and (2.18) we have

$$x_j(t) \cdot \dot{x}_j(t) = \prod_{m=0}^{j-1} (1 - g(x_m(t), \tau_m(t)), \sigma_0) x_j(t) f(x_j(t), x_{j+1}(t), \sigma_0) > 0. \quad (2.19)$$

Since $\delta > 0$ is arbitrary and I_δ goes to I in measure as $\delta \to 0$, by the continuity of $x_j \cdot \dot{x}_j$, we have $x_j(t) \cdot \dot{x}_j(t) \geq 0$ for every $t \in I$ and $j > N_1$. By (2.19) we know that $x_j \cdot \dot{x}_j \not\equiv 0$ on I with $j > N_1$. Therefore, $x_j \cdot x_j$ is a non-constant increasing continuous function. But this is impossible since $x_j \cdot x_j$ is continuous and periodic. This completes the proof. □

We now consider the periods of non-constant periodic solutions, where the delay coincides with the corresponding stationary value for every $t \in \mathbb{R}$.

Lemma 3 *Suppose system (1.1) satisfies (S6). Let (x, τ, σ, p) be a non-constant p-periodic solution of system (1.1). If $\tau(t) = \tau_\sigma$ for every $t \in \mathbb{R}$, then (x, τ, σ, p) is not τ_σ-periodic.*

Proof Suppose, by way of contradiction, that (x, τ, σ, p) is τ_σ-periodic. If $\tau(t) = \tau_\sigma$ for every $t \in \mathbb{R}$, then we have

$$\begin{cases} \dot{x}(t) = f(x(t), x(t), \sigma), \\ 0 = \dot{\tau}(t) = g(x(t), \tau_\sigma, \sigma). \end{cases} \quad (2.20)$$

It follows from (2.20) that

$$\ddot{\tau}(t) = \frac{\partial g}{\partial x}(x(t), \tau_\sigma, \sigma) \cdot f(x(t), x(t), \sigma) = 0. \quad (2.21)$$

Then by (S6) and (2.21), $x(t) = x_\sigma$ for every $t \in \mathbb{R}$. Thus, (x, τ, σ, p) is a constant periodic solution of (1.1). This is a contradiction. □

We now formulate the next assumption:

(S8) For every Hopf bifurcation point $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$, $mp \neq \tau$ for any $m \in \mathbb{N}$.

Theorem 4 *Assume that system (1.1) satisfies (S5–S8). Then for every solution $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$, there exist an open interval I and an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ such that every solution*

$$(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$$

satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I$.

Proof For a given $\sigma_0 \in \mathbb{R}$, if $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ is a constant periodic solution, then it is a Hopf bifurcation point of system (1.1) (See Lemma 4.3 of [7]). Thus the existence of an open interval I and an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ follows immediately from (S8) and Lemma 2.

If $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ is a non-constant periodic solution and $\tau_0(t) = \tau_{\sigma_0}$ for all $t \in \mathbb{R}$, then by Lemma 3, $(x_0, \tau_0, \sigma_0, p_0)$ is not τ_{σ_0}-periodic. The conclusion is implied by Lemma 2.

If $(x_0, \tau_0, \sigma_0, p_0)$ is a non-constant periodic solution and $\tau_0(t) \neq \tau_{\sigma_0}$ for some $t \in \mathbb{R}$. The conclusion follows from Theorem 3. □

2.3 Uniform Boundedness of Periods: Globally

We now start the process that uses the local exclusion of periods developed in the last subsection to construct a uniform upper bound for periods of solutions in the Fuller space. To achieve this goal, we need to "glue" the local exclusion of periods along the connected component.

We will show in the next section the validness of (1.9). The purpose of this subsection is to show that (1.10) is valid provided that (1.9) holds.

Theorem 5 *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the non-constant periodic solutions of system (1.2), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi;\ \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that system (1.1) satisfies (S5–S8). Then for every $(y_0, z_0, \sigma_0, p_0) \in C(y^*, z^*, \sigma^*, p^*)$, there exist an open interval I and an open neighborhood $U \ni (y_0, z_0, \sigma_0, p_0)$ such that $mp \neq z(t)$ for every solution $(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$, $m \in \mathbb{N}$ and $t \in I$.*

Proof Note that $p > 0$ for every solution (y, z, σ, p) in $C(y^*, z^*, \sigma^*, p^*)$. We show that the mapping

$$\begin{aligned} \iota : C(y^*, z^*, \sigma^*, p^*) &\to C(x^*, \tau^*, \sigma^*, p^*) \\ (y(\cdot), z(\cdot), \sigma, p) &\to \left(y\left(\frac{2\pi}{p}\cdot\right), z\left(\frac{2\pi}{p}\cdot\right), \sigma, p \right) \end{aligned} \tag{2.22}$$

is continuous, where $C(x^*, \tau^*, \sigma^*, p^*) \subseteq C(\mathbb{R};\ \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Indeed, if

$$\lim_{n\to+\infty} \|(y_n(\cdot), z_n(\cdot), \sigma_n, p_n) - (y_0(\cdot), z_0(\cdot), \sigma_0, p_0)\|_{C(\mathbb{R}/2\pi;\ \mathbb{R}^{N+1})\times\mathbb{R}^2} = 0,$$

then we have

$$\begin{aligned} &\|\iota(y_n(\cdot), z_n(\cdot), \sigma_n, p_n) - \iota(y_0(\cdot), z_0(\cdot), \sigma_0, p_0)\|_{C(\mathbb{R};\ \mathbb{R}^{N+1})\times\mathbb{R}} \\ &= \left|y_n\left(\frac{2\pi}{p_n}\cdot\right) - y_0\left(\frac{2\pi}{p_0}\cdot\right)\right|_C + \left|z_n\left(\frac{2\pi}{p_n}\cdot\right) - z_0\left(\frac{2\pi}{p_0}\cdot\right)\right|_C \\ &\quad + |\sigma_n - \sigma_0| + |p_n - p_0| \\ &\leq |y_n - y_0|_C + 2\pi|\dot{y}_0| \left|\frac{1}{p_n} - \frac{1}{p_0}\right| + |z_n - z_0|_C + 2\pi|\dot{z}_0| \left|\frac{1}{p_n} - \frac{1}{p_0}\right| \\ &\quad + |\sigma_n - \sigma_0| + |p_n - p_0| \\ &\to 0 \text{ as } n \to +\infty, \end{aligned}$$

where $|\cdot|_C$ denotes the supremum norm in either $C(\mathbb{R}/2\pi;\ \mathbb{R}^N)$ or $C(\mathbb{R}/2\pi;\ \mathbb{R})$. Therefore, $C(x^*, \tau^*, \sigma^*, p^*)$ is a connected component of periodic solutions of (1.1).

Let $(x_0, \tau_0, \sigma_0, p_0) = \iota(y_0, z_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$. Then, by Theorem 4, there exist an open interval I' and an open neighborhood $U' \ni (x_0, \tau_0, \sigma_0, p_0)$ such that every solution $(x, \tau, \sigma, p) \in U' \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I'$.

Since ι is continuous, we can choose an open set $U \subseteq C(\mathbb{R}/2\pi;\ \mathbb{R}^{N+1}) \times \mathbb{R}^2$ small enough so that $(y_0, z_0, \sigma_0, p_0) \in U \subseteq \iota^{-1}(U')$ and the open set

$$I := \bigcap_{\{p:(y, z, \sigma, p)\in U\}} \frac{p}{2\pi} \cdot I'$$

is nonempty. Then, by the definition of ι, $mp \neq z(t)$ for every $(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$, $m \in \mathbb{N}$ and $t \in I$. □

Lemma 4 (The generalized intermediate value theorem, [10]) *Let $f : X \to Y$ be a continuous map from a connected space X to a linearly ordered set Y with order topology. If $a, b \in X$ and $y \in Y$ lies between $f(a)$ and $f(b)$, then there exists $x \in X$ such that $f(x) = y$.*

Definition 2 Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (1.2), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Let $I \subset \mathbb{R}$ be an interval and U be a subset in $C(y^*, z^*, \sigma^*, p^*)$. We call $I \times (U \cap C(y^*, z^*, \sigma^*, p^*))$ a delay-period disparity set if every solution

$$(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$$

satisfies $mp \neq z(t)$ for every $t \in I$ and $m \in \mathbb{N}$. We call $I \times (U \cap C(y^*, z^*, \sigma^*, p^*))$ a delay-period disparity set at $(t_0, y_0, z_0, \sigma_0, p_0)$ if $(t_0, y_0, z_0, \sigma_0, p_0) \in I \times (U \cap C(y^*, z^*, \sigma^*, p^*))$.

In the remaining part of this subsection, the following assumption is sometimes needed:

(S9) Every periodic solution (x, τ, σ) of (1.1) satisfies $\tau(t) > 0$ for every $t \in \mathbb{R}$.

Lemma 5 *Suppose that system* (1.1) *satisfies* (*S5-S6*) *and* (x, τ, σ) *is a non-constant periodic solution. If*

(*i*) $\tau \not\equiv \tau_\sigma$ *and there exists* $t_0 \in \mathbb{R}$ *such that* $\tau(t_0) = \tau_\sigma$, *and*
(*ii*) (x, τ) *is* τ_σ*-periodic,*

then there exists $t_1 \in \mathbb{R}$ *such that* $\tau(t_1) > \tau_\sigma$.

Proof We prove by way of contradiction. Suppose that

$$\tau(t) \leq \tau_\sigma \text{ for every } t \in \mathbb{R}. \tag{2.23}$$

Then, since $\tau \not\equiv \tau_\sigma$, there exists $t^* \in \mathbb{R}$ such that $\tau(t^*) < \tau_\sigma$. We can choose a maximal interval $[a, b] \subset \mathbb{R}$ which contains t^* in the sense that

$$\begin{cases} \tau(t) < \tau_\sigma & \text{for any } t \in (a, b) \\ \tau(t) = \tau_\sigma & \text{for any } t = a \text{ and } t = b. \end{cases} \tag{2.24a}$$

If $\dot{\tau}(a) \neq 0$ or $\dot{\tau}(b) \neq 0$, then it follows from the local monotonicity of $\tau(t)$ (at a or b) that there exists $t_1 \in \mathbb{R}$ in some neighborhood of a or b such that $\tau(t_1) > \tau_\sigma$. This is a contradiction to (2.23).

If $\dot{\tau}(a) = \dot{\tau}(b) = 0$, then we have

$$g(x(a), \tau_\sigma, \sigma) = g(x(b), \tau_\sigma, \sigma) = 0. \tag{2.25}$$

We distinguish the following two cases:

Case 1. $x(a) \neq x_\sigma$ or $x(b) \neq x_\sigma$. Without loss of generality we suppose $x(a) \neq x_\sigma$. Then by (ii), we have

$$\ddot{\tau}(a) = \frac{\partial g}{\partial x}(x(a), \tau_\sigma, \sigma) f(x(a), x(a), \sigma). \tag{2.26}$$

It follows from (S6), (2.25) and (2.26) that $\ddot{\tau}(a) \neq 0$ holds. Therefore, we have $\dot{\tau}(t)$ is strictly monotonic in some neighborhood of a. Hence there exists $t_1 \in \mathbb{R}$ such that $\tau(t_1) > \tau_\sigma$. This is also a contradiction to (2.23).

Case 2. $x(a) = x(b) = x_\sigma$. By (S5), we have $\frac{\partial g}{\partial \tau}(x_\sigma, \tau_\sigma, \sigma) \neq 0$. Without loss of generality we assume that

$$\frac{\partial g}{\partial \tau}(x_\sigma, \tau_\sigma, \sigma) < 0. \tag{2.27}$$

Then by (2.24a), (2.25), (2.27) and the continuity of $x(t)$ and $\tau(t)$, we can choose $\epsilon > 0$ small enough so that

$$\dot{\tau}(t) = g(x(t), \tau(t), \sigma) > 0 \text{ for every } t \in (a, a+\epsilon) \cup (b-\epsilon, b). \tag{2.28}$$

Therefore, we have $\tau(a) < \tau(a+\epsilon)$. That is, there exists $t_1 = a + \epsilon$ such that $\tau(a) = \tau_\sigma < \tau(t_1)$. This is a contradiction to (2.23). The proof is complete. □

Lemma 6 *Suppose that* (1.1) *satisfies* (*S5–S9*). *Let* $C(y^*, z^*, \sigma^*, p^*)$ *be a connected component of the closure of all the nonconstant periodic solutions of system* (1.2)*, bifurcated from* $(y^*, z^*, \sigma^*, p^*)$ *in the Fuller space* $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. *Let* $I \subset \mathbb{R}$ *be an open interval and* $\bar{v} := (\bar{y}, \bar{z}, \bar{\sigma}, \bar{p}) \in C(y^*, z^*, \sigma^*, p^*)$. *If there is no delay-period disparity set at* $(t, \bar{u})$ *for any* $t \in I$*, then*

- (*i*) *there exists* $m \in \mathbb{N}$ *such that* $m\,\bar{p} = \bar{z}(t) = z_{\bar{\sigma}}$ *for every* $t \in I$;
- (*ii*) $\bar{v}$ *is a non-constant solution with* $\bar{z}(t) = z_{\bar{\sigma}}$ *for every* $t \in I$;
- (*iii*) *there exist an open interval* $I' \subseteq \mathbb{R}$ *and an open neighorhood* U' *of* $\bar{v}$ *so that* $I' \times (U' \cap C(y^*, z^*, \sigma^*, p^*))$ *is a delay-period disparity set with* $\bar{v} \in U' \cap C(y^*, z^*, \sigma^*, p^*)$, *and the inequality* $z_{\bar{\sigma}} < \bar{z}(t)$ *holds for every* $t \in I'$.

Proof (i) By Definition 2, for every $t \in I$, there exists $m \in \mathbb{N}$ such that $\bar{z}(t) = m\,\bar{p}$. Note that $\bar{z}(t)$ is continuous, $\bar{z}(t) = m\,\bar{p}$ for every $t \in I$. Then, for every $t \in I$, we have

$$\dot{\bar{y}}(t) = \frac{\bar{p}}{2\pi} f(\bar{y}(t), \bar{y}(t), \bar{\sigma}), \tag{2.29}$$

$$\dot{\bar{z}}(t) = \frac{\bar{p}}{2\pi} g(\bar{y}(t), m\bar{p}, \bar{\sigma}) = 0. \tag{2.30}$$

By (2.30), we have

$$\ddot{\bar{z}}(t) = \frac{\bar{p}^2}{4\pi^2}\frac{\partial g}{\partial x}(\bar{y}(t), m\bar{p}), \bar{\sigma}) \cdot f(\bar{y}(t), \bar{y}(t), \bar{\sigma}) = 0. \tag{2.31}$$

By (S6), (2.30) and (2.31), we have $\bar{y}(t) = y_{\bar{\sigma}}$ on I. Hence, by (S5) and by (2.30), we have $\bar{z}(t) = z_{\bar{\sigma}} = m\bar{p}$ on I. This finishes the proof of (i).

(ii) Note that the stationary solutions of (1.1) and (1.2) are equal. That is, $(x_\sigma, \tau_\sigma) = (y_\sigma, z_\sigma)$ for every $\sigma \in \mathbb{R}$.

If $\bar{v}$ is a constant solution, then by (i) we have $\bar{z}(t) = z_{\bar{\sigma}} = m\bar{p}$ and $\bar{y}(t) = y_{\bar{\sigma}}$ for all $t \in \mathbb{R}$. Then $(y_{\bar{\sigma}}, z_{\bar{\sigma}}, \bar{\sigma}, \bar{p})$ is a bifurcation point in $C(y^*, z^*, \sigma^*, p^*)$ which satisfies $z_{\bar{\sigma}} = m\bar{p}$ for some $m \in \mathbb{N}$. This contradicts with assumption (S8). So $\bar{v}$ is a non-constant solution with $\bar{z}(t) = z_{\bar{\sigma}}$ for all $t \in I$.

(iii) Now we show that there exists $t_0 \in \mathbb{R}$ such that $\bar{z}(t_0) \neq z_{\bar{\sigma}}$. If not, $\bar{z}(t) = z_{\bar{\sigma}}$ for all $t \in \mathbb{R}$, then

$$\begin{aligned}(\bar{x}(\cdot), \bar{\tau}(\cdot), \bar{\sigma}) : &= \left(\bar{y}\left(\frac{2\pi}{\bar{p}}\cdot\right), \bar{z}\left(\frac{2\pi}{\bar{p}}\cdot\right), \bar{\sigma}\right) = \left(\bar{y}\left(\frac{2\pi}{\bar{p}}\cdot\right), z_{\bar{\sigma}}, \bar{\sigma}\right)\\ &= \left(\bar{y}\left(\frac{2\pi}{\bar{p}}\cdot\right), \tau_{\bar{\sigma}}, \bar{\sigma}\right)\end{aligned}$$

is a solution of (1.1). Then, by Lemma 3, $(\bar{x}, \bar{\tau})$ is not $\tau_{\bar{\sigma}}$-periodic. Then we have $m\bar{p} \neq z_{\bar{\sigma}}$ for any $m \in \mathbb{N}$. This is a contradiction to (i).

Therefore, there exists $t_0 \in \mathbb{R}$ such that $\bar{z}(t_0) \neq z_{\bar{\sigma}}$. That is, $\bar{\tau}(\frac{\bar{p}}{2\pi}t_0) \neq \tau_{\bar{\sigma}}$. Note that, by (i), $(\bar{x}, \bar{\tau})$ is $\tau_{\bar{\sigma}}$-periodic and $\bar{\tau}(t) = \tau_{\bar{\sigma}}$ on $\frac{\bar{p}}{2\pi}I$. Then, by Lemma 5, there exists $t_1 \in \mathbb{R}$ such that

$$\bar{\tau}(t_1) > \tau_{\bar{\sigma}}. \tag{2.32}$$

By the continuity of $\bar{\tau}$ and by (2.32), there exists a finite interval $(a, b) \ni t_1$ so that for every $t \in (a, b)$

$$\bar{\tau}(t) > \tau_{\bar{\sigma}}. \tag{2.33}$$

We claim that there exists $t_0 \in (a, b)$ so that $\bar{v}$ is not $\bar{\tau}(t_0)$-periodic. Indeed, if not, $\bar{v}$ is $\bar{\tau}(t)$-periodic for every $t \in (a, b)$. Then by the continuity of $\bar{\tau}$ and by (2.33), there exist $t_1, t_2 \in (a, b)$ and an interval $(\bar{\tau}(t_1), \bar{\tau}(t_2))$ with $\bar{\tau}(t_2) > \bar{\tau}(t_1)$, so that $\bar{\tau}$ is p-periodic for all $p \in (\bar{\tau}(t_1), \bar{\tau}(t_2))$. Hence $\bar{v}$ is a constant solution. This is a contradiction with (ii) and the claim is proved.

Then, we have $\bar{\tau}(t_0) \neq m\bar{p}$ for all $m \in \mathbb{N}$. By Lemma 2, there exists an open interval $I_1 \ni t_0$ and an open neighborhood $U_1 \ni (\bar{x}, \bar{\tau}, \bar{\sigma}, \bar{p})$ such that every solution (x, τ, σ, p) of (1.1) in $U_1 \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I_1$. Note that $\bar{\tau}$ is continuous at $t = t_0$. We can therefore choose I_1 small enough so that (2.33) holds for all $t \in I_1$.

Let ι be the continuous mapping defined by (2.22). Then we can choose an open set $U' \subseteq C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ small enough so that $\bar{v} \in U' \subseteq \iota^{-1}(U_1)$ and

$$I' := \bigcap_{\{p:(y, z, \sigma, p)\in U'\}} \frac{p}{2\pi} \cdot I_1$$

is nonempty. It follows from the definition of ι that $mp \neq z(t)$ for every solution $(y, z, \sigma, p) \in U' \cap C(y^*, z^*, \sigma^*, p^*)$, $m \in \mathbb{N}$ and $t \in I'$. In particular, noting that (2.33) holds for all $t \in I_1$ and $I' \subseteq \frac{\bar{p}}{2\pi}I_1$, we have

$$\bar{z}(t) > z_{\bar{\sigma}} \tag{2.34}$$

for every $t \in I'$. This completes the proof. □

Now we are able to state our main result in this subsection.

Theorem 6 *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the non-constant periodic solutions of system (1.2), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that (1.1) satisfies (S5–S9). If $p^* < z^*$, then, for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, $p < z(t)$ for some $t \in \mathbb{R}$.*

Proof By Theorem 5 and (S8), there exist an open interval $I^* \subseteq \mathbb{R}$ and an open set U^* in $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that $I^* \times (U^* \cap C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set with $(y^*, z^*, \sigma^*, p^*) \in U^*$.

Let $A^* \ni (y^*, z^*, \sigma^*, p^*)$ be a connected component of $(U^* \cap C(y^*, z^*, \sigma^*, p^*))$. Then, $I^* \times A^*$ is connected in $\mathbb{R} \times C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Define $S : \mathbb{R} \times C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2 \to \mathbb{R}$ by

$$S(t, y, z, \sigma, p) = p - z(t).$$

Note that we have $p^* < z^*$. Then it follows that $S(t, y^*, z^*, \sigma^*, p^*) = p^* - z^* < 0$. Note that S is continuous. By Lemma 4, we have

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \tag{2.35}$$

for every $(t, y, z, \sigma, p) \in I^* \times A^*$, for otherwise, there exists $(t_0, y_0, z_0, \sigma_0, p_0) \in I^* \times A^*$ such that $p_0 = z_0(t_0)$ which contradicts the fact that $I^* \times A^*$ is a subset of the forbidden range of delay $I^* \times (U^* \cap C(y^*, z^*, \sigma^*, p^*))$.

Now we show that there exists a sequence of connected subsets of $C(y^*, z^*, \sigma^*, p^*)$, denoted by $\{A_n\}_{n=1}^{n_0}$, $n_0 \in \mathbb{N}$ or $n_0 = +\infty$, which satisfies that

(i) $A^* \subseteq A_1 \subset A_2 \subset \cdots \subset A_{n_0}$ and $\cup_{n=1}^{n_0} A_n = C(y^*, z^*, \sigma^*, p^*)$;
(ii) for every $(y, z, \sigma, p) \in A_n$ with $n \in \{1, 2, \ldots, n_0\}$, $p < z(t)$ at some $t \in \mathbb{R}$.

Let $A_1 := A^*$. If $A_1 = C(y^*, z^*, \sigma^*, p^*))$ then we are done by (2.35). If not, since the only sets that are both closed and open in the connected topological space $C(y^*, z^*, \sigma^*, p^*)$ are the empty set and the connected component $C(y^*, z^*, \sigma^*, p^*)$ itself, $A_1 \ni (y^*, z^*, \sigma^*, p^*)$ is not both closed and open. Then the boundary of A_1 in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$ is nonempty. That is,

$$\partial A_1 \neq \emptyset. \tag{2.36}$$

Let $\bar{v} = (\bar{y}, \bar{z}, \bar{\sigma}, \bar{p}) \in \partial A_1$. If there exists $t_1 \in I_1 := I^*$ and a delay-period disparity set $I' \times (U' \cap C(y^*, z^*, \sigma^*, p^*))$ such that $(t_1, \bar{v}) \in \bar{I}' \times (U' \cap C(y^*, z^*, \sigma^*, p^*)$, and if $A_{\bar{v}} \ni \bar{v}$ is the connected component of $U' \cap C(y^*, z^*, \sigma^*, p^*)$, then it is clear that $A_1 \cup A_{\bar{v}}$ is connected. As A_1 is closed we have $\bar{p} < \bar{z}(t_1)$. Then, by Lemma 4 we have

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \text{ for every } (t, y, z, \sigma, p) \in I' \times A_{\bar{v}}. \tag{2.37}$$

If, for any $t \in I_1$, there is no delay-period disparity set at $(t, \bar{u})$, then by Lemma 6, there exists a delay-period disparity set $I'' \times (U'' \cap C(y^*, z^*, \sigma^*, p^*)$ with $\bar{v} \in U' \cap C(y^*, z^*, \sigma^*, p^*)$ and

$$m\bar{p} = z_{\bar{\sigma}} < \bar{z}(t) \text{ for every } t \in I'' \text{ and } m \in \mathbb{N}. \tag{2.38}$$

Let $A_{\bar{v}} \ni \bar{v}$ be the connected component of $U'' \cap C(y^*, z^*, \sigma^*, p^*)$. It is clear that $A_1 \cup A_{\bar{v}}$ is connected. Then, by (2.38) and Lemma 4,

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \text{ for any } (t, y, z, \sigma, p) \in I'' \times A_{\bar{v}}. \tag{2.39}$$

By (2.37) and (2.39) we know that if $\bar{v} \in \partial A_1$, then there exists a delay-period disparity set $\tilde{I} \times (\tilde{U} \cap C(y^*, z^*, \sigma^*, p^*))$ with $A_{\bar{v}} \ni \bar{v}$ being the connected component of $\tilde{U} \cap C(y^*, z^*, \sigma^*, p^*)$ so that

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \text{ for any } (t, y, z, \sigma, p) \in \tilde{I} \times A_{\bar{v}}. \tag{2.40}$$

For every $\bar{v} \in \partial A_1$, we find a $A_{\bar{v}}$ satisfying (2.40). Then we define

$$A_2 = A_1 \cup \bigcup_{\bar{v} \in \partial A_1} A_{\bar{v}}.$$

It follows from (2.35), (2.37) and (2.39) that, for any $(y, z, \sigma, p) \in A_2$, $p < z(t)$ for some $t \in \mathbb{R}$. Note that for any $\bar{v} \in \partial A_1$, $A_1 \cup A_{\bar{v}}$ is connected. Therefore, A_2 is connected.

Note that the existence of A_2 only depends on the fact that $\partial A_1 \neq \emptyset$, in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$. Beginning with $n = 1$, we can always recursively construct a connected subset for each $n \geq 1, n \in \mathbb{N}$, with $\partial A_n \neq \emptyset$

$$A_{n+1} = A_n \cup \bigcup_{\bar{v} \in \partial A_n} A_{\bar{v}} \tag{2.41}$$

satisfying that for every $(y, z, \sigma, p) \in A_{n+1}$,

$$p < z(t) \text{ for some } t \in \mathbb{R}, \tag{2.42}$$

where $I_n \times (U_n \cap C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set at $(t, \bar{v}) \in I_n \times \partial A_n$ and $A_{\bar{v}}$ is the connected component of U_n.

If the construction in (2.41) stops at some $n_0 \in \mathbb{N}$ with $\partial A_{n_0} = \emptyset$. Then $A_{n_0} = C(y^*, z^*, \sigma^*, p^*)$ and we are done. If not, then $n_0 = +\infty$ and we obtain a sequence of sets $\{A_n\}_{n=1}^{+\infty}$ which is a totally ordered family of sets with respect to set inclusion relation "$\subseteq$". Note that $\cup_{n=1}^{+\infty} A_n$ is the upper bound of $\{A_n\}_{n=1}^{+\infty}$. Then by Zorn's lemma, there exists a maximal element A_∞ for the sequence $\{A_n\}_{n=1}^{+\infty}$.

Now we show that $\partial A_\infty = \emptyset$, in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$. Suppose not, then there exist $\bar{v} \in \partial A_\infty$ and $A_{\bar{v}}$, which is the connected component of U_∞, where $I_\infty \times (U_\infty \times C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set at $(t, \bar{v}) \in I_\infty \times \partial A_\infty$. We distinguish two cases:

Case 1. $A_{\bar{v}} \setminus A_\infty = \emptyset$ for all $\bar{v} \in \partial A_\infty$. Then A_∞ is a connected component of $C(y^*, z^*, \sigma^*, p^*)$. Recall that $C(y^*, z^*, \sigma^*, p^*)$ itself is a connected component of the closure of all the nonconstant periodic solutions of system (1.2). So we have $A_\infty = C(y^*, z^*, \sigma^*, p^*)$. That is $\partial A_\infty = \emptyset$. This is a contradiction.

Case 2. $A_{\bar{v}} \setminus A_\infty \neq \emptyset$. But this means $A_\infty \subset A_\infty \cup A_{\bar{v}}$ which contradicts the maximality of A_∞.

The contradictions show that $\partial A_\infty = \emptyset$, and hence $A_\infty = C(y^*, z^*, \sigma^*, p^*)$. Therefore, (2.42) holds for all $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$. This completes the proof. □

Theorem 7 *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the non-constant periodic solutions of system (1.2), bifurcated at $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that (1.1) satisfies (S5-S9). If there exists a continuous function $M_1 : \mathbb{R} \ni \sigma \to M_1(\sigma) > 0$ such that for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$ we have*

$$\|(y, z)\|_{C(\mathbb{R}; \mathbb{R}^{N+1})} \leq M_1(\sigma), \tag{2.43}$$

then $p^* < z^*$ implies that $p < M_1(\sigma)$ for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$.

Proof By Theorem 6, we have, for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, that $p < z(t)$ for some $t \in \mathbb{R}$. Then, by (2.43), we have $p < M_1(\sigma)$. □

3 Uniform Boundedness of Periodic Solutions

We refer to [13] for the concepts of balanced, convex and absorbing subsets and the Minkowski functional.

Lemma 7 *Let G be a convex absorbing subset of a locally convex linear topological space X which defines a Minkowski functional $p_G : X \to \mathbb{R}$ with $p_G(x) = \inf\{\alpha > 0 : \alpha^{-1}x := x/\alpha \in G\}$. For each $\gamma > 0$ define*

$$G^\gamma = \{x : p_G(x) < \gamma\}. \tag{3.1}$$

Then $x \in \partial G^\gamma$ *if and only if* $p_G(x) = \gamma$.

Proof It is clear that $G^\gamma = \gamma G$. By linearity, the Minkowski functional $p_{G^\gamma} : X \to \mathbb{R}$ determined by G^γ is well-defined. By (3.1) and by the definition of Minkowski functional, we have

$$\begin{aligned} x \in \partial G^\gamma &\iff p_{G^\gamma}(x) = 1 \\ &\iff \inf\{\alpha > 0 : x/\alpha \in G^\gamma\} = 1 \\ &\iff \inf\{\alpha > 0 : p_G(x/\alpha) < \gamma\} = 1 \\ &\iff \inf\{\alpha > 0 : p_G(x)/\gamma < \alpha\} = 1 \\ &\iff p_G(x) = \gamma. \end{aligned}$$

□

Lemma 8 *Let* G_1 *and* G_2 *be convex absorbing subsets of locally convex linear topological spaces* X_1 *and* X_2*, respectively. Let the Minkowski functionals associated with* G_1 *and* G_2 *be* $p_{G_1}(x)$ *and* $p_{G_2}(\tau)$*, respectively. Then the Minkowski functional defined by* $G = G_1 \times G_2$ *exists and satisfies*

$$p_G(x, \tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}.$$

Proof The existence of $p_G(x, \tau)$ is clear from the definition of a Minkowski functional. Let $A = \{\alpha : x/\alpha \in G_1\}$, $B = \{\alpha : \tau/\alpha \in G_2\}$. Then it is clear that $\inf A \cap B \geq \inf A$ and $\inf A \cap B \geq \inf B$. It follows that $\inf A \cap B \geq \max\{\inf A, \inf B\}$, that is

$$p_G(x, \tau) \geq \max\{p_{G_1}(x), p_{G_2}(\tau)\}. \tag{3.2}$$

On the other hand, if $\alpha_A = \inf A \geq \alpha_B = \inf B$, since G_1 and G_2 are absorbing, we have for every $\epsilon > 0$, $\alpha_A + \epsilon \in A$, $\alpha_A + \epsilon \in B$. Therefore, $\inf A \cap B \leq \alpha_A + \epsilon$. Similarly, if $\alpha_A = \inf A \leq \alpha_B = \inf B$, we have $\inf A \cap B \leq \alpha_B + \epsilon$. Hence we obtain $\inf A \cap B \leq \max\{\alpha_A, \alpha_B\} + \epsilon$. By the arbitrariness of $\epsilon > 0$, we get $\inf A \cap B \leq \max\{\alpha_A, \alpha_B\}$, that is

$$p_G(x, \tau) \leq \max\{p_{G_1}(x), p_{G_2}(\tau)\}. \tag{3.3}$$

By (3.2) and (3.3), we have

$$p_G(x, \tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}.$$

This completes the proof. □

An immediate corollary of Lemma 7 and Lemma 8 is the following

Corollary 1 *Let* G_1 *and* G_2 *be convex absorbing subsets of locally convex linear topological spaces* X_1 *and* X_2*, respectively. Let* $p_{G_1}(x)$ *and* $p_{G_2}(\tau)$ *be the Minkowski functionals associated with* G_1 *and* G_2*, respectively. Let* $G = G_1 \times G_2$*, and for every* $\gamma > 0$*, define*

$$\begin{aligned} G^\gamma &= \{(x, \tau) : p_G(x, \tau) < \gamma\}, \\ G_1^\gamma &= \{x : p_{G_1}(x) < \gamma\}, \\ G_2^\gamma &= \{\tau : p_{G_2}(\tau) < \gamma\}. \end{aligned}$$

Then $G^\gamma = G_1^\gamma \times G_2^\gamma$ *and* $\bar{G}^\gamma = \bar{G}_1^\gamma \times \bar{G}_2^\gamma$.

In this section, we use "·" to denote the usual inner product of an Euclidean space and use G^c and D^c to denote the complementary sets of G and D, respectively.

We can now state and prove the geometric conditions for uniform boundedness of the periodic solutions of (1.1) with $\sigma \in \Sigma$, where $\Sigma \subseteq \mathbb{R}$ is a given subset.

Theorem 8 *Suppose that $G_1 \subset \mathbb{R}^N$ and $G_2 \subset \mathbb{R}$ are bounded, balanced, convex and absorbing open subsets with associated Minkowski functionals $p_{G_1}(x)$ and $p_{G_2}(\tau)$. Let $G = G_1 \times G_2$ and $(\overline{x}, \overline{\tau}) = \frac{1}{p_G(x,\tau)}(x, \tau) \in \partial G$ for $(x, \tau) \neq 0$. Assume that there exists a vector-valued function $N : \partial G \setminus (\partial G_1 \times \partial G_2) \to \mathbb{R}^{N+1} \setminus \{0\}$ satisfying*

(i) : $\overline{G} \subseteq U_1 \cup U_2$, *where*

$$U_1 = \bigcap_{(x,\tau)\in\partial G\setminus(\partial G_1\times\partial G_2)} \{(u, v) : N(x, \tau)\cdot(u - x, v - \tau) \leq 0\};$$

$$U_2 = \bigcap_{(x,\tau)\in\partial G_1\times\partial G_2} \{(u, v) : x\cdot(u - x) \leq 0,\ \tau\cdot(v - \tau) \leq 0\};$$

(ii) : *$N(\overline{x}, \overline{\tau})\cdot(f(x, \widetilde{x}, \sigma), g(x, \tau, \sigma))$ is positive (or negative) for all $(x, \tau) \in G^c$ with $(\overline{x}, \overline{\tau}) \notin \partial G_1 \times \partial G_2$, and all $(\widetilde{x}, \tau) \in \mathbb{R}^N \times \mathbb{R}$ with $p_G(\widetilde{x}, \tau) \leq p_G(x, \tau)$ and $\sigma \in \Sigma$;*

(iii) : *$x\cdot f(x, \widetilde{x}, \sigma)$ and $\tau\cdot g(x, \tau, \sigma)$ are both positive (or negative) for all $(x, \tau) \in G^c$ with $(\overline{x}, \overline{\tau}) \in \partial G_1 \times \partial G_2$, and all $(\widetilde{x}, \tau) \in \mathbb{R}^N \times \mathbb{R}$ with $p_G(\widetilde{x}, \tau) \leq p_G(x, \tau)$ and $\sigma \in \Sigma$.*

Then the range of all the periodic solutions of (1.1) with $\sigma \in \Sigma$ is contained in G.

Remark 2 The prototype of the vector-valued function $N(x, \tau)$ is the (outer or inner) normal of G which is not defined on $\partial G_1 \times \partial G_2$. If G is a rectangle in a planar space, $\partial G_1 \times \partial G_2$ are four corner points of G. Conditions (ii)–(iii) of Theorem 8 require that the vector field determined by the right hand side of system (1.1) has positive (or negative) inner product with respect to the normal of a given rectangle G, where the vector field is evaluated at $(x, \tau) \in \mathbb{R}^{N+1}$ which satisfies $(x, \tau) \in G^c$ and $p_G(\widetilde{x}, \tau) \leq p_G(x, \tau)$.

Proof Letting $(x, \tau)(t) = (y, z)(\beta t)$ with a normalization parameter $\beta > 0$. we only need to consider the 2π-periodic solutions of the following system

$$\begin{cases} \dot{y}(t) = \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma), \\ \dot{z}(t) = \frac{1}{\beta} g(y(t), z(t), \sigma), \end{cases} \tag{3.4}$$

where $x \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$. It is clear that if $(x(t), \tau(t))$ and $(y(t), z(t))$ are solutions of (1.1) and (3.4), respectively, then $(x(t), \tau(t)) \in G$ for all $t \in \mathbb{R}$ if and only if $(y(t), z(t)) \in G$ for all $t \in \mathbb{R}$.

For simplicity, we denote $y(t - \beta z(t))$ by $\widetilde{y}(t)$ for each solution $(y(t), z(t))$ of (3.4). Let $(\overline{y}, \overline{z})$ be the positive constant multiple of (y, z) so that $(\overline{y}, \overline{z}) \in \partial G$. That is, for every $(y, z) \in \mathbb{R}^{N+1} \setminus \{0\}$, there exists $(\bar{y}, \bar{z}) \in \partial G$ so that $(y, z) = p_G(y, z)(\bar{y}, \bar{z})$.

Suppose there exists a 2π-periodic solution of (3.4) so that $(y(t_0), z(t_0)) \notin G$ for some $t_0 \in [0, 2\pi]$ and define the map $\gamma : \mathbb{R} \ni t \to p_G(y(t), z(t)) \in \mathbb{R}$. Since $\mathbb{R}^{N+1} \ni (y, z) \mapsto p_G(y, z) \in \mathbb{R}$ and $\mathbb{R} \ni t \mapsto (y(t), z(t)) \in \mathbb{R}^{N+1}$ are continuous, the map $\gamma : t \to p_G(y(t), z(t))$ is continuous and there exist $\gamma^* \geq 1$ and $t^* \in [0, 2\pi]$ such that

$$\gamma^* = p_G(y(t^*), z(t^*)) = \max_{t\in[0,2\pi]} p_G(y(t), z(t)). \tag{3.5}$$

Then, by Lemma 7 and (3.5), we have $(y(t^*), z(t^*)) \in \partial G^{\gamma^*}$ and $G^{\gamma(t)} \subseteq G^{\gamma^*}$ for all $t \in \mathbb{R}$. Therefore, by Corollary 1, $(y(t), z(t)) \in \bar{G}^{\gamma^*} = \bar{G}_1^{\gamma^*} \times \bar{G}_2^{\gamma^*}$ for all $t \in [0, 2\pi]$. In particular, by periodicity of $(y(t), z(t))$, we obtain $(y(t-\beta z(t)), z(t)) \in \bar{G}^{\gamma^*}$ for all $t \in [0, 2\pi]$ and $\beta > 0$. Therefore, we have

$$p_G(y(t^* - \beta z(t^*), z(t^*)) \le p_G(y(t^*), z(t^*)). \tag{3.6}$$

We first suppose that $(\bar{y}(t^*), \bar{z}(t^*)) = \frac{1}{p_G(y(t^*), z(t^*))}(y(t^*), z(t^*)) \in U_1$. Then, by (3.5), (3.6) and by assumption (ii), we have (we use the positiveness assumption in the proof. The proof is similar if we use the negativeness assumption. See Remark 3 for details.)

$$N\left(\bar{y}(t^*), \bar{z}(t^*)\right) \cdot \left[\frac{1}{\beta} f(y(t^*), y(t^* - \beta z(t^*)), \sigma), \frac{1}{\beta} g(y(t^*), z(t^*), \sigma)\right] > 0. \tag{3.7}$$

Let us write

$$\begin{bmatrix} y(t^*+h) \\ z(t^*+h) \end{bmatrix} = \begin{bmatrix} y(t^*) \\ z(t^*) \end{bmatrix} + \begin{bmatrix} \int_0^1 \dot{y}(t^*+sh)ds\, h \\ \int_0^1 \dot{z}(t^*+sh)ds\, h \end{bmatrix}, \tag{3.8}$$

and choose $h > 0$ small enough so that

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left[\frac{1}{\beta} f(y(t), y(t-\beta z(t)), \sigma), \frac{1}{\beta} g(y(t), z(t), \sigma)\right] > 0 \tag{3.9}$$

for $t^* \le t < t^* + h$. Then by (3.4), (3.8) and (3.9), we have

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot (y(t^*+h) - y(t^*), z(t^*+h) - z(t^*)) > 0. \tag{3.10}$$

Now we distinguish the following two cases in order to deduce contradictions:

Case 1. If $(y(t^*+h), z(t^*+h)) \in \bar{G}$, then $\gamma^{*-1}(y(t^*+h), z(t^*+h)) \in \bar{G}$ since $\gamma^* \ge 1$. Also, we have $(y(t^*), z(t^*)) = (\gamma^* \bar{y}(t^*), \gamma^* \bar{z}(t^*))$ with $(\bar{y}(t^*), \bar{z}(t^*)) \in \partial G$. Then by assumption (i), we have

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\gamma^{*-1} y(t^*+h) - \bar{y}(t^*), \gamma^{*-1} z(t^*+h) - \bar{z}(t^*)\right) \le 0. \tag{3.11}$$

On the other hand, we have by (3.10)

$$\begin{aligned} 0 &< N(\bar{y}(t^*), \bar{z}(t^*)) \cdot (y(t^*+h) - y(t^*), z(t^*+h) - z(t^*)) \\ &= \gamma^* N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\gamma^{*-1} y(t^*+h) - \bar{y}(t^*), \gamma^{*-1} z(t^*+h) - \bar{z}(t^*)\right), \end{aligned} \tag{3.12}$$

which contradicts (3.11).

Case 2. If $(y(t^*+h), z(t^*+h)) \notin \bar{G}$, then by (3.5), we have

$$1 \le \gamma_h = p_G(y(t^*+h), z(t^*+h)) \le p_G(y(t^*), z(t^*)) = \gamma^*. \tag{3.13}$$

Also, we have $(y(t^*+h), z(t^*+h)) = \gamma_h(\bar{y}(t^*+h), \bar{z}(t^*+h))$ with $(\bar{y}(t^*+h), \bar{z}(t^*+h)) \in \partial G$. By the convexity of $\bar{G}$ and by the inequality $\gamma_h/\gamma^* \le 1$, we have,

$$\left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h), \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h)\right) \in \bar{G}.$$

Then, by assumption (i), we have

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h) - \bar{y}(t^*), \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h) - \bar{z}(t^*)\right) \le 0. \tag{3.14}$$

On the other hand, we have by (3.10)

$$\begin{aligned} 0 &< N(\bar{y}(t^*), \bar{z}(t^*)) \cdot (y(t^*+h) - y(t^*), z(t^*+h) - z(t^*)) \\ &= \gamma^* N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h) - \bar{y}(t^*), \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h) - \bar{z}(t^*)\right), \end{aligned} \tag{3.15}$$

which contradicts (3.14).

Secondly, we suppose that $(\bar{y}(t^*), \bar{z}(t^*)) = \frac{1}{p_G(y(t^*), z(t^*))}(y(t^*), z(t^*)) \in U_2$. By assumption (iii), we have

$$\begin{cases} \bar{y}(t^*) \cdot \frac{1}{\beta} f(y(t^*), y(t^* - \beta z(t^*)), \sigma) > 0, \\ \bar{z}(t^*) \cdot \frac{1}{\beta} g(y(t^*), z(t^*), \sigma) > 0. \end{cases} \tag{3.16}$$

Therefore, we can choose $h > 0$ small enough so that for $t^* \le t < t^* + h$

$$\begin{cases} \bar{y}(t^*) \cdot \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma) > 0, \\ \bar{z}(t^*) \cdot \frac{1}{\beta} g(y(t), z(t), \sigma) > 0. \end{cases} \tag{3.17}$$

Then by (3.4), (3.8) and (3.17), we have

$$\begin{cases} \bar{y}(t^*) \cdot (y(t^*+h) - y(t^*)) > 0, \\ \bar{z}(t^*) \cdot (z(t^*+h) - z(t^*)) > 0. \end{cases} \tag{3.18}$$

We distinguish the following two cases in order to deduce contradictions:

Case 1′. If $(y(t^*+h), z(t^*+h)) \in \bar{G}$, then $\gamma^{*-1}(y(t^*+h), z(t^*+h)) \in \bar{G}$ since $\gamma^* \ge 1$. Also, we have $(y(t^*), z(t^*)) = (\gamma^*\bar{y}(t^*), \gamma^*\bar{z}(t^*))$ with $(\bar{y}(t^*), \bar{z}(t^*)) \in \partial G$. Then by assumption (i), we have

$$\begin{cases} \bar{y}(t^*) \cdot (\gamma^{*-1} y(t^*+h) - \bar{y}(t^*)) \le 0, \\ \bar{z}(t^*) \cdot (\gamma^{*-1} z(t^*+h) - \bar{z}(t^*)) \le 0. \end{cases} \tag{3.19}$$

On the other hand, we have by (3.18)

$$\begin{cases} \bar{y}(t^*) \cdot (y(t^*+h) - y(t^*)) = \gamma^*\bar{y}(t^*) \cdot (\gamma^{*-1} y(t^*+h) - \bar{y}(t^*)) > 0, \\ \bar{z}(t^*) \cdot (z(t^*+h) - z(t^*)) = \gamma^*\bar{z}(t^*) \cdot (\gamma^{*-1} z(t^*+h) - \bar{z}(t^*)) > 0, \end{cases} \tag{3.20}$$

which contradicts (3.19).

Case 2′. If $(y(t^*+h), z(t^*+h)) \notin \bar{G}$, then by (3.13) and the convexity of $\bar{G}$, we have,

$$\left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h), \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h)\right) \in \bar{G},$$

where $\gamma_h = p_G(y(t^*+h), z(t^*+h))$. Then, by assumption (i), we have

$$\begin{cases} \bar{y}(t^*) \cdot (\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h) - \bar{y}(t^*)) \le 0, \\ \bar{z}(t^*) \cdot (\frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h) - \bar{z}(t^*)) \le 0. \end{cases} \tag{3.21}$$

On the other hand, we have by (3.18)

$$\begin{cases} \bar{y}(t^*) \cdot (y(t^*+h) - y(t^*)) = \gamma^*\bar{y}(t^*) \cdot (\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h) - \bar{y}(t^*)) > 0, \\ \bar{z}(t^*) \cdot (z(t^*+h) - z(t^*)) = \gamma^*\bar{z}(t^*) \cdot (\frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h) - \bar{z}(t^*)) > 0, \end{cases} \tag{3.22}$$

which contradicts (3.21). Therefore, contradictions are obtained in all the cases and the proof is complete. □

Remark 3 If we use <0 instead of >0 in the inequality (3.7), we need to change (3.8) to be the difference between $(y(t^*), z(t^*))$ and $(y(t^*-h), z(t^*-h))$. That is,

$$\begin{bmatrix} y(t^*) \\ z(t^*) \end{bmatrix} = \begin{bmatrix} y(t^*-h) \\ z(t^*-h) \end{bmatrix} + \begin{bmatrix} \int_0^1 \dot{y}(t^*-sh)ds\, h \\ \int_0^1 \dot{z}(t^*-sh)ds\, h \end{bmatrix}.$$

Then the rest of the proof is similar.

Corollary 2 *Suppose that $G_1 \subset \mathbb{R}^N$ and $G_2 \subset \mathbb{R}$ are bounded, balanced, convex and absorbing open subsets which define the Minkowski functionals $p_{G_1}(x)$ and $p_{G_2}(\tau)$. Suppose $N : \partial G \setminus (\partial G_1 \times \partial G_2) \to \mathbb{R}^{N+1} \setminus \{0\}$ is the outer normal of G. Fix $\sigma \in \Sigma$ and let $G = G_1 \times G_2$ and*

$$F_{\max}(x, \sigma) = \max_{\{\tilde{x}:\, p_{G_1}(\tilde{x}) \le p_{G_1}(x)\}} x \cdot f(x, \tilde{x}, \sigma),$$
$$F_{\min}(x, \sigma) = \min_{\{\tilde{x}:\, p_{G_1}(\tilde{x}) \le p_{G_1}(x)\}} x \cdot f(x, \tilde{x}, \sigma).$$

Then the range of all the periodic solutions of (1.1) are contained in G if either of the following conditions (H1) or (H2) holds:

(*H1*) $F_{\max}(x, \sigma) < 0$ *for any* $x \in G_1^c$ *and* $\tau \cdot g(x, \tau) < 0$ *for any* $\tau \in G_2^c$, $x \in \mathbb{R}^N$.
(*H2*) $F_{\min}(x, \sigma) > 0$ *for any* $x \in G_1^c$ *and* $\tau \cdot g(x, \tau) > 0$ *for any* $\tau \in G_2^c$, $x \in \mathbb{R}^N$.

Proof We prove the conclusions by applying Theorem 8. By Corollary 1, there exist Minkowski functionals $p_G(x, \tau)$, $p_{G_1}(x)$ and $p_{G_2}(\tau)$ defined on $\mathbb{R}^N \times \mathbb{R}$, $\mathbb{R}^N$ and $\mathbb{R}$, respectively. For every $(x, \tau) \in G^c$, let $(\bar{x}, \bar{\tau}) = (x, \tau)/p_G(x, \tau) \in \partial G$. Recall that $N : \partial G \setminus (\partial G_1 \times \partial G_2) \to \mathbb{R}^{N+1} \setminus \{0\}$ is the outer normal of the convex set G. Then condition (i) of Theorem 8 is satisfied.

Suppose (H1) holds. Then we have

$$\begin{cases} x \cdot f(x, \tilde{x}, \sigma) < 0, \text{ for all } (x, \tilde{x}) \in G_1^c \times \mathbb{R}^N \text{ with } p_{G_1}(\tilde{x}) \le p_{G_1}(x), & (3.23a) \\ \tau \cdot g(x, \tau, \sigma) < 0, \text{ for all } \tau \in G_2^c,\ x \in \mathbb{R}^N. & (3.23b) \end{cases}$$

For every $(x, \tau) \in G^c$ with $p_G(\tilde{x}, \tau) \le p_G(x, \tau)$, let $(\bar{x}, \bar{\tau}) = (x, \tau)/p_G(x, \tau) \in \partial G$. Note that $\partial G = (G_1 \times \partial G_2) \cup (\partial G_1 \times G_2) \cup (\partial G_1 \times \partial G_2)$. We distinguish the following three cases:

Case 1. If $(\bar{x}, \bar{\tau}) \in G_1 \times \partial G_2$, then $N(\bar{x}, \bar{\tau}) = (0, \tau)/p_G(x, \tau) \neq 0$ is an outer normal of G. We claim that $\tau \in G_2^c$ holds.

Indeed, since $\bar{x} \in G_1$ then we have $p_{G_1}(\bar{x}) = p_{G_1}(x/p_G(x, \tau)) < 1$. Therefore, $p_{G_1}(x) < p_G(x, \tau)$. By Lemma 8, we know that $p_G(x, \tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}$. Then we have $p_{G_1}(x) < p_{G_2}(\tau)$ and $p_G(x, \tau) = p_{G_2}(\tau) > 1$. Then by Lemma 7, we have $\tau \in G_2^c$.

Then by (3.23b) we have

$$N(\bar{x}, \bar{\tau}) \cdot (f(x, \tilde{x}, \sigma), g(x, \tau, \sigma)) = \tau \cdot g(x, \tau, \sigma)/p_G(x, \tau) < 0.$$

Case 2. If $(\bar{x}, \bar{\tau}) \in \partial G_1 \times G_2$, then $N(\bar{x}, \bar{\tau}) = (x, 0)/p_G(x, \tau) \neq 0$ is an outer normal of G. We claim that $x \in G_1^c$ and $p_{G_1}(\tilde{x}) \le p_{G_1}(x)$ hold.

Indeed, since $\bar{\tau} \in G_2$ then we have $p_{G_2}(\bar{\tau}) = p_{G_2}(\tau/p_G(x, \tau)) < 1$. Therefore, $p_{G_2}(\tau) < p_G(x, \tau)$. By Lemma 8, we know that $p_G(x, \tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}$.

Then we have $p_{G_2}(\tau) < p_{G_1}(x)$ and $p_G(x, \tau) = p_{G_1}(x) > 1$. Then by Lemma 7, we have $x \in G_1^c$. Moreover, it follows again by Lemma 8 that $p_G(\tilde{x}, \tau) \le p_G(x, \tau)$ implies $p_{G_1}(\tilde{x}) \le p_{G_1}(x)$. This proves the claim.

By (3.23a), we have

$$N(\bar{x}, \bar{\tau}) \cdot (f(x, \tilde{x}, \sigma), g(x, \tau, \sigma)) = x \cdot f(x, \tilde{x}, \sigma)/p_G(x, \tau) < 0.$$

From Case 1 and Case 2, we know that $N(\bar{x}, \bar{\tau}) \cdot (f(x, \tilde{x}, \sigma), g(x, \tau, \sigma))$ is negative definite for all $(x, \tau) \in G^c$ and $\sigma \in \Sigma$ with $(\overline{x}, \overline{\tau}) \notin \partial G_1 \times \partial G_2$, and all $(\widetilde{x}, \tau) \in \mathbb{R}^N \times \mathbb{R}$ with $p_G(\widetilde{x}, \tau) \le p_G(x, \tau)$. That is, condition (ii) of Theorem 8 is satisfied.

Case 3. If $(\bar{x}, \bar{\tau}) \in \partial G_1 \times \partial G_2$, we claim that $(x, \tau) \in G_1^c \times G_2^c$ and $p_{G_1}(\tilde{x}) = p_{G_1}(x)$ hold.

Indeed, since $(\bar{x}, \bar{\tau}) \in \partial G_1 \times \partial G_2$, then we have $p_{G_1}(\bar{x}) = p_{G_1}(x/p_G(x, \tau)) = 1$ and $p_{G_2}(\bar{\tau}) = p_{G_2}(\tau/p_G(x, \tau)) = 1$. Therefore, $p_G(x, \tau) = p_{G_1}(x) = p_{G_2}(\tau)$. Since $(x, \tau) \in G^c$, we have $p_{G_1}(x) = p_{G_2}(\tau) = p_G(x, \tau) > 1$. Then by Lemma 7, we have $(x, \tau) \in G_1^c \times G_2^c$. Moreover, it follows again by Lemma 8 that $p_G(\tilde{x}, \tau) \le p_G(x, \tau)$ implies $p_{G_1}(\tilde{x}) \le p_{G_1}(x)$. This proves the claim.

Then by (3.23a) and (3.23b) we have

$$x \cdot f(x, \tilde{x}, \sigma) < 0 \text{ and } \tau \cdot g(x, \tau, \sigma) < 0.$$

>From Case 3, for all $(x, \tau) \in G^c$ and $\sigma \in \Sigma$ with $(\overline{x}, \overline{\tau}) \notin \partial G_1 \times \partial G_2$, and all $(\widetilde{x}, \tau) \in \mathbb{R}^N \times \mathbb{R}$ with $p_G(\widetilde{x}, \tau) \le p_G(x, \tau)$. That is, condition (iii) of Theorem 8 is satisfied.

It follows from Theorem 8 that the range of all the periodic solutions of (1.1) with $\sigma \in \Sigma$ is contained in G. Similarly, if (H2) holds, we can obtain from Theorem 8 the same conclusion. This completes the proof. □

4 Global Continuation of Rapidly Oscillating Periodic Solutions: An Example

In this section we illustrate our general results by applying them to the study of the global continua of rapidly oscillating periodic solutions for the following differential equations with state-dependent delay,

$$\begin{cases} \dot{x}(t) = -\mu x(t) + \sigma^2 b(x(t - \tau(t))), \\ \dot{\tau}(t) = 1 - h(x(t)) \cdot (1 + \tanh \tau(t)), \end{cases} \tag{4.1}$$

where $\tanh(\tau) = (e^{2\tau} - 1)/(e^{2\tau} + 1)$ and $\mu > 0$ is a constant. We make the following assumptions:

- (α_1) $b, h : \mathbb{R} \to \mathbb{R}$ are C^2 functions with $b'(0) = -1$;
- (α_2) There exist $h_0 < h_1$ in $(1/2, 1)$ such that $h_1 > h(x) > h_0$ for all $x \in \mathbb{R}$;
- (α_3) b is decreasing on $\mathbb{R}$;
- (α_4) $xb(x) < 0$ for $x \neq 0$, and there exists a continuous function $M : \mathbb{R} \ni \sigma \to M(\sigma) \in (0, +\infty)$ so that

$$\frac{b(x)}{x} > -\frac{\mu}{\sigma^2}$$

  for every $x \in \mathbb{R}$ with $|x| \ge M(\sigma)$;
- (α_5) There exists $M_0 > 0$ such that $|b'(x)| < M_0$ for every $x \in \mathbb{R}$;
- (α_6) $h'(x) = 0$ only if x satisfies $-\mu x + \sigma^2 b(x) = 0$.

Remark 4 We use $\tanh(\tau)$ just for the sake of simplicity. Other types of functions can be used with minor changes of our arguments below.

We start with the uniform boundedness of periodic solutions $(x(t),\ \tau(t))$ of (4.1).

Lemma 9 *Assume* (α_1)–(α_4) *hold. Then the range of every periodic solution* $(x,\ \tau)$ *of* (4.1) *with* $\sigma \in \mathbb{R}$ *is contained in*

$$\Omega_1 = (-M(\sigma),\ M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2}\right).$$

Proof If $\sigma = 0$, the only periodic solution is $\left(0,\ -\frac{\ln(2h(0)-1)}{2}\right)$. By (α_2) and by (α_3), we have $0 < -\frac{\ln(2h(0)-1)}{2} < -\frac{\ln(2h_0-1)}{2}$ and $0 \in (-M(0),\ M(0))$. It follows that

$$\left(0,\ -\frac{\ln(2h(0) - 1)}{2}\right) \in (-M(0),\ M(0)) \times \left(0,\ -\frac{\ln(2h_0 - 1)}{2}\right). \tag{4.2}$$

Now we assume $\sigma \neq 0$. If $x > 0$, then, by assumption (α_3), we have

$$\begin{aligned}\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) &= \max_{y\in\{y:|y|\le|x|\}} -\sigma^2 x^2 \left(\frac{\mu}{\sigma^2} - \frac{b(y)}{x}\right)\\ &= -\sigma^2 x^2 \left(\frac{\mu}{\sigma^2} - \frac{b(-x)}{x}\right).\end{aligned}$$

Then by (α_4) we have

$$\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) = -\sigma x^2 \left(\frac{\mu}{\sigma^2} - \frac{b(-x)}{x}\right) < 0$$

for every $x \in \mathbb{R}$ with $x \geq M(\sigma)$. It follows that

$$\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) < 0 \text{ for } x \geq M(\sigma). \tag{4.3}$$

Similarly, if $x < 0$, then by assumption (α_3), we have

$$\begin{aligned}\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) &= \max_{y\in\{y:|y|\le|x|\}} -x^2 \left(\mu - \frac{\sigma^2 b(y)}{x}\right)\\ &= -\sigma^2 x^2 \left(\frac{\mu}{\sigma^2} - \frac{b(-x)}{x}\right).\end{aligned}$$

By (α_4) we have

$$\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) = -\sigma^2 x^2 \left(\frac{\mu}{\sigma^2} - \frac{b(-x)}{x}\right) < 0$$

for every x with $x < -M(\sigma)$.

Therefore, we have

$$\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) < 0 \text{ for } x \leq -M(\sigma). \tag{4.4}$$

Then by (4.3) and (4.4) we have

$$\max_{y\in\{y:|y|\le|x|\}} x \cdot (-\mu x + \sigma^2 b(y)) < 0 \text{ if } x \notin (-M(\sigma),\ M(\sigma)). \tag{4.5}$$

It is clear from (α_2) that for all $x \in \mathbb{R}$,

$$\lim_{\tau \to \pm\infty} \tau \cdot (1 - h(x)(1 + \tanh \tau)) < 0.$$

To obtain an upper bound for τ, where (x, τ) is a periodic solution of (4.1), we introduce the following change of variable:

$$z(t) = \tau(t) + \frac{\ln(2h_0 - 1)}{4}. \tag{4.6}$$

Then system (4.1) is transformed to

$$\begin{cases} \dot{x}(t) = -\mu x(t) + \sigma^2 b\left(x\left(t - z(t) + \frac{\ln(2h_0-1)}{4}\right)\right), \\ \dot{z}(t) = 1 - h(x(t))\left(1 + \tanh\left(z(t) - \frac{1}{4}\ln(2h_0 - 1)\right)\right). \end{cases} \tag{4.7}$$

By (α_2) and the monotonicity of $\tanh \tau$, we have, for every $z \leq \frac{\ln(2h_0-1)}{4} < 0$ and for all $x \in \mathbb{R}$,

$$\begin{aligned} & z \cdot \left(1 - h(x)\left(1 + \tanh\left(z - \frac{1}{4}\ln(2h_0 - 1)\right)\right)\right) \\ & \leq z \cdot (1 - h(x)(1 + \tanh(0))) \\ & < z \cdot (1 - h_1) \\ & < 0. \end{aligned} \tag{4.8}$$

Similarly, by (α_2) and the monotonicity of $\tanh \tau$, for every $z \geq -\frac{\ln(2h_0-1)}{4} > 0$ and for all $x \in \mathbb{R}$, we have

$$\begin{aligned} & z \cdot \left(1 - h(x)\left(1 + \tanh\left(z - \frac{1}{4}\ln(2h_0 - 1)\right)\right)\right) \\ & < z \cdot \left(1 - h(x)\left(1 + \tanh\left(-\frac{1}{2}\ln(2h_0 - 1)\right)\right)\right) \\ & = z \cdot \left(1 - h(x)\left(1 + \frac{1 - h_0}{h_0}\right)\right) \\ & = z \cdot \left(1 - \frac{h(x)}{h_0}\right) \\ & < 0. \end{aligned} \tag{4.9}$$

Then, by (4.8) and (4.9), we have, for any $z \notin \left(\frac{\ln(2h_0-1)}{4}, -\frac{\ln(2h_0-1)}{4}\right)$ and for all $x \in \mathbb{R}$,

$$z \cdot \left(1 - h(x)\left(1 + \tanh\left(z - \frac{1}{4}\ln(2h_0 - 1)\right)\right)\right) < 0. \tag{4.10}$$

Thus it follows from Corollary 2, (4.5) and (4.10) that the range of all the periodic solutions (x, z) of (4.7) are contained in $(-M(\sigma), M(\sigma)) \times \left(\frac{\ln(2h_0-1)}{4}, -\frac{\ln(2h_0-1)}{4}\right)$. Then, by (4.6), all periodic solutions (x, τ) of (4.1) with $\sigma \neq 0$ are contained in

$$\Omega_1 = (-M(\sigma), M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2}\right). \tag{4.11}$$

Then by (4.2) and (4.11), the proof is complete. □

Now we consider the global Hopf bifurcation problem of system (4.1) under the assumptions (α_1)–(α_6). By (α_4), $(x, \tau) = (0, \tau^*)$ is the only stationary solution of (4.1), where $\tau^* = -\frac{1}{2}\ln(2h(0) - 1) > 0$. Freezing the state-dependent delay $\tau(t)$ at τ^* for the term $x(t - \tau(t))$ of (4.1) and linearizing the resulting system with constant delay at the stationary solution $(0, \tau^*)$, we obtain the following formal linearization of system (4.1)

$$\begin{cases} \dot{X}(t) = -\mu X(t) - \sigma^2 X(t - \tau^*), \\ \dot{T}(t) = -\rho X(t) - qT(t), \end{cases} \tag{4.12}$$

where

$$\rho = \frac{h'(0)}{h(0)},\ q = 2 - \frac{1}{h(0)} > 0. \tag{4.13}$$

In the following, we regard σ as the bifurcation parameter. We obtain the characteristic equation of the linear system corresponding to (4.12)

$$(\lambda + \mu + \sigma^2 e^{-\tau^*\lambda})(\lambda + q) = 0. \tag{4.14}$$

Since the zero of $\lambda + q = 0$ is $-q$ which is real, Hopf bifurcation points are related to zeros of only the first factor $(\lambda + \mu + \sigma^2 e^{-\tau^*\lambda})$. To locate local Hopf bifurcation points we let $\lambda = i\beta$, $\beta > 0$, in $\lambda + \mu + \sigma^2 e^{-\tau^*\lambda} = 0$ and express the resulting equation in terms of its real and imaginary parts as

$$\begin{cases} \beta = \sigma^2 \sin(\tau^*\beta), \\ \mu = -\sigma^2 \cos(\tau^*\beta). \end{cases} \tag{4.15}$$

We illustrate in the following lemma the close relation between (4.15) and the following equations

$$\begin{cases} \tan \tau^*\beta = -\frac{\beta}{\mu}, \\ \quad\beta^2 = \sigma^4 - \mu^2. \end{cases} \tag{4.16}$$

Lemma 10 *We have the following conclusions:*

(i) *All the positive solutions of (4.15) can be represented by an infinite sequence* $\{\beta_n\}_{n=1}^{+\infty}$ *which satisfies* $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots$, $\lim_{n\to+\infty}\beta_n = +\infty$ *and*

$$\beta_n \in \left(\frac{(4n-3)\pi}{2\tau^*}, \frac{(4n-2)\pi}{2\tau^*}\right) \text{ for } n \geq 1.$$

(ii) $\pm i\beta_n$ *are characteristic values of the stationary solution* $(0, \tau^*, \sigma_n)$, *where*

$$\sigma_n = \pm\left(\beta_n^2 + \mu^2\right)^{1/4}.$$

If $\sigma \neq \sigma_n$, *then the stationary solution* $(0, \tau^*, \sigma)$ *has no purely imaginary characteristic value.*

(iii) *Let* $\lambda_n(\sigma) = u_n(\sigma) + iv_n(\sigma)$ *be the root of (4.14) for* σ *close to* σ_n *such that* $u_n(\sigma_n) + iv_n(\sigma_n) = i\beta_n$. *Then*

$$u_n'(\sigma)\Big|_{\sigma=\sigma_n} = \frac{2}{\sigma_n}\frac{\left(\mu^2 + \beta_n^2\right)\tau^* + \mu}{(1 + \mu\tau^*)^2 + (\beta_n\tau^*)^2}.$$

Proof (i) We first claim that (4.16) has infinitely many solutions $0 < \beta_1 < \beta_2 < \cdots < \beta_j < \cdots$ such that $\lim_{j\to+\infty} \beta_j = +\infty$ and

$$\beta_j \in \left(\frac{(2j-1)\pi}{2\tau^*}, \frac{2j\pi}{2\tau^*}\right) \text{ for } j \geq 1. \tag{4.17}$$

Note that the function $z = \tan \tau^* \beta$ is a strictly increasing 1-1 mapping from the open interval $\left(\frac{(2j-1)\pi}{2\tau^*}, \frac{2j\pi}{2\tau^*}\right)$ to $(-\infty, 0)$ with $\tan(\tau^* \frac{2j\pi}{2\tau^*}) = 0$ and $\lim_{\theta\to\left(\tau^* \frac{(2j-1)\pi}{2\tau^*}\right)^+} \tan(\theta) = -\infty$ for every $j \geq 1$. Then, it has a unique intersection with the straight line $z = -\beta/\mu$, $\mu > 0$, in the strip area $\left(\frac{(2n-1)\pi}{2\tau^*}, \frac{2n\pi}{2\tau^*}\right) \times (-\infty, 0)$ on the (β, z)-plane. Thus the claim follows.

By assuming $-\beta/\sigma^2 = \sin\theta$ and $-\mu/\sigma^2 = \cos\theta$ in the second equation of (4.16), then we have $\tan\theta = -\tan\tau^*\beta$ and hence $\theta = k\pi - \tau^*\beta$ for $k \in \mathbb{Z}$. Depending on whether k is even or odd, the set of solutions $\{\beta_j\}_{j=1}^{+\infty}$ of (4.16) can be categorized into two classes which solve the following equations (I) and (II), respectively. Note that $\beta > 0$, $\mu > 0$ and $\sigma \neq 0$. There is no common solution for (I) and (II).

$$\text{(I)} \begin{cases} \beta/\sigma^2 = \sin\tau^*\beta, \\ \mu/\sigma^2 = -\cos\tau^*\beta, \end{cases} \qquad \text{(II)} \begin{cases} \beta/\sigma^2 = -\sin\tau^*\beta, \\ \mu/\sigma^2 = \cos\tau^*\beta. \end{cases} \tag{4.18}$$

It is clear that equation (I) is exactly (4.15). To prove (i), we only need to identify solutions of (I) from $\{\beta_j\}_{j=1}^{+\infty}$ which satisfies (4.17).

If $j = 2m$ for some $m \in \mathbb{N}$, then by (4.17) we have $\tau^*\beta_{2m} \in (2m\pi - \pi/2, 2m\pi)$, $\sin(\tau^*\beta_{2m}) < 0$ and $\cos(\tau^*\beta_{2m}) > 0$. Then β_{2m} is not the solution of (I) but the solution of (II) since we have $\beta_{2m}/\sigma^2 > 0$ and $\beta_{2m}/\sigma^2 \neq \sin\tau^*\beta_{2m}$. If $j = 2m-1$ for some $m \in \mathbb{N}$, then by (4.17) we have $\tau^*\beta_{2m-1} \in (2m\pi - 3\pi/2, 2m\pi - \pi)$ and $\sin(\tau^*\beta_{2m-1}) > 0$. Then β_{2m-1} is not the solution of (II) but the solution of (I) since we have $\beta_{2m-1}/\sigma^2 > 0$ and $\beta_{2m-1}/\sigma^2 \neq -\sin(\tau^*\beta_{2m-1})$.

Note that $j \in \mathbb{N}$ is an arbitrary odd number. The set of positive solutions of (4.15) can be represented by the infinite sequence $\{\beta_n\}_{n=1}^{+\infty}$ where β_n satisfies

$$\beta_n \in \left(\frac{(4n-3)\pi}{2\tau^*}, \frac{(4n-2)\pi}{2\tau^*}\right), \text{ for } n \geq 1.$$

This completes the proof of (i).

The conclusion (ii) follows from the second equation of (4.15).

To prove (iii), let $F(\lambda, \sigma) = \lambda + \mu + \sigma^2 e^{-\tau^*\lambda}$. Then we have

$$\left.\frac{\partial F}{\partial\lambda}\right|_{\lambda=i\beta_n,\,\sigma=\sigma_n} = 1 - \sigma_n^2\tau^* e^{-\tau^* i\beta_n} \neq 0.$$

By the implicit function theorem, there exists a differentiable function $\sigma \to \lambda_n(\sigma) = u_n(\sigma) + iv_n(\sigma)$ which is a root of (4.14) for σ close to σ_n with $u_n(\sigma_n) + iv_n(\sigma_n) = i\beta_n$. Note that $\lambda_n(\sigma) \to i\beta_n \neq q$ as $\sigma \to \sigma_n$. Now we substitute λ by $\lambda_n(\sigma) = u_n(\sigma) + iv_n(\sigma)$ into $\lambda + \mu + \sigma^2 e^{-\tau^*\lambda} = 0$ and obtain

$$u_n(\sigma) + iv_n(\sigma) + \mu + \sigma^2 e^{-\tau^*(u_n(\sigma)+iv_n(\sigma))} = 0.$$

Differentiating both sides of the above equation with respect to σ and then substituting $\sigma = \sigma_n$, we have

$$\begin{cases} \left(1 - \sigma_n^2\tau^*\cos(\tau^*\beta_n)\right) u_n'(\sigma_n) - \sigma_n^2\tau^*\sin(\tau^*\beta_n) v_n'(\sigma_n) = -2\sigma_n\cos(\tau^*\beta_n), \\ \sigma_n^2\tau^*\sin(\tau^*\beta_n) u_n'(\sigma_n) + (1 - \sigma_n^2\tau^*\cos(\tau^*\beta_n)) v_n'(\sigma_n) = 2\sigma_n\sin(\tau^*\beta_n). \end{cases} \tag{4.19}$$

Note that from (i), we have $\tau^* > 0$, $\sigma_n \neq 0$ and $\beta_n > 0$ for every $n \geq 1$. We combine (4.19) with (4.15) to obtain

$$u_n'(\sigma)\Big|_{\sigma=\sigma_n} = \frac{2}{\sigma_n}\frac{(\mu^2+\beta_n^2)\tau^*+\mu}{(1+\mu\tau^*)^2+(\beta_n\tau^*)^2}.$$

This completes the proof. □

Now we are able to state our main results.

Theorem 9 *Assume* $(\alpha_1$–$\alpha_6)$ *hold. Let* $\beta_n \in \left(\frac{(4n-3)\pi}{2\tau^*}, \frac{(4n-2)\pi}{2\tau^*}\right)$, $n \geq 1$, *be given in (i) of Lemma 10. Let* $\sigma_n = \pm(\mu^2+\beta_n)^{1/4}$ *for* $n \geq 1$. *Then*

(a) There exists an unbounded connected component $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ *of the closure of all the nonconstant periodic solutions of system* (4.1)*, bifurcated from* $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ *in the Fuller space where* σ *satisfies* $sgn(\sigma_n)\sigma > 0$.

(b) $(0, \tau^*, \sigma_1, \frac{2\pi}{\beta_1}) \notin C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ *for every* $n \geq 2$.

(c) For every $n \geq 2$*, the projection of* $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ *onto the parameter space* $\mathbb{R}$ *is unbounded in* $(0, +\infty)$ *if* $\sigma_n > 0$ *and is unbounded in* $(-\infty, 0)$ *if* $\sigma_n < 0$.

Proof (a) We prove by applying Theorem 1. We first verify assumptions (S1–S3). It is clear that (α_2) and (α_1) imply (S1) and (S2). Let us check (S3). Indeed, noticing that $\sigma_n = \pm(\mu^2+\beta_n^2)^{1/4}$, $b'(0) = -1$ and $\beta_n > 0$, we have

$$\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right)\left[-\mu\theta_1+\sigma^2 b(\theta_2)\right]_{\sigma=\sigma_n,\,\theta_1=\theta_2=0} = -\mu-\sigma_n^2 < 0. \tag{4.20}$$

Also, it follows from $\tau^* = -\frac{\ln(2h(0)-1)}{2}$ that

$$\frac{\partial}{\partial\gamma_2}(1-h(\gamma_1)(1+\tanh(\gamma_2))\Big|_{\sigma=\sigma_n,\,\gamma_1=0,\,\gamma_2=\tau^*} = -h(0)\cdot\frac{4e^{2\tau^*}}{(e^{2\tau^*}+1)^2} < 0. \tag{4.21}$$

Therefore, condition (S3) is satisfied by system (4.1).

We note from Lemma 10 (i), (ii) and (iii) that every center (including those with $\sigma < 0$) of system (4.12) is isolated. We now calculate the crossing number of $(0, \tau^*, \sigma_n, \beta_n)$. Let $u_n(\sigma) + iv_n(\sigma)$ be the characteristic value of (4.12) such that $u_n(\sigma_n) + iv_n(\sigma_n) = i\beta_n$. By (iv) of Lemma 10, we have

$$\begin{aligned}\frac{d}{d\sigma}u_n(\sigma)\Big|_{\sigma=\sigma_n} &= u_n'(\sigma_n)|_{\sigma=\sigma_n}\\ &= \frac{2}{\sigma_n}\frac{(\mu^2+\beta_n^2)\tau^*+\mu}{(1+\mu\tau^*)^2+(\beta_n\tau^*)^2}.\end{aligned} \tag{4.22}$$

That is, $\frac{d}{d\sigma}u_n(\sigma)\big|_{\sigma=\sigma_n}$ has the same sign as σ_n since $\tau^* > 0$ and $\mu > 0$. We note from (1.7) that the crossing number $\gamma\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ counts the difference, when σ varies from σ_n^- to σ_n^+, of the number of imaginary characteristic values with positive real parts in a small neighborhood of $i\beta_n$ in the complex plane, where $\sigma_n^- < \sigma_n < \sigma_n^+$ are numbers in a small neighborhood of σ_n. Then by (4.22) the crossing number of the isolated center $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ in the Fuller space $C(\mathbb{R}; \mathbb{R}^2) \times \mathbb{R}^2$ satisfies

$$\gamma\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right) = -\text{sgn}(\sigma_n) \text{ for every } n \in \mathbb{N}. \tag{4.23}$$

Then by Theorem 1, there exists a connected component $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$ of the closure of all the nonconstant periodic solutions of system (4.1), bifurcated from the stationary solution $(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n})$ in the Fuller space. Note that there is no nonconstant periodic solution for the system (4.1) if $\sigma = 0$ since in this case x satisfies a scalar ordinary differential equation. Moreover, there is no bifurcation point at $\sigma = 0$. Therefore, $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$ is located in the Fuller space where σ satisfies $\mathrm{sgn}(\sigma_n)\sigma > 0$.

To prove the unboundedness of $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$ in the Fuller space, we apply the global Hopf bifurcation Theorem 2 to exclude the case that there are finitely many bifurcation points in $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$.

Now we suppose there are finitely many bifurcation points $\{(0,\,\tau^*,\,\sigma_{n_j},\,\frac{2\pi}{\beta_{n_j}})\}_{j=1}^q$, $q \in \mathbb{N}$, in $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$. We know that $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$ is located in the Fuller space where σ satisfies $\mathrm{sgn}(\sigma_n)\sigma > 0$. Then the bifurcation points $\left\{\left(0,\,\tau^*,\,\sigma_{n_j},\,\frac{2\pi}{\beta_{n_j}}\right)\right\}_{j=1}^q$ satisfies $\mathrm{sgn}(\sigma_n)\sigma_{n_j} > 0$ for all $j \in \{1,\,2,\,\ldots,\,q\}$.

Let ϵ_{n_j} be the value of

$$\mathrm{sgn}\,\det\begin{bmatrix}\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right)\widetilde{f}(\theta_1,\,\theta_2,\,\sigma) & 0\\ \frac{\partial}{\partial\gamma_1}\widetilde{g}(\gamma_1,\,\gamma_2,\,\sigma) & \frac{\partial}{\partial\gamma_2}\widetilde{g}(\gamma_1,\,\gamma_2,\,\sigma)\end{bmatrix}$$

evaluated at $(\theta_1,\,\theta_2,\,\sigma) = (0,\,0,\,\sigma_{n_j})$ and $(\gamma_1,\,\gamma_2,\,\sigma) = (0,\,\tau^*,\,\sigma_{n_j})$, where

$$\widetilde{f}(\theta_1,\,\theta_2,\,\sigma) = \left[-\mu\theta_1 + \sigma^2 b(\theta_2)\right],\quad \widetilde{g}(\gamma_1,\,\gamma_2,\,\sigma) = (1 - h(\gamma_1)(1 + \tanh(\gamma_2)).$$

Then by (4.20 and (4.21) we have

$$\epsilon_{n_j} = 1 \text{ for all } j = 1,\,2,\,\ldots,\,q. \tag{4.24}$$

By (4.23) and (4.24) we have

$$\sum_{j=1}^{q}\epsilon_{n_j}\gamma(\left(0,\,\tau^*,\,\sigma_{n_j},\,\frac{2\pi}{\beta_{n_j}}\right) = -q\,\mathrm{sgn}(\sigma_n) \neq 0. \tag{4.25}$$

Note that (α_5) and (α_6) implies (S4). Then by Theorem 2, (4.25) is a contradiction. The unboundedness of $C\left(0,\,\tau^*,\,\sigma_n,\,\frac{2\pi}{\beta_n}\right)$ follows.

(b) In order to verify assumption (S8) we claim that the virtual period p_n of every bifurcation point $(0,\,\tau^*,\,\sigma_n,\,2\pi/\beta_n)$ satisfies

$$mp_n \neq \tau^* \text{ for every } m \in \mathbb{N}. \tag{4.26}$$

Suppose that there exist $m_0,\,n_0 \in \mathbb{N}$ so that $m_0 p_{n_0} = m_0 \cdot 2\pi/\beta_{n_0} = \tau^*$. We note that

$$\beta_n \in \left(\frac{(4n-3)\pi}{2\tau^*},\,\frac{(4n-2)\pi}{2\tau^*}\right) \text{ for all } n \geq 1. \tag{4.27}$$

Then we have

$$4n_0 - 3 < 4m_0 < 4n_0 - 2.$$

This is a contradiction and the claim is proved.

We note that, by (4.27), a sufficient condition for $p_n = \frac{2\pi}{\beta_n} < \tau^*$, is that $\frac{2\pi}{\beta_n} < 4\tau^*/(4n-3) < \tau^*$, that is, $n \geq 7/4$. Therefore, every $(0, \tau^*, \sigma_n, p_n)$ with $n \geq 2$ is a bifurcation point of system (4.1) satisfying

$$p_n < \tau^* \text{ for all } n \geq 2. \tag{4.28}$$

For the bifurcation point $(0, \tau^*, \sigma_1, p_1)$ we can conclude from (4.27) that

$$2\tau^* < p_1 < 4\tau^*. \tag{4.29}$$

We want to obtain the uniform boundedness of period in $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ with $n \geq 2$. We only need to check the conditions (S5–S9) for applying Theorem 6 and Theorem 7.

It is clear that (α_4), (4.26) and (4.21) imply (S7), (S8) and (S5), respectively. Also we conclude from (S2), (S4) and Lemma 4.2 in [7] that

$$p > 0 \tag{4.30}$$

for every $(x, \tau, \sigma, p) \in C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$. Also, by Lemma 9, we have

$$0 < \tau(t) < -\frac{1}{2}\ln(2h_0 - 1) \tag{4.31}$$

for every $t \in \mathbb{R}$ and hence (S9) is satisfied. To check (S6), we let

$$\begin{cases} 1 - h(x)(1 + \tanh\tau) = 0, \\ (1 + \tanh\tau)h'(x)\left(-\mu x + \sigma^2 b(x)\right) = 0. \end{cases} \tag{4.32}$$

Then, by (α_1), (α_4) and (α_6), the solutions of (4.32) are stationary solutions of (4.1). This verifies (S6).

Therefore, we can use Theorem 6, Theorem 7, (4.28), (4.30) and (4.31) to conclude that there exists some $t \in \mathbb{R}$ so that

$$0 < p < \tau(t) < -\frac{1}{2}\ln(2h_0 - 1) \tag{4.33}$$

for every $(x, \tau, \sigma, p) \in C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ with $n \geq 2$. Then by (4.29) and (4.33) we know that $\left(0, \tau^*, \sigma_1, \frac{2\pi}{\beta_1}\right) \notin C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ for every $n \geq 2$. This proves (b).

(c) Let Σ be the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the σ-parameter space $\mathbb{R}$. By (a), we know that $\Sigma \subseteq (0, +\infty)$ if $\sigma_n > 0$ and $\Sigma \subseteq (-\infty, 0)$ if $\sigma_n < 0$. By Lemma 9, we know that for every $\sigma \in \Sigma$, there exists a constant $M_n(\sigma) > 0$ such that

$$\|(x, \tau)\|_{C(\mathbb{R}; \mathbb{R}^{N+1})} \leq M_n(\sigma), \tag{4.34}$$

where (x, τ, σ, p) is the solution associated with σ in $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ and $M_n : \mathbb{R} \ni \sigma \to M_n(\sigma) \in (0, +\infty)$ is a continuous function on $\mathbb{R}$.

We know from (4.33) that the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the p-parameter space $\mathbb{R}$ is bounded. If Σ is bounded, then it follows from (a) that the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the (x, τ)-space $C(\mathbb{R}; \mathbb{R}^{N+1})$ must be unbounded in the supremum norm. But by the continuity of M_n on $\mathbb{R}$ and by (4.34), the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the (x, τ)-space $C(\mathbb{R}; \mathbb{R}^{N+1})$ is uniformly bounded with respect to $\sigma \in \Sigma$. This is a contradiction and the proof is complete. □

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