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PERIODIC SOLUTIONS AND THE GLOBAL ATTRACTOR IN A SYSTEM OF DELAY DIFFERENTIAL EQUATIONS*

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Abstract. We consider a system of delay differential equations with a delayed excitatory feedback loop and instantaneous damping. We develop the Cao–Krisztin–Walther technique and establish the existence and uniqueness of periodic solutions with prescribed oscillation frequencies (characterized by the values of a discrete Lyapunov functional). We then use the Poincaré-Bendixson theorem due to Mallet-Paret and Sell to show that the global attractor of such a system is the union of the unstable sets of stationary points and periodic orbits.

Key words. delay differential equation, discrete Lyapunov functional, global attractor, periodic orbit, Poincaré-Bendixson theorem

AMS subject classifications. 34K13, 34C12

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1. Introduction.

Consider the following system of delay differential equations:

$$\begin{cases} \dot{x}^0(t) = -\mu x^0(t) + f(x^1(t)), \\ \dot{x}^1(t) = -\mu x^1(t) + f(x^2(t)), \\ \dot{x}^2(t) = -\mu x^2(t) + f(x^0(t-1)), \end{cases} \tag{1.1}$$

where $\mu > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuously differentiable function, normalized so that $f(0) = 0$. Such a system describes the computational performance of a feedback loop of three identical saturating amplifiers (neurons) with excitatory interaction [9, 10], with the delay incorporated to account for the finite switching speed of amplifiers (see, for example, [17, 24]). In applications to associative information processing where the network is triggered by an appropriate external stimulus and relaxes towards the attractor that encodes previously stored memories [8], it is important to describe completely the structure of the global attractor and, in particular due to the monotone feedback structures, to describe the existence, uniqueness/nonuniqueness, and stability of equilibria and periodic solutions.

So our goal here is to characterize the uniqueness, absence, and existence of periodic orbits with prescribed oscillation frequencies, and to describe the relationship between the system's global attractor and the unstable sets of the stationary points and periodic orbits. This study is heavily inspired by the previous studies of [3, 15, 4] for scalar delay differential equations or systems of coupled delay differential equations

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with $\mathcal{Z}_2$-symmetry. In particular, the following scalar delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)), \tag{1.2}$$

also arising from other biological and/or physical problems, has been discussed extensively and intensively in the literature (see [3, 11, 12, 13, 14, 15, 16, 18, 21] and references therein). Cao [3] and Krisztin and Walther [15] established the uniqueness of periodic solutions with prescribed oscillation frequencies when f represents either a negative or a positive feedback. Some of their results have then been extended, by Chen and Wu [4], to the following coupled system of delayed differential equations:

$$\begin{cases} \dot{x}^0(t) = -\mu x^0(t) + f(x^1(t)), \\ \dot{x}^1(t) = -\mu x^1(t) + f(x^0(t-1)). \end{cases} \tag{1.3}$$

A major observation in [4] is that the characteristic equation of the linearized system of (1.3) is just the product of the characteristic equation of the linearized equation of (1.2) with a positive feedback and the characteristic equation of the linearized equation of (1.2) with a negative feedback. This observation seems to be the key in [4] to apply the results in [3, 15] to obtain the uniqueness and absence of periodic orbits of (1.3). This observation is no longer true for (1.1). As will be shown in Appendix A, the characteristic equation of the linearized system of (1.3) is much more complicated.

A main technical tool to be used in our study is the discrete Lyapunov functional for a cyclic system of delay differential equations developed by Mallet-Paret and Sell [19, 20], and some of the properties of such a functional will be summarized in section 2. One of our major results is about the existence, uniqueness, and absence of periodic solutions in a certain level set of the discrete Lyapunov functional, and this requires the extension of the Cao–Krisztin–Walther technique [3, 15] to systems of three coupled equations and some continuation arguments (see section 3). We do not provide much information on the priori estimates of solutions of (1.1) like that in [1, 2, 4, 15, 18, 23]; we are nevertheless able to describe the global attractor as the union of the unstable sets of the stationary points and periodic orbits in section 4. In Appendix A, we present some technical results about the distribution of the roots of the characteristic equation of the linearized system of (1.1) around 0. One of the main results is that both the real and imaginary parts of the roots are well ordered. Though we have modified the phase space for (1.1), the basic theory of (1.1) is unchanged. For clarity and readers' convenience, some basic results used in the main body of the paper are proved in Appendix B.

Unfortunately, as discussed in section 3, we cannot prove the uniqueness of periodic orbits in the level set where the value of the discrete Lyapunov functional is 2. However, we believe that such a periodic orbit (if exists) should be unique. This is supported by the results for (1.2) and (1.3) and by the proof for the existence of such periodic orbits. How to prove the uniqueness of such a periodic orbit seems to be very challenging.

To conclude this section, we mention that the arguments in sections 2–4 can be easily modified to study the following general system of delay differential equations:

$$\begin{cases} \dot{x}^0(t) = -\mu x^0(t) + f(x^1(t)), \\ \dot{x}^1(t) = -\mu x^1(t) + f(x^2(t)), \\ \qquad \vdots \\ \dot{x}^{n-1} = -\mu x^{n-1}(t) + f(x^n(t)), \\ \dot{x}^n(t) = -\mu x^n(t) + f(x^0(t-1)). \end{cases} \tag{1.4}$$

Some results on periodic solutions to (1.4) will be listed in section 5. For simplicity of presentation, we focus only on (1.1) rather than (1.4).

2. Preliminaries. We first introduce some notations. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$ stand for the sets of all nonnegative integers, integers, reals, nonnegative reals, and complex numbers, respectively. Simple closed curves are continuous maps c from a compact interval $[a, b] \subseteq \mathbb{R}$ into $\mathbb{R}^n$ so that $c|[a, b)$ is injective and $c(a) = c(b)$. The set of the values of a simple closed curve c, or trace, is denoted by $|c|$ or c for simplicity. The Jordan curve theorem guarantees that the complement of the trace of a simple closed curve c in $\mathbb{R}^2$ consists of two nonempty connected open sets, one bounded and the other unbounded, with $|c|$ being the boundary of each of them. We denote the bounded component by $\text{int}(c)$ and the unbounded one by $\text{ext}(c)$. The interior and the boundary of a subset M of a topological space are denoted by M° and ∂M, respectively.

For μ and f, we make the following assumption:

(H1) f is odd and continuously differentiable, $f'(\xi) > 0$ for all $\xi \in \mathbb{R}$, $f - \mu Id$ has exactly one positive zero ξ^+, and $f'(\xi^+) < \mu < f'(0)$.

Obviously, assumption (H1) implies that $f - \mu Id$ has exactly one negative zero $-\xi^+$, which is denoted by ξ^-. Moreover, $f'(\xi^-) = f'(\xi^+)$ and $|f(\xi)| < \mu|\xi|$ for all $|\xi| > \xi^+$.

The natural phase space for (1.1) is the Banach space $C = C(\mathbb{K}, \mathbb{R})$ equipped with the supremum norm $||\cdot||$, where $\mathbb{K} = [-1, 0] \cup \{1, 2\}$ (see Mallet-Paret and Sell [19]). Let $C^1 = \{\phi \in C : \phi|_{[-1,0]}$ is continuously differentiable$\}$ be the Banach space with the C^1-norm $||\phi||_1 = \max\{\sup\{|\dot{\phi}(\theta)| : \theta \in [-1, 0]\}, ||\phi||\}$. The space C is a slight modification of the standard phase space $C([-1, 0], \mathbb{R}^3)$ found in Hale and Verduyn Lunel [7] and is adapted to suit (1.1) in which a time lag appears only in the last equation.

For a given interval I, let $I + [-1, 0] = \{t + \theta : t \in I \text{ and } \theta \in [-1, 0]\}$. We say that x is a D-type function on I if $x = (x^0, x^1, x^2)^{\text{tr}}$ such that $x^0 \in C(I + [-1, 0], \mathbb{R})$ and x^1, $x^2 \in C(I, \mathbb{R})$. In particular, if $I = (-\infty, a)$ for some $a \in \mathbb{R}$, then a D-type function x on I is just a vector function from I into $\mathbb{R}^3$. For a given D-type function x on the interval I and, for each $t \in I$, $x_t \in C$ is defined by

$$x_t(\theta) = \begin{cases} x_t^0(\theta) = x^0(t + \theta), & \theta \in [-1, 0], \\ x^1(t), & \theta = 1, \\ x^2(t), & \theta = 2. \end{cases}$$

For a given interval I, a solution of (1.1) on I is a D-type function $x = (x^0, x^1, x^2)^{\text{tr}}$ such that $x^0|_I$, x^1, $x^2 \in C^1(I, \mathbb{R})$ and they satisfy (1.1) on I. For convenience of notations, we shall often write $x^3(t) = x^0(t - 1)$, although we emphasize that x^3 is not one of the coordinate functions in (1.1).

Though the phase space for (1.1) has been modified to be C, the basic theory will not change. For clarification and the readers' convenience, some basic results are summarized below while the proofs are left to Appendix B.

Let $E \subseteq C$ denote the set of equilibria for (1.1). Of course, each element of E is a constant function on $\mathbb{K}$. For each $\mathbf{a} = (a^0, a^1, a^2)^{\text{tr}} \in \mathbb{R}^3$, $\hat{\mathbf{a}} \in C$ is defined as $\hat{\mathbf{a}}(\theta) = a^0$ for all $\theta \in [-1, 0]$, $\hat{\mathbf{a}}(1) = a^1$ and $\hat{\mathbf{a}}(2) = a^2$. Clearly, assumption (H1) implies that $E = \{\widehat{(0, 0, 0)^{\text{tr}}}, \widehat{\xi_-}, \widehat{\xi_+}\}$, where $\xi_\pm = (\xi^\pm, \xi^\pm, \xi^\pm)^{\text{tr}}$. For simplicity of notation, we shall write $\hat{\eta} = \widehat{(\eta, \eta, \eta)^{\text{tr}}}$ for any $\eta \in \mathbb{R}$. Thus, $E = \{\widehat{0}, \widehat{\xi^-}, \widehat{\xi^+}\}$.

For each $\phi \in C$ and $t_0 \in \mathbb{R}$, there exists a unique solution $x_\phi = (x_\phi^0, x_\phi^1, x_\phi^2)^{\text{tr}}$ of (1.1) on $[t_0, \infty)$ satisfying $(x_\phi)_{t_0} = \phi$. The solution depends continuously on the initial data. Two solutions x and y of (1.1) must be identical if $x_t = y_t \in C$ for some t in their common domain. It follows that for every $\phi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}^3$ of (1.1) with $x_0 = \phi$. We denote such a solution on $\mathbb{R}$ also by x_ϕ.

The map $F : \mathbb{R}_+ \times C \ni (t, \phi) \mapsto (x_\phi)_t \in C$ is a continuous semiflow. All maps $F(t, \cdot) : C \to C$ are injective whenever $t \geq 0$ and are conditionally completely continuous whenever $t \geq 1$ (see [6, p. 13] for the definition of the conditionally completely continuous map). Thus the semiflow $F(t, \cdot) : C \to C$ is conditionally completely continuous for $t \geq 1$ (see [6, p. 36] for the definition of the conditionally completely continuous semiflow or semigroup). Moreover, for each $t \geq 1$, $F(t, \cdot) : C \to C^1$ is continuous.

A set $B \subseteq C$ is said to be invariant (respectively, positively invariant) if, for every $\psi \in B$, $F(t, B) \subseteq B$ for all $t \in \mathbb{R}$ (respectively, for all $t \in \mathbb{R}_+$). It is not difficult to show that B is invariant if and only if for every $\psi \in B$ there is a solution $x : \mathbb{R} \to \mathbb{R}^3$ with $x_0 = \psi$ and $x_t \in B$ for all $t \in \mathbb{R}$.

For each $\phi \in C$, let $\gamma^+(\phi) = \{F(t, \phi) : t \in \mathbb{R}_+\}$ be the positive orbit through ϕ. For a solution $x : \mathbb{R} \to \mathbb{R}^3$ of (1.1) with $x_0 = \phi$, let $\gamma(\phi) = \{x_t : t \in \mathbb{R}\}$ be the full orbit through ϕ. Obviously, $\gamma^+(\phi)$ is positively invariant and $\gamma(\phi)$ is invariant provided that it exists.

By applying Lemma 3.2.1 and Corollary 3.2.2 in [6] and the fact that the semiflow $F(t, \cdot) : C \to C$ is conditionally completely continuous for each $t \geq 1$, we can obtain that, for every bounded solution x_ϕ, the ω-limit set

$$\omega(\phi) \triangleq \left\{\psi \in C : \begin{array}{l} \text{There exists a sequence } \{t_n\} \subset \mathbb{R}_+ \text{ such} \\ \text{that } t_n \to \infty \text{ and } F(t_n, \phi) \to \psi \text{ as } n \to \infty \end{array}\right\}$$

is nonempty. In fact, $\omega(\phi)$ is compact, connected, and invariant. For every bounded solution $x : \mathbb{R} \to \mathbb{R}^3$, the α-limit set

$$\alpha(x) = \left\{\psi \in C : \begin{array}{l} \text{There exists a sequence } \{t_n\} \subset (-\infty, 0] \text{ such} \\ \text{that } t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to \infty \end{array}\right\}$$

is also nonempty, compact, connected, and invariant. In the following, we also write $\omega(x_0) = \omega(x)$ and $\alpha(x_0) = \alpha(x)$ whenever $x : \mathbb{R} \to \mathbb{R}^3$ is a bounded solution of (1.1) on $\mathbb{R}$.

Let $C_+ = \{\phi \in C : \phi(\theta) \geq 0 \text{ for all } \theta \in \mathbb{K}\}$. Then C_+ is a positive cone of C and induces the pointwise ordering $\geq$ on C; that is, for ϕ, $\psi \in C$, we say that $\phi \geq \psi$ if $\phi - \psi \in C_+$. For $-\infty < a < b < \infty$, we set $C_{a,b} \triangleq \{\phi \in C : a < \phi(\theta) < b \text{ for all } \theta \in \mathbb{K}\}$.

Definition 2.1. *For a continuous semiflow H on C, H is said to be a monotone semiflow provided that $H(t, \phi) - H(t, \psi) \in C_+$ whenever $\phi - \psi \in C_+$ and $t \in [0, \infty)$. The monotone semiflow H is said to be an eventually strongly monotone semiflow with respective to the pointwise ordering provided that there exists $T > 0$ such that $H(t, \phi) - H(t, \psi) \in (C_+)^\circ$ whenever $\phi - \psi \in C_+ \setminus \{\widehat{0}\}$ and $t \in [T, \infty)$.*

We can easily show the following proposition.

Proposition 2.2. *Assume that $x : \mathbb{R} \to \mathbb{R}^3$ is a solution of* (1.1). *Then $y = (y^0, y^1, y^2)^{\mathrm{tr}} : \mathbb{R} \ni t \mapsto (x^1(t), x^2(t), x^0(t-1))^{\mathrm{tr}} \in \mathbb{R}^3$ is also a solution of* (1.1).

Proposition 2.3. *Under the assumption* (H1), *the following statements are true.*

(i) *The semiflow F is an eventually strongly monotone semiflow with respect to the pointwise ordering.*

(ii) *The sets C_+ and $-C_+$ are positively invariant, and $F(t, C_+ \setminus \{\widehat{0}\}) \subseteq (C_+)^\circ$, $F(t, -C_+ \setminus \{\hat{0}\}) \subseteq (-C_+)^\circ$ for all $t > 4$.*

(iii) *If* $-\infty < a < \xi^- < \xi^+ < b < \infty$, *then* $C_{a,b}$ *is positively invariant and for each* $\phi \in C$ *there exists* $t^* > 0$ *such that* $F([t^*, \infty), \phi) \subseteq C_{a,b}$.

Proof. By Corollary 5.3.5 in Smith [22], we know that the semiflow F is an eventually strongly monotone semiflow with respect to the pointwise ordering. In particular, $F(t, \phi) - F(t, \psi) \in (C_+)^\circ$ whenever $t > 4$ and ϕ, $\psi \in C$ such that $\phi - \psi \in C_+ \setminus \{0\}$. Thus (i) holds.

It follows from the above discussions and $0 \in E$ that (ii) holds.

Finally, (iii) can be proved by applying an argument similar to that of Proposition 2.1 in [15]. ∎

In what follows, we shall always assume that (H1) holds.

Remark 2.1. Consider the scalar delay differential equation

$$\dot{u} = -\mu u(t) + f(u(t - r)), \tag{2.1}$$

where $r > 0$. Obviously, for a given real function u defined on some interval, we know that u satisfies (2.1) if and only if u satisfies the equation $\frac{\mathrm{d}(e^{\mu t} u(t))}{\mathrm{d}t} = e^{\mu t} f(u(t - r))$. This, combined with the method of steps, gives that for each $\varphi \in C([-r, 0], \mathbb{R}_+)$, there exists $u : [-r, \infty) \to \mathbb{R}$ such that u satisfies (2.1) with $u|_{[-r,0]} = \varphi$. In particular, if $\varphi \in C([-r, 0], \mathbb{R}_+)$ and $\varphi \neq 0$, then, for every solution u of (2.1) defined on $[-r, \infty)$ with $u|_{[-r,0]} = \varphi$, we have $u(t) > 0$ for all large t. Analogously, if $\varphi \in C([-r, 0], (-\infty, 0])$ and $\varphi \neq 0$, then, for every solution u of (2.1) defined on $[-r, \infty)$ with $u|_{[-r,0]} = \varphi$, we have $u(t) < 0$ for all large t.

By applying Proposition 2.3(iii) and a similar argument as those in [16, Chapter 17] or in [15], one can obtain the existence of a global attractor of the semiflow F. This global attractor is a nonempty compact and invariant set $A \subseteq C$ which attracts each bounded subset $B \subseteq C$ in the sense that for any $\varepsilon > 0$ there exists $T^* = T^*(\varepsilon, A, B)$ such that $F(t, B)$ belongs to the ε-neighborhood of A for all $t \geq T^*$. The global attractor A is uniquely determined.

PROPOSITION 2.4. *The global attractor* A *has the following properties:*

(i) $A = \{\phi \in C :$ *There is a bounded solution* $x : \mathbb{R} \to \mathbb{R}^3$ *of* (1.1) *with* $x_0 = \phi\}$.

(ii) $A \subseteq \{\phi \in C : \xi^- \leq \phi(\theta) \leq \xi^+$ *for all* $\theta \in \mathbb{K}\}$.

(iii) A *is a compact subset of* C^1. *Moreover,* C *and* C^1 *define the same topology on* A.

(iv) *The map* $F : \mathbb{R}_+ \times C \ni (t, \phi) \mapsto (x_\phi)_t$ *can be extended to a continuous flow* $F_A : \mathbb{R} \times A \to A$ *such that, for each* $\phi \in A$ *and each* $t \in \mathbb{R}$, $F_A(t, \phi) = x_t$, *where* $x : \mathbb{R} \to \mathbb{R}^3$ *is the unique solution of* (1.1) *satisfying* $x_0 = \phi$.

Proof. (i) Suppose $\phi \in A$. Obviously, there is a bounded solution $x : \mathbb{R} \to \mathbb{R}^3$ with $x_0 = \phi$ and $x_t \in A$ for all $t \in \mathbb{R}$ since A is a compact and invariant set of C. On the other hand, if $\phi \in C$ is given so that there exists a bounded solution $x : \mathbb{R} \to \mathbb{R}^3$ with $x_0 = \phi$, then $Cl(\gamma(\phi))$ is a compact and invariant subset of C. It follows from the attractivity of A that $Cl(\gamma(\phi)) \subseteq A$. In particular, $\phi \in A$. This proves the statement (i).

(ii) Let $a_n = \xi^- - \frac{1}{n}$ and $b_n = \xi^+ + \frac{1}{n}$ for all $n \in \mathbb{N} \setminus \{0\}$. By Proposition 2.3(iii), we obtain that $A \subseteq C_{a_n, b_n}$ for all $n \in \mathbb{N} \setminus \{0\}$. Letting n tend to ∞, we have $A \subseteq C_{\xi^-, \xi^+}$, that is, $A \subseteq \{\phi \in C : \xi^- \leq \phi(\theta) \leq \xi^+$ for all $\theta \in \mathbb{K}\}$.

(iii) Let $M_1 = \sup\{| - \mu\eta + f(\xi)| : \eta, \xi \in [\xi^-, \xi^+]\}$, $M_2 = \sup\{| - \mu\eta + f'(\xi)\zeta| : \xi \in [\xi^-, \xi^+]$ and $\eta, \zeta \in [-M_1, M_1]\}$ and $M^* = \max\{M_1, M_2\}$. For any $\phi \in A$, by the invariance of A, there exists a full solution $x_\phi : \mathbb{R} \to \mathbb{R}^3$ of (1.1) with $(x_\phi)_0 = \phi$. Since f is continuously differentiable, it follows from (1.1) and the choice of M^* that $|\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}| \leq M^*$ and $|\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2}| \leq M^*$ for all $\theta \in [-1, 0]$. By the arbitrariness of $\phi \in A$ and the Arzèla-Ascoli theorem, we obtain that A is a compact subset of C^1.

Define $\iota : C^1 \to C$ by $\iota(\phi) = \phi$ for all $\phi \in C^1$. Obviously, ι is a continuous map. It follows from the compactness of A in C^1 that $\iota|_A$ is a closed map. Then $\iota|_A$ is a homeomorphism. This implies that C and C^1 define the same topology on A.

(iv) follows from the last assertion in Theorem 3.4.2 in [6]. ∎

PROPOSITION 2.5. *If $\phi \in C_+ \setminus \{\widehat{0}\}$, then $\omega(\phi) = \{\widehat{\xi^+}\}$. Analogously, if $\phi \in (-C_+ \setminus \{\widehat{0}\})$, then $\omega(\phi) = \{\widehat{\xi^-}\}$.*

Proof. We only prove the case where $\phi \in C_+ \setminus \{\widehat{0}\}$ as the other case can be dealt with similarly. By Proposition 2.3(ii), there exist $t^* > 4$ and $\eta \in (0, \xi^+)$ such that $F(t^*, \phi) > \hat{\eta}$. It follows from (H1) that there exists $\eta^* \in (0, \eta)$ such that $f(\xi) > \mu\eta^*$ for all $\xi \in [\eta^*, \infty)$. Then, by Corollary 5.2.2 in Smith [22], there exists $\chi = (\chi^0, \chi^1, \chi^2)^{\mathrm{tr}} \in \mathbb{R}^3$ such that for all $j \in \{0, 1, 2\}$, $\chi^j \geq \eta$ and $\lim_{t\to\infty}(x_{\widehat{\eta^*}})^j(t) = \chi^j$. In view of the definition of $\omega(\widehat{\eta^*})$, we know $\omega(\widehat{\eta^*}) = \{\widehat{\chi}\} \subseteq E$. This, combined with $E = \{\widehat{0}, \widehat{\xi^-}, \widehat{\xi^+}\}$, yields $\omega(\widehat{\eta^*}) = \{\widehat{\xi_+}\}$. By the choice of η^*, we have $F(t^*, \phi) > \widehat{\eta^*}$. Proposition 2.3(i) implies $F(t + t^*, \phi) \geq F(t, \widehat{\eta^*})$ for all $t \in \mathbb{R}$. By letting $t \to \infty$, we have $\omega(\phi) \geq \widehat{\xi_+}$. This, combined with Proposition 2.4(ii) and the fact that $\omega(\phi) \subseteq A$, gives $\omega(\phi) = \{\widehat{\xi_+}\}$. ∎

In the following, we summarize some properties of a discrete Lyapunov functional V and we refer to [19, 20] for more details.

Let $\mathbb{I} \subseteq \mathbb{K}$ be a nonempty subset of $\mathbb{K}$ and let $\phi \in C$ such that $\phi|_{\mathbb{I}}$ is not identically zero. Define the number of sign changes, $\mathrm{sc}(\phi; \mathbb{I})$, by $\mathrm{sc}(\phi; \mathbb{I}) = 0$ if $\phi|_{\mathbb{I}}$ is nonnegative or nonpositive and otherwise by

$$\mathrm{sc}(\phi; \mathbb{I}) = \sup \left\{ k \in \mathbb{N} \setminus \{0\} : \begin{array}{l} \text{There exists a strictly increasing finite sequence} \\ (\theta^j)_0^k \text{ in } \mathbb{I} \text{ with } \phi(\theta^{j-1})\phi(\theta^j) < 0 \text{ for all } 1 \leq j \leq k \end{array} \right\}.$$

We denote $\mathrm{sc}(\phi; \mathbb{K})$ by $\mathrm{sc}(\phi)$. In particular, $\mathrm{sc}(\phi) = 0$ for all $\phi \in (C_+ \cup (-C_+)) \setminus \{\widehat{0}\}$. Then, for any $\phi \in C \setminus \{\widehat{0}\}$, define

$$V(\phi) = \begin{cases} \mathrm{sc}(\phi) & \text{if } \mathrm{sc}(\phi) \in 2\mathbb{N} \cup \{\infty\}, \\ \mathrm{sc}(\phi) + 1 & \text{if } \mathrm{sc}(\phi) \in 2\mathbb{N} + 1. \end{cases}$$

In order to state the properties of V, we set

$$R = \left(\bigcap_{i=-1}^{2} S^i \right) \cap S^*,$$

where

$$\begin{aligned} S^{-1} &= \{\phi \in C^1 : \text{if } \phi(-1) = 0, \text{ then } \phi(2)\dot{\phi}(-1) < 0\}, \\ S^0 &= \{\phi \in C^1 : \text{if } \phi(0) = 0, \text{ then } \dot{\phi}(0)\phi(1) > 0\}, \\ S^1 &= \{\phi \in C^1 : \text{if } \phi(1) = 0, \text{ then } \phi(0)\phi(2) < 0\}, \\ S^2 &= \{\phi \in C^1 : \text{if } \phi(2) = 0, \text{ then } \phi(1)\phi(-1) < 0\}, \\ S^* &= \{\phi \in C^1 : \text{if } \phi(\theta) = 0 \text{ for some } \theta \in [-1, 0], \text{ then } \dot{\phi}(\theta) \neq 0\}, \end{aligned}$$

and define several maps as follows: for each $j \in \{0, 1, 2\}$, $\pi^j : C \ni \phi \mapsto (\phi(j), \phi(j+1))^{\mathrm{tr}} \in \mathbb{R}^2$. Notice that $\phi(3) = \phi(-1)$.

PROPOSITION 2.6.

(i) *For every $\phi \in C \setminus \{\widehat{0}\}$ and for every sequence $(\phi_n)_0^\infty$ in $C \setminus \{\widehat{0}\}$ with $\phi_n \to \phi$ as $n \to \infty$, $V(\phi) \leq \liminf_{n\to\infty} V(\phi_n)$.*

(ii) *For every* $\phi \in R$ *and for every sequence* $(\phi_n)_0^\infty$ *in* $C^1\backslash\{\widehat{0}\}$ *with* $\|\phi_n - \phi\|_{C^1} \to 0$ *as* $n \to \infty$, $V(\phi) = \lim_{n\to\infty} V(\phi_n) < \infty$.

(iii) *Let* $I \subset \mathbb{R}$ *be an interval,* $\mu \in \mathbb{R}_+$, *and* b_0, b_1, $b_2 : I \to (0, \infty)$ *be continuous. Suppose that* $z = (z^0, z^1, z^2)^{\mathrm{tr}}$ *is such that* $z^0 : I + [-1, 0] \to \mathbb{R}$ *and* z^1, $z^2 : I \to \mathbb{R}$ *so that* $z|_I$ *is continuously differentiable with*

$$\begin{cases} \dot{z}^0(t) = -\mu z^0(t) + b_0(t) z^1(t), \\ \dot{z}^1(t) = -\mu z^1(t) + b_1(t) z^2(t), \\ \dot{z}^2(t) = -\mu z^2(t) + b_2(t) z^0(t-1) \end{cases} \tag{2.2}$$

for $\inf I < t \in I$, *and* $z(t) \neq 0$ *for some* $t \in I + [-1, 0]$. *Then the map* $I \ni t \mapsto V(z_t) \in 2\mathbb{N} \cup \{\infty\}$ *is decreasing. Furthermore, we have*

(a) *if* $t \in I$, $t - 3 \in I$ *and either* $z^j(t) = z^{j+1}(t) = 0$ *for some* $j \in \{0, 1, 2\}$ *or* $z^j(t) = 0$, $z^{j-1}(t) z^{j+1}(t) > 0$ *for some* $j \in \{1, 2\}$, *then* $V(z_t) = \infty$ *or* $V(z_{t-3}) > V(z_t)$;

(b) *if* $t \in I$ *with* $t - 4 \in I$ *and* $V(z_{t-4}) = V(z_t)$, *then* $z_t \in R$.

Observe that the linear variational systems

$$\begin{cases} \dot{v}^0(t) = -\mu v^0(t) + f'(x^1(t)) v^1(t), \\ \dot{v}^1(t) = -\mu v^1(t) + f'(x^2(t)) v^2(t), \\ \dot{v}^2(t) = -\mu v^2(t) + f'(x^0(t-1)) v^0(t-1), \end{cases}$$

along solutions x of (1.1) are of the form of (2.2) as well as the systems satisfied by the weighted differences $y = (1/c)(x - \hat{x})$, where $c \neq 0$, on a common domain of solutions x, $\hat{x}$ of (1.1),

$$\begin{cases} \dot{y}^0(t) = -\mu y^0(t) + (\int_0^1 f'((1-s)\hat{x}^1(t) + s x^1(t)) ds) y^1(t), \\ \dot{y}^1(t) = -\mu y^1(t) + (\int_0^1 f'((1-s)\hat{x}^2(t) + s x^2(t)) ds) y^2(t), \\ \dot{y}^2(t) = -\mu y^2(t) + (\int_0^1 f'((1-s)\hat{x}^0(t-1) + s x^0(t-1)) ds) y^0(t-1). \end{cases}$$

Remark 2.2. Assume that $x : \mathbb{R} \to \mathbb{R}^3$ is a solution of (1.1). Then by Proposition 2.4 (i) and (iii), for each sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$, there exist a subsequence $\{s_n\}$ of $\{t_n\}$ and $\psi \in \omega(x)$ such that x_{s_n} tends to ψ in C^1 as $n \to \infty$. Moreover, if there exists $k \in \mathbb{N}$ such that $V(x_t) = 2k$ for all $t \in \mathbb{R}$, then, by Proposition 2.6(iii)(b), $x_t \in R$ for all $t \in \mathbb{R}$, and hence $V(\psi) = 2k$ for all $\psi \in \omega(x) \setminus \{\widehat{0}\}$ follows from Proposition 2.6(ii).

Remark 2.3. For each $m \in \mathbb{N}$, assume that $x_m : \mathbb{R} \to \mathbb{R}^3$ is a continuously differentiable map. If, for any compact interval I of $\mathbb{R}$, $\{x_m|_I\}_{m\in\mathbb{N}}$, $\{\dot{x_m}|_I\}_{m\in\mathbb{N}}$, and $\{\ddot{x_m}|_I\}_{m\in\mathbb{N}}$ are all bounded maps on I, then by the Arzèla-Ascoli theorem and the Cantor diagonalization process, there exist a subsequence of $\{x_m\}_{m\in\mathbb{N}}$, say $\{x_{m_k}\}_{k\in\mathbb{N}}$, and a map $x : \mathbb{R} \to \mathbb{R}^3$ such that $x_{m_k} \to x$ and $\dot{x_{m_k}} \to \dot{x}$ uniformly in any compact interval of $\mathbb{R}$ as $k \to \infty$.

We conclude this section with the following corollary of the general Poincaré-Bendixson theorem for monotone cyclic feedback systems due to Mallet-Paret and Sell [20].

Proposition 2.7. *For each* $\phi \in C$, $\omega(\phi)$ *is either a single nonconstant periodic orbit or for each solution* $y : \mathbb{R} \to \mathbb{R}^3$ *of* (1.1) *in* $\omega(\phi)$, *the sets* $\omega(y)$ *and* $\alpha(y)$ *consist of equilibria of* F. *An analogous statement holds for* α*-limit sets of bounded solutions of* (1.1) *on* $\mathbb{R}$.

3. Uniqueness, absence, and existence of periodic orbits. In this section, we first focus on the uniqueness and absence of periodic orbits with prescribed oscillation frequencies, i.e., in given level sets of the discrete Lyapunov functional V. Our approach uses the techniques in [3, 15], where oscillating periodic solutions of scalar delay equations are studied. For this purpose, it is necessary to consider the following system

$$\begin{cases} \dot{z}^0(t) = -\mu z^0(t) + g(z^1(t)), \\ \dot{z}^1(t) = -\mu z^1(t) + g(z^2(t)), \\ \dot{z}^2(t) = -\mu z^2(t) + g(z^0(t-\tau)), \end{cases} \tag{3.1}$$

where $\mu > 0$, $\tau > 0$ and the C^1-function $g : \mathbb{R} \to \mathbb{R}$ is odd and satisfies $g'(x) > 0$ for all $x \in \mathbb{R}$. If z is a solution of (3.1), then the function w given by $w(t) = z(\tau t)$ is a solution of

$$\begin{cases} \dot{w}^0(t) = -\tau\mu w^0(t) + \tau g(w^1(t)), \\ \dot{w}^1(t) = -\tau\mu w^1(t) + \tau g(w^2(t)), \\ \dot{w}^2(t) = -\tau\mu w^2(t) + \tau g(w^0(t-1)). \end{cases} \tag{3.2}$$

PROPOSITION 3.1. *Let $w : \mathbb{R} \to \mathbb{R}^3$ be a nonconstant periodic solution of* (3.2) *with the minimal period $T_w > 0$. Then one of the following statements is true:*

(i) *$w^j(t) > 0$ for all $j \in \{0, 1, 2\}$ and $t \in \mathbb{R}$.*

(ii) *$w^j(t) < 0$ for all $j \in \{0, 1, 2\}$ and $t \in \mathbb{R}$.*

(iii) *For each $j \in \{0, 1, 2\}$, w^j has a zero.*

Additionally, if $g - \mu Id$ has exactly one zero $\eta^- \in (-\infty, 0)$ and exactly one zero $\eta^+ \in (0, \infty)$, $g'(\eta^-) < \mu$ and $g'(\eta^+) < \mu$, then, for each $j \in \{0, 1, 2\}$, w^j has a zero.

Proof. Suppose that statement (iii) is not true. Then there exists $j_0 \in \{0, 1, 2\}$ such that w^{j_0} has no zero. By Proposition 2.2, without loss of generality, we can assume $j = 0$. We shall finish the proof by distinguishing two cases.

Case 1. $w^0(t) > 0$ for all $t \in \mathbb{R}$. It follows from the third equation of (3.2) that

$$\dot{w}^2(t) > -\tau\mu w^2(t). \tag{3.3}$$

Pick up $t^* \in [0, T_w]$ such that $w^2(t^*) = \inf\{w^2(t) : t \in \mathbb{R}\}$. Obviously, $\dot{w}^2(t^*) = 0$. Then, by (3.3), $w^2(t^*) > 0$, and hence $w^2(t) \geq w^2(t^*) > 0$ for all $t \in \mathbb{R}$. Similarly, with the help of $w^2(t) > 0$ for all $t \in \mathbb{R}$ and the second equation of (3.2), we can show that $w^1(t) > 0$ for all $t \in \mathbb{R}$. Therefore, statement (i) holds.

Case 2. $w^0(t) < 0$ for all $t \in \mathbb{R}$. Similar arguments to those in Case 1 will show that statement (ii) holds.

Now, suppose that $g - \mu Id$ has exactly one zero $\eta^- \in (-\infty, 0)$ and exactly one zero $\eta^+ \in (0, \infty)$, $g'(\eta^-) < \mu$ and $g'(\eta^+) < \mu$. We show that neither statement (i) nor statement (ii) holds. By way of contradiction, if statement (i) holds, then, by Proposition 2.5, $\omega(w_0) = \{\widehat{\eta^+}\}$, a contradiction as w is a nonconstant periodic solution of (3.2). Thus statement (i) is not true. Similarly, statement (ii) cannot be true. Therefore, w^j has a zero for all $j \in \{0, 1, 2\}$. ∎

We now give a priori information on periodic solutions of (3.2) which follows from the general results in [20] for certain systems of delay differential equations.

PROPOSITION 3.2. *Let $w : \mathbb{R} \to \mathbb{R}^3$ be a nonconstant periodic solution of* (3.2) *with the minimal period $T_w > 0$. Then the following statements are true.*

(i) *For each $j \in \{0, 1, 2\}$, $c^j_w : [0, T_w] \ni t \mapsto \pi^j(w_t) \in \mathbb{R}^2$ and $C^j_w : [0, T_w] \ni t \mapsto (w^j(t), \dot{w}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ are two simple closed curves.*

(ii) *For each* $j \in \{0, 1, 2\}$*, there are* $t_0^j \in \mathbb{R}$ *and* $t_1^j \in (t_0^j, t_0^j + T_w)$ *such that* $0 < \dot{w}^j(t)$ *for all* $t_0^j < t < t_1^j$*,* $w^j(\mathbb{R}) = [w^j(t_0^j), w^j(t_1^j)]$*,* $\dot{w}^j(t) < 0$ *for all* $t_1^j < t < t_0^j + T_w$*.*

(iii) *If* w^0 *has a zero, then* $w(t + \frac{T_w}{2}) = -w(t)$ *for all* $t \in \mathbb{R}$*. Moreover,* $0 \in \mathrm{int}(c_w^j)$ *and* $0 \in \mathrm{int}(C_w^j)$ *for all* $j \in \{0, 1, 2\}$*.*

(iv) *There exists* $k \in \mathbb{N}$ *such that* $\{w_t : t \in \mathbb{R}\} \subset V^{-1}(2k)$*.*

(v) *For every nonconstant periodic solution* $y : \mathbb{R} \to \mathbb{R}^3$ *of* (3.2) *with the minimal period* $T_y > 0$ *and* $y_t \neq w_s$ *for all* t*,* s *in* $\mathbb{R}$*, we have* $|c_y^j| \cap |c_w^j| = \emptyset$ *and* $|C_y^j| \cap |C_w^j| = \emptyset$ *for all* $j \in \{0, 1, 2\}$*.*

Remark 3.1. For $j \in \{0, 1, 2\}$ and t_0^j and t_1^j in Proposition 3.2(ii), we have $w^j(t_0^j) \neq 0$ and $w^j(t_1^j) \neq 0$. This follows from Proposition 3.2(iv) in combination with the last statement in Proposition 2.6 and the definition of R.

Remark 3.2. Assume that $w : \mathbb{R} \to \mathbb{R}^3$ is a nonconstant periodic solution of (3.2) with the minimal period $T_w > 0$. If w^0 has a zero, then for each $j \in \{0, 1, 2\}$, by Proposition 3.1 and Proposition 3.2(ii) and (iii), there exists a $t_2^j \in [0, T_w]$ such that $w^j(t_2^j) = 0$ and $\dot{w}^j(t_2^j) > 0$. Moreover, the following statements are true.

(i) $(-1)^l w^j(t) > 0$ for all $l \in \mathbb{Z}$ and $t \in (t_2^j + \frac{l}{2}T_w, t_2^j + \frac{l+1}{2}T_w)$.

(ii) $\{t \in \mathbb{R} : w^j(t) = 0\} = \{t_2^j + \frac{l}{2}T_w : l \in \mathbb{Z}\}$.

(iii) All zeros of w^j are simple, in particular, $(-1)^l \dot{w}^j(t_2^j + \frac{l}{2}T_w) > 0$ for all $l \in \mathbb{Z}$.

Remark 3.3. Assume that $w : \mathbb{R} \to \mathbb{R}^3$ is a nonconstant periodic solution of (3.2) with the minimal period $T_w > 0$. For each $j \in \{0, 1, 2\}$, choose $t_3^j \in [0, T_w]$ such that $w^j(t_3^j) = \max\{w^j(t) : t \in \mathbb{R}\}$. If w^0 has a zero, then by Proposition 3.1, Proposition 3.2(ii) and (iii), $(-1)^l \dot{w}^j(t) < 0$ for all $l \in \mathbb{Z}$ and $t \in (t_3^j + \frac{l}{2}T_w, t_3^j + \frac{l+1}{2}T_w)$.

Observe that for every solution $z : \mathbb{R} \to \mathbb{R}^3$ of (3.1) and for the corresponding solution $w : \mathbb{R} \ni t \mapsto z(\tau t) \in \mathbb{R}^3$ of (3.2) and for every $j \in \{0, 1, 2\}$, we have

$$\{(z^j(t), \dot{z}^j(t))^{\mathrm{tr}} : t \in \mathbb{R}\} = \left\{ \begin{pmatrix} w^j(t) \\ \frac{1}{\tau}\dot{w}^j(t) \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Note that for every solution $w : \mathbb{R} \to \mathbb{R}^3$ of (3.2) and for every $t \in \mathbb{R}$, the values $w^j(t)$ and $\dot{w}^j(t)/\tau$ uniquely determine $w^{j+1}(t)$. Here, as before, $w^3(t) = w^0(t-1)$. Proposition 3.2 combined with these facts yields the following result.

COROLLARY 3.3. *Let* $z : \mathbb{R} \to \mathbb{R}^3$ *be a nonconstant periodic solution of* (3.1) *with the minimal period* $T_z > 0$*. Then the following statements are true.*

(i) *For each* $j \in \{0, 1, 2\}$*, the map* $Z^j : [0, T_z] \ni t \mapsto (w^j(t), \dot{w}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ *is a simple closed curve.*

(ii) *If* z^0 *has a zero, then* $0 \in \mathrm{int}(Z^j)$ *for all* $j \in \{0, 1, 2\}$*.*

(iii) *Let* $x : \mathbb{R} \to \mathbb{R}^3$ *be a nonconstant periodic solution of* (3.1) *with the minimal period* $T_x > 0$*. Suppose that the functions* $w : \mathbb{R} \ni t \mapsto z(\tau t) \in \mathbb{R}^3$ *and* $y : \mathbb{R} \ni t \mapsto x(\tau t) \in \mathbb{R}^3$ *satisfy* $y_t \neq w_s$ *for all* t*,* $s \in \mathbb{R}$*. Then, for every* $j \in \{0, 1, 2\}$*, the traces of* Z^j *and of the simple closed curve* $X^j : [0, T_x] \ni t \mapsto (x^j(t), \dot{x}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ *are disjoint. An analogous statement holds if* $(z^j(t), \dot{z}^j(t))^{\mathrm{tr}}$ *and* $(x^j(t), \dot{x}^j(t))^{\mathrm{tr}}$ *are replaced with* $\pi^j(z_t)$ *and* $\pi^j(x_t)$*, respectively.*

PROPOSITION 3.4. *Let* $w : \mathbb{R} \to \mathbb{R}^3$ *be a nonconstant periodic solution of* (3.2) *with the minimal period* $T_w > 0$*. If* $V(w_t) = 2k + 2$ *for some* $k \in \mathbb{N}$ *and for all* $t \in \mathbb{R}$*, then there exists* $\alpha \in [0, T_w)$ *such that* $(w^1(t), w^2(t), w^0(t-1)) = (w^0(t+\alpha), w^1(t+\alpha), w^2(t+\alpha))$ *for all* $t \in \mathbb{R}$*.*

Proof. Define $v : \mathbb{R} \ni t \mapsto (w^1(t), w^2(t), w^0(t-1))^{\mathrm{tr}} \in \mathbb{R}^3$. By Proposition 2.2, v is also a nonconstant periodic solution of (3.2) with the minimal period T_w. By Proposition 3.1 and the fact that $V(w_t) = 2k+2 > 0$, we know that, for any $j \in \{0, 1, 2\}$, w^j and v^j have a zero. Suppose the desired result is not true. Then $w_s \neq v_t$ for all s, $t \in \mathbb{R}$. By Proposition 3.2(iii), $0 \in \mathrm{int}(c_v^j)$ and $0 \in \mathrm{int}(c_w^j)$ for each $j \in \{0, 1, 2\}$. Then by Proposition 3.2(v), for each $j \in \{0, 1, 2\}$, either $c_w^j \subsetneqq \mathrm{int}(c_v^j)$ or $c_v^j \subsetneqq \mathrm{int}(c_w^j)$. Without loss of generality, we may assume that $c_w^0 \subsetneqq \mathrm{int}(c_v^0)$. It follows from the definitions of c_w^0 and c_v^0 that $w^0(\mathbb{R}) \subsetneqq v^0(\mathbb{R}) = w^1(\mathbb{R})$ and $w^1(\mathbb{R}) \subsetneqq v^1(\mathbb{R}) = w^2(\mathbb{R})$. The latter, combined with the fact that either $c_w^1 \subsetneqq \mathrm{int}(c_v^1)$ or $c_v^1 \subsetneqq \mathrm{int}(c_w^1)$, implies $c_w^1 \subsetneqq \mathrm{int}(c_v^1)$. Otherwise, we have $c_v^1 \subsetneqq \mathrm{int}(c_w^1)$, which implies that $v^1(\mathbb{R}) = w^2(\mathbb{R}) \subsetneqq w^1(\mathbb{R})$, a contradiction. It follows from $c_w^1 \subsetneqq \mathrm{int}(c_v^1)$ that $w^2(\mathbb{R}) \subsetneqq v^2(\mathbb{R}) = w^0(\mathbb{R})$. Therefore, we have obtained that $w^0(\mathbb{R}) \subsetneqq w^1(\mathbb{R}) \subsetneqq w^2(\mathbb{R}) \subsetneqq w^0(\mathbb{R})$. This is a contradiction and hence the proof is complete. ∎

PROPOSITION 3.5. *Let $w : \mathbb{R} \to \mathbb{R}^3$ be a nonconstant periodic solution of* (3.2) *with the minimal period $T_w > 0$. If $V(w_t) = 2k+2$ for some $k \in \mathbb{N}$ and for all $t \in \mathbb{R}$, then there exists $\delta \in (-\frac{1}{2}, 1]$ such that the following results are true.*

(i) $1 = kT_w + \delta T_w$.

(ii) $(w^1(t), w^2(t), w^0(t-1)) = (w^0(t+\alpha), w^1(t+\alpha), w^2(t+\alpha))$ *for all* $t \in \mathbb{R}$, *where* $\alpha = \frac{1-\delta}{3} T_w$.

Proof. By Proposition 3.1 and the definition of V, we know that w^j has a zero for all $j \in \{0, 1, 2\}$ since $V(w_t) = 2k+2 > 0$ for all $t \in \mathbb{R}$.

First, we show (i) by way of contradiction. Suppose that there is no $\delta \in (-\frac{1}{2}, 1]$ such that $1 = kT_w + \delta T_w$. Then it is easy to see that either $\frac{1}{T_w} > k+1$ or $\frac{1}{T_w} \leq k - \frac{1}{2}$. If the former holds, then $1 > (k+1)T_w$. By Proposition 3.2(ii), we may pick up $s^* \in \mathbb{R}$ such that $w^0(s^*) = 0$ and $\dot{w}^0(s^*) > 0$. Let $t^* = s^* + \min\{\frac{T_w}{2}, \frac{1-(k+1)T_w}{2}\}$ and $t^{**} = \max\{t^* - 1, s^* - (k+1+\frac{1}{2})T_w\}$. Then $t^* \in (s^*, s^* + \frac{T_w}{2}]$, $t^* - 1 \leq s^* - \frac{1+(k+1)T_w}{2} \leq s^* - (k+1)T_w$ and thus $\max\{t^* - 1, s^* - (k+1+\frac{1}{2})T_w\} = t^{**} < s^* - (k+1)T_w$. It follows from Proposition 3.2(ii)–(iii) that $w^0(t) > 0$ for all $t \in (s^*, t^*)$, $w^0(t) < 0$ for all $t \in (s^* - (l+\frac{1}{2})T_w, s^* - lT_w)$ and all $l \in \{0, 1, 2, \ldots, k\}$, $w^0(t) > 0$ for all $t \in (s^* - (l+1)T_w, s^* - (l+\frac{1}{2})T_w)$ and all $l \in \{0, 1, 2, \ldots, k\}$, and $w^0(t) < 0$ for all $t \in (t^{**}, s^* - (k+1)T_w)$. Then $w^0(\min\{t^*, s^* + \frac{T_w}{4}\}) > 0$, $w^0(t^{**}) < 0$ and $(-1)^{l+1} w^0(s^* - \frac{(2l+1)T_w}{4}) > 0$ for all $l \in \{0, 1, 2, \ldots, 2k+1\}$. By the definitions of sc and V, $\mathrm{sc}(w_{t^*}; [-1, 0]) \geq 2k+3$ and hence $V(w_{t^*}) \geq 2k+4$, a contradiction. If the latter holds, then, by a similar argument with obvious modifications, we can obtain $V(w_{\tilde{t}}) \leq 2k$ for some $\tilde{t} \in \mathbb{R}$, a contradiction. Therefore, $1 = kT_w + \delta T_w$ for some $\delta \in (-\frac{1}{2}, 1]$ and (i) is proved.

Now, we prove (ii). By Proposition 3.4, there exists $\alpha \in [0, T_w)$ such that $w^1(t) = w^0(t+\alpha)$, $w^2(t) = w^1(t+\alpha)$, and $w^0(t-1) = w^2(t+\alpha)$ for all $t \in \mathbb{R}$. It follows that $w^0(t-1) = w^0(t+3\alpha)$ for all $t \in \mathbb{R}$. Since w is periodic with the minimal period T_w, there exists an integer m such that $3\alpha + 1 = mT_w$. We distinguish three cases to show that $m = k+1$.

Case 1. $\delta \in (-\frac{1}{2}, 0]$. Then $m = \frac{1+3\alpha}{T_w} = \frac{kT_w + \delta T_w + 3\alpha}{T_w}$ implies that $m \in \{k, k+1, k+2\}$. We now show, by way of contradiction, that $m = k+1$. Suppose that $m \neq k+1$. Then $m = k+2$ or $m = k$. If $m = k+2$, then $\alpha = \frac{2-\delta}{3} T_w$ and hence $w^1(t) = w^0(t + \frac{2-\delta}{3} T_w) = w^0(t - \frac{1+\delta}{3} T_w)$ for all $t \in \mathbb{R}$. From (3.2) we have $\dot{w}^0(t) = -\tau\mu w^0(t) + \tau f(w^0(t - \frac{1+\delta}{3} T_w))$ for all $t \in \mathbb{R}$. By Proposition 3.2(ii)–(iii), there exists $r_1 \in \mathbb{R}$ such that $w^0(t) > 0$ for all $t \in (r_1, r_1 + \frac{1}{2} T_w)$. Since $\frac{1+\delta}{3} \in [0, \frac{1}{2}]$, we have $w^0(t) > 0$ for all $t \in (r_1, r_1 + \frac{1+\delta}{3} T_w)$. This, combined with Remark 2.1, gives that $w^0(t) > 0$ for all large t. It follows from the periodicity of w that $w^0(t) > 0$ for all $t \in \mathbb{R}$, a contradiction. If $m = k$, then $\alpha = -\frac{\delta}{3}T_w$ and hence $w^1(t) = w^0(t - \frac{\delta}{3}T_w)$ and $w^2(t) = w^0(t - \frac{2\delta}{3}T_w)$ for all $t \in \mathbb{R}$. By Proposition 3.2(ii)–(iii), there exists $s_1 > 0$ such that $w^0(s_1) = 0$ and $\dot{w}^0(s_1) > 0$. Let $s_2 = s_1 + \frac{1+2\delta}{4}T_w$. Then, by Proposition 3.2(ii)–(iii) and the choices of s_1 and s_2, we know that $w^0(t) > 0$ for all $t \in (s_1, s_2)$, $w^0(t) > 0$ for all $t \in (s_2 - 1, s_1 - \frac{2k-1}{2}T_w)$, $(-1)^l w^0(t) < 0$ for all $t \in (s_1 - (\frac{l+1}{2})T_w, s_1 - \frac{lT_w}{2})$ and all $l \in \{0, 1, 2, \dots, 2k-2\}$. Moreover, $w^1(s_2) = w^0(s_2 - \frac{\delta}{3}T_w) > 0$ and $w^2(s_2) = w^0(s_2 - \frac{2\delta}{3}T_w) > 0$ since $s_2 - \frac{\delta}{3}T_w \in [s_1, s_1 + \frac{T_w}{2}]$ and $s_2 - \frac{2\delta}{3}T_w \in [s_1, s_1 + \frac{T_w}{2}]$. It follows from the definitions of sc and V that $\mathrm{sc}(w_{s_2}) = 2k$ and thus $V(w_{s_2}) = 2k$, a contradiction. Therefore, $m = k+1$.

Case 2. $\delta \in (0, \frac{1}{2}]$. In this case, as before, we can see that $m \in \{k+1, k+2, k+3\}$. Again, we show by way of contradiction that $m = k+1$. Suppose that $m \neq k+1$. Then $m \in \{k+2, k+3\}$. It follows from $\alpha \in \{\frac{2-\delta}{3}T_w, \frac{3-\delta}{3}T_w\}$ that $\alpha - T_w \in [-\frac{1}{2}T_w, 0]$. Thus $w^1(t) = w^0(t+\alpha) = w^0(t + \alpha - T_w)$ for all $t \in \mathbb{R}$. From (3.2) we have $\dot{w}^0(t) = -\tau\mu w^0(t) + \tau f(w^0(t - (T_w - \alpha)))$ for all $t \in \mathbb{R}$. By an argument similar to that in excluding $m = k+2$ in Case 1, we can get $w^j(t) > 0$ for all $j \in \{0, 1, 2\}$ and $t \in \mathbb{R}$, a contradiction. Therefore, $m = k+1$.

Case 3. $\delta \in (\frac{1}{2}, 1]$. In this case, again we have $m \in \{k+1, k+2, k+3\}$. We still use by way of contradiction to show that $m = k+1$. Suppose that $m \neq k+1$. Then $m \in \{k+2, k+3\}$. First, suppose $m = k+3$. Then $\alpha = \frac{3-\delta}{3}T_w$ and hence $\alpha - \frac{1}{2}T_w = -\frac{\delta}{3}T_w \in [-\frac{T_w}{2}, 0]$. Again, an argument similar to that in excluding $m = k+2$ in Case 1 will produce a contradiction. Now, suppose $m = k+2$. By Proposition 3.2(ii)–(iii), there exists $t_1 > 0$ such that $w^0(t_1) = 0$ and $\dot{w}^0(t_1) > 0$. Let $t_2 = t_1 + \frac{\delta}{2}T_w$. Then, by the choices of t_1 and t_2, we know that $w^0(t) > 0$ for all $t \in (t_1, t_2)$, $w^0(t) > 0$ for all $t \in (t_2 - 1, t_1 - (k+2)T_w)$, $(-1)^l w^0(t) < 0$ for all $t \in (t_1 - (\frac{l+1}{2})T_w, t_1 - \frac{lT_w}{2})$ and all $l \in \{0, 1, 2, \dots, 2k+3\}$. It follows from the definitions of sc and V that $\mathrm{sc}(w_{t_2}; [-1, 0]) \geq 2k+5$ and thus $V(w_{t_2}) \geq 2k+6$, a contradiction. Therefore, $m = k+1$. ∎

Conversely, let $w : \mathbb{R} \to \mathbb{R}^3$ be a nonconstant periodic solution of (3.2) with the minimal period $T_w > 0$. Suppose that w^0 has a zero and there exists $k \in \mathbb{N}$ such that $w^1(t) = w^0(t + \alpha)$ for all $t \in \mathbb{R}$, where $\alpha = \frac{(k+1)T_w - 1}{3} \in [0, T_w)$. Then, by Proposition 3.2(iv) and the fact that w^0 has a zero, we can infer $V(w_t) = 2 + 2l$ for all $t \in \mathbb{R}$ and some $l \in \mathbb{N}$. Thus Proposition 3.5 tells us that $(w^1(t), w^2(t), w^0(t-1)) = (w^0(t + \alpha^*), w^1(t + \alpha^*), w^2(t + \alpha^*))$ for all $t \in \mathbb{R}$, where $\alpha^* = \frac{(l+1)T_w - 1}{3} \in [0, \frac{T_w}{2}]$. We claim that $l = k$. If not, then $|\alpha - \alpha^*| \in (0, T_w)$. It follows from $w^1(t) = w^0(t + \alpha) = w^0(t + \alpha^*)$ for all $t \in \mathbb{R}$ that $w^0(t) = w^0(t + |\alpha - \alpha^*|)$ for all $t \in \mathbb{R}$. This implies that $|\alpha - \alpha^*|$ is also a period of w, which contradicts with the fact that T_w is the minimal period of w. Therefore, $l = k$ and we have proved the following result.

COROLLARY 3.6. *Let $w : \mathbb{R} \to \mathbb{R}^3$ be a nonconstant periodic solution of* (3.2) *with the minimal period $T_w > 0$. If w^0 has a zero and there exists $k \in \mathbb{N}$ such that $\alpha = \frac{(k+1)T_w - 1}{3} \in [0, T_w)$ and $w^1(t) = w^0(t + \alpha)$ for all $t \in \mathbb{R}$, then $V(w_t) = 2k + 2$ for all $t \in \mathbb{R}$.*

For each $\theta \in [0, 2\pi)$, define $l(\theta) = \{r(\cos\theta, \sin\theta)^{\mathrm{tr}} \in \mathbb{R}^2 : r \geq 0\}$.

PROPOSITION 3.7. *Let $z : \mathbb{R} \to \mathbb{R}^3$ be a nonconstant periodic solution of* (3.1) *with the minimal period $T > 0$. Assume that z^0 has a zero. For a given $j \in \{0, 1, 2\}$, let a maximum $a \in \mathbb{R}$ of z^j be given. Then the functions*

$$\psi^j : [0, 2\pi) \ni \theta \mapsto \inf\{t \in (a, a+T] : (z^j(t), \dot{z}^j(t))^{\mathrm{tr}} \in l(\theta)\} \in \mathbb{R}$$

and

$$\Psi^j : [0, 2\pi) \ni \theta \mapsto \sup\{t \in (a, a+T] : (z^j(t), \dot{z}^j(t))^{\mathrm{tr}} \in l(\theta)\} \in \mathbb{R}$$

are strictly decreasing.

Proof. We only show that ψ^j is strictly decreasing as the proof for Ψ^j to be strictly decreasing is similar. Observe that $w : \mathbb{R} \ni t \mapsto z(\tau t) \in \mathbb{R}^3$ is a nonconstant periodic solution of (3.2). By Remarks 3.2 and 3.3, there exist reals a_1, a_2, a_3 such that $a < a_1 < a_2 < a_3 < a + T$ and

$$\begin{aligned}
&z^j(a_1) = \dot{z}^j(a_2) = z^j(a_3) = 0,\\
&z^j(t) > 0 \text{ for } a \le t < a_1 \text{ and } a_3 < t \le a + T,\\
&z^j(t) < 0 \text{ for } a_1 < t < a_3,\\
&\dot{z}^j(t) < 0 \text{ for } a < t < a_2, \text{ and}\\
&\dot{z}^j(t) > 0 \text{ for } a_2 < t < a + T.
\end{aligned}$$

For a given $\theta \in (0, \pi/2)$, observe that

$$(z^j(t), \dot{z}^j(t))^{\mathrm{tr}} \in l(\theta) \qquad \text{for some } t \in (a, a+T]$$

if and only if

$$t \in (a_3, a+T) \quad \text{and} \quad \theta = \arctan \frac{\dot{z}^j(t)}{z^j(t)}.$$

Clearly, $\psi^j(0) = a + T$, $\psi^j(\pi/2) = a_3$, and $\psi^j(\theta) \in (a_3, a+T)$ for all $\theta \in (0, \frac{\pi}{2})$. The function $(a_3, a+T) \ni t \mapsto \arctan \frac{\dot{z}^j(t)}{z^j(t)} \in \mathbb{R}$ is continuous with range in $(0, \pi/2)$. Moreover, $\lim_{t \to a_3^+} \arctan \frac{\dot{z}^j(t)}{z^j(t)} = \frac{\pi}{2}$ and $\lim_{t \to (a+T)^-} \arctan \frac{\dot{z}^j(t)}{z^j(t)} = 0$. It follows that

$$\psi^j(\theta) = \min \left\{ t \in (a_3, a+T) : \theta = \arctan \frac{\dot{z}^j(t)}{z^j(t)} \right\}.$$

Let θ_1 and θ_2 be given in $(0, \pi/2)$ with $\theta_1 < \theta_2$. By way of contradiction, we show $\psi^j(\theta_1) > \psi^j(\theta_2)$. If not, we have $\psi^j(\theta_1) \le \psi^j(\theta_2)$. Then there exist t_1 and t_2 in $(a_3, a+T)$ such that

$$t_l = \psi^j(\theta_l) \quad \text{and} \quad \theta_l = \arctan \frac{\dot{z}^j(t_l)}{z^j(t_l)}, \qquad l = 1, 2.$$

Since $t_1 = \psi^j(\theta_1) \le \psi^j(\theta_2) = t_2$ and $\theta_1 < \theta_2$, we have $t_1 < t_2$. This, combined with the facts that $\lim_{t \to a_3^+} \arctan \frac{\dot{z}^j(t)}{z^j(t)} = \frac{\pi}{2}$ and $0 < \arctan \frac{\dot{z}^j(t_1)}{z^j(t_1)} < \arctan \frac{\dot{z}^j(t_2)}{z^j(t_2)} < \frac{\pi}{2}$, implies the existence of $t_3 \in (a_3, t_1)$ such that $\arctan \frac{\dot{z}^j(t_3)}{z^j(t_3)} = \arctan \frac{\dot{z}^j(t_2)}{z^j(t_2)} = \theta_2$. This contradicts with the choice of $\psi^j(\theta_2)$ and t_2. It follows that ψ^j is strictly decreasing on $[0, \pi/2]$. Similarly, one can show that ψ^j is strictly decreasing on each of the three intervals, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, and $[3\pi/2, 2\pi)$. This completes the proof. ∎

The next result is the key to obtaining results on uniqueness of periodic orbits and results on absence of rapidly oscillating periodic solutions. It is analogous to earlier results in [3, 15] on periodic solutions of some scalar delay differential equations.

PROPOSITION 3.8. *Let $k \in \mathbb{N} \setminus \{0, 1\}$ and $\tau \ge 1$. Suppose that f satisfies* (H1) *and*

(H2) *the function* $(0,\infty) \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)}$ *is strictly decreasing.*

Also suppose $g : \mathbb{R} \to \mathbb{R}$ *is an odd and continuously differentiable function satisfying*

(H3) $g'(0) = f'(0)$, *and* $g(\xi) > f(\xi)$ *and* $\frac{g'(\xi)}{g(\xi)} > \frac{f'(\xi)}{f(\xi)}$ *for all* $\xi \in (0,\infty)$.

Let x *be a nonconstant periodic solution of* (1.1) *with the minimal period* $T_x > 0$ *and let* z *be a nonconstant periodic solution of* (3.1) *with the minimal period* $T_z > 0$. *Suppose* $w : \mathbb{R} \ni t \mapsto z(\tau t) \in \mathbb{R}^3$ *satisfies* $V(x_t) = V(w_t) = 2k$ *for all* $t \in \mathbb{R}$. *Define* $X^j : [0, T_x] \ni t \mapsto (x^j(t), \dot{x}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ *and* $Z^j : [0, T_z] \ni t \mapsto (z^j(t), \dot{z}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ *for all* $j \in \{0, 1, 2\}$. *If* $|Z^j| \subset |X^j| \cup \mathrm{ext}(X^j)$ *and* $r|Z^j| \subset \mathrm{ext}(X^j)$ *for all* $r \in (1,\infty)$ *and for all* $j \in \{0, 1, 2\}$, *then* $|Z^j| \cap |X^j| = \emptyset$ *for all* $j \in \{0, 1, 2\}$.

Proof. By applying Proposition 3.4, we remark that there exist $\alpha^* \in [0, T_x)$ and $\beta^* \in [0, T_z)$ such that $x^2(t) = x^1(t+\alpha^*) = x^0(t+2\alpha^*)$ and $z^2(t) = z^1(t+\beta^*) = z^0(t+2\beta^*)$ for all $t \in \mathbb{R}$. Thus, by the definitions of X^j and Z^j, we have $|X^0| = |X^1| = |X^2|$ and $|Z^0| = |Z^1| = |Z^2|$.

By way of contradiction, we may assume that $|Z^j| \cap |X^j| \neq \emptyset$ for some $j \in \{0, 1, 2\}$. This, combined with the above discussions, implies $|Z^0| \cap |X^0| \neq \emptyset$.

First, it follows from Proposition 3.1 and the facts that $k \in \mathbb{N} \setminus \{0, 1\}$ and $V(x_t) = V(w_t) = 2k$ (≥ 4) for all $t \in \mathbb{R}$ that $x^0(t^*) = 0$ for some $t^* \in \mathbb{R}$ and $z^0(s^*) = 0$ for some $s^* \in \mathbb{R}$; i.e., x^0 and z^0 have zeros. Then Corollary 3.3(ii) implies $0 \in \mathrm{int}(X^0)$ and $0 \in \mathrm{int}(Z^0)$. Note that $|Z^0| \subseteq |X^0| \cup \mathrm{ext}(X^0)$ implies that $|X^0| \subseteq |Z^0| \cup \mathrm{int}(Z^0)$. Thus, for each $\theta \in [0, 2\pi)$, any point of $l(\theta) \cap |Z^0|$ is not closer to $0 \in \mathbb{R}^2$ than any point of $l(\theta) \cap |X^0|$. Using $|Z^0| \cap |X^0| \neq \emptyset$ and a translation if necessary, without loss of generality, we may assume $X^0(0) = Z^0(0)$, i.e.,

$$x^0(0) = z^0(0) \quad \text{and} \quad \dot{x}^0(0) = \dot{z}^0(0). \tag{3.4}$$

We distinguish two cases to complete the proof.

Case 1. $\dot{x}^0(0) = \dot{z}^0(0) = 0$. Then $c = x^0(0) = z^0(0) \neq 0$ since $0 \in \mathrm{int}(X^0)$ and $0 \in \mathrm{int}(Z^0)$. Without loss of generality, we assume $c > 0$ as the proof for the case where $c < 0$ is similar. Proposition 3.2(iii) yields that x and z have the special symmetry $x^0(t + T_x/2) = -x^0(t)$ and $z^0(t + T_z/2) = -z^0(t)$ for all $t \in \mathbb{R}$. This, combined with Remarks 3.2 and 3.3, implies

$$\begin{array}{lcl} c = \max\limits_{t\in\mathbb{R}} x^0(t) = \max\limits_{t\in\mathbb{R}} z^0(t), & & -c = \min\limits_{t\in\mathbb{R}} x^0(t) = \min\limits_{t\in\mathbb{R}} z^0(t), \\ \dot{x}^0(t) > 0 \quad \text{for } -\frac{T_x}{2} < t < 0, & & \dot{z}^0(t) > 0 \quad \text{for } -\frac{T_z}{2} < t < 0, \\ x^0(-\frac{T_x}{2}) = -c \text{ and } \dot{x}^0(-\frac{T_x}{2}) = 0, & \text{and} & z^0(-\frac{T_z}{2}) = -c \text{ and } \dot{z}^0(-\frac{T_z}{2}) = 0. \end{array}$$

Let $T^* = \min\{T_x, T_z\}$. We claim that $z^0(s) \leq x^0(s)$ for $-T^*/2 \leq s \leq 0$.

We now prove the claim. Let $(x^0)^{-1}$ and $(z^0)^{-1}$ be the inverses of the functions $[-\frac{T_x}{2}, 0] \ni t \mapsto x^0(t) \in \mathbb{R}$ and $[-\frac{T_z}{2}, 0] \ni t \mapsto z^0(t) \in \mathbb{R}$, respectively. Then the domain of $(x^0)^{-1}$ is the same as that of $(z^0)^{-1}$, which is $[-c, c]$. The functions

$$\phi_x^0 : [-c, c] \ni u \mapsto \dot{x}^0((x^0)^{-1}(u)) \in \mathbb{R}$$

and

$$\phi_z^0 : [-c, c] \ni u \mapsto \dot{z}^0((z^0)^{-1}(u)) \in \mathbb{R}$$

satisfy $\phi_x^0(-c) = \phi_x^0(c) = \phi_z^0(-c) = \phi_z^0(c) = 0$, and $\phi_x^0(u) > 0$ and $\phi_z^0(u) > 0$ for all $u \in (-c, c)$. The arcs $\Omega_x = \{X^0(t) : t \in [-\frac{T_x}{2}, 0]\}$ and $\Omega_z = \{Z^0(t) : t \in [-\frac{T_z}{2}, 0]\}$

coincide with the graphs $\{(u,\phi_x^0(u))^{\mathrm{tr}} : u\in[-c,c]\}$ and $\{(u,\phi_z^0(u))^{\mathrm{tr}} : u\in[-c,c]\}$, respectively. From the special symmetry of x and z we obtain $|X^0| = \Omega_x\cup(-\Omega_x)$ and $|Z^0| = \Omega_z\cup(-\Omega_z)$. Hence

$$\mathrm{int}(X^0) = \{(u,v)^{\mathrm{tr}} : u\in(-c,c), -\phi_x^0(-u) < v < \phi_x^0(u)\}.$$

From $|Z^0|\subset|X^0|\cup\mathrm{ext}(X^0)$, we conclude

$$\phi_x^0(u)\le\phi_z^0(u) \qquad \text{for } -c\le u\le c. \tag{3.5}$$

The functions x and z satisfy

$$\dot{x^0}(t) = \phi_x^0(x^0(t)) \qquad \text{for all } t\in[-\tfrac{T_x}{2},0]$$

and

$$\dot{z^0}(t) = \phi_z^0(z^0(t)) \qquad \text{for all } t\in[-\tfrac{T_z}{2},0].$$

For $-T_z/2 < s_1 < s_2 < 0$, the last equation and the inequality $\dot{z^0}(t) > 0$ for $-T_z/2 < t < 0$ combined yield

$$\int_{z^0(s_1)}^{z^0(s_2)}\frac{du}{\phi_z^0(u)} = \int_{s_1}^{s_2}\frac{\dot{z^0}(t)}{\phi_z^0(z^0(t))}dt = s_2 - s_1.$$

Similarly,

$$\int_{x^0(s_1)}^{x^0(s_2)}\frac{du}{\phi_x^0(u)} = s_2 - s_1 \quad \text{for } \tfrac{-T_x}{2} < s_1 < s_2 < 0.$$

By the continuity of z and x at 0, we have

$$\int_{z^0(s)}^{c}\frac{du}{\phi_z^0(u)} = -s \quad \text{for } \tfrac{-T_z}{2} < s\le 0$$

and

$$\int_{x^0(s)}^{c}\frac{du}{\phi_x^0(u)} = -s \quad \text{for } \tfrac{-T_x}{2} < s\le 0.$$

Thus, for $-T^*/2 < s \le 0$, we obtain immediately that

$$\int_{z^0(s)}^{c}\frac{du}{\phi_z^0(u)} = \int_{x^0(s)}^{c}\frac{du}{\phi_x^0(u)}.$$

With the help of (3.5), we know $z^0(s)\le x^0(s)$ for $\frac{-T^*}{2} < s\le 0$. Because of continuity, we easily see that the claim holds.

If $T_z > T_x$, then from the claim above and from $x^0(-T_x/2) = -c$ we obtain $z^0(-T_x/2)\le x^0(-T_x/2) = -c$. This is impossible since $-T_z/2 < -T_x/2 < 0$, $z^0(-T_z/2) = -c$, and $\dot{z^0}(t) > 0$ for $-T_z/2 < t < 0$. Thus $T_z\le T_x$.

Proposition 3.5 and the facts that $V(x_t) = V(w_t) = 2k$ for all $t\in\mathbb{R}$ imply that there exist δ_x, $\delta_z\in(-\frac{1}{2},1]$ such that $1 = (k-1)T_x + \delta_x T_x = (k-1)\frac{T_z}{\tau} + \delta_z\frac{T_z}{\tau}$, and $x^1(t) = x^0(t+\alpha_x)$, and $z^1(t) = z^0(t+\alpha_z)$ for all $t\in\mathbb{R}$, where $\alpha_x = \frac{1-\delta_x}{3}T_x$

and $\alpha_z = \frac{1-\delta_z}{3}T_z$. Let $\alpha_x^* = \alpha_x - \frac{T_x}{2}$ and $\alpha_z^* = \alpha_z - \frac{T_z}{2}$. Then $\alpha_x^* = \frac{(2k-3)T_x-2}{6}$ and $\alpha_z^* = \frac{(2k-3)T_z-2\tau}{6}$. Thus $0 \geq \alpha_x^* \geq \alpha_z^* \geq -\frac{T^*}{2}$ since k belongs to $\mathbb{N}\setminus\{0,1\}$. Consequently,

$$z^1(0) = z^0(\alpha_z) = -z^0(\alpha_z^*) \geq -x^0(\alpha_z^*) \geq -x^0(\alpha_x^*) = x^0(\alpha_x) = x^1(0). \tag{3.6}$$

Using (1.1), (3.1), $\dot{z}^0(0) = \dot{x}^0(0) = 0$, and $z^0(0) = x^0(0) = c > 0$, we obtain $z^1(0) > 0$, $x^1(0) > 0$, and $g(z^1(0)) = f(x^1(0))$. It follows from (H3) that

$$f(x^1(0)) = g(z^1(0)) > f(z^1(0)).$$

As f is monotone, this implies $z^1(0) < x^1(0)$, which contradicts (3.6).

Case 2. $\dot{z}^0(0) = \dot{x}^0(0) \neq 0$. Without loss of generality, we may assume $\dot{z}^0(0) = \dot{x}^0(0) > 0$ as the proof for the case where $\dot{z}^0(0) = \dot{x}^0(0) < 0$ is similar. Then there exists $\varepsilon > 0$ such that $\dot{z}^0(t) \neq 0$ and $\dot{x}^0(t) \neq 0$ for $t \in (-\varepsilon, \varepsilon)$, and hence there is $\delta > 0$ such that there are inverses $(z^0)^{-1} : (d-\delta, d+\delta) \to \mathbb{R}$ and $(x^0)^{-1} : (d-\delta, d+\delta) \to \mathbb{R}$ of the restrictions of z^0 and x^0 to the open intervals in $(-\varepsilon, \varepsilon)$, respectively, where $d = z^0(0) = x^0(0)$. The maps

$$\eta_z : (d-\delta, d+\delta) \ni u \mapsto \dot{z}^0((z^0)^{-1}(u)) \in \mathbb{R}$$

and

$$\eta_x : (d-\delta, d+\delta) \ni u \mapsto \dot{x}^0((x^0)^{-1}(u)) \in \mathbb{R}$$

are C^1-smooth since x and z are C^2-smooth by (1.1) and (3.1). We have $\eta_z(d) = \dot{z}^0(0) = \dot{x}^0(0) = \eta_x(d) \neq 0$ and, for all $u \in (d-\delta, d+\delta)$,

$$\begin{aligned}
\eta_x'(u) &= \ddot{x}^0((x^0)^{-1}(u))\frac{d}{du}(x^0)^{-1}(u) = \ddot{x}^0((x^0)^{-1}(u))\frac{1}{\dot{x}^0((x^0)^{-1}(u))},\\
\eta_z'(u) &= \ddot{z}^0((z^0)^{-1}(u))\frac{d}{du}(z^0)^{-1}(u) = \ddot{z}^0((z^0)^{-1}(u))\frac{1}{\dot{z}^0((z^0)^{-1}(u))}.
\end{aligned}$$

In particular,

$$\eta_x'(d) = \frac{\ddot{x}^0(0)}{\dot{x}^0(0)} \qquad \text{and} \qquad \eta_z'(d) = \frac{\ddot{z}^0(0)}{\dot{z}^0(0)}. \tag{3.7}$$

The sets $\{(u, \eta_x(u))^{\mathrm{tr}} : u \in (d-\delta, d+\delta)\}$ and $\{(u, \eta_z(u))^{\mathrm{tr}} : u \in (d-\delta, d+\delta)\}$ are graph representations of pieces of $|X^0|$ and $|Z^0|$, respectively. It is not difficult to show that there exists $\gamma > 0$ such that the sets $\{(u,v)^{\mathrm{tr}} : u \in (d-\frac{\delta}{2}, d+\frac{\delta}{2}), \eta_x(u) - \gamma < v < \eta_x(u)\}$ and $\{(u,v)^{\mathrm{tr}} : u \in (d-\frac{\delta}{2}, d+\frac{\delta}{2}), \eta_x(u) < v < \eta_x(u) + \gamma\}$ belong to different connected components of $\mathbb{R}^2 \setminus |X^0|$. Hence, using $|Z^0| \subset |X^0| \cup \mathrm{ext}(X^0)$ and $(d, \eta_z(d)) = (d, \eta_x(d))$, we obtain $\eta_z'(d) = \eta_x'(d)$. This, combined with $\dot{z}^0(0) = \dot{x}^0(0) \neq 0$ and (3.7), implies $\ddot{z}^0(0) = \ddot{x}^0(0)$. Differentiating (1.1) and (3.1) yields

$$\begin{aligned}
\ddot{x}^0(t) &= -\mu\dot{x}^0(t) + f'(x^1(t))\dot{x}^1(t),\\
\ddot{z}^0(t) &= -\mu\dot{z}^0(t) + g'(z^1(t))\dot{z}^1(t).
\end{aligned}$$

From $\ddot{z}^0(0) = \ddot{x}^0(0)$ and $\dot{z}^0(0) = \dot{x}^0(0)$, we get

$$g'(z^1(0))\dot{z}^1(0) = f'(x^1(0))\dot{x}^1(0). \tag{3.8}$$

Evaluating (1.1) and (3.1) at $t = 0$ and using (3.4), one has $g(z^1(0)) = f(x^1(0))$. It follows that either ($z^1(0) > 0$ and $x^1(0) > 0$) or ($z^1(0) < 0$ and $x^1(0) < 0$) or ($z^1(0) = x^1(0) = 0$). In the following, we continue our case by case discussions.

Case 2.1. $z^1(0) > 0$ and $x^1(0) > 0$. Then $f(x^1(0)) = g(z^1(0)) > f(z^1(0))$ gives

$$0 < z^1(0) < x^1(0). \tag{3.9}$$

This, combined with (H2) and (H3), gives

$$z^1(0)\frac{g'(z^1(0))}{g(z^1(0))} > z^1(0)\frac{f'(z^1(0))}{f(z^1(0))} > x^1(0)\frac{f'(x^1(0))}{f(x^1(0))} > 0. \tag{3.10}$$

On the other hand, (3.9) combined with (3.8) and $g(z^1(0)) = f(x^1(0))$ produces

$$z^1(0)\frac{g'(z^1(0))\dot{z}^1(0)}{g(z^1(0))z^1(0)} = x^1(0)\frac{f'(x^1(0))\dot{x}^1(0)}{f(x^1(0))x^1(0)}. \tag{3.11}$$

According to (3.10) and (3.11), we can distinguish three subsubcases to finish the discussion on Case 2.1.

Case 2.1.1. $\dot{z}^1(0) = \dot{x}^1(0) = 0$. Then we have $Z^1(0) \notin \mathrm{int}(X^1)$ and $Z^1(0) \in \{(u,0)^{\mathrm{tr}} \in \mathbb{R}^2 : u > 0\}$. It is easy to see from Proposition 3.2 that $\{(u,0)^{\mathrm{tr}} \in \mathbb{R}^2 : 0 \le u < x^1(0)\} \subset \mathrm{int}(X^1)$. Consequently, $z^1(0) \ge x^1(0)$, a contradiction to (3.9).

Case 2.1.2. $0 < \dot{z}^1(0)/z^1(0) < \dot{x}^1(0)/x^1(0)$. First, choose $a \in \mathbb{R}$ so that $z^1(a) = \max_{t\in\mathbb{R}} z^1(t)$ and $0 \in (a, a + T_z]$. Select a_1, a_2, a_3 so that $a < a_1 < a_2 < a_3 < a + T_z$ and

$$\begin{aligned}
&z^1(t) > 0 \text{ for } a < t < a_1 \text{ and for } a_3 < t \le a + T_z,\\
&z^1(a_1) = \dot{z}^1(a_2) = z^1(a_3) = 0, \qquad \dot{z}^1(t) < 0 \text{ for } a < t < a_2,\\
&z^1(t) < 0 \text{ for } a_1 < t < a_3, \qquad \dot{z}^1(t) > 0 \text{ for } a_2 < t < a + T_z.
\end{aligned}$$

Define $\theta_z = \arctan\frac{\dot{z}^1(0)}{z^1(0)}$ and $\theta_x = \arctan\frac{\dot{x}^1(0)}{x^1(0)}$. Then $0 < \theta_z < \theta_x < \pi/2$, $Z^1(0) \in l(\theta_z)$ and $X^1(0) \in l(\theta_x)$. Let $t_* = \psi^1(\theta_x)$. Observe $t_* \in [a_3, a + T_z)$. Recall that, for $\theta \in [0, 2\pi)$, any point of $|Z^1| \cap l(\theta)$ is not closer to the origin than any point of $|X^1| \cap l(\theta)$. Consequently, $z^1(t_*) \ge x^1(0) > 0$. The monotonicity of ψ^1 and the fact $(z^1(0), \dot{z}^1(0))^{\mathrm{tr}} \in |Z^1| \cap l(\theta_z)$ yield $0 \ge \psi^1(\theta_z) > \psi^1(\theta_x) = t_*$. As z^1 is strictly increasing on $[a_3, a + T_z]$ and $a_3 \le t_* < 0 \le a + T_z$, we infer $z^1(t_*) < z^1(0)$. The last inequality and $z^1(t_*) \ge x^1(0) > 0$ together imply $z^1(0) > x^1(0)$, a contradiction to (3.9).

Case 2.1.3. $\dot{x}^1(0)/x^1(0) < \dot{z}^1(0)/z^1(0) < 0$. Define $\theta_z = \arctan\frac{\dot{z}^1(0)}{z^1(0)} + 2\pi$ and $\theta_x = \arctan\frac{\dot{x}^1(0)}{x^1(0)} + 2\pi$. Then $3\pi/2 < \theta_x < \theta_z < 2\pi$, $Z^1(0) \in l(\theta_z)$ and $X^1(0) \in l(\theta_x)$. We now choose a, a_1, a_2, a_3 as in Case 2.1.2 and apply Proposition 3.7 to z^1 as in Case 2.1.2. Let $t^* = \Psi^1(\theta_x)$. Then $t^* \in (a, a_1]$. Analogously to Case 2.1.2, we find $z^1(t^*) \ge x^1(0) > 0$. The monotonicity of Ψ^1 and $Z^1(0) \in |Z^1| \cap l(\theta_z)$ combined yield $t^* = \Psi^1(\theta_x) > \Psi^1(\theta_z) \ge 0$. Now we can use the inequality $a < 0 < t^* \le a_1$ and the fact that z^1 is strictly decreasing on $(a, a_1]$ to obtain $z^1(0) > z^1(t^*)$. Thus $z^1(0) > x^1(0)$, a contradiction to (3.9).

Case 2.2. $z^1(0) < 0$ and $x^1(0) < 0$. Arguments similar to those used in Case 2.1 will lead to a contradiction.

Case 2.3. $z^1(0) = x^1(0) = 0$. Then (3.8) and $g'(0) = f'(0) \neq 0$ combined imply $\dot{z}^1(0) = \dot{x}^1(0)$. Note that $\dot{z}^1(0) = \dot{x}^1(0) \neq 0$ since $0 \in \operatorname{int}(Z^1)$ and $0 \in \operatorname{int}(X^1)$. This, combined with (1.1) and (3.1), gives $z^2(0)x^2(0) > 0$.

Define $v = (v^0, v^1, v^2)^{\mathrm{tr}} : \mathbb{R} \ni t \mapsto (x^1(t), x^2(t), x^0(t-1))^{\mathrm{tr}} \in \mathbb{R}^3$ and $w = (w^0, w^1, w^2)^{\mathrm{tr}} : \mathbb{R} \ni t \mapsto (z^1(t), z^2(t), z^0(t-1))^{\mathrm{tr}} \in \mathbb{R}^3$. Then $(v^0(0), \dot{v}^0(0))^{\mathrm{tr}} = (w^0(0), \dot{w}^0(0))^{\mathrm{tr}}$, $\dot{v}^0(0) = \dot{w}^0(0) \neq 0$ and $w^1(0)v^1(0) > 0$. By replacing z and x with w and v, respectively, we easily check that all the conditions of Proposition 3.8 hold. Moreover, by applying the same discussions as those in Cases 2.1 and 2.2, we shall also arrive at a contradiction in Case 2.3. This completes the proof. $\square$

Recall from Proposition 3.2 that for any nonconstant periodic solution $x : \mathbb{R} \to \mathbb{R}^3$ of (1.1) there exists $k \in \mathbb{N}$ such that $V(x_t) = 2k$ for all $t \in \mathbb{R}$. For a given $k \in \mathbb{N}$, we say that (1.1) has a periodic orbit in $V^{-1}(2k)$ if it has a nonconstant periodic solution $x : \mathbb{R} \to \mathbb{R}^3$ with $V(x_t) = 2k$ for all $t \in \mathbb{R}$.

In order to introduce quantities, expressed explicitly in terms of μ and $f'(0)$, that characterize the uniqueness and absence of periodic solutions of (1.1), we need information on the distribution of the solutions to the characteristic equation

$$(\lambda + \tau\mu)^3 - (\tau f'(0))^3 e^{-\lambda} = 0 \tag{3.12}$$

of the linear system

$$\begin{cases} \dot{x}^0(t) = -\tau\mu x^0(t) + \tau f'(0)x^1(t) \\ \dot{x}^1(t) = -\tau\mu x^1(t) + \tau f'(0)x^2(t) \\ \dot{x}^2(t) = -\tau\mu x^2(t) + \tau f'(0)x^0(t-1) \end{cases} \tag{3.13}$$

with parameter $\tau > 0$. The discussion is given in Appendix A.

PROPOSITION 3.9. *Suppose that f satisfies the assumptions* (H1) *and* (H2)*. Let $\beta > 1$ and $x : \mathbb{R} \to \mathbb{R}^3$ be a solution of* (1.1)*. Define $g : \mathbb{R} \ni \xi \mapsto \beta f(\frac{\xi}{\beta}) \in \mathbb{R}$. Then g satisfies the assumption* (H3) *and $z : \mathbb{R} \ni t \mapsto \beta x(t) \in \mathbb{R}^3$ is a solution of* (3.1) *with $\tau = 1$.*

Proof. It is easy to see that z is a solution of (3.1), g is an odd continuously differentiable function, and $g'(0) = f'(0)$. By $\lim_{\xi \to 0}(\xi f'(\xi)/f(\xi)) = 1$ and (H2), we obtain

$$\frac{\xi f'(\xi)}{f(\xi)} < 1 \qquad \text{for all } \xi \in (0, \infty). \tag{3.14}$$

For each given $\xi > 0$, the function $(0, \infty) \ni u \mapsto u f(\frac{\xi}{u}) \in \mathbb{R}$ is strictly increasing since its derivative, $f(\frac{\xi}{u})[1 - \frac{(\xi/u)f'(\xi/u)}{f(\xi/u)}]$, is larger than 0 by (3.14). Thus $g(\xi) = \beta f(\frac{\xi}{\beta}) > f(\xi)$ for all $\xi \in (0, \infty)$ as $\beta > 1$. Moreover, using (H2) and $\beta > 1$, we obtain

$$\frac{g'(\xi)}{g(\xi)} = \frac{1}{\xi} \frac{(\xi/\beta)f'(\xi/\beta)}{f(\xi/\beta)} > \frac{1}{\xi} \frac{\xi f'(\xi)}{f(\xi)} = \frac{f'(\xi)}{f(\xi)}$$

for all $\xi \in (0, \infty)$. Therefore, g satisfies (H3). $\square$

For each $(k, j) \in (\mathbb{N} \times \{0, 1, 2\}) \setminus \{(0, 0)\}$, let $\tau_{k,j} = \frac{6k\pi + 2j\pi - 3 \arccos \frac{\mu}{f'(0)}}{\sqrt{[f'(0)]^2 - \mu^2}}$.

Now we are ready to present the first main result of this section.

THEOREM 3.10. *Suppose* (H1) *and* (H2) *hold. Then the following two statements are true.*

(i) *For every $k \in \mathbb{N} \setminus \{0, 1\}$, system* (1.1) *has at most one periodic orbit in $V^{-1}(2k)$.*

(ii) *If* $(k,j) \in \{(0,0)\} \cup \{(k,j) \in (\mathbb{N} \times \{0,1,2\}) \setminus \{(0,0),(0,1)\} : \tau_{k,j} \geq 1\}$, *then system* (1.1) *has no periodic orbit in* $V^{-1}(6k+2j)$.

Proof. (i) By way of contradiction, suppose that there exist two nonconstant periodic solutions $x : \mathbb{R} \to \mathbb{R}^3$ and $y : \mathbb{R} \to \mathbb{R}^3$ of (1.1) with the minimal periods $T_x > 0$ and $T_y > 0$, respectively, satisfying $\{x_t : t \in [0,T_x]\} \cap \{y_t : t \in [0,T_y]\} = \emptyset$ and $V(x_t) = V(y_t) = 2k > 0$ for all $t \in \mathbb{R}$. Then, by Proposition 3.1 and the definition of V, x^j and y^j have a zero for all $j \in \{0,1,2\}$.

For each $j \in \{0,1,2\}$, define $X^j : [0,T_x] \ni t \mapsto (x^j(t), \dot{x}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ and $Y^j : [0,T_y] \ni t \mapsto (y^j(t), \dot{y}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$. We remark that Proposition 3.4 implies $|X^0| = |X^1| = |X^2|$ and $|Y^0| = |Y^1| = |Y^2|$. Since x^0 and y^0 have zeros, by Corollary 3.3(ii), $0 \in \mathrm{int}(X^0)$ and $0 \in \mathrm{int}(Y^0)$. Moreover, the last statement of Corollary 3.3(iii) implies $|X^0| \cap |Y^0| = \emptyset$. Thus either $|X^0| \subset \mathrm{int}(Y^0)$ or $|Y^0| \subset \mathrm{int}(X^0)$. It suffices to consider the case where $|Y^0| \subset \mathrm{int}(X^0)$ since the other case can be handled similarly. Suppose $|Y^0| \subset \mathrm{int}(X^0)$. Then $\rho|Y^0| \subset \mathrm{ext}(X^0)$ for all sufficiently large $\rho > 0$. Let

$$\beta = \inf\{\rho \geq 0 : \rho'|Y^0| \subset \mathrm{ext}(X^0) \text{ for all } \rho' \in (\rho, \infty)\}.$$

Then $\beta > 1$, $\rho|Y^0| \subset \mathrm{ext}(X^0)$ for all $\rho \in (\beta, \infty)$, $\beta|Y^0| \subset |X^0| \cup \mathrm{ext}(X^0)$, and $\beta|Y^0| \cap |X^0| \neq \emptyset$. This, combined with the facts that $|X^0| = |X^1| = |X^2|$ and $|Y^0| = |Y^1| = |Y^2|$, implies that $\beta|Y^j| \subset |X^j| \cup \mathrm{ext}(X^j)$ and $\rho|Y^j| \subset \mathrm{ext}(X^j)$ for all $\rho \in (\beta, \infty)$ and $j \in \{0,1,2\}$. By Proposition 3.9, the function $g : \mathbb{R} \ni \xi \mapsto \beta f(\frac{\xi}{\beta}) \in \mathbb{R}$ satisfies (H3) and $z : \mathbb{R} \ni t \mapsto \beta y(t) \in \mathbb{R}^3$ is a T_y-periodic solution of (3.1) with $\tau = 1$. Clearly, $V(z_t) = 2k$ for all $t \in \mathbb{R}$. By applying Proposition 3.8, we obtain that $\beta|Y^j| \cap |X^j| = \emptyset$ for all $j \in \{0,1,2\}$. This contradicts with $\beta|Y^0| \cap |X^0| \neq \emptyset$ and hence we have proved (i).

(ii) We finish the proof of statement (ii) by distinguishing two cases.

Case 1. $(k,j) = (0,0)$. By way of contradiction, we show that (1.1) has no nonconstant periodic solutions in $V^{-1}(0)$. Suppose that there exists a nonconstant periodic solution $x : \mathbb{R} \to \mathbb{R}^3$ of (1.1) such that $V(x_t) = 0$ for all $t \in \mathbb{R}$. It follows from the definition of V that $x^0(t) \neq 0$ for all $t \in \mathbb{R}$. Then, by Proposition 3.1, either ($x^j(t) > 0$ for all $j \in \{0,1,2\}$ and $t \in \mathbb{R}$) or ($x^j(t) < 0$ for all $j \in \{0,1,2\}$ and $t \in \mathbb{R}$). Therefore, Proposition 2.5 implies that either $x_t \equiv \widehat{\xi^+}$ or $x_t \equiv \widehat{\xi^-}$, a contradiction with the fact that $x : \mathbb{R} \to \mathbb{R}^3$ is a nonconstant periodic solution of (1.1).

Case 2. $(k,j) \in \mathbb{N} \times \{0,1,2\} \setminus \{(0,0),(0,1)\}$ such that $\tau_{k,j} \geq 1$. Again, we show by way of contradiction that (1.1) has no periodic solutions in $V^{-1}(6k+2j)$. Suppose that $x : \mathbb{R} \to \mathbb{R}^3$ is a nonconstant periodic solution of (1.1) with the minimal period $T_x > 0$ and $V(x_t) = 6k + 2j$ (≥ 4) for all $t \in \mathbb{R}$. Define $w : \mathbb{R} \to \mathbb{R}^3$ such that $w(t) = X(k,j,1,0,\tau_{k,j})(t)$ for all $t \in \mathbb{R}$ (see Appendix A, p. 59 for the definition of $X(k,j,1,0,\tau_{k,j})$). Then, by Theorems A.13 and A.16, w is a nontrivial periodic solution of (3.2) with $\tau = \tau_{k,j}$ and $g(\xi) = f'(0)\xi$ for all $\xi \in \mathbb{R}$, and $V(w_t) = 6k + 2j$ for all $t \in \mathbb{R}$. It follows that $y : \mathbb{R} \ni t \mapsto w(\frac{t}{\tau_{k,j}}) \in \mathbb{R}$ is a nontrivial periodic solution of (3.1) with $\tau = \tau_{k,j}$ and $g(\xi) = f'(0)\xi$ for all $\xi \in \mathbb{R}$. Denote the minimal period of y by T_y. For each $j \in \{0,1,2\}$, define $X^j : [0,T_x] \ni t \mapsto (x^j(t), \dot{x}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$ and $Y^j : [0,T_y] \ni t \mapsto (y^j(t), \dot{y}^j(t))^{\mathrm{tr}} \in \mathbb{R}^2$. From Proposition 3.1 and the definition of V, we know that x^0 and y^0 have zeros. By Corollary 3.3(ii), $0 \in \mathrm{int}(X^0)$ and $0 \in \mathrm{int}(Y^0)$. Let

$$\beta = \inf\{\rho \geq 0 : \rho'|Y^0| \subset \mathrm{ext}(X^0) \text{ for all } \rho' \in (\rho, \infty)\}.$$

Note that Proposition 3.4 yields $|X^0| = |X^1| = |X^2|$ and $|Y^0| = |Y^1| = |Y^2|$. This, combined with the definition of β, implies that for all $j \in \{0,1,2\}$,

$$\beta|Y^j| \subset |X^j| \cup \operatorname{ext}(X^j) \quad \text{and} \quad r\beta|Y^j| \subset \operatorname{ext}(X^j) \quad \text{for } r > 1,$$

and $\beta|Y^0| \cap |X^0| \neq \emptyset$. Clearly, $g'(0) = f'(0)$. The fact that $\lim_{\xi\to 0} \frac{\xi f'(\xi)}{f(\xi)} = 1$ and (H2) combined yield $\frac{\xi f'(\xi)}{f(\xi)} < 1$ for $\xi > 0$, or equivalently, $(\frac{f(\xi)}{\xi})' < 0$ for $\xi > 0$. Using this and the equation $\lim_{\xi\to 0} \frac{f(\xi)}{\xi} = f'(0)$, we obtain $g(\xi) = f'(0)\xi > f(\xi)$ for $\xi > 0$. Moreover, $\frac{g'(\xi)}{g(\xi)} = \frac{1}{\xi} > \frac{f'(\xi)}{f(\xi)}$ for $\xi > 0$. Applying Proposition 3.8 with $\tau = \tau_{k,j}$, $g(\xi) = f'(0)\xi$ and $z(t) = \beta y(t)$, we get a contradiction. ∎

In the remaining part of this section, we study the existence of periodic orbits of (1.1). For this purpose, we introduce the following parameterized scalar delay differential equation

$$\dot{x}(t) = -\tau\mu x(t) + \tau f(x(t-1)) \tag{3.15}$$

and parameterized system of delay differential equations

$$\begin{cases} \dot{x}^0(t) = -\tau\mu x^0(t) + \tau f(x^1(t)), \\ \dot{x}^1(t) = -\tau\mu x^1(t) + \tau f(x^2(t)), \\ \dot{x}^2(t) = -\tau\mu x^2(t) + \tau f(x^0(t-1)) \end{cases} \tag{3.16}$$

with parameter τ (> 0), where $\mu > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and continuously differentiable function satisfying assumptions (H1) and (H2).

The following result comes from [15].

PROPOSITION 3.11. *If $\tau > \frac{\tau_{1,0}}{3}$, then there exists a nonconstant periodic orbit $\mathcal{O}(\tau)$ of* (3.15) *such that the minimal period $T(\tau)$ of $\mathcal{O}(\tau)$ belongs to* $(1,2)$.

PROPOSITION 3.12. *If $\tau > \tau_{1,0}$, then there exists a unique nonconstant periodic orbit $\mathcal{O}_{1,0}(\tau)$ of* (3.16) *such that $V(\psi) = 6$ for all $\psi \in \mathcal{O}_{1,0}(\tau)$ and the minimal period $T_{1,0}(\tau)$ of $\mathcal{O}_{1,0}(\tau)$ belongs to $(\frac{1}{3}, \frac{2}{3})$.*

Proof. For each $\tau \in (\frac{\tau_{1,0}}{3}, \infty)$, by Proposition 3.11, we may choose a nonconstant periodic solution $p(\tau) : \mathbb{R} \to \mathbb{R}$ of (3.15) such that $(p(\tau))_t \in \mathcal{O}(\tau)$ for all $t \in \mathbb{R}$ and the minimal period $T(\tau)$ of $p(\tau)$ belongs to $(1,2)$.

For a given $\tau > \tau_{1,0}$, define $p_{1,0}(\tau) : \mathbb{R} \to \mathbb{R}^3$ by $(p_{1,0}(\tau))^j(t) = p(\frac{\tau}{3})(3t-j)$ for all $(t,j) \in \mathbb{R} \times \{0,1,2\}$. It is easy to check that $p_{1,0}(\tau)$ is a nonconstant periodic solution of (3.16) with the minimal period $T_{1,0}(\tau) = \frac{T(\tau)}{3} \in (\frac{1}{3}, \frac{2}{3})$. Then $(p_{1,0}(\tau))^1(t) = (p_{1,0}(\tau))^0(t-\frac{1}{3}) = (p_{1,0}(\tau))^0(t + \frac{3T_{1,0}(\tau)-1}{3})$ and $\frac{3T_{1,0}(\tau)-1}{3} \in (0, T_{1,0}(\tau))$. Moreover, by Proposition 3.1 and (H1), $(p_{1,0}(\tau))^0$ has a zero. By applying Corollary 3.6, we know that $V((p_{1,0}(\tau))_t) = 6$ for all $t \in \mathbb{R}$. On the other hand, the uniqueness of a nonconstant periodic orbit in $V^{-1}(6)$ follows from Theorem 3.10. Let $\mathcal{O}_{1,0}(\tau) = \{(p_{1,0}(\tau))_t : t \in \mathbb{R}\}$. Then $\tau > \tau_{1,0}$, $\mathcal{O}_{1,0}(\tau)$ is the unique nonconstant periodic orbit of (3.16) such that $V(\psi) = 6$ for all $\psi \in \mathcal{O}_{1,0}(\tau)$ and the minimal period $T_{1,0}(\tau)$ of $\mathcal{O}_{1,0}(\tau)$ belongs to $(\frac{1}{3}, \frac{2}{3})$. This completes the proof. ∎

PROPOSITION 3.13. *$T_{1,0} : (\tau_{1,0}, \infty) \to (0,\infty)$ is a continuous function. Moreover, $T_{1,0}(\tau) \to \frac{2\pi}{b_{1,0}}$ as $\tau \to (\tau_{1,0})^+$.*

Proof. Define $T : [\tau_{1,0}, \infty) \to \mathbb{R}$ by $T(\tau_{1,0}) = \frac{2\pi}{b_{1,0}}$ and $T(\tau) = T_{1,0}(\tau)$ for all $\tau \in (\tau_{1,0}, \infty)$. We claim that T is a continuous function on $[\tau_{1,0}, \infty)$. If the claim is true, then it follows immediately that $T_{1,0}$ is continuous and $T_{1,0}(\tau) \to \frac{2\pi}{b_{1,0}}$ as $\tau \to (\tau_{1,0})^+$.

In the following, we will prove the claim by way of contradiction. Suppose that the claim is not true. Then there exists a sequence $\{\tau_m\} \subset [\tau_{1,0}, \infty)$ such that $\lim_{m\to\infty} \tau_m = \tau^* \in [\tau_{1,0}, \infty)$ and $T(\tau_m)$ does not tend to $T(\tau^*)$ as $m \to \infty$. Note that $T([\tau_{1,0}, \infty)) \subseteq [\frac{1}{3}, \frac{2}{3}]$ as $T((\tau_{1,0}, \infty)) \subset (\frac{1}{3}, \frac{2}{3})$ by Proposition 3.12 and $\frac{2\pi}{b_{1,0}} \in (\frac{1}{3}, \frac{4}{9})$. By taking a subsequence if necessary, without loss of generality, we assume that $\{\tau_m\} \subset (\tau_{1,0}, \infty)$ and there exists $T_0 \neq T(\tau^*)$ such that $T_0 \in [\frac{1}{3}, \frac{2}{3}]$ and $T(\tau_m) \to T_0$ as $m \to \infty$. By Propositions 3.1 and 3.12, for each $m \in \mathbb{N}$, we choose the solution $x_m : \mathbb{R} \to \mathbb{R}^3$ of (3.16) such that $(x_m)_0 \in \mathcal{O}_{1,0}(\tau_m)$ and $(x_m)^0(0) = \max\{x_m^0(t) : t \in \mathbb{R}\}$. By Proposition 2.4 (ii), for each $m \in \mathbb{N}$, $(x_m)^j(t) \in [\xi^-, \xi^+]$ for all $j \in \{0, 1, 2\}$ and $t \in \mathbb{R}$. Let $a^* = \sup\{| - \tau_m\mu a + \tau_m f(b)| : a, b \in [\xi^-, \xi^+] \text{ and } m \in \mathbb{N}\}$ and $b^* = \sup\{| - \tau_m\mu a + \tau_m f'(d)b| : d \in [\xi^-, \xi^+] \text{ and } a, b \in [-a^*, a^*] \text{ and } m \in \mathbb{N}\}$. It follows from (3.16) that $|(\dot{x_m})^j(t)| \leq a^*$ and $|(\ddot{x_m})^j(t)| \leq b^*$ for all $m \in \mathbb{N}$, $j \in \{0, 1, 2\}$ and $t \in \mathbb{R}$. Thus, by Remark 2.3, by passing to a subsequence if necessary, we can assume that there exists a map $x : \mathbb{R} \to \mathbb{R}^3$ such that $x_m \to x$ and $\dot{x_m} \to \dot{x}$ uniformly in any compact interval of $\mathbb{R}$ as $m \to \infty$. We shall finish the proof by distinguishing two cases.

Case 1. x is not zero. Obviously, $x^0(0) = \max\{x^0(t) : t \in \mathbb{R}\} \neq 0$ follows from the fact that $x_m^0(0) = \max\{x_m^0(t) : t \in \mathbb{R}\}$ for all $m \in \mathbb{N}$. We first prove the following statements:

(i) $x(t) = x(t - T_0)$ and $x(t) = -x(t - \frac{T_0}{2})$ for all $t \in \mathbb{R}$.

(ii) x is a nonconstant periodic solution of (3.16) with $\tau = \tau^*$.

(iii) $\dot{x}^0(t) < 0$ for all $t \in (0, \frac{T_0}{2})$ and $\dot{x}^0(t) > 0$ for all $t \in (\frac{T_0}{2}, T_0)$.

(iv) The minimal period of x is T_0.

To verify (i), for any $t \in \mathbb{R}$, let $I^* = [t - 1 - 2T_0, t]$. Then the above discussions imply that $\{x_m|_{I^*}\}_{m\in\mathbb{N}}$ is equicontinuous on I^* and $x_m|_{I^*}$ tends uniformly to $x|_{I^*}$ as $m \to \infty$. Thus, for any $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $|x_m(t - T(\tau_m)) - x_m(t - T_0)| < \frac{\varepsilon}{3}$ and $|x_m(t - T_0) - x(t - T_0)| < \frac{\varepsilon}{3}$ whenever $m > n_1$. It follows that $|x_m(t) - x(t - T_0)| = |x_m(t - T(\tau_m)) - x(t - T_0)| \leq |x_m(t - T(\tau_m)) - x_m(t - T_0)| + |x_m(t - T_0) - x(t - T_0)| < \varepsilon$ whenever $m > n_1$. This means that $x_m(t) \to x(t - T_0)$ as $m \to \infty$ and hence $x(t) = x(t - T_0)$. Similarly, by using $x_m(t) = -x_m(t + \frac{T(\tau_m)}{2})$, we can show that $x(t) = -x(t - \frac{T_0}{2})$. Since t is arbitrary, we have proved (i).

From statement (i) and the fact that x is not zero, we know that x is a nonconstant periodic map. Moreover, since $x_m \to x$ and $\dot{x_m} \to \dot{x}$ uniformly in any compact interval of $\mathbb{R}$ as $m \to \infty$, we easily see that x is a nonconstant periodic solution of (3.16) with $\tau = \tau^*$. This proves (ii).

Observe that $\dot{x_m}|_{I^*}$ tends uniformly to $\dot{x}|_{I^*}$ as $m \to \infty$, where $I^* = [0, T_0]$. By Remark 2.3, we have $\dot{x_m}^0(t) < 0$ for all $t \in (0, \frac{T(\tau_m)}{2})$ and $\dot{x_m}^0(t) > 0$ for all $t \in (\frac{T(\tau_m)}{2}, T(\tau_m))$. These facts, combined with the fact $T(\tau_m) \to T_0$ as $m \to \infty$, show that $\dot{x}^0(t) \leq 0$ for all $t \in (0, \frac{T_0}{2})$ and $\dot{x}^0(t) \geq 0$ for all $t \in (\frac{T_0}{2}, T_0)$. Again by statement (ii) and Remark 2.3, we obtain that $\dot{x}^0(t) < 0$ for all $t \in (0, \frac{T_0}{2})$ and $\dot{x}^0(t) > 0$ for all $t \in (\frac{T_0}{2}, T_0)$, verifying (iii).

By statements (i) and (iii), we know that the minimal period of x is T_0.

Statements (ii) and (iv) imply that x is a nonconstant periodic solution of (3.16) with $\tau = \tau^*$ such that the minimal period of x is T_0.

By Proposition 3.2(iv) and statement (ii), we know $V(x_t) = 2k$ for some $k \in \mathbb{N}$ and all $t \in \mathbb{R}$. This, combined with Proposition 2.6(iii)(b), implies that $x_t \in R$ for all $t \in \mathbb{R}$. Thus, for each $t \in \mathbb{R}$, by Proposition 2.6(ii), we obtain that $V(x_t) = \lim_{m\to\infty} V((x_m)_t) = 6$ since $(x_m)_t \to x_t$ and $(\dot{x_m})_t \to (\dot{x})_t$ as $m \to \infty$. Thus by

Theorem 3.10 and Proposition 3.12, we obtain that $\tau^* > \tau_{1,0}$ and the minimal period $T_0 = T(\tau^*)$, a contradiction.

Case 2. $x \equiv 0$. In this case, x_m uniformly tends to zero in any compact subset of $\mathbb{R}$ as $m \to \infty$. Let $M_m = \max\{|(x_m)^0(t)| : t \in \mathbb{R}\}$ and $y_m(t) = \frac{x_m(t)}{M_m}$ for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Then $|(y_m)^j(t)| \le 1$ for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$. We can easily check that $\lim_{m\to\infty} M_m = 0$, and $V((y_m)_t) = 6$ for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$. It follows from (3.16) that, for each $m \in \mathbb{N}$, y_m satisfies the following equation:

$$\begin{cases} \dot{y}^0(t) = -\tau_m \mu y^0(t) + \tau_m \frac{f(M_m y^1(t))}{M_m}, \\ \dot{y}^1(t) = -\tau_m \mu y^1(t) + \tau_m \frac{f(M_m y^2(t))}{M_m}, \\ \dot{y}^2(t) = -\tau_m \mu y^2(t) + \tau_m \frac{f(M_m y^0(t-1))}{M_m}. \end{cases} \tag{3.17}$$

For each $m \in \mathbb{N}$, define $f_m : \mathbb{R} \to \mathbb{R}$ by $f_m(y) = \frac{f(M_m y)}{M_m}$ for all $y \in \mathbb{R}$ and define $f_0 : \mathbb{R} \to \mathbb{R}$ by $f_0(y) = f'(0)y$ for all $y \in \mathbb{R}$. Thus, $f_m \to f_0$ and $f'_m \to f'_0$ uniformly in any compact subset of $\mathbb{R}$ as $m \to \infty$.

Let $a^{**} = \sup\{| - \tau_m \mu a + \tau_m f_m(b)| : a, b \in [-1, 1] \text{ and } m \in \mathbb{N}\}$ and $b^{**} = \sup\{| - \tau_m \mu a + \tau f'_m(d) b| : d \in [-1, 1] \text{ and } a, b \in [-a^{**}, a^{**}] \text{ and } m \in \mathbb{N}\}$. It follows from (3.16) that $|(\dot{y_m})^j(t)| \le a^{**}$ and $|(\ddot{y_m})^j(t)| \le b^{**}$ for all $m \in \mathbb{N}$, $j \in \{0, 1, 2\}$ and $t \in \mathbb{R}$. Then, by passing to a subsequence if necessary, Remark 2.3 shows there exists a function $y : \mathbb{R} \to \mathbb{R}^3$ such that $y_m \to y$ and $\dot{y}_m \to \dot{y}$ uniformly in any compact subset of $\mathbb{R}$ as $m \to \infty$. Thus, y is a nonconstant periodic solution of (3.13) with $\tau = \tau^*$, and the minimal period of y is T_0. Again, by Proposition 2.6 (ii), we obtain that $V(y_t) = \lim_{m\to\infty} V((y_m)_t) = 6$ for all $t \in \mathbb{R}$. By Theorems A.13 and A.16 and the fact that $V(x_t) = 6$ for all $t \in \mathbb{R}$, we know that $\tau^* = \tau_{1,0}$ and $T_0 = \frac{2\pi}{b_{1,0}}$, which contradict with $T_0 \ne T(\tau^*)$.

Therefore, we have proved the claim. This completes the proof. ∎

Proposition 3.14. $\liminf_{\tau\to\infty} T_{1,0}(\tau) = \frac{1}{3}$.

Proof. By way of contradiction, assume $\liminf_{\tau\to\infty} T_{1,0}(\tau) \ne \frac{1}{3}$. Then, by Proposition 3.12, $\liminf_{\tau\to\infty} T_{1,0}(\tau) > \frac{1}{3}$. Thus there exist $T^* \in (\frac{1}{3}, \frac{2}{3})$ and $\tau^{**} > \tau_{1,0}$ such that $T_{1,0}(\tau) > T^*$ for all $\tau \in (\tau^{**}, \infty)$.

Let $M = \sup\{f'(x) : x \in \mathbb{R}\}$. By (H2), for $\xi > \xi^+$, we have $\frac{f'(\xi^+)}{\mu} = \frac{\xi^+ f'(\xi^+)}{f(\xi^+)} > \frac{\xi f'(\xi)}{f(\xi)}$, which implies that $f'(\xi) < \frac{f'(\xi^+)}{\mu} \frac{f(\xi)}{\xi} < \frac{f'(\xi^+)}{\mu} \cdot \mu = f'(\xi^+)$. It follows that $M \in (0, \infty)$ and $|f(x)| \le M|x|$ for all $x \in \mathbb{R}$. Furthermore, let $k = \frac{f'(0)+\mu}{2}$. Then $k > \mu$ as $f'(0) > \mu$. It follows from $f'(0) > \mu$ and $f'(x) > 0$ that there exists $y^* \in (0, \frac{\xi^+}{2(1+\frac{M}{\mu})})$ such that $|f(x)| \ge k|x|$ for all $x \in [-y^*, y^*]$ and $|f(x)| \ge ky^*$ for all $x \in (-\infty, -y^*] \cup [y^*, \infty)$. Let $A^* = \frac{1}{\mu} \ln(\frac{ky^* + \xi^+(2\mu+2M)}{(k-\mu)y^*})$ and $\tau^* = \max\{\frac{2A^*}{T^* - \frac{1}{3}}, 1 + \tau_{1,0}, 1 + \tau^{**}\}$.

Assume that $p : \mathbb{R} \to \mathbb{R}^3$ is a periodic solution of (3.16) with $\tau = \tau^*$ such that $p^0(0) = \max\{|(p^0(t)| : t \in \mathbb{R}\}$ and $V(p_t) = 6$ for all $t \in \mathbb{R}$. Then by Proposition 3.12, we have $p(t) = p_{1,0}(\tau^*)(t + t^*)$ for all $t \in \mathbb{R}$ and some $t^* \in \mathbb{R}$, and the minimal period of p is $T_{1,0}(\tau^*)$. Define $z : \mathbb{R} \to \mathbb{R}$ by $z(t) = p^0(t)$ for all $t \in \mathbb{R}$. Let $T = T_{1,0}(\tau^*)$. Then $z(0) = \max\{|(z(t)| : t \in \mathbb{R}\}$, $T \in (T^*, \frac{2}{3})$ and the minimal period of z is T. By Proposition 3.5, we infer that z satisfies the following equation:

$$\dot{z}(t) = -\tau^* \mu z(t) + \tau^* f\left(z\left(t - \frac{1}{3}\right)\right). \tag{3.18}$$

Let $\xi^* = \frac{\mu y^* z(0)}{(2\mu+2M)\xi^+}$. Since $z(0) \le \xi^+$ (by Proposition 2.4(ii)) and $y^* < \xi^+$, we know that $\xi^* < \min\{y^*, z(0)\}$. It follows that $|f(x)| \ge k|x|$ for all $x \in [-\xi^*, \xi^*]$ and $|f(x)| \ge k\xi^*$ for all $x \in (-\infty, -\xi^*] \cup [\xi^*, \infty)$. By the fact that $\xi^* < z(0)$, there exist T_1 and T_2 such that $T_1 = \sup\{t \in (-\infty, 0) : z(t) = \xi^*\}$ and $T_2 = \inf\{t \in (0, \infty) : z(t) = \xi^*\}$. Using Remark 3.3, we know that $z(t) \ge \xi^*$ for all $t \in [T_1, T_2]$ and $z(t) \le \xi^*$ for all $t \in [T_2, T_1 + T]$. Moreover, $T_2 < T_1 + \frac{T}{2}$ since $z(T_1 + \frac{T}{2}) = -z(T_1) = -\xi^*$.

We now prove that $T_2 - T_1 \ge T - \frac{1}{3}$. If this is not true, then $T_2 - T_1 < T - \frac{1}{3}$. It follows that $-T_1 < T - \frac{1}{3} < \frac{1}{3}$ and $T_2 + \frac{1}{3} < T + T_1 < T$. Thus $-\frac{1}{3} + [T_2 + \frac{1}{3}, T] \subseteq [T_2, T_1 + T]$, which implies that $z(t - \frac{1}{3}) \le \xi^*$ for all $t \in [T_2 + \frac{1}{3}, T]$. It follows from (3.18) that

$$\dot{z}(t) = -\tau^*\mu z(t) + \tau^* f(z(t - \tfrac{1}{3})) \le -\tau^*\mu z(t) + M\tau^*\xi^*,$$

or

$$\frac{\mathrm{d}(z(t)e^{\mu\tau^* t})}{\mathrm{d}t} \le \frac{M\xi^*}{\mu}\frac{\mathrm{d}(e^{\mu\tau^* t})}{\mathrm{d}t}$$

for all $t \in [T_2 + \frac{1}{3}, T]$. Integrating this differential inequality gives us

$$\begin{aligned}
z(T) &\le z(T_2 + \tfrac{1}{3})e^{-\mu\tau^*(T - T_2 - \frac{1}{3})} + \tfrac{M\xi^*}{\mu} - \tfrac{M\xi^*}{\mu}e^{-\mu\tau^*(T - T_2 - \frac{1}{3})} \\
&\le z(T_2 + \tfrac{1}{3})e^{-\mu\tau^*(T - T_2 - \frac{1}{3})} + \tfrac{M\xi^*}{\mu} \\
&\le \xi^* e^{-\mu\tau^*(T - T_2 - \frac{1}{3})} + \frac{M\xi^*}{\mu} \\
&\le (1 + \tfrac{M}{\mu})\xi^* \\
&= \frac{y^* z(0)}{2\xi^+} \\
&\le \frac{z(0)}{2},
\end{aligned}$$

a contradiction to $z(T) = z(0) > 0$. This proves $T_2 - T_1 \ge T - \frac{1}{3}$ and hence $T_2 - T_1 \ge T^* - \frac{1}{3}$.

Let $S^* = \frac{A^*}{\tau^*} + T_1 + \frac{1}{3}$. Note that $T^* + \frac{1}{3} \le 2T$ and $T < \frac{2}{3}$. Also recall that $T_2 \le T_1 + \frac{T}{2}$. By a simple computation, we obtain that $S^* > T_1 + \frac{1}{3} > T_1 + \frac{T}{2} \ge T_2$ and $S^* \le \frac{T^* - \frac{1}{3}}{2} + T_1 + \frac{1}{3} \le \frac{2T - \frac{1}{3} - \frac{1}{3}}{2} + T_1 + \frac{1}{3} = T + T_1 \le T_2 + \frac{1}{3}$. In particular, $S^* \in [T_1 + \frac{1}{3}, T_2 + \frac{1}{3}]$. Using (3.18), for all $t \in [T_1 + \frac{1}{3}, T_2 + \frac{1}{3}]$, we have

$$\dot{z}(t) = -\tau^*\mu z(t) + \tau^* f(z(t - \tfrac{1}{3})) \ge -\tau^*\mu z(t) + k\tau^*\xi^*$$

and hence

$$\frac{\mathrm{d}(z(t)e^{\mu\tau^* t})}{\mathrm{d}t} \ge \frac{k\xi^*}{\mu}\frac{\mathrm{d}(e^{\mu\tau^* t})}{\mathrm{d}t}.$$

Integrating the above differential inequality yields, for all $t \in [T_1 + \frac{1}{3}, T_2 + \frac{1}{3}]$, that

$$\begin{aligned}
z(t) &\ge z\left(T_1 + \tfrac{1}{3}\right) e^{-\mu\tau^*\left(t - T_1 - \frac{1}{3}\right)} + \tfrac{k\xi^*}{\mu} - \tfrac{k\xi^*}{\mu}e^{-\mu\tau^*\left(t - T_1 - \frac{1}{3}\right)} \\
&\ge -\left(z(0) + \tfrac{k\xi^*}{\mu}\right) e^{-\mu\tau^*\left(t - T_1 - \frac{1}{3}\right)} + \tfrac{k\xi^*}{\mu}.
\end{aligned}$$

In particular,

$$\begin{aligned} z(S^*) &\geq -\left(z(0)+\tfrac{k\xi^*}{\mu}\right)e^{-\mu\tau^*\left(S^*-T_1-\frac{1}{3}\right)}+\tfrac{k\xi^*}{\mu} \\ &= -\left(z(0)+\tfrac{k\xi^*}{\mu}\right)e^{-\mu A^*}+\tfrac{k\xi^*}{\mu} \\ &= -\left(\tfrac{(2\mu+2M)\xi^+\xi^*}{\mu y^*}+\tfrac{k\xi^*}{\mu}\right)e^{-\mu A^*}+\tfrac{k\xi^*}{\mu}. \end{aligned}$$

It follows from the definition of A^* and the above inequality that $z(S^*)=z(\frac{A^*}{\tau^*}+T_1+\frac{1}{3})\geq\xi^*$. This, combined with the fact that $S^*>T_2$, implies $\frac{A^*}{\tau^*}+T_1+\frac{1}{3}\geq T+T_1$. Thus $T\leq\frac{A^*}{\tau^*}+\frac{1}{3}<T^*$, a contradiction. This completes the proof. ∎

PROPOSITION 3.15. *Assume that $\tau^*>0$ and $k^*\in\mathbb{N}\setminus\{0\}$ satisfy*

(H4) $\frac{(2k^*+2)\pi-3\arccos\frac{\mu}{f'(0)}}{\sqrt{[f'(0)]^2-\mu^2}}\geq\tau^*>\frac{2k^*\pi-3\arccos\frac{\mu}{f'(0)}}{\sqrt{[f'(0)]^2-\mu^2}}$.

Then, for each positive integer $l\leq k^*$, we have the following results.

(i) *If $l\geq 2$, then there exists one and only one nonconstant periodic orbit $\mathcal{O}_l(\tau^*)$ of* (3.16) *with $\tau=\tau^*$ such that $V(\psi)=2l$ for all $\psi\in\mathcal{O}_l(\tau^*)$ and the minimal period $T_l(\tau^*)>0$.*

(ii) *If $l=1$, then there exists at least one nonconstant periodic orbit of* (3.16) *with $\tau=\tau^*$ in the level set $V^{-1}(2)$: that is, for any such orbit $\mathcal{O}_1(\tau^*)$, $V(\psi)=2$ for all $\psi\in\mathcal{O}_1(\tau^*)$ and the minimal period $T_1(\tau^*)>0$.*

Proof. Let $\kappa=\{1,2,\ldots,k^*\}$, $D_1=(1,\infty)$, and $D_l=(\frac{1}{l},\frac{2}{2l-3})$ for all $l\in\kappa\setminus\{1\}$. We now claim that there exists k^* nonconstant periodic orbits $\{\mathcal{O}_l:l\in\kappa\}$ of (3.16) such that the minimal period of $\mathcal{O}_l$ is $T_l\in D_l$, where $l\in\kappa$.

Now suppose $l\in\kappa$. Then there exists $(k_l,j_l)\in\mathbb{N}\times\{0,1,2\}$ such that $l=3k_l+j_l$ and $\tau^*>\tau_{k_l,j_l}$. Let $\omega_l(\gamma)=1+(l-3)T_{1,0}(\gamma)$, $T_l(\gamma)=\frac{T_{1,0}(\gamma)}{\omega_l(\gamma)}$ and $p_l(\gamma)(t)=p_{1,0}(\gamma)(\omega_l(\gamma)t)$ for all $t\in\mathbb{R}$ and $\gamma>\tau_{1,0}$. A simple computation shows that $p_l(\gamma):\mathbb{R}\to\mathbb{R}^3$ is a periodic solution of (3.16) with $\tau=\gamma\omega_l(\gamma)$. Proposition 3.12 implies that the function $h:(\tau_{1,0},\infty)\ni\gamma\mapsto\gamma\omega_l(\gamma)\in\mathbb{R}$ is continuous and satisfies $\lim_{\gamma\to(\tau_{1,0})^+}h(\gamma)=\tau_{k_l,j_l}$. Moreover, Proposition 3.14 implies $\limsup_{\gamma\to\infty}h(\gamma)=\infty$. Hence, by $\tau^*>\tau_{k_l,j_l}$, there exists a $\gamma^*\in(\tau_{1,0},\infty)$ such that $h(\gamma^*)=\tau^*$. Then $\omega_l(\gamma^*)>0$. Thus $p_l(\gamma^*):\mathbb{R}\to\mathbb{R}^3$ is a periodic solution of (3.16) with $\tau=\tau^*$ such that its minimal period is $T_l(\gamma^*)$. Moreover, by Proposition 3.5 and the definition of $p_l(\gamma^*)$, we obtain

$$\begin{aligned} (p_l(\gamma^*))^1(t) &= (p_{1,0}(\gamma^*))^1(\omega_l(\gamma^*)t) \\ &= (p_{1,0}(\gamma^*))^0\left(\omega_l(\gamma^*)t+\frac{3T_{1,0}(\gamma^*)-1}{3}\right) \\ &= (p_{1,0}(\gamma^*))^0\left(\omega_l(\gamma^*)\left[t+\frac{3T_{1,0}(\gamma^*)-1}{3\omega_l(\gamma^*)}\right]\right) \\ &= (p_l(\gamma^*))^0\left(t+\frac{3T_{1,0}(\gamma^*)-1}{3\omega_l(\gamma^*)}\right) \\ &= (p_l(\gamma^*))^0\left(t+\frac{3T_{1,0}(\gamma^*)-(\omega_l(\gamma^*)-(l-3)T_{1,0}(\gamma^*))}{3\omega_l(\gamma^*)}\right) \\ &= (p_l(\gamma^*))^0\left(t+\frac{lT_{1,0}(\gamma^*)-\omega_l(\gamma^*)}{3\omega_l(\gamma^*)}\right) \\ &= (p_l(\gamma^*))^0\left(t+\frac{lT_l(\gamma^*)-1}{3}\right). \end{aligned}$$

Obviously, $\frac{lT_l(\gamma^*)-1}{3} = \frac{3T_{1,0}(\gamma^*)-1}{3\omega_l(\gamma^*)} = T_l(\gamma^*) - \frac{1}{3\omega_l(\gamma^*)} \in (0, T_l(\gamma^*))$ follows from $\omega_l(\gamma^*) > 0$. So, by Corollary 3.6, we have $V((p_l(\gamma^*))_t) = 2l$ for all $t \in \mathbb{R}$. Now, we show $T_l(\gamma^*) \in D_l$. First, by Proposition 3.5 there exists a $\delta_1 \in (-\frac{1}{2}, 1]$ such that $1 = (l-1)T_l(\gamma^*) + \delta_1 T_l(\gamma^*)$, which implies that $T_l(\gamma^*) \geq \frac{1}{l}$. If $T_l(\gamma^*) = \frac{1}{l}$, then it follows from $T_l(\gamma^*) = \frac{1}{l} = \frac{T_{1,0}(\gamma^*)}{1+(l-3)T_{1,0}(\gamma^*)}$ that $T_{1,0}(\gamma^*) = \frac{1}{3}$, a contradiction to $T_{1,0}(\gamma^*) \in (\frac{1}{3}, \frac{2}{3})$. Thus $T_l(\gamma^*) > \frac{1}{l}$. On the other hand, by Proposition 3.5 again, there exists $\delta_2 \in (-\frac{1}{2}, 1]$ such that $1 = 2T_{1,0}(\gamma^*) + \delta_2 T_{1,0}(\gamma^*)$. Then $\omega_l(\gamma^*) = 2T_{1,0}(\gamma^*) + \delta_2 T_{1,0}(\gamma^*) + (l-3)T_{1,0}(\gamma^*) > (2 - \frac{1}{2} + l - 3)T_{1,0}(\gamma^*) = (l - \frac{3}{2})T_{1,0}(\gamma^*)$. Thus, $T_l(\gamma^*) < \frac{T_{1,0}(\gamma^*)}{(l-\frac{3}{2})T_{1,0}(\gamma^*)} = \frac{2}{2l-3}$ for $l \geq 2$. It follows that $T_l(\gamma^*) \in D_l$ for $l \geq 1$. So far we have shown the existence of at least one periodic orbit in $V^{-1}(2l)$ for $l \in \kappa$. This, combined with Theorem 3.10 and the arbitrariness of l, shows that statements (i) and (ii) hold. This completes the proof. ∎

We now present the second main result of this section.

THEOREM 3.16. *Suppose* (H1) *and* (H2) *hold. Then the following statements are true.*

(i) *If* $\tau_{0,1} \in (0,1)$, *then system* (1.1) *has at least one periodic orbit in* $V^{-1}(2)$.

(ii) *For every* $(k,j) \in \{(k,j) \in \mathbb{N} \times \{0,1,2\} \setminus \{(0,0),(0,1)\} : \tau_{k,j} \in (0,1)\}$, *system* (1.1) *has a unique periodic orbit in* $V^{-1}(6k+2j)$.

(iii) *For every* $(k,j) \in \{(0,0)\} \bigcup \{(k,j) \in \mathbb{N} \times \{0,1,2\} \setminus \{(0,0),(0,1)\} : \tau_{k,j} \geq 1\}$, *system* (1.1) *has no nonconstant periodic orbit in* $V^{-1}(6k+2j)$.

Proof. Let $\tau^* = 1$ and $k^* = \sup\{k \in \mathbb{N} : \tau^* > \frac{2k\pi - 3\arccos\frac{\mu}{f'(0)}}{\sqrt{[f'(0)]^2 - \mu^2}}\}$.

(i) Since $\tau_{0,1} \in (0,1)$, we know that $l = 1 \leq k^*$. Then conclusion (i) follows from Proposition 3.15(ii).

(ii) For every $(k,j) \in \{(0,0)\} \bigcup \{(k,j) \in \mathbb{N} \times \{0,1,2\} \setminus \{(0,0),(0,1)\} : \tau_{k,j} \in (0,1)\}$, we have $3k + j \leq k^*$. Then by Proposition 3.15(i), system (1.1) has a unique periodic orbit in $V^{-1}(6k+2j)$.

(iii) Obviously, conclusion (iii) follows from Theorem 3.10(ii). ∎

According to Theorems 3.16, A.12, and A.13(ii), we know that the number of the nonconstant periodic orbits is larger than or equal to the number of the roots with positive imaginary parts and positive real parts of the characteristic equation (3.12) with $\tau = 1$.

4. Structure of the global attractor.

In this section, we describe the structure of the global attractor A. For $K \subseteq A$, define $W^u(K) = \{\phi \in A : \alpha(\phi) \subseteq K\}$.

PROPOSITION 4.1. *Let* $x : \mathbb{R} \to \mathbb{R}^3$ *be a nonzero solution of* (1.1). *If there exists* $k \in \mathbb{N}$ *such that* $V(x_t) = 2k$ *for all* $t \in \mathbb{R}$, *then* $x_t \in R$ *for all* $t \in \mathbb{R}$. *In particular, for any* $j \in \{0,1,2\}$, $\pi^j(x_t) \neq (0,0)^{\mathrm{tr}}$ *for all* $t \in \mathbb{R}$, *and if* $t_0 \in \mathbb{R}$ *is a zero of* x^j, *then it must be simple.*

Proof. It follows from Proposition 2.6(iii)(b) and the fact that $V(x_t) = 2k$ for all $t \in \mathbb{R}$ that $x_t \in R$ for all $t \in \mathbb{R}$. This, combined with the definition of R, yields the conclusion and hence the proof is complete. ∎

PROPOSITION 4.2. *Let* $x : \mathbb{R} \to \mathbb{R}^3$ *be a nonzero solution of* (1.1) *and let* $\chi = \{t \in \mathbb{R} : x^0(t) = 0\}$. *If* $\omega(x) = \alpha(x) = \{\widehat{0}\}$ *and there exists* $k \in \mathbb{N}$ *such that* $V(x_t) = 2k$ *for all* $t \in \mathbb{R}$, *then* $\chi \neq \emptyset$ *and* $\inf\{t : t \in \chi\} = -\infty$.

Proof. First, we claim that, for any $j \in \{0,1,2\}$, there exists no $T \in \mathbb{R}$ such that $x^j(t)x^{j+1}(t) < 0$ for all $t \in (-\infty, T)$. Recall that $x^3(t) = x^0(t-1)$. If the claim is not true, then there exist $j \in \{0,1,2\}$ and $T \in \mathbb{R}$ such that $x^j(t)x^{j+1}(t) < 0$ for all $t \in (-\infty, T)$. It suffices to consider the case where $x^0(t) > 0$ and $x^1(t) < 0$

for all $t \in (-\infty, T)$ since the proofs for other cases are similar. By (1.1), $\dot{x}^0(t) = -\mu x^0(t) + f(x^1(t)) < 0$ for all $t \in (-\infty, T)$. This implies that $x^0(t) \geq x^0(T-1) > 0$ for all $t \in (-\infty, T-1]$, a contradiction to $\alpha(x) = \{\widehat{0}\}$. This proves the claim.

Now, we prove Proposition 4.2. Obviously, it suffices to show that $\inf\{t : t \in \chi \cup \{0\}\} = -\infty$. By way of contradiction, suppose $\inf\{t : t \in \chi \cup \{0\}\} = T^* \in (-\infty, 0]$. Then either ($x^0(t) > 0$ for all $t \in (-\infty, T^*)$) or ($x^0(t) < 0$ for all $t \in (-\infty, T^*)$). Since $-x$ satisfies all the conditions of Proposition 4.2, without loss of generality, we can assume that $x^0(t) > 0$ for all $t \in (-\infty, T^*)$. It follows from (1.1) that $\frac{\mathrm{d}x^2(t)e^{\mu t}}{\mathrm{d}t} = e^{\mu t} f(x^0(t-1)) > 0$ for all $t \in (-\infty, T^* + 1)$, which indicates that $x^2(t)e^{\mu t}$ is strictly increasing on $(-\infty, T^*+1)$. If there exists $t^* \in (-\infty, T^*+1)$ such that $x^2(t^*) = 0$ then $x^2(t) < 0$ for all $t \in (-\infty, t^*)$. It follows that $x^2(t)x^0(t-1) < 0$ for all $t \in (-\infty, t^*)$, a contradiction to the above claim. Thus, $x^2(t) > 0$ for all $t \in (-\infty, T^*+1)$. Then it follows from (1.1) that $\frac{\mathrm{d}x^1(t)e^{\mu t}}{\mathrm{d}t} = e^{\mu t} f(x^2(t)) > 0$ for all $t \in (-\infty, T^*+1)$, which implies that $x^1(t)e^{\mu t}$ is strictly increasing on $(-\infty, T^*+1)$. Similarly as before, we have $x^1(t) > 0$ for all $t \in (-\infty, T^*+1)$. Thus $x_t \in (C_+)^\circ$ for all $t \in (-\infty, T^*)$. This, combined with Proposition 2.5, gives $\omega(x) = \{\widehat{\xi_+}\}$, a contradiction and hence the proof is complete. ∎

PROPOSITION 4.3. *Let $x : \mathbb{R} \to \mathbb{R}^3$ be a nonzero solution of* (1.1)*. If there exists $k \in \mathbb{N}$ such that $V(x_t) = 2k$ for all $t \in \mathbb{R}$ and $V(x_t - x_s) = 2k$ for all $t \neq s$, then either $\omega(x) \neq \{\widehat{0}\}$ or $\alpha(x) \neq \{\widehat{0}\}$.*

Proof. By way of contradiction, suppose $\omega(x) = \alpha(x) = \{\widehat{0}\}$. Proposition 4.1 and Proposition 4.2 ensure that x^0 have zeros and all zeros are simple. Moreover, $\inf\{t \in \mathbb{R} : x^0(t) = 0\} = -\infty$. Choose $t_0 \in (-\infty, 0]$ such that $x^0(t_0) = 0$ and $\dot{x}^0(t_0) > 0$. Note that $\{t \in (-\infty, t_0] : x^0(t) = 0\}$ is a discrete set of $\mathbb{R}$. Arranging these zeros of x^0 in a decreasing order gives a strictly decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = -\infty$ and $\dot{x}^0(t_{2n})\dot{x}^0(t_{2n+1}) < 0$ for all $n \in \mathbb{N}$. As $\dot{x}^0(t_0) > 0$, it follows from (1.1) and $\dot{x}^0(t_{2n})\dot{x}^0(t_{2n+1}) < 0$ for all $n \in \mathbb{N}$ that $x^1(t_{2n+1}) < 0$ and $x^1(t_{2n}) > 0$ for all $n \in \mathbb{N}$.

Define $c : \mathbb{R} \ni t \mapsto \pi^0(x_t) \in \mathbb{R}^2$. Proposition 4.1 and the facts that $V(x_t) = 2k$ for all $t \in \mathbb{R}$ and $V(x_t - x_s) = 2k$ for $t \neq s$ imply that $x_t \in R$ for all $t \in \mathbb{R}$ and $x_t - x_s \in R$ for $t \neq s$, and thus $c(t) \neq c(s) \neq 0$ for $t \neq s$. Choose $\rho > 0$ so that $||c(0)|| > \rho$. It follows from $\omega(x) = \alpha(x) = \{\widehat{0}\}$ that $\lim_{t\to\infty} c(t) = \lim_{t\to-\infty} c(t) = 0$. Then there exists a positive integer n_0 such that $x^1(t_{2n_0}) > x^1(t_{2n_0-2})$ and $||c(t)|| < \rho$ for $t \leq t_{2n_0}$. Let $a = t_{2n_0-2}$, $b = t_{2n_0}$ and $C : [a-1, b] \to \mathbb{R}^2$ be such that $C(t) = c(t)$ for all $t \in [a, b]$ and $C(t) = (a-t)c(b) + (t-a+1)c(a)$ for all $t \in [a-1, a]$. Then C is a simple closed curve in $\mathbb{R}^2$. In fact, in the (x, y)-plane, C consists of the line segment $C([a-1, a])$ on the positive y-axis, the curve $C([a, t_{2n_0-1}])$ is completely on the right-hand side of the y-axis and the curve $C([t_{2n_0-1}, b])$ is completely on the left-hand side of the y-axis. By the definition of C, we have $0 \in \operatorname{int}(C)$. On the other hand, from $\lim_{t\to\infty} c(t) = 0$ and $||c(0)|| > \rho$, we can deduce that there exists $s_1 > 0$ such that $c(s_1) \in C((a-1, a))$ because $c(s_1) \notin c([a, b])$. Let $t^* = \sup\{t \in \mathbb{R} : c(t) \in C((a-1, a))\}$. Then $t^* \geq s_1$ and $t^* < \infty$ as $\lim_{t\to\infty} c(t) = 0$. Moreover, $t^* \geq s_1 > 0$ implies that $t^* \notin [a, b]$ and hence $c(t^*) \in C((a-1, a))$. Let $\delta = \min_{t\in[a,b]} \|c(t) - c(t^*)\|$. Then $\delta > 0$ and $N_{\frac{\delta}{2}}(c(t^*)) \cap \{(\xi_1, \xi_2)^{\mathrm{tr}} \in \mathbb{R}^2 : \xi_1 > 0, \xi_2 > 0\} \subset \operatorname{ext}(C)$, where $N_{\frac{\delta}{2}}(c(t^*)) = \{(\xi_1, \xi_2)^{\mathrm{tr}} \in \mathbb{R}^2 : \|(\xi_1, \xi_2)^{\mathrm{tr}} - c(t^*)\| \leq \frac{\delta}{2}\}$. Moreover, it follows from the definition of C that $x^0(t^*) = 0$ and $x^1(t^*) > 0$. By (1.1), we have $\dot{x}^0(t^*) > 0$. Then there exists $\varepsilon > 0$ such that $x^0(t) > 0$, $x^1(t) > 0$ and $(x^0(t), x^1(t))^{\mathrm{tr}} \in N_{\frac{\delta}{2}}(c(t^*))$ for all $t \in (t^*, t^* + \varepsilon)$. That is, $c(t) \in \operatorname{ext}(C)$ for all $t \in (t^*, t^* + \varepsilon)$. This, combined with

$\lim_{t\to\infty} c(t) = 0$, implies that there exists $t^{**} \geq t^* + \varepsilon$ such that $c(t^{**}) \in C((a-1, a))$, a contradiction with the choice of t^*. ∎

PROPOSITION 4.4. *Let $x : \mathbb{R} \to \mathbb{R}^3$ and $y : \mathbb{R} \to \mathbb{R}^3$ be two nonzero solutions of* (1.1) *with $y_t \in \alpha(x)$ for all $t \in \mathbb{R}$. If $\alpha(y) = \{\widehat{0}\}$, then there exists $k \in \mathbb{N}$ such that $V(y_t) = 2k$ for all $t \in \mathbb{R}$ and $V(y_t - y_s) = 2k$ for $t \neq s$.*

Proof. Clearly, y is not a periodic solution of (1.1). Let $k_\sigma = \frac{\sup_{t\in\mathbb{R}} V(y_{t+\sigma}-y_t)}{2}$, where $\sigma > 0$. We claim $k_\sigma < \infty$ for all $\sigma \in (0, \infty)$. Otherwise, there exist a $\sigma_0 > 0$ and a sequence $\{t_n\}_0^\infty$ with $\lim_{n\to\infty} t_n = -\infty$ such that $\lim_{n\to\infty} V(y_{\sigma_0+t_n} - y_{t_n}) = \infty$. On the other hand, it follows from $y_0 \in \alpha(x)$ that there exists a sequence $\{s_m\}_0^\infty$ with $\lim_{m\to\infty} s_m = -\infty$ such that $\lim_{m\to\infty} x_{s_m} = y_0$. Thus, for each $t \in \mathbb{R}$, by Proposition 2.4(iii), $x_{t+s_m} \to y_t$ in C^1 as $m \to \infty$. Then $\lim_{m\to\infty}(x_{\sigma_0+s_m+t_n} - x_{s_m+t_n}) = y_{\sigma_0+t_n} - y_{t_n}$ and Proposition 2.6(i) implies

$$V(y_{\sigma_0+t_n} - y_{t_n}) \leq \liminf_{m\to\infty} V(x_{\sigma_0+s_m+t_n} - x_{s_m+t_n}).$$

Therefore, $\lim_{t\to-\infty} V(x_{\sigma_0+t} - x_t) = \infty$. However, by Corollary 4.6 in [20] and Proposition 2.6(iii), $V(y_{\sigma_0+t} - y_t)$ is constant for all large t and thus $y_{\sigma_0+t^*} - y_{t^*} \in R$ for some large $t^* > 0$. Hence by Proposition 2.6(ii), $V(y_{\sigma_0+t^*} - y_{t^*}) = \lim_{m\to\infty} V(x_{\sigma_0+t^*+s_m} - x_{t^*+s_m}) = \lim_{t\to-\infty} V(x_{\sigma_0+t} - x_t) = \infty$, a contradiction to $V(y_{\sigma_0+t^*} - y_{t^*}) < \infty$. This proves the claim.

For a given $\sigma > 0$, it follows from Proposition 2.6 (iii) and the above claim that there exists $t_1 > 0$ such that $V(y_{\sigma+t} - y_t) = 2k_\sigma$ for $t < -t_1$ and $V(y_{\sigma+t} - y_t)$ is constant for $t > t_1$. Hence, by Proposition 2.6 (iii)(b), we have $y_{\sigma+t} - y_t \in R$ for $|t| > t_1 + 4$. Let $\{s_m\}_0^\infty$ be a sequence as in the above paragraph. Applying Proposition 2.6 (ii), we obtain $\lim_{m\to\infty} V(x_{\sigma+s_m+t} - x_{s_m+t}) = V(y_{\sigma+t} - y_t)$ for $|t| > t_1 + 4$. On the other hand, by Proposition 2.6 (iii), we can also obtain that for each $\sigma > 0$, $V(x_{\sigma+s} - x_s)$ is nonincreasing and thus $\lim_{s\to-\infty} V(x_{\sigma+s} - x_s) = \lim_{m\to\infty} V(x_{\sigma+s_m+t} - x_{s_m+t}) = V(y_{\sigma+t} - y_t)$ for $|t| > t_1 + 4$. Hence $V(y_{\sigma+t} - y_t) = \lim_{s\to-\infty} V(x_{\sigma+s} - x_s)$ for $|t| > t_1 + 4$. Again, Proposition 2.6(iii) implies that $V(y_{\sigma+t} - y_t)$ is nonincreasing and thus for each $\sigma > 0$, $V(y_{\sigma+t} - y_t) = \lim_{s\to-\infty} V(x_{\sigma+s} - x_s) = 2k_\sigma$ for all $t \in \mathbb{R}$. This, combined with Proposition 2.6(iii)(b), implies $y_{\sigma+t} - y_t \in R$ for all $t \in \mathbb{R}$ and $\sigma > 0$. By Proposition 2.4(iii), we know that $y_\sigma - y_0$ varies continuously in $\sigma \in (0, \infty)$, in the C^1 topology. So, by Proposition 2.6(ii), we infer that k_σ varies continuously in $\sigma \in (0, \infty)$. This, combined with the fact that $k_\sigma \in \mathbb{N}$ for all $\sigma \in (0, \infty)$, yields that k_σ is independent of σ. Let $k = k_\sigma$ for some $\sigma \in (0, \infty)$. In view of $\widehat{0} \in \alpha(y)$ and Proposition 2.6(i), $V(y_t) \leq 2k$ for all $t \in \mathbb{R}$. On the other hand, since $V(y_t)$ is constant for all large t, we can deduce $y_t \in R$ for all large t and thus, by Proposition 2.6(ii), $V(y_t) = \lim_{s\to-\infty} V(y_t - y_{t+s}) = 2k$ for all large t. Therefore, by Proposition 2.6(iii), we have $V(y_t) = 2k$, for all $t \in \mathbb{R}$. ∎

THEOREM 4.5. $A = \{\widehat{\xi^-}, \widehat{\xi^+}\} \cup W^u(\{\widehat{0}\}) \cup \{W^u(\mathcal{O}) :$ *$\mathcal{O}$ is a nonconstant periodic orbit of* $(1.1)\}$.

Proof. Note that $\widehat{0}$, $\widehat{\xi^-}$, $\widehat{\xi^+}$ are the only stationary points of (1.1) with $\widehat{\xi^-}$ and $\widehat{\xi^+}$ being locally asymptotically stable (see Theorem B.5). Let $\phi \in A \setminus \{\widehat{\xi^-}, \widehat{\xi^+}\}$. Then there is a solution $x : \mathbb{R} \to \mathbb{R}^3$ such that $x_0 = \phi$ and $x_t \in A$ for all $t \in \mathbb{R}$. By Proposition 2.7, we can deduce that either $\alpha(x)$ is a periodic orbit $\mathcal{O}$ of (1.1) or for each solution $y : \mathbb{R} \to \mathbb{R}^3$ with $y_t \in \alpha(x)$, the sets $\alpha(y)$, $\omega(y) \subseteq \{\widehat{0}, \widehat{\xi^-}, \widehat{\xi^+}\}$. If the former holds, then $\phi \in W^u(\mathcal{O})$. If the latter holds, then by the locally asymptotical stability of $\widehat{\xi^-}$ and $\widehat{\xi^+}$, we have $\alpha(x) \cap \{\widehat{0}, \widehat{\xi^-}, \widehat{\xi^+}\} = \{\widehat{0}\}$. We claim $\alpha(x) = \{\widehat{0}\}$. Otherwise, there exists a nonzero solution $z : \mathbb{R} \to \mathbb{R}^3$ in $\alpha(x)$ such that $\alpha(z) = \omega(z) = \{\widehat{0}\}$. Note that, by Proposition 4.4, there exists $k \in \mathbb{N}$ such that $V(z_t) = V(z_t - z_s) = 2k$ for $t \neq s$, a contradiction to Proposition 4.3. Therefore, $\alpha(x) = \{\widehat{0}\}$, that is, $\phi \in W^u(\{\widehat{0}\})$. This completes the proof. $\square$

5. Discussion and generalization. In sections 2–4, we studied the cyclic system (1.1) and obtained results on the existence, absence, and uniqueness of periodic orbits in the level sets of a discrete Lyapunov functional V, and we described the structure of the global attractor. Unfortunately, we cannot prove the uniqueness of periodic orbits in $V^{-1}(2)$. This is due to the fact that we can only prove Proposition 3.8 for $k \in \mathbb{N} \setminus \{0, 1\}$. The proof of Proposition 3.8 is quite similar to that of Proposition 3.4 in Krisztin and Walther [15]. When proving Proposition 3.8, we need $k \in \mathbb{N} \setminus \{0, 1\}$ to guarantee that $\alpha_x^* \geq \alpha_z^*$ in Case 1. Unlike in the proof of Proposition 3.4 in Krisztin and Walther [15], instead of

$$\left(k - \tfrac{1}{2}\right) T_x < 1 < kT_x \qquad \text{and} \qquad \left(k - \tfrac{1}{2}\right) T_z < \alpha < kT_z, \tag{5.1}$$

we can only have

$$\left(k - \tfrac{3}{2}\right) T_x < 1 < kT_x \qquad \text{and} \qquad \left(k - \tfrac{3}{2}\right) T_z < \alpha < kT_z. \tag{5.2}$$

The inequalities (5.1) play a key role in the proof of Proposition 3.4 in [15] to get a contradiction in Case 1. The weaker version (5.2) we obtained forced us to modify the argument in [15] in order. Unfortunately, this modification does not allow us to obtain the contradiction for the case where $k = 1$. We thus leave the following as a future project.

CONJECTURE 5.1. *System* (1.1) *has at most one periodic orbit in* $V^{-1}(2)$.

To conclude this section, we mention that the arguments in this paper can be modified to study the general system (1.4), that is, the system

$$\begin{cases} \dot{x}^0(t) = -\mu x^0(t) + f(x^1(t)), \\ \dot{x}^1(t) = -\mu x^1(t) + f(x^2(t)), \\ \qquad \vdots \\ \dot{x}^{n-1}(t) = -\mu x^{n-1}(t) + f(x^n(t)), \\ \dot{x}^n(t) = -\mu x^n(t) + f(x^0(t-1)), \end{cases} \tag{5.3}$$

where f satisfies the same assumptions as those in sections 2–4 and $n \in \mathbb{N}$ with $x^1(t) = x^0(t-1)$ if $n = 0$.

Let $\mathbb{K}_n = [-1, 0] \cup \{1, 2, \ldots, n\}$ and $C_n = C(\mathbb{K}_n, \mathbb{R})$. Note that $\mathbb{K}_0 = [-1, 0]$. Define sc_n, $V_n : C_n \setminus \{0\} \to \mathbb{N} \cup \{\infty\}$, respectively, by

$$\mathrm{sc}_n(\phi) = \begin{cases} 0 & \text{if } \phi \text{ is nonnegative or nonpositive} \\ \sup \left\{ k \in \mathbb{N} \setminus \{0\} : \begin{array}{l} \text{there exists a strictly increasing} \\ \text{finite sequence } (\theta^j)_0^k \subset \mathbb{K}_n \\ \text{with } \phi(\theta^{j-1})\phi(\theta^j) < 0 \text{ for all } 1 \leq j \leq k \end{array} \right\} \end{cases}$$

and

$$V_n(\phi) = \begin{cases} \mathrm{sc}_n(\phi) & \text{if } \mathrm{sc}_n(\phi) \in 2\mathbb{N} \cup \{\infty\}, \\ \mathrm{sc}_n(\phi) + 1 & \text{if } \mathrm{sc}_n(\phi) \in 2\mathbb{N} + 1 \end{cases}$$

for $\phi \in C_n \setminus \{0\}$.

The only challenge in generalizing the arguments to (5.3) is to establish an analogue of Proposition 3.5, that is, to establish the following result: *Let $w : \mathbb{R} \to \mathbb{R}^{n+1}$ be a nonconstant periodic solution of*

$$\begin{cases} \dot{x}^0(t) = -\mu x^0(t) + f(x^1(t)), \\ \dot{x}^1(t) = -\mu x^1(t) + f(x^2(t)), \\ \qquad \vdots \\ \dot{x}^{n-1}(t) = -\mu x^{n-1}(t) + f(x^n(t)), \\ \dot{x}^n(t) = -\mu x^n(t) + f(x^0(t-1)) \end{cases}$$

with the minimal period $T_w > 0$, where $\mu > 0$, $\tau > 0$, and the C^1-function $g : \mathbb{R} \to \mathbb{R}$ is odd and $g'(\xi) > 0$ for all $\xi \in \mathbb{R}$. If $V_n(w_t) = 2k + 2$ for some $k \in \mathbb{N}$ and for all $t \in \mathbb{R}$, then there exists $\delta \in (\frac{1-n}{2}, 1]$ such that the following results are true.

(i) $1 = kT_w + \delta T_w$.

(ii) $(w^1(t), w^2(t), \ldots, w^n(t), w^0(t-1)) = (w^0(t+\alpha), w^1(t+\alpha), \ldots, w^{n-1}(t+\alpha), w^n(t+\alpha))$ *for all $t \in \mathbb{R}$, where $\alpha = \frac{1-\delta}{n+1} T_w$.*

The difficult part in establishing the above result is determining how to keep track of the sign changes due to the large number of coordinates. The modification for the proofs of other results is minor. We list some results for (5.3) without detailed proofs.

PROPOSITION 5.1. *Let $x : \mathbb{R} \to \mathbb{R}^{n+1}$ be a nonconstant periodic solution of* (5.3) *with the minimal period $T_x > 0$.*

(i) *For each $j \in \{0, 1, 2, \ldots, n\}$, there exist $t_0^j \in \mathbb{R}$ and $t_1^j \in (t_0^j, t_0^j + T_x)$ such that $0 < \dot{x}^j(t)$ for all $t_0^j < t < t_1^j$, $x^j(\mathbb{R}) = [x^j(t_0^j), x^j(t_1^j)]$, $\dot{x}^j(t) < 0$ for all $t_1^j < t < t_0^j + T_x$.*

(ii) *If x^0 has a zero, then $x(t + \frac{T_x}{2}) = -x(t)$ for all $t \in \mathbb{R}$.*

(iii) *There exists $k \in \mathbb{N}$ such that $\{x_t : t \in \mathbb{R}\} \subset V_n^{-1}(2k)$.*

PROPOSITION 5.2. *Let $x : \mathbb{R} \to \mathbb{R}^{n+1}$ be a nonconstant periodic solution of* (5.3) *with the minimal period $T_x > 0$. If $V_n(x_t) = 2k + 2$ for some $k \in \mathbb{N}$, then there exists $\delta_x \in (\frac{1-n}{2}, 1]$ such that $1 = kT_x + \delta_x T_x$ and $(x^1(t), x^2(t), \ldots, x^n(t), x^0(t-1)) = (x^0(t+\alpha_x), x^1(t+\alpha_x), \ldots, x^{n-1}(t+\alpha_x), x^n(t+\alpha_x))$ for all $t \in \mathbb{R}$, where $\alpha_x = \frac{1-\delta_x}{n+1} T_x$.*

Let $\mathcal{A}_n = (\mathbb{N} \times \{0, 1, 2, \ldots, n\}) \setminus \{(0, 0)\}$, $\mathcal{B}_n = \{(k, j) \in \mathcal{A}_n : 2(n+1)k + 2j < n+1\}$, and $\tau_{k,j;n} = \frac{2(n+1)k\pi + 2j\pi - (n+1)\arccos\frac{\mu}{f'(0)}}{\sqrt{[f'(0)]^2 - \mu^2}}$ for each $(k, j) \in \mathcal{A}_n$.

Then the generalization of Theorem 3.16 to (5.3) is as follows.

THEOREM 5.3.

(i) *If $(k, j) \in \mathcal{B}_n$ is given so that $\tau_{k,j;n} < 1$, then system* (5.3) *has at least one periodic orbit in $V_n^{-1}(2(n+1)k + 2j)$.*

(ii) *For every $(k, j) \in \{(k, j) \in \mathcal{A}_n \setminus \mathcal{B}_n : \tau_{k,j;n} \in (0, 1)\}$, system* (5.3) *has a unique periodic orbit in $V_n^{-1}(2(n+1)k + 2j)$.*

(iii) *For every $(k, j) \in \{(0, 0)\} \cup \{(k, j) \in \mathcal{A}_n : \tau_{k,j;n} \geq 1\}$, system* (5.3) *has no nonconstant periodic orbit in $V_n^{-1}(2(n+1)k + 2j)$.*

Appendix A. Zeros of the characteristic equation. The purpose of this appendix is to analyze the distribution of the roots of

$$(\zeta + \tau\mu)^3 - (\tau f'(0))^3 e^{-\zeta} = 0, \tag{A.1}$$

where $\tau > 0$ and $f'(0) > \mu$. Our main results, among others, claim that the real parts and imaginary parts of roots of (A.1) are well ordered. Let $\lambda = \frac{\zeta + \tau\mu}{3}$. Then (A.1) reduces to

$$\lambda^3 - \beta^3 e^{-3\lambda} = 0, \tag{A.2}$$

where $\beta = \frac{\tau f'(0)e^{\frac{\tau\mu}{3}}}{3} > 0$. In the remaining part of this appendix, we study the general case of (A.2), namely, we only assume $\beta > 0$.

Lemma A.1. *Any root of* (A.2) *is simple.*

Proof. By way of contradiction, suppose that (A.2) has a root λ_0, which is not simple. Then we have

$$\lambda_0^3 - \beta^3 e^{-3\lambda_0} = 0 \tag{A.3}$$

and

$$3\lambda_0^2 + 3\beta^3 e^{-3\lambda_0} = 0. \tag{A.4}$$

Multiply (A.3) by 3 and add it to (A.4) to obtain $3\lambda_0^3 + 3\lambda_0^2 = 0$. Then $\lambda_0 = 0$ or $\lambda_0 = -1$. However, it is easy to see that neither $\lambda_0 = 0$ nor $\lambda_0 = -1$ satisfies (A.3), a contradiction. This completes the proof. $\square$

Observe that if λ is a root of (A.2), then so is $\overline{\lambda}$. Thus, in the remaining part of this appendix, we shall focus on roots of (A.2) with nonnegative imaginary parts. Also observe that λ is a root of (A.2) if and only if λ is a root of one of the following three equations:

$$\lambda - \beta e^{-\lambda} = 0, \tag{A.5}$$

$$\lambda - \beta e^{-\lambda + \frac{2\pi}{3}i} = 0, \tag{A.6}$$

$$\lambda - \beta e^{-\lambda + \frac{4\pi}{3}i} = 0. \tag{A.7}$$

Lemma A.2. *Equation* (A.2) *has a root* $\lambda = iv$ *with* $v \geq 0$ *if and only if there exists* $k \in \mathbb{N}$ *such that* $\beta = \frac{2k\pi}{3} + \frac{\pi}{6}$ *and* $v = \beta$.

Proof. If $\beta = \frac{2k\pi}{3} + \frac{\pi}{6}$ for some $k \in \mathbb{N}$, then we can easily check that $i\beta$ is a root of (A.2). On the other hand, if $\lambda = iv$ with $v \geq 0$ is a root of (A.2), then $-iv^3 = \beta^3 e^{-3iv}$. Separating the real and imaginary parts gives $\cos(3v) = 0$ and $v^3 = \beta^3 \sin(3v)$. It follows that $v = \beta$ and there exists a $k \in \mathbb{N}$ such that $3v = 2k\pi + \frac{\pi}{2}$ as $\sin(3v) = 1$, that is, $v = \beta = \frac{2k\pi}{3} + \frac{\pi}{6}$. $\square$

Lemma A.3. *Suppose* $\beta < \frac{\pi}{6}$. *If* λ *with* $\mathrm{Im}(\lambda) > 0$ *is a root of* (A.2), *then* $\mathrm{Re}(\lambda) < 0$.

Proof. By way of contradiction, suppose $\mathrm{Re}(\lambda) \geq 0$. Write $\lambda = u + iv$. Then, from $|\lambda|^3 = \beta^3 |e^{-3\lambda}| = \beta^3 e^{-3u}$, we have $(u^2 + v^2)e^{2u} = \beta^2$. It follows that $0 \leq u \leq \beta$ and $0 \leq v \leq \beta$. First, assume that λ satisfies (A.5). Then $v + \beta e^{-u} \sin v = 0$ and hence $1 \leq e^u = -\beta \frac{\sin v}{v} < 0$, a contradiction. Next, assume that λ satisfies (A.6). Then $u = \beta e^{-u} \cos(v - \frac{2\pi}{3}) < 0$, a contradiction. Similarly, if λ satisfies (A.7), then we shall have $u < 0$, a contradiction again. This completes the proof. $\square$

Lemma A.4. *If* $m \in \mathbb{R}_+$ *is given then, for every* $\beta \geq \sqrt{m^2 + \frac{1}{e^2}}$, (A.2) *has no roots in the strip* $B_m \equiv \{u + iv : 0 \leq v \leq m, u < 0\}$.

Proof. Suppose that the result is not true. Then there exist $\beta \geq \sqrt{m^2 + \frac{1}{e^2}}$ and $\lambda = u + iv \in B_m$ such that λ satisfies (A.2). Thus, $\beta = \sqrt{(v^2 + u^2)e^{2u}} < \sqrt{v^2 + u^2 e^{2u}} \leq \sqrt{m^2 + \sup\{u^2 e^{2u} : u \leq 0\}} = \sqrt{m^2 + \frac{1}{e^2}}$. This contradicts the choice of β. $\square$

Lemma A.5. *Let* I *be a compact subset of* $(0, \infty)$ *and* B *be a horizontal strip in the complex plane. Then the real parts of roots of* (A.2) *in* B *are bounded from above and below for all* $\beta \in I$.

Proof. Otherwise, there exist a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ in I and a sequence $\{\lambda_n = u_n+iv_n : u_n, v_n \in \mathbb{R}, n \in \mathbb{N}\}$ in B such that λ_n satisfies (A.2) with $\beta = \beta_n$, and either $\lim_{n\to\infty} u_n = \infty$ or $\lim_{n\to\infty} u_n = -\infty$. Thus, $\beta_n = \sqrt{((v_n)^2 + (u_n)^2)e^{2u_n}}$. Note that $\lim_{u_n\to\infty} \sqrt{((v_n)^2 + (u_n)^2)e^{2u_n}} = \infty$ and $\lim_{u_n\to-\infty} \sqrt{((v_n)^2 + (u_n)^2)e^{2u_n}} = 0$, a contradiction to the facts that $\beta_n \in I$ and $I \subseteq (0,\infty)$ is compact. ∎

Define $\sum_0^0 = \{u+iv : v = 0 \text{ and } u \in \mathbb{R}\} = \mathbb{R}$, $\sum_0^1 = \{u+iv : v \in (0, \frac{2\pi}{3}) \text{ and } u \in \mathbb{R}\}$, $\sum_0^2 = \{u+iv : v \in (\frac{\pi}{3}, \frac{4\pi}{3}) \text{ and } u \in \mathbb{R}\}$, $\sum_k^0 = \{u+iv : v \in ((2k-1)\pi, 2k\pi) \text{ and } u \in \mathbb{R}\}$, $\sum_k^1 = \{u+iv : v \in ((2k-\frac{1}{3})\pi, (2k+\frac{2}{3})\pi) \text{ and } u \in \mathbb{R}\}$, and $\sum_k^2 = \{u+iv : v \in ((2k+\frac{1}{3})\pi, (2k+\frac{4}{3})\pi) \text{ and } u \in \mathbb{R}\}$ for $k \in \mathbb{N}\setminus\{0\}$.

LEMMA A.6. *If λ with $\mathrm{Im}(\lambda) \geq 0$ is a root of* (A.2)*, then one of the following statements is true.*

(i) *If λ satisfies* (A.5)*, then $\lambda \in \sum_k^0$ for some $k \in \mathbb{N}$.*

(ii) *If λ satisfies* (A.6)*, then $\lambda \in \sum_k^1$ for some $k \in \mathbb{N}$.*

(iii) *If λ satisfies* (A.7)*, then $\lambda \in \sum_k^2$ for some $k \in \mathbb{N}$.*

Proof. If λ satisfies (A.5), then (i) follows directly from Theorem XI.3.1 of Diekmann et al. [5]. If $\lambda = u+iv$ with $v \geq 0$ satisfies (A.6), then $v = -\beta e^{-u}\sin(v - \frac{2\pi}{3})$ and hence $\sin(v - \frac{2\pi}{3}) < 0$, which implies that $\lambda \in \sum_k^1$ for some $k \in \mathbb{N}$. Similarly, if λ satisfies (A.7), then $\lambda \in \sum_k^2$ for some $k \in \mathbb{N}$. This completes the proof. ∎

LEMMA A.7. *For each given $k \in \mathbb{N}$ and $\beta \in (0,\infty)$, the following statements are true.*

(i) *Equation* (A.5) *has a simple and unique root $\lambda_{k,0}$ in $\sum_k^0$.*

(ii) *Equation* (A.6) *has a simple and unique root $\lambda_{k,1}$ in $\sum_k^1$.*

(iii) *Equation* (A.7) *has a simple and unique root $\lambda_{k,2}$ in $\sum_k^2$.*

Proof. Because of Lemma A.1, we only need to show the existence and uniqueness of a root in $\sum_k^j$, $k \in \mathbb{N}$ and $j \in \{0,1,2\}$.

First, we prove (i). If $k \geq 1$, then (i) follows directly from Theorem XI.3.1 of Diekmann et al. [5]. Now, suppose $k = 0$. Define $\xi : \mathbb{R} \to \mathbb{R}$ by $\xi(\lambda) = \lambda - \beta e^{-\lambda}$. Then $\xi(0) = -\beta < 0$, $\lim_{\lambda\to\infty}\xi(\lambda) = \infty$, and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\xi(\lambda) = 1 + \beta e^{-\lambda} > 0$ for all $\lambda \in \mathbb{R}$. According to the definitions of $\sum_0^0$ and ξ, we know that (A.5) has a simple and unique root $\lambda_{0,0}$ in $\sum_0^0$. This proves (i).

Next, we prove (ii) ((iii) can be proved similarly).

In view of Lemma A.6, (A.6) has no roots on the boundary of $\sum_k^1$. We claim that the number of roots of (A.6) in $\sum_k^1$ is finite and independent of β. Indeed, for any compact interval $I_1 \subseteq (0,\infty)$, by Lemma A.5, there exist $a_1, b_1 \in \mathbb{R}$ such that for each $\beta \in I_1$, all the roots of (A.6) in $\sum_k^1$ shall be in $\{u+iv \in \sum_k^1 : u \in (a_1, b_1) \text{ and } v \in \mathbb{R}\}$. Let $B^* = \{u+iv \in \sum_k^1 : u \in (a_1,b_1) \text{ and } v \in \mathbb{R}\}$. It follows from Lemma XI.2.8 of Diekmann et al. [5] that for any $\beta^* \in I_1$, there exists an open interval $V_{\beta^*} \subseteq (0,\infty)$ such that $\beta^* \in V_{\beta^*}$ and the number of roots of (A.6) in B^* is constant for all $\beta \in V_{\beta^*}$. This and the compactness of I imply that the number of roots of (A.6) in B^* is constant for all $\beta \in I_1$. Therefore, the number of roots of (A.6) in $\sum_k^1$ is constant for all $\beta \in (0,\infty)$. This proves the claim.

Let $\tilde{\beta} = 2k\pi + \frac{\pi}{6}$. Then it follows from Lemma A.2 that (A.6) has a unique root $i\tilde{\beta} \in \sum_k^1$ on the imaginary axis when $\beta = \tilde{\beta}$. Using the above claim, we know that, for $\beta = \tilde{\beta}$, the number of roots of (A.6) in $\sum_k^1$ is finite. Then there exists $\delta > 0$ such that, for $\beta = \tilde{\beta}$, any root other than $i\tilde{\beta}$ has a real part bigger than δ or smaller than $-\delta$. It follows that, for $\beta = \tilde{\beta}$, the number of the roots of (A.6) in $\sum_k^1$ is the sum of 1 and the number of roots in $\sum_k^1 \cap \{u+iv : |u| > \delta\}$.

Let $B_- = \sum_k^1 \cap \{u + iv : u < 0\}$. For any compact interval $I_2 \subseteq [\tilde{\beta}, \infty)$, by Lemma A.5, there exist a_2, $b_2 \in \mathbb{R}$ such that for each $\beta \in I_2$, all the roots of (A.6) in B_- shall be in $\{u + iv \in B_- : u \in (a_2, b_2) \text{ and } v \in \mathbb{R}\}$. Let $B^{**} = \{u + iv \in \sum_k^1 : u \in (a_2, b_2) \text{ and } v \in \mathbb{R}\}$. A similar argument as that in the proof of the above claim yields that the number of roots of (A.6) in B^{**} is constant for all $\beta \in I_2$. Then the number of roots of (A.6) in B_- is constant for all $\beta \in [\tilde{\beta}, \infty)$. This and Lemma A.4 imply that the number of roots of (A.6) in B_- is zero for all $\beta \in [\tilde{\beta}, \infty)$. In particular, when $\beta = \tilde{\beta}$, the number of roots of (A.6) in $\sum_k^1 \cap \{u + iv : u < -\delta\}$ is zero. Similarly, with the help of Lemma A.3, one can show that the number of roots of (A.6) in $\sum_k^1 \cap \{u + iv : u > \delta\}$ is zero when $\beta = \tilde{\beta}$. Therefore, the number of roots of (A.6) in $\sum_k^1$ is 1 and this proves (ii). ☐

For every $k \in \mathbb{N}$, $j \in \{0, 1, 2\}$ and $\beta \in (0, \infty)$, let us denote the unique root in Lemma A.7 by $\lambda_{k,j}(\beta)$. Define the functions $u_{k,j}$, $v_{k,j} : (0, \infty) \to \mathbb{R}$ by $u_{k,j}(\beta) = \mathrm{Re}(\lambda_{k,j}(\beta))$ and $v_{k,j}(\beta) = \mathrm{Im}(\lambda_{k,j}(\beta))$ for all $\beta \in (0, \infty)$, respectively.

According to Lemmas A.6 and A.7, we know that for each given $\beta \in (0, \infty)$, (A.5) has a simple and unique root $\lambda_{0,0}(\beta)$ in $\{u + iv : v \in (-\pi, \pi) \text{ and } u \in \mathbb{R}\}$.

Define $D_0^0 = (-\pi, \pi)$, $D_0^1 = (0, \frac{2\pi}{3})$, $D_0^2 = (\frac{\pi}{3}, \frac{4\pi}{3})$, and, for every $k \in \mathbb{N} \setminus \{0\}$, $D_k^0 = ((2k-1)\pi, 2k\pi)$, $D_k^1 = ((2k - \frac{1}{3})\pi, (2k + \frac{2}{3})\pi)$, and $D_k^2 = ((2k + \frac{1}{3})\pi, (2k + \frac{4}{3})\pi)$.

LEMMA A.8. *For each given* $(k, j) \in \mathbb{N} \times \{0, 1, 2\}$*, the functions* $u_{k,j}$ *and* $v_{k,j}$ *are continuously differentiable functions on* $(0, \infty)$*. Moreover, for all* $\beta \in (0, \infty)$*,*

$$\frac{\mathrm{d}u_{k,j}(\beta)}{\mathrm{d}\beta} = \frac{(v_{k,j}(\beta))^2 + u_{k,j}(\beta)(1 + u_{k,j}(\beta))}{\beta[(v_{k,j}(\beta))^2 + (1 + u_{k,j}(\beta))^2]}$$

and

$$\frac{\mathrm{d}v_{k,j}(\beta)}{\mathrm{d}\beta} = \frac{v_{k,j}(\beta)}{\beta[(v_{k,j}(\beta))^2 + (1 + u_{k,j}(\beta))^2]}.$$

Proof. For $j \in \{0, 1, 2\}$, let $\delta_j = \frac{2j\pi}{3}$. Define $G = (G_1, G_2) : (0, \infty) \times \mathbb{R} \times D_k^j \to \mathbb{R}^2$ by $G_1(\beta, u, v) = u - \beta e^{-u} \cos(v - \delta_j)$ and $G_2(\beta, u, v) = v + \beta e^{-u} \sin(v - \delta_j)$ for all $(\beta, u, v) \in (0, \infty) \times \mathbb{R} \times D_k^j$. Simple computations give us

$$\begin{aligned}
\frac{\partial G_1}{\partial u} &= 1 + \beta e^{-u} cos(v - \delta_j), \\
\frac{\partial G_1}{\partial v} &= \beta e^{-u} sin(v - \delta_j), \\
\frac{\partial G_2}{\partial u} &= -\beta e^{-u} sin(v - \delta_j), \\
\frac{\partial G_2}{\partial v} &= 1 + \beta e^{-u} cos(v - \delta_j),
\end{aligned}$$

and hence the determinant of the Jacobian of G is $1 + \beta^2 e^{-2u} + 2\beta e^{-u} \cos(v - \delta_j) > 0$ for all $(\beta, u, v) \in (0, \infty) \times \mathbb{R} \times D_k^j$. Note that $G(\beta, u_{k,j}(\beta), v_{k,j}(\beta)) = 0$ for all $\beta \in (0, \infty)$. Applying the implicit function theorem, we can deduce that, for each given $\tilde{\beta} \in (0, \infty)$, there exist an open interval $V \subseteq (0, \infty)$ and two continuously differentiable functions $p : V \to \mathbb{R}$, $q : V \to D_k^j$ such that $\tilde{\beta} \in V$, $p(\tilde{\beta}) = u_{k,j}(\tilde{\beta})$, $q(\tilde{\beta}) = v_{k,j}(\tilde{\beta})$ and $G(\beta, p(\beta), q(\beta)) = 0$ for all $\beta \in V$. This and the definition of G imply that, for each $\beta \in V$, $p(\beta) + iq(\beta)$ is a root of (A.j+5) in $\sum_k^j$. It follows from Lemma A.7 that $p(\beta) = u_{k,j}(\beta)$ and $q(\beta) = v_{k,j}(\beta)$. Therefore, $u_{k,j}$ and $v_{k,j}$ are continuously differentiable functions on $(0,\infty)$. The formulas for $\frac{\mathrm{d}u_{k,j}(\beta)}{\mathrm{d}\beta}$ and $\frac{\mathrm{d}v_{k,j}(\beta)}{\mathrm{d}\beta}$ can be easily found by using a simple computation. This completes the proof. ∎

LEMMA A.9. *$u_{k,j}(\beta) > u_{k+1,j}(\beta)$ for all $\beta \in (0,\infty)$, $k \in \mathbb{N}$ and $j \in \{0,1,2\}$.*

Proof. For the case where $j = 0$, the conclusion follows from Theorem XI.3.12 of Diekmann et al. [5].

It suffices to consider the case where $j = 1$ since the case where $j = 2$ can be proved by a similar argument. We first claim that there are no $k \in \mathbb{N}$ and $\beta \in (0,\infty)$ such that $u_{k,1}(\beta) = u_{k+1,1}(\beta)$. If this is not true, then there exist $\tilde{k} \in \mathbb{N}$ and $\tilde{\beta} \in (0,\infty)$ such that $u_{\tilde{k},1}(\tilde{\beta}) = u_{\tilde{k}+1,1}(\tilde{\beta})$. It follows from (A.6) that

$$\cos\left(v_{\tilde{k},1}(\tilde{\beta}) - \frac{2\pi}{3}\right) = \cos\left(v_{\tilde{k}+1,1}(\tilde{\beta}) - \frac{2\pi}{3}\right)$$

and

$$v_{\tilde{k}+1,1}(\tilde{\beta})\sin\left(v_{\tilde{k},1}(\tilde{\beta}) - \frac{2\pi}{3}\right) = v_{\tilde{k},1}(\tilde{\beta})\sin\left(v_{\tilde{k}+1,1}(\tilde{\beta}) - \frac{2\pi}{3}\right).$$

Note that $\sin(v_{\tilde{k},1}(\tilde{\beta}) - \frac{2\pi}{3}) \neq 0$ as $v_{\tilde{k},1}(\tilde{\beta}) = -\beta e^{-u_{\tilde{k},1}(\tilde{\beta})}\sin(v_{\tilde{k},1}(\tilde{\beta}) - \frac{2\pi}{3}) \neq 0$. It follows that $v_{\tilde{k},1}(\tilde{\beta}) = v_{\tilde{k}+1,1}(\tilde{\beta})$, a contradiction, and this proves the claim.

Let $M_k = \{\beta > 0 : u_{k,1}(\beta) > u_{k+1,1}(\beta)\}$ for $k \in \mathbb{N}$. Then, by the above claim, we know that M_k is a closed subset of $(0,\infty)$. Indeed, suppose that $\{\beta_n^*\} \subset M_k$ and $\lim_{n\to\infty}\beta_n^* = \beta^* \in (0,\infty)$. Then it follows from $u_{k,1}(\beta_n^*) > u_{k+1,1}(\beta_n^*)$ and the continuity of both $u_{k,1}$ and $u_{k+1,1}$ that $u_{k,1}(\beta^*) \geq u_{k+1,1}(\beta^*)$. This, with the help of the claim, implies that $u_{k,1}(\beta^*) > u_{k+1,1}(\beta^*)$, namely, $\beta^* \in M_k$. This shows that M_k is a closed subset of $(0,\infty)$. On the other hand, also by the continuity of $u_{k,1}$ and $u_{k+1,1}$, we know that M_k is an open subset of $(0,\infty)$.

Next we claim $M_k \neq \emptyset$. Let $\beta_k = 2k\pi + \frac{\pi}{6}$ for all $k \in \mathbb{N}$. It follows from Lemma A.2 that $u_{k,1}(\beta_k) = 0 = u_{k+1,1}(\beta_{k+1})$. Moreover, by Lemma A.8, we have

$$\frac{\mathrm{d}u_{k+1,1}(\beta_{k+1})}{\mathrm{d}\beta} = \frac{(v_{k+1,1}(\beta_{k+1}))^2 + u_{k+1,1}(\beta_{k+1})(1 + u_{k+1,1}(\beta_{k+1}))}{\beta_{k+1}[(v_{k+1,1}(\beta_{k+1}))^2 + (1 + u_{k+1,1}(\beta_{k+1}))^2]} > 0.$$

Then there exists $a^* \in (\beta_k, \beta_{k+1})$ such that $u_{k+1,1}(\beta) < 0$ for all $\beta \in (a^*, \beta_{k+1})$. We now prove that $u_{k+1,1}(\beta_k) < 0$. If this is not true, then $u_{k+1,1}(\beta_k) \geq 0$. It follows that there exists $b^* \in [\beta_k, a^*)$ such that $u_{k+1,1}(b^*) = 0$. This and Lemma A.2 imply that $v_{k+1,1}(b^*) = b^* \in \{\frac{2l\pi}{3} + \frac{\pi}{6} : l \in \mathbb{N}\}$. Using the definition of $\sum_k^1$, we have $v_{k+1,1}(b^*) = b^* = \beta_{k+1}$, a contradiction with the choice of b^*. This proves that $u_{k+1,1}(\beta_k) < 0$. Therefore, $\beta_k \in M_k$.

To summarize, we have shown that $M_k \subset (0,\infty)$ is a nonempty, closed and open subset of $(0,\infty)$. As a result, $M_k = (0,\infty)$ and hence the proof is complete. ∎

THEOREM A.10. *$u_{k,0}(\beta) > u_{k,1}(\beta) > u_{k,2}(\beta) > u_{k+1,0}(\beta)$ for all $\beta \in (0,\infty)$ and $k \in \mathbb{N}$.*

Proof. First, for each given $\beta \in (0,\infty)$, we claim that if λ^0, λ^1 and λ^2 respectively satisfy (A.5), (A.6), and (A.7) with $\mathrm{Im}(\lambda^j) \geq 0$ for all $j \in \{0,1,2\}$, then $\mathrm{Re}(\lambda^{j^*}) \neq \mathrm{Re}(\lambda^{j^{**}})$ for all $(j^*, j^{**}) \in \{(0,1),(0,2),(1,2)\}$. Otherwise, there exists $(j^*, j^{**}) \in \{(0,1),(0,2),(1,2)\}$ such that $\mathrm{Re}(\lambda^{j^*}) = \mathrm{Re}(\lambda^{j^{**}})$. It follows from (A.5), (A.6), and (A.7) that

$$\cos\left(-\frac{2j^*\pi}{3} + \mathrm{Im}(\lambda^{j^*})\right) = \cos\left(-\frac{2j^{**}\pi}{3} + \mathrm{Im}(\lambda^{j^{**}})\right) \tag{A.8}$$

and

$$\mathrm{Im}(\lambda^{j^{**}})\sin\left(-\frac{2j^*\pi}{3}+\mathrm{Im}(\lambda^{j^*})\right)=\mathrm{Im}(\lambda^{j^*})\sin\left(-\frac{2j^{**}\pi}{3}+\mathrm{Im}(\lambda^{j^{**}})\right). \tag{A.9}$$

It follows from (A.8) that

$$\left|\sin\left(-\frac{2j^*\pi}{3}+\mathrm{Im}(\lambda^{j^*})\right)\right|=\left|\sin\left(-\frac{2j^{**}\pi}{3}+\mathrm{Im}(\lambda^{j^{**}})\right)\right|. \tag{A.10}$$

If one of $\sin(-\frac{2j^*\pi}{3}+\mathrm{Im}(\lambda^{j^*}))$ and $\sin(-\frac{2j^{**}\pi}{3}+\mathrm{Im}(\lambda^{j^{**}}))$ is zero, then so is the other and hence it follows from $\mathrm{Im}(\lambda^{j^*})=-\beta e^{-\mathrm{Re}(\lambda^{j^*})}\sin(-\frac{2j^*\pi}{3}+\mathrm{Im}(\lambda^{j^*}))$ and $\mathrm{Im}(\lambda^{j^{**}})=-\beta e^{-\mathrm{Re}(\lambda^{j^{**}})}\sin(-\frac{2j^{**}\pi}{3}+\mathrm{Im}(\lambda^{j^{**}}))$ that $\mathrm{Im}(\lambda^{j^*})=\mathrm{Im}(\lambda^{j^{**}})=0$. If neither $\sin(-\frac{2j^*\pi}{3}+\mathrm{Im}(\lambda^{j^*}))$ nor $\sin(-\frac{2j^{**}\pi}{3}+\mathrm{Im}(\lambda^{j^{**}}))$ is zero, then it follows from (A.9) and (A.10) that $\mathrm{Im}(\lambda^{j^*})=\mathrm{Im}(\lambda^{j^{**}})$. Therefore, we have shown $\mathrm{Im}(\lambda^{j^*})=\mathrm{Im}(\lambda^{j^{**}})$ and hence $\lambda^{j^*}=\lambda^{j^{**}}$. Then it follows from $\lambda^{j^*}=\beta e^{-\lambda^{j^*}+i\frac{2j^*\pi}{3}}$ and $\lambda^{j^{**}}=\beta e^{-\lambda^{j^{**}}+i\frac{2j^{**}\pi}{3}}$ that $e^{i\frac{2(j^*-j^{**})\pi}{3}}=1$, which is impossible as $(j^*,j^{**})\in\{(0,1),(0,2),(1,2)\}$. This proves the claim.

Let $N_k^0=\{\beta>0: u_{k,0}(\beta)>u_{k,1}(\beta)\}$, $N_k^1=\{\beta>0: u_{k,1}(\beta)>u_{k,2}(\beta)\}$, and $N_k^2=\{\beta>0: u_{k,2}(\beta)>u_{k+1,0}(\beta)\}$ for all $k\in\mathbb{N}$. Then similar arguments as those in the proof of Lemma A.9 will yield that, for every $(k,j)\in\mathbb{N}\times\{0,1,2\}$, N_k^j is both open and closed in $(0,\infty)$. We shall show that, for every $(k,j)\in\mathbb{N}\times\{0,1,2\}$, $N_k^j\neq\emptyset$ and hence $N_k^j=(0,\infty)$.

Let $\beta_k^j=2k\pi+\frac{2j\pi}{3}-\frac{\pi}{2}$ for all $(k,j)\in\mathbb{N}\times\{0,1,2\}$. We show that, for every $(k,j)\in\mathbb{N}\times\{0,1,2\}$, $N_k^j\neq\emptyset$ by distinguishing three cases.

Case 1. $j=0$. First, suppose $k=0$. Note that $\lambda_{0,0}(1)\in\mathbb{R}_+$. Then, from (A.5) and (A.6), we have $\lambda_{0,0}(1)-e^{-\lambda_{0,0}(1)}=0$ and $u_{0,1}(1)=e^{-u_{0,1}(1)}\cos(-\frac{2\pi}{3}+v_{0,1}(1))<e^{-u_{0,1}(1)}$. Since the curve $y=x-e^{-x}$ is strictly increasing on $\mathbb{R}$, we must have $\lambda_{0,0}(1)>u_{0,1}(1)$. In other words, $1\in N_0^0$ and hence $N_0^0\neq\emptyset$. Next, suppose $k\geq 1$. Then $\lambda_{k,0}(\beta_k^0)=i\beta_k^0$ and $u_{k,0}(\beta_k^0)=0$. On the other hand, it follows from Lemma A.7 and (A.6) that $v_{k,1}(\beta_k^0)\in(2k\pi-\frac{1}{3}\pi,2k\pi+\frac{2\pi}{3})$ and $v_{k,1}(\beta_k^0)=-(2k\pi-\frac{\pi}{2})e^{-u_{k,1}(\beta_k^0)}\sin(v_{k,1}(\beta_k^0)-\frac{2\pi}{3})\leq(2k\pi-\frac{\pi}{2})e^{-u_{k,1}(\beta_k^0)}$. Then, $e^{u_{k,1}(\beta_k^0)}\leq\frac{2k\pi-\frac{\pi}{2}}{2k\pi-\frac{\pi}{3}}<1$, and hence $u_{k,1}(\beta_k^0)<0=u_{k,0}(\beta_k^0)$. This means that $\beta_k^0\in N_k^0$, and hence $N_k^0\neq\emptyset$.

Case 2. $j=1$. Similar arguments as those for the case where $k\geq1$ in Case 1 will show that $\beta_k^1\in N_k^1$, and hence $N_k^1\neq\emptyset$.

Case 3. $j=2$. Suppose $k\in\mathbb{N}$. Then $u_{k,2}(\beta_k^2)=0$. On the other hand, it follows from Lemma A.7 and (A.5) that $v_{k+1,0}(\beta_k^2)\in(2k\pi+\pi,2k\pi+2\pi)$ and $v_{k+1,0}(\beta_k^2)=-\beta_k^2e^{-u_{k+1,0}(\beta_k^2)}\sin(v_{k+1,0}(\beta_k^2)-\frac{4\pi}{3})\leq\beta_k^2e^{-u_{k+1,0}(\beta_k^2)}$. It follows that $e^{u_{k+1,0}(\beta_k^2)}\leq\frac{\beta_k^2}{2k\pi+\pi}<1$, and hence $u_{k+1,0}(\beta_k^2)<0$. This means that $\beta_k^2\in N_k^2$, and hence $N_k^2\neq\emptyset$.

To summarize, we have shown that $N_k^j\subset(0,\infty)$ is a nonempty, closed and open subset of $(0,\infty)$. As a result, $N_k^j=(0,\infty)$ for all $(k,j)\in\mathbb{N}\times\{0,1,2\}$, and hence the proof is complete. ∎

As a corollary of Theorem A.10, we present the following theorem.

THEOREM A.11. *$v_{k,0}(\beta)<v_{k,1}(\beta)<v_{k,2}(\beta)<v_{k+1,0}(\beta)$ for all $\beta\in(0,\infty)$ and $k\in\mathbb{N}$.*

Proof. We first claim that, for each given $\beta\in(0,\infty)$, if λ^* and λ^{**} satisfy (A.2) with $\mathrm{Re}(\lambda^*)>\mathrm{Re}(\lambda^{**})$ and $\mathrm{Im}(\lambda^*)$, $\mathrm{Im}(\lambda^{**})\in[1,\infty)$, then $\mathrm{Im}(\lambda^*)<\mathrm{Im}(\lambda^{**})$.

Suppose that this is not true. Then $\mathrm{Im}(\lambda^*) \geq \mathrm{Im}(\lambda^{**})$. It follows from (A.2) that $\{[\mathrm{Re}(\lambda^*)]^2 + [\mathrm{Im}(\lambda^*)]^2\}e^{2\mathrm{Re}(\lambda^*)} = \beta^2$ and $\{[\mathrm{Re}(\lambda^{**})]^2 + [\mathrm{Im}(\lambda^{**})]^2\}e^{2\mathrm{Re}(\lambda^{**})} = \beta^2$. Define $H : \mathbb{R} \times [1,\infty) \to \mathbb{R}$ by $H(u,v) = (u^2+v^2)e^{2u} - \beta^2$ for all $(u,v) \in \mathbb{R} \times (1,\infty)$. Then

$$\frac{\partial H(u,v)}{\partial u} = (2u^2+2u+v^2)e^{2u} \geq (2u^2+2u+1)e^{2u} \geq \frac{1}{2}e^{2u} > 0$$

and

$$\frac{\partial H(u,v)}{\partial v} = 2ve^{2u} > 0$$

for all $(u,v) \in \mathbb{R} \times [1,\infty)$. It follows that $0 = H(\mathrm{Re}(\lambda^*), \mathrm{Im}(\lambda^*)) > H(\mathrm{Re}(\lambda^{**}), \mathrm{Im}(\lambda^*)) > H(\mathrm{Re}(\lambda^{**}), \mathrm{Im}(\lambda^{**})) = 0$, a contradiction. This proves the claim.

Now, we are ready to finish the proof. Obviously, $v_{0,0}(\beta) < v_{0,1}(\beta)$ for all $\beta \in (0,\infty)$ by the definitions of $v_{0,0}$ and $v_{0,1}$. It is also easy to see that the other inequalities hold with the help of Theorem A.10, the definitions of $v_{k,j}$, and the above claim. This completes the proof. ∎

For each $(k,j) \in \mathbb{N} \times \{0,1,2\}$, define the function $\zeta_{k,j} : (0,\infty) \to \mathbb{C}$ by $\zeta_{k,j}(\tau) = -\tau\mu + 3\lambda_{k,j}(\frac{\tau f'(0)e^{\frac{\tau\mu}{3}}}{3})$ for all $\tau \in (0,\infty)$. By applying Lemma A.7, Theorems A.10 and A.11, the definitions of all the $\zeta_{k,j}$, and the fact that $f'(0) > \mu$, we have the following theorem.

THEOREM A.12. *For each $\tau \in (0,\infty)$, the zeros of* (A.1) *are given by a positive real $\zeta_{0,0}(\tau)$ and a sequence of complex conjugate pairs*

$$\{\zeta_{k,j}(\tau), \overline{\zeta_{k,j}(\tau)}\}_{(k,j)\in\mathbb{N}\times\{0,1,2\}\setminus\{(0,0)\}}$$

with $\zeta_{k,j}(\tau) \in 3\sum_k^j$, $\mathrm{Re}(\zeta_{k,0}(\tau)) > \mathrm{Re}(\zeta_{k,1}(\tau)) > \mathrm{Re}(\zeta_{k,2}(\tau)) > \mathrm{Re}(\zeta_{k+1,0}(\tau))$ and $\mathrm{Im}(\zeta_{k,0}(\tau)) < \mathrm{Im}(\zeta_{k,1}(\tau)) < \mathrm{Im}(\zeta_{k,2}(\tau)) < \mathrm{Im}(\zeta_{k+1,0}(\tau))$ for all $k \in \mathbb{N}$ and $j \in \{0,1,2\}$. These are all simple zeros and (A.1) *has no other zeros.*

For each given $(k,j) \in (\mathbb{N}\times\{0,1,2\})\setminus\{(0,0)\}$, let $b_{k,j} = 6k\pi + 2j\pi - 3\arccos\frac{\mu}{f'(0)}$ and $\tau_{k,j} = \frac{b_{k,j}}{\sqrt{[f'(0)]^2-\mu^2}}$. Then, by Lemma A.8 and Theorem A.12, we have the following theorem.

THEOREM A.13. *Equation* (A.1) *has a purely imaginary root ζ if and only if there exists $(k,j) \in (\mathbb{N}\times\{0,1,2\})\setminus\{(0,0)\}$ such that $\tau = \tau_{k,j}$ and $\zeta \in \{\pm\zeta_{k,j}(\tau_{k,j})\} = \{\pm ib_{k,j}\}$. Moreover, For each $(k,j) \in (\mathbb{N}\times\{0,1,2\})\setminus\{(0,0)\}$, the following statements are true.*

(i) $\frac{\mathrm{dRe}(\zeta_{k,j}(\tau))}{\mathrm{d}\tau}|_{\tau=\tau_{k,j}} > 0$.

(ii) $\mathrm{Re}(\zeta_{k,j}(\tau)) < 0$ *if and only if* $\tau < \tau_{k,j}$, *and* $\mathrm{Re}(\zeta_{k,j}(\tau)) \geq 0$ *if and only if* $\tau \geq \tau_{k,j}$.

Proof. Suppose that $\tau = \tau_{k,j}$ and $\zeta \in \{\pm\zeta_{k,j}(\tau_{k,j})\} = \{\pm ib_{k,j}\}$ for some $(k,j) \in (\mathbb{N}\times\{0,1,2\})\setminus\{(0,0)\}$. Then we can easily check that ζ is a purely imaginary root of (A.1). On the other hand, let ζ be a purely imaginary root of (A.1). Without loss of generality, we may assume that $\zeta = ib$ for some $b \geq 0$. Obviously, $b > 0$. It follows from (A.1) that $\mu = f'(0)\cos(\frac{b}{3} - \frac{2j\pi}{3})$ and $b = -\tau f'(0)\sin(\frac{b}{3} - \frac{2j\pi}{3})$ for some $j \in \{0,1,2\}$. Thus there exists $k \in \mathbb{N}$ such that $(k,j) \neq (0,0)$, $b = b_{k,j} = 6k\pi + 2j\pi - 3\arccos\frac{\mu}{f'(0)}$, $\tau = \tau_{k,j} = \frac{b_{k,j}}{\sqrt{[f'(0)]^2-\mu^2}}$.

Now, we prove (i). For given $(k,j) \in (\mathbb{N} \times \{0,1,2\}) \setminus \{(0,0)\}$, it follows from Lemma A.8 and the definition of $\zeta_{k,j}$ that

$$\frac{\mathrm{dRe}(\zeta_{k,j}(\tau))}{\mathrm{d}\tau} = -\mu + 3\frac{(v_{k,j}(\beta))^2 + u_{k,j}(\beta)(1+u_{k,j}(\beta))}{\beta[(v_{k,j}(\beta))^2 + (1+u_{k,j}(\beta))^2]} \times \left(\frac{f'(0)e^{\frac{\tau\mu}{3}}}{3} + \frac{\tau f'(0)e^{\frac{\tau\mu^2}{3}}}{9} \right),$$

where $\beta = \frac{\tau f'(0)e^{\frac{\tau\mu}{3}}}{3}$. It follows from the definitions of $\zeta_{k,j}$ and $\tau_{k,j}$ that, for $\tau^* = \tau_{k,j}$ and $\beta^* = \frac{\tau_{k,j} f'(0)e^{\frac{\tau_{k,j}\mu}{3}}}{3}$, we have $\mathrm{Re}(u_{k,j}(\beta^*)) = \frac{\tau^*\mu}{3}$, and

$$\begin{aligned}
&\frac{\mathrm{dRe}(\zeta_{k,j}(\tau))}{\mathrm{d}\tau}|_{\tau=\tau^*} \\
&= -\mu + 3\frac{(v_{k,j}(\beta^*))^2 + u_{k,j}(\beta^*)(1+u_{k,j}(\beta^*))}{\beta^*[(v_{k,j}(\beta^*))^2 + (1+u_{k,j}(\beta^*))^2]} \times \left(\frac{f'(0)e^{\frac{\tau^*\mu}{3}}}{3} + \frac{\tau^* f'(0)e^{\frac{\tau\mu^2}{3}}}{9} \right) \\
&\geq -\mu + \frac{(v_{k,j}(\beta^*))^2 + \frac{\tau^*\mu}{3}(1+\frac{\tau^*\mu}{3})}{\beta[(v_{k,j}(\beta^*))^2 + (1+\frac{\tau^*\mu}{3})^2]} \times (f'(0)e^{\frac{\tau^*\mu}{3}} + \mu\beta^*) \\
&\geq -\mu + \min\left\{ \frac{v^2 + \frac{\tau^*\mu}{3}(1+\frac{\tau^*\mu}{3})}{\beta[v^2 + (1+\frac{\tau^*\mu}{3})^2]} : v \in (0,\infty) \right\} \times (f'(0)e^{\frac{\tau^*\mu}{3}} + \mu\beta) \\
&\geq -\mu + \frac{\frac{\tau^*\mu}{3}}{\beta(1+\frac{\tau^*\mu}{3})} \times (f'(0)e^{\frac{\tau^*\mu}{3}} + \mu\beta^*) \\
&> 0.
\end{aligned}$$

The last inequality comes from the definitions of τ^* and β^*.

Finally, we prove (ii). Let $\mathrm{Re}(\zeta_{k,j}(\tau)) < 0$. Suppose $\tau_{k,j} \leq \tau$. Then $\tau_{k,j} < \tau$ since $\mathrm{Re}(\zeta_{k,j}(\tau)) \neq 0$. From the above discussion, we have $\mathrm{Re}(\zeta_{k,j}(\tau_{k,j})) = 0$ and $\frac{\mathrm{dRe}(\zeta_{k,j}(\tau))}{\mathrm{d}\tau}|_{\tau=\tau_{k,j}} > 0$. It follows that there exists $a^* \in (\tau_{k,j}, \tau)$ such that $\mathrm{Re}(\zeta_{k,j}(a^*)) = 0$. Hence $\zeta_{k,j}(a^*)$ is also a purely imaginary root of (A.1), a contradiction. On the other hand, if $\tau < \tau_{k,j}$, then we can also show that $\mathrm{Re}(\zeta_{k,j}(\tau)) < 0$. Indeed, by way of contradiction, assume that $\mathrm{Re}(\zeta_{k,j}(\tau)) \geq 0$. Then similarly we can deduce that there exists $a^{**} \in [\tau, \tau_{k,j})$ such that $\mathrm{Re}(\zeta_{k,j}(a^{**})) = 0$. Again $\zeta_{k,j}(a^{**})$ is also a purely imaginary root of (A.1), a contradiction.

This completes the proof. ∎

The following lemma is important to study the value of the discrete Lyapunov functional for the solution of (3.13).

LEMMA A.14. *Assume that $a : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with minimal period $T > 0$ such that $a(0) = 0$, $a(t) = -a(t + \frac{T}{2})$ and $a(t) > 0$ for all $t \in (0, \frac{T}{2})$. Define $x = (x^0, x^1, x^2)^{\mathrm{tr}} : \mathbb{R} \to \mathbb{R}^3$ such that $x^2(t) = x^1(t+\alpha) = x^0(t+2\alpha) = a(t+2\alpha)$ for all $t \in \mathbb{R}$ and some $\alpha \in \mathbb{R}$. If there exists $l \in \mathbb{N}$ such that $V(x_t) = 2l$ for all $t \in \mathbb{R}$, then the following statements are true.*

(i) *If $\alpha = -\frac{1}{3}$ and there exists $k \in \mathbb{N} \setminus \{0\}$ such that $3kT > 1 > (3k - \frac{3}{2})T$, then $l = 3k$ and thus $V(x_t) = 6k$ for all $t \in \mathbb{R}$.*

(ii) *If $\alpha = -\frac{1}{3} + \frac{T}{3}$ and there exists $k \in \mathbb{N}$ such that $(3k+1)T > 1 > (3k - \frac{1}{2})T$, then $l = 3k+1$ and thus $V(x_t) = 6k+2$ for all $t \in \mathbb{R}$.*

(iii) *If $\alpha = -\frac{1}{3} + \frac{2T}{3}$ and there exists $k \in \mathbb{N}$ such that $(3k+2)T > 1 > (3k + \frac{1}{2})T$, then $l = 3k+2$ and thus $V(x_t) = 6k+4$ for all $t \in \mathbb{R}$.*

Proof. We only prove (ii) since the other parts can be proved similarly. We distinguish two cases to finish the proof.

Case 1. $k = 0$. In this case, we have $T > 1$. We distinguish two subcases to finish the proof.

Case 1.1. $T \geq 2$. Then $\alpha \in (\frac{T}{6}, \frac{T}{3})$. It follows that $x^1(0) = a(\alpha) > 0$. This, combined with the definition of sc and $x^0(t) < 0$ for all $t \in (-1, 0)$, yields $\mathrm{sc}(x_0) \in \{1, 2\}$. Thus $V(x_0) = 2$ and $l = 1$.

Case 1.2. $T \in (1, 2)$. Then $\alpha \in (0, \frac{T}{6})$. It follows that $x^1(0) > 0$ and $x^2(0) > 0$. This, combined with the definition of sc, $x^0(t) < 0$ for all $t \in (-\frac{T}{2}, 0)$ and $x^0(t) > 0$ for all $t \in (-1, -\frac{T}{2})$, yields $\mathrm{sc}(x_0) = 2$. Thus $V(x_0) = 2$ and $l = 1$.

Case 2. $k \geq 1$. Let $\alpha^* = \alpha + kT$. Then $\alpha^* \in (0, \frac{T}{2})$ and $x^2(t) = x^1(t + \alpha^*) = x^0(t + 2\alpha^*) = a(t + 2\alpha^*)$ for all $t \in \mathbb{R}$. We distinguish three subcases to complete the proof.

Case 2.1. $1 \in ((3k + \frac{1}{2})T, (3k + 1)T)$. In this case, $\alpha^* \in (0, \frac{T}{6})$. Then α^*, $2\alpha^* \in (0, \frac{T}{2})$ and hence $x^1(0) > 0$ and $x^2(0) > 0$. A simple computation gives $\mathrm{sc}(x_0) = 6k + 2$. So $V(x_0) = 6k + 2$ and $l = 3k + 1$.

Case 2.2. $1 \in (3kT, (3k + \frac{1}{2})T]$. In this case, $\alpha^* \in [\frac{T}{6}, \frac{T}{3})$. Then $x^0(\alpha^*) > 0$ and $x^2(\alpha^*) \leq 0$. By a simple computation, we get $\mathrm{sc}(x_{\alpha^*}) \in \{6k + 1, 6k + 2\}$. So $V(x_{\alpha^*}) = 6k + 2$ and $l = 3k + 1$.

Case 2.3. $1 \in ((3k - \frac{1}{2})T, 3kT]$. In this case, $\alpha^* \in [\frac{T}{3}, \frac{T}{2})$. Then $\alpha^* \in (0, \frac{T}{2})$, $2\alpha^* \in (\frac{T}{2}, T)$, and hence $x^1(0) > 0$ and $x^2(0) < 0$. Again, by a simple computation, we get $\mathrm{sc}(x_0) = 6k + 1$. So $V(x_0) = 6k + 2$ and $l = 3k + 1$. $\square$

Let

$$\mathscr{A} = \{(k, j, a, b, \tau) \in \mathbb{N} \times \{0, 1, 2\} \times \mathbb{R} \times \mathbb{R} \times (0, \infty) : (a, b) \in A_{k,j}\},$$

where

$$A_{k,j} = \begin{cases} \mathbb{R} \times \mathbb{R}, & \text{if } (k, j) \in (\mathbb{N} \times \{0, 1, 2\}) \setminus \{(0, 0)\}, \\ \mathbb{R} \times \{\frac{\pi}{2}\}, & \text{if } (k, j) = (0, 0). \end{cases}$$

For each $(k, j, a, b, \tau) \in \mathscr{A}$, define $X(k, j, a, b, \tau) : \mathbb{R} \to \mathbb{R}^3$ by

$$\begin{aligned}
(X(k, j, a, b, \tau))^0(t) &= a e^{\mathrm{Re}(\zeta_{k,j}(\tau))t} \sin(\mathrm{Im}(\zeta_{k,j}(\tau))t + b), \\
(X(k, j, a, b, \tau))^1(t) &= a e^{\mathrm{Re}(\zeta_{k,j}(\tau))(t - \frac{1}{3})} \sin\left(\mathrm{Im}(\zeta_{k,j}(\tau))\left(t - \frac{1}{3}\right) + b + \frac{2j\pi}{3}\right), \\
(X(k, j, a, b, \tau))^2(t) &= a e^{\mathrm{Re}(\zeta_{k,j}(\tau))(t - \frac{1}{3})} \sin\left(\mathrm{Im}(\zeta_{k,j}(\tau))\left(t - \frac{1}{3}\right) + b + \frac{4j\pi}{3}\right).
\end{aligned}$$

In particular, $(X(0, 0, a, \frac{\pi}{2}, \tau))^0(t) = a e^{(\zeta_{0,0}(\tau))t}$, $(X(0, 0, a, \frac{\pi}{2}, \tau))^1(t) = a e^{(\zeta_{0,0}(\tau))(t - \frac{1}{3})}$, and $(X(0, 0, a, \frac{\pi}{2}, \tau))^2(t) = a e^{(\zeta_{0,0}(\tau))(t - \frac{1}{3})}$ for all $\tau \in (0, \infty)$ and $t \in \mathbb{R}$. According to the above definition, we can easily check that $X(k, j, a, b, \tau)(\cdot)$ is a solution of (3.13) for all $(\tau, k, j, a, b) \in \mathscr{A}$.

For $\tau \in (0, \infty)$, let $P_{0,0}(\tau)$ and $P_{k,j}(\tau)$ $((k, j) \in (\mathbb{N} \times \{0, 1, 2\}) \setminus \{(0, 0)\})$ be the realified generalized eigenspaces of the generator of the semigroup given by the linear system (3.13) for the spectral sets $\{\zeta_{0,0}(\tau)\}$ and $\{\zeta_{k,j}(\tau), \overline{\zeta_{k,j}(\tau)}\}$, respectively. By a simple computation, we have the following result.

PROPOSITION A.15. *For each* $(\tau, k, j) \in (0, \infty) \times \mathbb{N} \times \{0, 1, 2\}$,

$$P_{k,j}(\tau) = \{(X(k, j, a, b, \tau))_0 : (a, b) \in A_{k,j}\}.$$

THEOREM A.16. *If* $(\tau, k, j) \in (0, \infty) \times \mathbb{N} \times \{0, 1, 2\}$ *and* $\phi \in P_{k,j}(\tau) \setminus \{0\}$, *then there exist* $a \in \mathbb{R} \setminus \{0\}$ *and* $b \in \mathbb{R}$ *such that*

$$(X(k, j, a, b, \tau))_0 = \phi \quad \text{and} \quad V((X(k, j, a, b, \tau))_t) = 6k + 2j \quad \text{for all } t \in \mathbb{R}.$$

Proof. By Proposition A.15, there exist a, $b \in \mathbb{R}$ such that the solution of (3.13) through ϕ is $X(k,j,a,b,\tau)$ with $(X(k,j,a,b,\tau))_0 = \phi$. If $k = j = 0$, then $b = \frac{\pi}{2}$. In view of the definition of $X(0,0,a,\frac{\pi}{2},\tau)$ and V, $V((X(0,0,a,\frac{\pi}{2},\tau))_t) = 0$ for all $t \in \mathbb{R}$. Now suppose $(k,j) \neq (0,0)$. Since $X(k,j,a,b,\tau)$ is the multiplication of a periodic function with a positive (or negative) function, $V((X(k,j,a,b,\tau))_t)$ must be periodic with a period $\frac{2\pi}{\mathrm{Im}(\zeta_{k,j}(\tau))}$. Moreover, Proposition 2.6 (iii) tells us that $V((X(k,j,a,b,\tau))_t)$ is nonincreasing in t. Thus $V((X(k,j,a,b,\tau))_t)$ is constant and there exists $l \in \mathbb{N}$ such that $V((X(k,j,a,b,\tau))_t) = 2l$ for all $t \in \mathbb{R}$. For $(k,j,a,b,\tau) \in \mathscr{A}$, define $\widetilde{X}(k,j,a,b,\tau) : \mathbb{R} \to \mathbb{R}^3$ by

$$
\begin{aligned}
(\widetilde{X}(k,j,a,b,\tau))^0(t) &= \sin(\mathrm{Im}(\zeta_{k,j}(\tau))t + b),\\
(\widetilde{X}(k,j,a,b,\tau))^1(t) &= \sin\left(\mathrm{Im}(\zeta_{k,j}(\tau))\left(t - \frac{1}{3}\right) + b + \frac{2j\pi}{3}\right),\\
(\widetilde{X}(k,j,a,b,\tau))^2(t) &= \sin\left(\mathrm{Im}(\zeta_{k,j}(\tau))\left(t - \frac{1}{3}\right) + b + \frac{4j\pi}{3}\right).
\end{aligned}
$$

Then we easily see that $V((\widetilde{X}(k,j,a,b,\tau))_t) = V((X(k,j,a,b,\tau))_t) = 2l$ for $t \in \mathbb{R}$. Applying Lemma A.14, we have $V((\widetilde{X}(k,j,a,b,\tau))_t) = 6k + 2j$ and it follows that $V((X(k,j,a,b,\tau))_t) = 6k + 2j$ for $t \in \mathbb{R}$. This completes the proof. ∎

Appendix B. Several basic results on (1.1). Though the natural phase space $C([-1,0],\mathbb{R}^3)$ is replaced with $C = C(\mathbb{K},\mathbb{R})$, there is no change in the basic theory on (1.1) with this obvious modification. In this appendix, for clarification and for the readers' convenience, we provide the precise explanations and detailed proofs for some of the results mentioned in the main text.

A simple computation reveals that x satisfies (1.1) if and only if x satisfies the following system of equations:

$$
\text{(B.1)} \qquad \begin{cases} \frac{\mathrm{d}x^0(t)e^{\mu t}}{\mathrm{d}t} = e^{\mu t} f(x^1(t)),\\ \frac{\mathrm{d}x^1(t)e^{\mu t}}{\mathrm{d}t} = e^{\mu t} f(x^2(t)),\\ \frac{\mathrm{d}x^2(t)e^{\mu t}}{\mathrm{d}t} = e^{\mu t} f(x^0(t-1)). \end{cases}
$$

According to the method of steps, we can easily deduce results on the existence, uniqueness, and continuous dependence on the initial value $\phi \in C$ of solutions to (1.1). Precisely, for each $\phi \in C$ and $t_0 \in \mathbb{R}$, there exists a unique solution $x_\phi = (x_\phi^0, x_\phi^1, x_\phi^2)^{\mathrm{tr}}$ to (1.1) on $[t_0,\infty)$; that is, $x_\phi^0 \in C([t_0-1,\infty),\mathbb{R})$ and $x_\phi^0|_{[t_0,\infty)}$, x_ϕ^1, $x_\phi^2 \in C^1([t_0,\infty),\mathbb{R})$ such that $(x_\phi)_{t_0} = \phi$ and $\dot{x}_\phi^i$ $(i = 0, 1, 2)$ satisfy (1.1) on $[t_0,\infty)$. The solution is also denoted by $x(\cdot,\phi,t_0)$. Moreover, $x(\cdot,\phi,t_0)$ depends continuously on $\phi \in C$.

PROPOSITION B.1. *Let x and y be two solutions of* (1.1) *on a closed interval I with $\sup I = \infty$. If $x_{t^*} = y_{t^*}$ for some $t^* \in I$, then $x_t = y_t$ for all $t \in I$.*

Proof. Obviously, it follows from the uniqueness of solution that $x_t = y_t$ for all $t \in [t^*,\infty)$. We shall distinguish two cases to finish the proof.

Case 1. There exists $a \in \mathbb{R}$ such that $I = [a,\infty)$. Let $s^* = \sup\{t \in [a,t^*), x_t \neq y_t\}$. Then $x_{s^*} = y_{s^*}$ and hence $x_t = y_t$ for $t \geq s^*$ by the above discussion. We claim that $s^* = a$. If $s^* \neq a$, then $s^* > a$. Let $t^{**} = \max\{s^*, a+1\}$. It follows from $x_t = y_t$ for all $t \geq t^{**}$ that $x^0(t) = y^0(t)$ for all $t \geq t^{**} - 1$. Then the first equation of (B.1) implies that $f(x^1(t)) = f(y^1(t))$ for all $t \geq t^{**} - 1$. Thus $x^1(t) = y^1(t)$ for all $t \geq t^{**} - 1$ as f is strictly increasing. Similarly, from the other two equations in (B.1), we can obtain $x^2(t) = y^2(t)$ for all $t \geq t^{**} - 1$ and $x^0(t) = y^0(t)$ for all $t \geq t^{**} - 2$. It follows that $x_t = y_t$ for all $t \geq t^{**} - 1$, a contradiction to the choice of t^*. This proves the claim and hence $x_t = y_t$ for all $t \in [a, \infty)$.

Case 2. $I = \mathbb{R}$. For any $b \in \mathbb{R}$, from the discussion of Case 1, we know that $x_t = y_t$ for all $t \in [b, \infty)$. Because of the arbitrariness of b, $x_t = y_t$ for all $t \in \mathbb{R}$.

This completes the proof. ∎

Define the map $F : \mathbb{R}_+ \times C \ni (t, \phi) \mapsto (x_\phi)_t \in C$.

PROPOSITION B.2. *The following statements about the map F hold.*

(i) *The map $F : \mathbb{R}_+ \times C \ni (t, \phi) \mapsto (x_\phi)_t \in C$ is a continuous semiflow on C.*
(ii) *All maps $F(t, \cdot) : C \to C$ are injective whenever $t \geq 0$.*
(iii) *All maps $F(t, \cdot) : C \to C$ are conditionally completely continuous whenever $t \geq 1$.*
(iv) *For each $t \geq 1$, all maps $F(t, \cdot) : C \to C^1$ are continuous.*

Proof. (i) follows immediately from the uniqueness and the continuous dependence on the initial data of solutions of (1.1).

(ii) follows from Proposition B.1.

(iii) follows from Remark 2.3, and (1.1).

(iv) follows from (1.1), Remark 2.3, and the definition of $F(\cdot, \cdot)$.

This completes the proof. ∎

In order to derive locally asymptotic stability of the equilibria of (1.1), we introduce the following system of delay differential equations,

$$\begin{cases} \dot{y}^0(t) = -\mu y^0(t) + f(y^1(t - \frac{1}{3})), \\ \dot{y}^1(t) = -\mu y^1(t) + f(y^2(t - \frac{1}{3})), \\ \dot{y}^2(t) = -\mu y^2(t) + f(y^0(t - \frac{1}{3})). \end{cases} \tag{B.2}$$

The natural phase space for (B.2) is the Banach space $Y = C([-\frac{1}{3}, 0], \mathbb{R}^3)$, equipped with the supremum norm $|| \cdot ||_Y$. For a given interval I, let $[-\frac{1}{3}, 0] + I = \{t + \theta : t \in I \text{ and } \theta \in [-\frac{1}{3}, 0]\}$. For a continuous function $y : [-\frac{1}{3}, 0] + I \to \mathbb{R}^3$ on I and $t \in I$, define $P(y, t) \in Y$ by $P(y, t)(\theta) = y(t + \theta)$ for all $\theta \in [-\frac{1}{3}, 0]$.

For a given interval I, we say that a continuous function $y : [-\frac{1}{3}, 0] + I \to \mathbb{R}^3$ is a solution of (B.2) on I if y^0, y^1, and y^2 are continuously differentiable on the interval I and satisfy (B.2) on I. We denote the solution semiflow of (B.2) by $G(t, \psi)$, where $\psi \in Y$ and $G(0, \psi) = \psi$.

Define $M : C \to Y$ by

$$M(\phi)(\theta) = (\phi^0(\theta), (x_\phi)^1(\theta + \tfrac{1}{3}), (x_\phi)^2(\theta + \tfrac{1}{3}))^{\mathrm{tr}} \qquad \text{for } \theta \in [-\tfrac{1}{3}, 0] \text{ and } \phi \in C.$$

Obviously, $\{M(\widehat{0}), M(\widehat{\xi^+}), M(\widehat{\xi^-})\}$ is the set of equilibria of (B.2).

PROPOSITION B.3. *For any $\phi \in C$ and $t \in \mathbb{R}_+$, we have $G(t, M(\phi)) = M(F(t, \phi))$.*

Proof. Define x, $y : [-\frac{1}{3}, \infty) \to \mathbb{R}^3$ by

$$x(t) = \begin{cases} M(F(t, \phi))(0) & t \in [0, \infty), \\ M(\phi)(t) & t \in [-\frac{1}{3}, 0] \end{cases}$$

and

$$y(t) = \begin{cases} G(t, M(\phi))(0) & t \in [0, \infty), \\ M(\phi)(t) & t \in [-\frac{1}{3}, 0]. \end{cases}$$

It suffices to prove that $x(t) = y(t)$ for all $t \in [-\frac{1}{3}, \infty)$. Obviously, by the definition of y, we know that y satisfies (B.2) with the initial value $M(\phi)$. On the other hand, from the definitions of x, $F(t,\phi)$, and M, we obtain that $x(t) = ((x_\phi)^0(t), (x_\phi)^1(t+\frac{1}{3}), (x_\phi)^2(t+\frac{2}{3}))^{\rm tr}$ for all $t \in [-\frac{1}{3}, \infty)$. This, combined with the fact that x_ϕ satisfies (1.1) with the initial value ϕ, implies that x also satisfies (B.2) with the initial value $M(\phi)$. Therefore, by the uniqueness of solution to (B.2), we infer that $x(t) = y(t)$ for all $t \in [-\frac{1}{3}, \infty)$. This completes the proof. ∎

PROPOSITION B.4. *$M(\widehat{\xi^+})$ and $M(\widehat{\xi^-})$ are locally asymptotically stable.*

Proof. Recall that $f'(\xi^+) = f'(\xi^-)$. Then linerization of (B.2) around $M(\widehat{\xi^\pm})$ yields the delay system

$$\begin{cases} \dot{y}^0(t) = -\mu y^0(t) + f'(\xi^+) y^1(t-\frac{1}{3}), \\ \dot{y}^1(t) = -\mu y^1(t) + f'(\xi^+) y^2(t-\frac{1}{3}), \\ \dot{y}^2(t) = -\mu y^2(t) + f'(\xi^+) y^0(t-\frac{1}{3}) \end{cases}$$

with the characteristic equation

$$(\zeta + \tau\mu)^3 - (\tau f'(\xi^+))^3 e^{-\zeta} = 0. \tag{B.3}$$

Since $f'(\xi^+) \in (0, \mu)$, it is easy to see that (B.3) has a negative solution. Let $\lambda = \frac{\zeta+\tau\mu}{3}$ and $\beta = \frac{\tau f'(\xi^+) e^{\frac{\tau\mu}{3}}}{3}$. Then (B.3) becomes (A.2). It follows from Lemma A.7, Theorems A.10 and A.11, and the fact that (B.3) has a negative root that all the roots of (B.3) have negative real parts. This and the general results in [7] show that $M(\widehat{\xi^\pm})$ are locally asymptotically stable. This completes the proof. ∎

Given $\varepsilon > 0$, $\varphi^* \in C$ and $\psi^* \in Y$, let $B(\varphi^*, \varepsilon) = \{\varphi \in C : ||\varphi - \varphi^*|| < \varepsilon\}$ and $B^Y(\psi^*, \varepsilon) = \{\psi \in Y : ||\psi - \psi^*||_Y < \varepsilon\}$.

THEOREM B.5. *The equilibria $\widehat{\xi^+}$ and $\widehat{\xi^-}$ of* (1.1) *are locally asymptotically stable.*

Proof. We only prove that $\widehat{\xi^+}$ is locally asymptotically stable since the proof for the stability of $\widehat{\xi^-}$ is similar.

First, we prove that $\widehat{\xi^+}$ is locally stable. For any $\varepsilon > 0$, by Proposition B.4, there exists $\delta_1 > 0$ such that $G(t,\psi) \in B^Y(M(\widehat{\xi^+}), \varepsilon)$ for all $t \in \mathbb{R}_+$ and $\psi \in B^Y(M(\widehat{\xi^+}), \delta_1)$. From the continuity of M, there exists $\delta \in (0, \delta_1)$ such that $M(\phi) \in B^Y(M(\widehat{\xi^+}), \delta_1)$ for all $\phi \in B(\widehat{\xi^+}, \delta)$. This, combined with Proposition B.3, gives us that $M(F(t,\phi)) \in B^Y(M(\widehat{\xi^+}), \varepsilon)$ for all $t \in \mathbb{R}_+$ and $\phi \in B(\widehat{\xi^+}, \delta)$. It follows from the definition of M and the continuity of $F(\cdot,\cdot)$ that $F(t,\phi) \in B(\widehat{\xi^+}, \varepsilon)$ for all $t \in \mathbb{R}_+$ and $\phi \in B(\widehat{\xi^+}, \delta)$; i.e., $\widehat{\xi^+}$ is locally stable.

Next, we show that $\widehat{\xi^+}$ is attractive. Proposition B.4 implies that there exists $\delta^* > 0$ such that $\lim_{t\to\infty} ||G(t,\psi) - M(\widehat{\xi^+})|| = 0$ for all $\psi \in B^Y(M(\widehat{\xi^+}), \delta^*)$. From Proposition B.3 and the continuity of M, there exists $\delta^{**} \in (0, \delta^*)$ such that $\lim_{t\to\infty} ||M(F(t,\phi)) - M(\widehat{\xi^+})|| = 0$ for all $\phi \in B(\widehat{\xi^+}, \delta^{**})$. It follows from the definition of M and the continuity of $F(\cdot,\cdot)$ that $\lim_{t\to\infty} ||F(t,\phi) - \widehat{\xi^+}|| = 0$ for all $\phi \in B(\widehat{\xi^+}, \delta^{**})$; i.e., $\widehat{\xi^+}$ is attractive.

In summary, by the above discussions, we have shown that $\widehat{\xi^+}$ is locally asymptotically stable. ∎

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REFERENCES

[1] O. Arino, *A note on "The discrete Lyapunov function ..."*, J. Differential Equations, 104 (1993), pp. 169–181.

[2] Y. Cao, *The discrete Lyapunov function for scalar differential delay equations*, J. Differential Equations, 87 (1990), pp. 365–390.

[3] Y. Cao, *Uniqueness of periodic solution for differential delay equations*, J. Differential Equations, 128 (1996), pp. 46–57.

[4] Y. Chen and J. Wu, *Existence and attraction of a phase-locked oscillation in a delayed network of two neurons*, Differential Integral Equations, 14 (2001), pp. 1181–1236.

[5] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay Equations: Functional, Complex, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.

[6] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys Monogr. 25, Amer. Math. Soc., Providence, RI, 1988.

[7] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Springer-Verlag, New York, 1993.

[8] A. V. M. Herz, *Global analysis of recurrent neural networks*, in Models of Neural Networks, Vol. 3, E. Domany, J. L. van Hemmen, and K. Schulten, eds., Springer-Verlag, New York, 1996

[9] J. J. Hopfield, *Neural networks and physical systems with emergent collective computational abilities*, Proc. Natl. Acad. Sci. USA, 79 (1982), pp. 2554–2558.

[10] J. J. Hopfield, *Neurons with graded response have collective computational properties like two-stage neurons*, Proc. Natl. Acad. Sci. USA, 81 (1984), pp. 3088–3092.

[11] J. L. Kaplan and J. A. Yorke, *On the stability of a periodic solution of a differential delay equation*, SIAM J. Math. Anal., 6 (1975), pp. 268–282.

[12] J. L. Kaplan and J. A. Yorke, *On the nonlinear differential delay equation $x'(t) = -f(x(t), x(t-1))$*, J. Differential Equations, 23 (1977), pp. 293–314.

[13] T. Krisztin, *The unstable set of zero and the global attractor for delayed monotone positive feedback*, Discrete Contin. Dyn. Syst., Added Volume, (2001), pp. 229–240.

[14] T. Krisztin, *Unstable sets of periodic orbits and the global attractor for delayed feedback*, Fields Inst. Commun., 29 (2001), pp. 267–296.

[15] T. Krisztin and H.-O. Walther, *Unique periodic orbits for delayed positive feedback and the global attractor*, J. Dynam. Differential Equations, 13 (2001), pp. 1–57.

[16] T. Krisztin, H.-O. Walther, and J. Wu, *Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback*, Amer. Math. Soc., Providence, RI, 1999.

[17] C. M. Marcus and R. M. Westervelt, *Stability of analog neural networks with delay*, Phys. Rev. A (3), 39 (1989), pp. 347–359.

[18] J. Mallet-Paret, *Morse decompositions for delay-differential equations*, J. Differential Equations, 72 (1988), pp. 270–315.

[19] J. Mallet-Paret and G. Sell, *Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions*, J. Differential Equations, 125 (1996), pp. 385–440.

[20] J. Mallet-Paret and G. Sell, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay*, J. Differential Equations, 125 (1996), pp. 441–489.

[21] R. D. Nussbaum, *Uniqueness and nonuniqueness for periodic solutions of $x'(t) = -g(x(t-1))$*, J. Differential Equations, 34 (1979), pp. 25–54.

[22] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Amer. Math. Soc., Providence, RI, 1995.

[23] H.-O. Walther, *A theorem on the amplitudes of periodic solutions of differential delay equations with applications to bifurcation*, J. Differential Equations, 29 (1978), pp. 396–404.

[24] J. Wu, *Symmetric functional-differential equations and neural networks with memory*, Trans. Amer. Math. Soc., 350 (1998), pp. 4799–4838.