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WELL-DEFINED SOLVABILITY AND SPECTRAL PROPERTIES OF ABSTRACT HYPERBOLIC EQUATIONS WITH AFTEREFFECT

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Abstract. We study functional differential equations with unbounded operator coefficients in Hilbert spaces such that the principal part of the equation is an abstract hyperbolic equation perturbed by terms with delay and terms containing Volterra integral operators. The well-posed solvability of initial boundary-value problems for the specified problems in weighted Sobolev spaces on the positive semi-axis is established.

Our concern is spectra of operator-valued functions that are symbols of the specified equations in the autonomous case. In particular, the spectra of the Gurtin–Pipkin equation is studied, which is an integrodifferential equation modelling the heat propagation in media with memory.

CONTENTS

Introduction

Many papers are nowadays devoted to the solvability and asymptotic behavior of functional differential and integrodifferential equations in Banach and Hilbert spaces (see [2–4, 10, 11, 13–15, 18–25] and references therein).

In this paper, functional differential and integrodifferential equations with unbounded operator coefficients are studied in Hilbert spaces. The principal part of the considered equation is an abstract hyperbolic equation perturbed by terms with delay and terms containing Volterra integral operators.

We study the case of variable delays, while the majority of papers of this direction, known to the authors, treat the case of constant delays (see [2–4, 10, 11]) and bounded operator coefficients at delay terms (see [24, 25] and references therein).

We prove that initial boundary-value problems in weighted Sobolev spaces on the positive semi-axis are well posed for the specified equations and study spectra of operator-valued functions that are symbols of the specified equations in the autonomous case.

Finally, certain results are applied to the Gurtin–Pipkin equations, which are integrodifferential equations modelling the finite-speed heat propagation in media with memory. Results of the present paper are natural generalizations of the corresponding results of [22] (the extension to the case of Volterra integral operators) and [21, 23] (the extension to the case of series of discrete delays).

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1. Notation, Definitions, and Main Results

Let H be a separable Hilbert space and A be a positive self-adjoint operator acting in H such that A is invertible and its inverse is bounded. We introduce on $\mathrm{Dom}(A^\beta)$, which is the domain of the operator A^β, $\beta > 0$, the norm $\|\cdot\|_\beta = \|A^\beta \cdot\|$ equivalent to the graph norm of A^β. Then $\mathrm{Dom}(A^\beta)$ becomes a Hilbert space.

Let $L_{2,\gamma}((a,b),X)$, $-\infty \le a < b \le +\infty$, denote the space of measurable functions that take values in a Hilbert space X. The norm in $L_{2,\gamma}((a,b),X)$ is introduced as follows:

$$\|f\|_{L_{2,\gamma}((a,b),X)} \equiv \left(\int_a^b \exp(-2\gamma t)\|f(t)\|_X^2 \, dt \right)^{1/2}, \quad \gamma \ge 0.$$

Let $W_{2,\gamma}^m((a,b),A^m)$, $m \in \mathbb{N}$, denote the space of functions that take values in H such that $A^{pm}u^{(1-p)m} \in L_{2,\gamma}((a,b),H)$, $(p = 0,1)$. The norm in $W_{2,\gamma}^m((a,b),A^m)$ is introduced as follows:

$$\|u\|_{W_{2,\gamma}^m((a,b),A^m)} \equiv \left(\|u^{(m)}\|_{L_{2,\gamma}((a,b),H)}^2 + \|A^m u\|_{L_{2,\gamma}((a,b),H)}^2 \right)^{1/2}.$$

We use $u^{(m)}(t)$ to denote $\dfrac{d^m}{dt^m}u(t)$, $m \in \mathbb{N}$, throughout the paper. In [12, Chap. 1], a more complete description of $W_{2,\gamma}^m((a,b),A^m)$ is presented. In the sequel, we omit the corresponding index if $\gamma = 0$.

Consider the following initial boundary-value problem for the functional differential equation

$$u^{(2)}(t) + A^2u(t) + \sum_{j=1}^{\infty} \left[B_j(t)Au(g_j(t)) + D_j(t)u^{(1)}(g_j(t)) \right] + \int_{-\infty}^{t} K(t-s)Au(s)\,ds + \int_{-\infty}^{t} Q(t-s)u^{(1)}(s)\,ds = f(t), \quad t > 0, \quad (1.1)$$

with the initial condition

$$u(t) = \varphi(t), \quad t \in \mathbb{R}_- = (-\infty, 0]. \quad (1.2)$$

Here $B_j(t)$ and $D_j(t)$, $j = 1,2,\ldots$, are functions with values in the ring of bounded operators acting in H, while $K(t)$ and $Q(t)$ are functions with values in the ring of bounded operators acting in H and being Bochner-integrable over $\mathbb{R}_+$ with the weight $e^{-\varkappa t}$:

$$\int_0^{+\infty} \exp(-\varkappa t)\|K(t)\|\,dt < +\infty, \qquad \int_0^{+\infty} \exp(-\varkappa t)\|Q(t)\|\,dt < +\infty. \quad (1.3)$$

The scalar real-valued functions $g_j(t)$ are supposed to be continuously differentiable on $\mathbb{R}_+$ and such that $g_j(t) \le t$ and $g_j^{(1)}(t) > 0$, $t \in \mathbb{R}_+$ $(j = 1,2,\ldots)$.

The functions inverse to g_j are denoted by g_j^{-1}, $j = 1,2,\ldots$. In the sequel, we assume that there exist $\gamma_0, \gamma_1 \in \mathbb{R}$ such that $f \in L_{2,\gamma_0}(\mathbb{R}_+, H_1)$ and $\varphi \in W_{2,\gamma_1}^2(\mathbb{R}_-, A^2)$.

Definition 1.1. *We say that a function u is a strong solution of problem* (1.1),(1.2) *if there exists $\gamma \ge 0$ such that u belongs to $W_{2,\gamma}^2(\mathbb{R}_+, A^2)$, u satisfies Eq.* (1.1) *a.e. on $\mathbb{R}_+$, and u satisfies* (1.2).

The following theorem on the well-posed solvability of problem (1.1),(1.2) is one of the main results of the present paper.

Theorem 1.1. *Let $B_j(t)$, $D_j(t)$, $\tilde{B}_j(t) = AB_j(t)A^{-1}$, and $\tilde{D}_j(t) = AD_j(t)A^{-1}$ be strongly continuous functions with values in the ring of bounded operators acting in H. Let there exist a nonnegative δ*

such that the following inequalities hold:

$$\sum_{j\geq 1}\sup_{t\in\mathbb{R}_+}\exp(-\delta(t-g_j(t)))\left[\sup_{t\in\mathbb{R}_+}\frac{\|B_j(t)\|}{(g_j^{(1)}(t))^{1/2}}+\sup_{t\in\mathbb{R}_+}\frac{\|D_j(t)\|}{(g_j^{(1)}(t))^{1/2}}\right]<+\infty \tag{1.4}$$

and

$$\sum_{j\geq 1}\sup_{t\in\mathbb{R}_+}\exp(-\delta(t-g_j(t)))\left[\sup_{t\in\mathbb{R}_+}\frac{\|\tilde{B}_j(t)\|}{(g_j^{(1)}(t))^{1/2}}+\sup_{t\in\mathbb{R}_+}\frac{\|\tilde{D}_j(t)\|}{(g_j^{(1)}(t))^{1/2}}\right]<+\infty. \tag{1.5}$$

Let $t-g_j(t)\geq\alpha_0=\mathrm{const}>0$, $t\geq 0$, $j=1,2,\ldots$. *Let* $K(t)$, $Q(t)$, $\tilde{K}(t)=AK(t)A^{-1}$, *and* $\tilde{Q}(t)=AQ(t)A^{-1}$ *be strongly continuous functions with values in the ring of bounded operators acting in* H *and satisfying* (1.3), *and let there exist a nonnegative* $\varkappa_1$ *such that*

$$\int_0^{+\infty}\exp(-\varkappa_1 t)\|\tilde{K}(t)\|\,dt<+\infty,\qquad \int_0^{+\infty}\exp(-\varkappa_1 t)\|\tilde{Q}(t)\|\,dt<+\infty. \tag{1.6}$$

Then there exist nonnegative γ_0 *and* γ_1 *such that, for all* $\varphi\in W^2_{2,\gamma_1}(\mathbb{R}_-,A^2)$ *and* $f\in L_{2,\gamma_0}(\mathbb{R}_+,H_1)$, *there exists a constant* $\gamma^*\geq\max(\gamma_0,\gamma_1)$ *such that, for any* $\gamma>\gamma^*$, *problem* (1.1),(1.2) *has a unique solution* $u\in W^2_{2,\gamma}(\mathbb{R}_+,A^2)$ *satisfying the inequality*

$$\|u\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)}\leq d\left(\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)}+\|\varphi\|^2_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}\right)^{1/2}, \tag{1.7}$$

where the constant d *depends neither on* f *nor on* φ.

Remark 1.1. Conditions (1.4),(1.5) are essential for the well-posed solvability of problem (1.1),(1.2). In [22], corresponding examples are presented.

If the coefficients $B_j(t)$ and $D_j(t)$ do not depend on t (i.e., $B_j(t)\equiv B_j$ and $D_j(t)\equiv D_j$) and the delays $g_j(t)$ are of the form $g_j(t)=t-h_j$ with constants $0<h_1<h_2<\ldots$, then we consider the operator-valued function

$$\mathcal{L}(\lambda)=\lambda^2 I+A^2+\sum_{j=1}^{\infty}(B_jA+\lambda D_j)\exp(-\lambda h_j)+\hat{K}(\lambda)A+\lambda\hat{Q}(\lambda), \tag{1.8}$$

where $\hat{K}(\lambda)$ and $\hat{Q}(\lambda)$ are the Laplace transforms of the operator-valued functions $K(t)$ and $Q(t)$, while I is the identity operator in H. Note that the operator-valued function $\mathcal{L}(\lambda)$ is the symbol (an analog of the characteristic polynomial) of Eq. (1.1) in the autonomous case.

Suppose that there exists a constant ν_0 such that

$$\sum_{j=1}^{\infty}\exp(-\nu_0 h_j)(\|B_j\|+\|D_j\|)<+\infty. \tag{1.9}$$

The operator-valued function $\mathcal{L}(\lambda)$ is estimated as follows.

Proposition 1.1. *Let condition* (1.9) *be satisfied and the operator-valued functions* $K(t)$ *and* $Q(t)$ *satisfy condition* (1.3). *Then there exists a positive* $\varkappa^*$ *such that the operator-valued function* $\mathcal{L}(\lambda)$ *satisfies the inequality*

$$\|\lambda\mathcal{L}^{-1}(\lambda)\|+\|A\mathcal{L}^{-1}(\lambda)\|<\frac{\mathrm{const}}{\mathrm{Re}\,\lambda},\quad \mathrm{Re}\,\lambda>\varkappa^*. \tag{1.10}$$

We consider the autonomous case in more detail.

Together with problem (1.1), (1.2), we consider the functional differential equation

$$u^{(2)}(t) + A^2u(t) + \sum_{j=1}^{N} \Big[B_jAu(t-h_j) + D_ju^{(1)}(t-h_j)\Big] + \int_{-\infty}^{t} K(t-s)A^2u(s)\,ds + \int_{-\infty}^{t} Q(t-s)u^{(1)}(s)\,ds = f(t), \qquad t>0, \quad (1.11)$$

with the initial condition

$$u(t) = \varphi(t), \quad t \in \mathbb{R}_- = (-\infty, 0]. \quad (1.12)$$

Theorem 1.2. *Let the operators B_j, D_j, $\tilde{B}_j = AB_jA^{-1}$, and $\tilde{D}_j = AD_jA^{-1}$, $j = 1,2,\dots,N$, be bounded in H. Let $K \in W_1^1(\mathbb{R}_+)$ and $Q \in L_1(\mathbb{R}_+)$ be scalar complex-valued functions. Let there exist real γ_0 and γ_1 such that the vector-valued functions f and φ belong to $L_{2,\gamma_0}(\mathbb{R}_+, H_1)$ and $W^3_{2,\gamma_1}(\mathbb{R}_-, A^3)$, respectively. Then there exists $\gamma^* > \max(\gamma_0, \gamma_1)$ such that, for any $\gamma > \gamma^*$, problem (1.11),(1.12) has a unique solution $u \in W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ satisfying the estimate*

$$\|u\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \le d_1 \left(\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|^2_{W^3_{2,\gamma}(\mathbb{R}_-,A^3)}\right)^{1/2}, \quad (1.13)$$

where the constant d_1 depends neither on f nor on φ.

Note that there are differences in the setting of problems (1.1),(1.2) and (1.11),(1.12). In particular, the integral terms are different: in Eq. (1.1), only the operator A to the power one is allowed under quite general assumptions for the operator-valued function $K(t)$, while in Eq. (1.11), the operator A to the power two is allowed, but the restrictions for the kernel $K(t)$ are much stricter.

Equation (1.11) is related to applications: if $B_j \equiv D_j \equiv 0$, $j = 1,2,\dots,N$, then it is an abstract form of the Gurtin–Pipkin integrodifferential equation modelling the finite-speed heat propagation in media with memory. That integrodifferential equation is deduced in [6].

Equations of the above type are currently investigated by many authors (see [7, 17] and references therein).

We impose the following assumptions.

(1) The operators B_j and D_j are identically zeros, $j = 1,2,\dots,N$, and $Q(t) \equiv 0$;
(2) the operator A has a compact inverse A^{-1};
(3) the real-valued function $K(t)$ admits the representation

$$K(t) = -\sum_{j=1}^{\infty} c_j \exp(-\alpha_j t),$$

where $c_j > 0$, $\alpha_j > 0$, $j = 1,2,\dots$, $0 < \alpha_1 < \alpha_2 \dots < \alpha_n < \dots$, $\alpha_j \to +\infty$ as $j \to +\infty$, and

$$\sum_{j=1}^{\infty} c_j < +\infty, \qquad \sum_{j=1}^{\infty} \frac{c_j}{\alpha_j} < 1. \quad (1.14)$$

Note that a detailed structure of the spectrum of the operator-valued function $\mathcal{L}(\lambda)$ can be described under the above assumptions.

We also note that conditions (1.14) imply that $K \in W_1^1(\mathbb{R}_+)$ and $\|K\|_{L_1(\mathbb{R}_+)} < 1$.

Let $\{a_j\}_{j=1}^{\infty}$ be eigenvalues of the operator A ($Ae_j = a_je_j$) numbered according to the increasing order (the multiplicity is taken into account): $0 < a_1 \le a_2 \le \dots \le a_n \le \dots$, $a_n \to +\infty$ as $n \to +\infty$. The corresponding eigenvectors $\{e_j\}_{j=1}^{\infty}$ form an orthonormal basis of the space H.

Now we consider the structure of the spectrum of the operator-valued function

$$\mathcal{L}_1(\lambda) = \lambda^2 I + A^2 + \hat{K}(\lambda)A^2,$$

where $\hat{K}(\lambda)$ is the Laplace transform of the function K.

In the considered case, Eq. (1.10) can be decomposed into a countable set of scalar integrodifferential equations

$$u_n^{(2)}(t) + a_n^2 u_n(t) - \int_{-\infty}^{t} \sum_{k=1}^{\infty} c_k e^{-\alpha_k (t-s)} a_n^2 u_n(s)\, ds = f_n(t), \qquad t > 0, \tag{1.15}$$

where $u_n(t) = (u(t), e_n)$ and $f_n = (f(t), e_n)$, $n = 1, 2, \dots$ Those equations are projections of Eq (1.10) onto the one-dimensional spaces spanned by vectors $\{e_n\}$.

Using the Laplace transform, we naturally arrive at the countable set of meromorphic functions $f_n(\lambda)$:

$$f_n(\lambda) = \lambda^2 + a_n^2 - a_n^2 \left(\sum_{k=1}^{\infty} \frac{c_k}{\lambda + \alpha_k} \right), \ n = 1, 2, \dots, \tag{1.16}$$

which are symbols of the integrodifferential equations given by (1.15).

The spectrum of the operator-valued function $\mathcal{L}_1(\lambda)$ is described as follows.

Theorem 1.3. *The spectrum of the operator-function $\mathcal{L}_1(\lambda)$ coincides with the closure of the union of the sets of zeros for the functions $\{f_n(\lambda)\}_{n=1}^{\infty}$. The zeros of the meromorphic function $f_n(\lambda)$ form a countable set of real roots $\left\{\lambda_{n,k}\right\}_{n=1}^{\infty}$ satisfying the inequalities*

$$\begin{gathered} -a_k < \lambda_{n,k} < x_k < -a_{k-1} \quad (a_0 = 0), \\ \lambda_{n,k} = x_k + \underline{O}\left(\frac{1}{a_n^2}\right), \quad k \in \mathbb{N}, \quad k > 1, \quad a_n \to +\infty, \end{gathered} \tag{1.17}$$

where x_k are the real zeros of the function

$$g(\lambda) = 1 - \sum_{k=1}^{\infty} \frac{c_k}{\lambda + \alpha_k}, \tag{1.18}$$

and a pair of complex-conjugate roots $\left\{\lambda_n^{\pm}\right\}_1^{\infty}$, $\lambda_n^{+} = \overline{\lambda_n^{-}}$, such that

$$\lambda_n^{\pm} = -\frac{1}{2} \sum_{k=1}^{\infty} c_k \pm i a_n + \underline{O}\left(\frac{1}{a_n^2}\right), \quad a_n \to +\infty. \tag{1.19}$$

Thus, the spectrum $\sigma(\mathcal{L}_1)$ of the operator-valued function $\mathcal{L}_1(\lambda)$ is representable as

$$\sigma(\mathcal{L}_1) \equiv \overline{\left(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{\lambda_{n_k}\} \right)} \bigcup \left(\bigcup_{n=1}^{\infty} \lambda_n^{\pm} \right),$$

where $\lim\limits_{n\to\infty} \lambda_{nk} = x_k$, $k = 1, 2, \dots$.

The proof of Theorem 1.3 is given in the third part of this paper.

Remark 1.2. The spectrum of the operator-valued function $\mathcal{L}(\lambda)$ is located in the left-hand semi-plane $\{\lambda : \operatorname{Re} \lambda < 0\}$ if

$$\sum_{j=1}^{\infty} \frac{c_j}{\alpha_j} < 1.$$

If

$$\sum_{j=1}^{\infty} \frac{c_j}{\alpha_j} > 1,$$

then the accumulation point x_1 of the poles, which are eigenvalues $\{\lambda_{n1}\}_{n=1}^{\infty}$ of the operator-valued function $\mathcal{L}$, is located in the right-hand semi-plane $\{\lambda : \operatorname{Re}\lambda > 0\}$; this corresponds to the instability case.

2. Main Results: Proofs

Several auxiliary assertions precede the proofs of the main results.

In the sequel, norms of operators acting in $L_2(\mathbb{R}_+, H)$ and $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ are denoted by $|||\cdot|||$ and $|||\cdot|||_W$, respectively.

Lemma 2.1. *Let operators L and M be defined as follows*:

$$(Lz)(t) = \int_0^t \exp(-\gamma(t-s))A^{-1}\sin(A(t-s))z(s)\,ds, \quad t \geq 0,$$

and

$$(Mz)(t) = \int_0^t \exp(-\gamma(t-s))\cos(A(t-s))z(s)\,ds, \quad t \geq 0.$$

Then their norms satisfy the estimates

$$|||AL||| \leq \frac{1}{2\gamma}, \quad |||M||| \leq \frac{1}{\gamma}, \quad \gamma > 0. \tag{2.1}$$

The proof of Lemma 2.1 is given in [22]. Consider the following operators:

$$(\mathcal{S}_{g_j}u)(t) = \begin{cases} u(g_j(t)) & \text{for } t \text{ such that } g_j(t) \geq 0, \\ 0 & \text{for } t \text{ such that } g_j(t) < 0 \end{cases}$$

and

$$(\mathcal{T}^{g_j}u)(t) = \begin{cases} 0 & \text{for } t \text{ such that } g_j(t) \geq 0, \\ u(g_j(t)) & \text{for } t \text{ such that } g_j(t) < 0. \end{cases}$$

Obviously, the following relation holds:

$$u(g_j(t)) = (\mathcal{S}_{g_j}u)(t) + (\mathcal{T}^{g_j}u)(t), \quad t \geq 0.$$

Using the introduced operators, we represent Eq. (1.1) as follows:

$$u^{(2)}(t) + A^2u(t) + \sum_{j=1}^{\infty}\Big[B_j(t)S_{g_j}(Au)(t) + D_j(t)S_{g_j}(u^{(1)})(t)\Big] + \int_0^t K(t-s)Au(s)\,ds + \int_0^t Q(t-s)u^{(1)}(s)\,ds = f_1(t), \quad t > 0, \tag{2.2}$$

where $f_1(t) = f(t) - q_1(t) - q_2(t)$, while

$$\begin{aligned} q_1(t) &= f(t) - \sum_{j=1}^{\infty}\big[B_j(t)T^{g_j}(A\varphi)(t) + D_j(t)T^{g_j}(\varphi^{(1)})(t)\big], \\ q_2(t) &= -\int_{-\infty}^{0} K(t-s)A\varphi(s)\,ds - \int_{-\infty}^{0} Q(t-s)\varphi^{(1)}(s)\,ds, \quad t > 0, \end{aligned} \tag{2.3}$$

because $u(t) = \varphi(t)$ for $t < 0$.

The next step is to study the well-posed solvability of Eq. (2.2) with the initial conditions

$$u(+0) = \varphi(-0) = \varphi_0, \quad u^{(1)}(+0) = \varphi^{(1)}(-0) = \varphi_1. \tag{2.4}$$

It is important that Eq. (2.2) is a linear equation containing the Volterra operators. Such objects were discussed in [1, Chap. I].

To pass from Problem (2.2)–(2.4) to a problem with homogeneous (zero) initial conditions, we introduce a new unknown function $\omega(t)$ as follows:

$$u(t) = \cos(At)\varphi_0 + A^{-1}\sin(At)\varphi_1 + \omega(t).$$

Then, taking into account that Eq. (2.2) is linear, we obtain the following problem for the function $\omega(t)$:

$$\omega^{(2)}(t) + A^2\omega(t) + \sum_{j=1}^{\infty}\left[B_j(t)S_{g_j}(A\omega)(t) + D_j(t)S_{g_j}(\omega^{(1)})(t)\right] + \int_0^t K(t-s)A\omega(s)\,ds + \int_0^t Q(t-s)\omega^{(1)}(s)\,ds = f_2(t), \quad t > 0, \quad (2.5)$$

where $f_2(t) = f_1(t) - h(t)$,

$$h(t) = -\sum_{j=1}^{\infty}\left[B_j(t)S_{g_j}(Ap)(t) + D_j(t)S_{g_j}(p^{(1)})(t)\right] - \int_0^t K(t-s)Ap(s)\,ds + \int_0^t Q(t-s)p^{(1)}(s)\,ds, \quad (2.6)$$

and

$$p(t) = A^{-1}\sin(At)\varphi_1 + \cos(At)\varphi_0,$$

with the initial conditions

$$\omega(+0) = 0, \quad \omega^{(1)}(+0) = 0. \quad (2.7)$$

We seek a solution of problem (2.5)–(2.7) in the form $\omega(t) = \exp(\gamma t)v(t)$ with new unknown function $v(t)$. Then we obtain the following problem for the function $v(t)$:

$$v^{(2)}(t) + 2\gamma v^{(1)}(t) + (A^2 + \gamma^2 I)v(t) + \int_0^t \exp(-\gamma(t-s))K(t-s)Av(s)\,ds + \int_0^t \exp(-\gamma(t-s))Q(t-s)(v^{(1)}(s) + \gamma v(s))\,ds + \sum_{j=1}^{\infty}\exp(-\gamma(t-g_j(t)))\left[B_j(t)S_{g_j}(Av)(t) + D_j(t)S_{g_j}(v^{(1)} + \gamma v)(t)\right] = \exp(-\gamma t)f_2(t), \quad t > 0, \quad (2.8)$$

$$v(+0) = 0, \quad v^{(1)}(+0) = 0. \quad (2.9)$$

The solution of problem (2.8),(2.9) is sought in the form $v = Lz$ with a new unknown function z. Substituting that function into (2.8), we obtain the integrodifferential equation

$$z(t) + \int_0^t \exp(-\gamma(t-s))K(t-s)A(Lz)(s)\,ds + \int_0^t \exp(-\gamma(t-s))Q(t-s)(Mz)(s)\,ds + \sum_{j=1}^{\infty}\exp(-\gamma(t-g_j(t)))\left[B_j(t)S_{g_j}(ALz)(t) + D_j(t)S_{g_j}(Mz)(t)\right] = \exp(-\gamma t)f_2(t), \quad t \in \mathbb{R}_+, \quad (2.10)$$

which is solvable if and only if problem (2.8),(2.9) is solvable.

Our aim is to prove that Eq. (2.10) is well posed in the space $L_2(\mathbb{R}_+, H_1)$. To do so, we prove that the norms of integral operators on the left-hand side of (2.10) can be made small by selecting sufficiently large $\gamma > 0$.

To prove that result, we need the following assertion.

Proposition 2.1. *Let $g(t)$ be a continuously differentiable real-valued function such that $g(t) \le t$ and $g^{(1)}(t) > 0$ for $t \in \mathbb{R}_+$. Let $B(t)$ be a strongly continuous function with values in the ring of bounded operators in H such that*

$$\sup_{t\in[g^{-1}(0),+\infty]} \left(\|B(t)\|^2 \frac{1}{g^{(1)}(t)} \right) = b_0 < +\infty. \tag{2.11}$$

Then the operator $(Sv)(t) = B(t)(S_g v)(t)$, where S_g is the operator of internal superposition

$$(\mathcal{S}_g v)(t) = \begin{cases} v(g(t)) & \text{for } t \text{ such that } g(t) \ge 0, \\ 0 & \text{for } t \text{ such that } g(t) < 0, \end{cases}$$

is bounded in the space $L_2(\mathbb{R}_+, H)$ and its norm satisfies the estimate

$$\|\|S\|\| \le \sqrt{b_0}. \tag{2.12}$$

Proof. The proof of Proposition 2.1 is given in [18]. Actually, it is reduced to a change of variables. In [1, Chap. I, pp. 20–28], similar assertions are presented. Now we pass from Eq. (2.10) to the following equation with respect to the new unknown function $y(t) = Az(t)$:

$$y(t) + \int_0^t \exp(-\gamma(t-s)) AK(t-s)A^{-1}(Ly)(s)\,ds + \int_0^t \exp(-\gamma(t-s)) AQ(t-s)A^{-1}(My)(s)\,ds$$
$$+ \sum_{j=1}^{\infty} \exp(-\gamma(t-g_j(t))) \left[\tilde{B}_j S_{g_j}(ALy)(t) + \tilde{D}_j S_{g_j}(My)(t) \right] = \exp(-\gamma t) A f_2(t), \quad t \in \mathbb{R}_+. \tag{2.13}$$

This equation is studied in the space $L_2(\mathbb{R}_+, H)$.

By virtue of Proposition 2.1, Lemma 2.1, and conditions (1.4) and (1.5), we obtain the estimate

$$\left\| \sum_{j=1}^{\infty} \exp(-\gamma(t-g_j(t))) \left[\tilde{B}_j(t) S_{g_j}(ALy)(t) + \tilde{D}_j(t) S_{g_j}(My)(t) \right] \right\|_{L_2(\mathbb{R}_+,H)}$$
$$\le \sum_{j\ge 1} \sup_{t\in\mathbb{R}_+} \exp(-\gamma(t-g_j(t))) \left[\sup_{t\in\mathbb{R}_+} \frac{\|\tilde{B}_j(t)\|}{(g_j^{(1)}(t))^{1/2}} + \sup_{t\in\mathbb{R}_+} \frac{\|\tilde{D}_j(t)\|}{(g_j^{(1)}(t))^{1/2}} \right] \cdot \frac{1}{\gamma} \|y\|_{L_2(\mathbb{R}_+,H)}, \tag{2.14}$$

$\gamma > \delta > 0$.

For the second and third terms on the left-hand side of (2.13), we obtain the inequalities

$$\left\| \int_0^t \exp(-\gamma(t-s)) \tilde{K}(t-s)(Ly)(s)\,ds \right\|_{L_2(\mathbb{R}_+,H)} \le \sup_{\nu\in\mathbb{R}} \|\tilde{K}(\gamma+i\nu)\| \cdot \frac{1}{2\gamma} \|y\|_{L_2(\mathbb{R}_+,H)}, \quad \gamma > \varkappa_1, \tag{2.15}$$

and

$$\left\| \int_0^t \exp(-\gamma(t-s)) \tilde{Q}(t-s)(My)(s)\,ds \right\|_{L_2(\mathbb{R}_+,H)} \le \sup_{\nu\in\mathbb{R}} \|\tilde{Q}(\gamma+i\nu)\| \cdot \frac{1}{\gamma} \|y\|_{L_2(\mathbb{R}_+,H)}, \quad \gamma > \varkappa_1. \tag{2.16}$$

Taking into account the relations

$$\sup_{\nu\in\mathbb{R}} \|\hat{\tilde{K}}(\gamma+i\nu)\| \to 0, \quad \sup_{\nu\in\mathbb{R}} \|\hat{\tilde{Q}}(\gamma+i\nu)\| \to 0, \quad \gamma \to +\infty \tag{2.17}$$

(they are well-known properties of the Laplace transforms of the functions $\tilde{K}(t)$ and $\tilde{Q}(t)$), as well as inequalities (2.1) and (2.14), we obtain the unique solvability of Eq. (2.13) in the space $L_2(\mathbb{R}_+, H)$ for sufficiently large $\gamma \geq 0$ because the norms of all operators (apart from the first one) at the left-hand side of Eq. (2.13) can be made strictly less than one. It follows from the conditions imposed on the right-hand side $f(t)$ of Eq. (1.1), from the representations given by (2.3) and (2.6), and conditions imposed on the initial function φ that the vector-valued function $\exp(-\gamma t)Af(t)$ belongs to $L_2(\mathbb{R}_+, H)$ for $\gamma \geq \gamma^*$, where $\gamma^* > \max(\gamma_0, \gamma_1)$.

Using (1.4)–(1.6), one can directly check that

$$\|Af_2\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \leq \text{const}\left(\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}\right), \tag{2.18}$$

where the constant depends neither on f nor on φ.

Indeed, by virtue of the representation

$$e^{-\gamma t}Aq_2(t) = -\int_{-\infty}^{0} \tilde{K}(t-s)e^{-\gamma(t-s)}(e^{-\gamma s}A^2\varphi(s))\,ds - \int_{-\infty}^{0} \tilde{Q}(t-s)e^{-\gamma(t-s)}(e^{-\gamma s}A\varphi^{(1)}(s))\,ds,$$

by conditions (1.6), and by the Hausdorff–Young inequality, we obtain the estimate

$$\|q_2\|_{L_{2,\gamma}(\mathbb{R}_+,H_1)} \leq \text{const}\,\|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}, \quad \gamma > \varkappa_1, \tag{2.19}$$

where the constant does not depend on φ.

On the other hand, using (1.5) and a change of variables, one can easily check that

$$\|q_1\|_{L_{2,\gamma}(\mathbb{R}_+,H_1)} \leq \text{const}\,\|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}, \quad \gamma > \delta, \tag{2.20}$$

where the constant does not depend on φ. Hence, by virtue of (2.19), (2.20), and (2.3), we see that f_1 belongs to $L_{2,\gamma}(\mathbb{R}_+, H_1)$ and

$$\|f_1\|_{L_{2,\gamma}(\mathbb{R}_+,H_1)} \leq \text{const}(\|f\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}), \quad \gamma > \max(\delta, \varkappa_1), \tag{2.21}$$

where the constant depends neither on f nor on φ.

Using (2.6), the trace theorem, and (1.5), (1.6), one can directly check that the function $h(t)$ belongs to $L_{2,\gamma}(\mathbb{R}_+, H_1)$ and satisfies the inequality

$$\|h\|_{L_{2,\gamma}(\mathbb{R}_+,H_1)} \leq \text{const}\,\|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}, \quad \gamma > \max(\delta, \varkappa_1), \tag{2.22}$$

where the constant does not depend on φ. Combining that with estimates given by (2.21) and (2.22), we obtain the desired estimate (2.18). On the other hand, by virtue of inequality (2.18) and the well-posed solvability of Eq. (2.13), we obtain the estimate

$$\|y\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \leq c_1\big(\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}\big), \tag{2.23}$$

where the constant c_1 depends neither on f nor on φ.

Finally, by virtue of the representations $v = Lz$ and $y = Az$ and Lemma 2.1, we see that (under the above assumptions) the function v belongs to $W^2_2(\mathbb{R}_+, A^2)$ and the estimate

$$\|v\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \leq c_2\big(\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}\big) \tag{2.24}$$

is valid, where the constant c_2 depends neither on f nor on φ. The latter estimate immediately implies (1.7), which is the desired estimate. □

We pass to the proof of Proposition 2.1.

Proof of proposition 2.1. The spectrum of the operator A lies in the semi-axis $\mathbb{R}_+$. Applying the well-known resolvent estimate in terms of the distances to the spectra of the normal operators $\pm iA$, we obtain

$$\|\lambda(\lambda^2 I + A^2)^{-1}\| \leq |\lambda|\|(\lambda I + iA)^{-1}\|\|(\lambda I - iA)^{-1}\| \leq (\operatorname{Re}\lambda)^{-1}, \quad \operatorname{Re}\lambda > 0. \tag{2.25}$$

Then, by virtue of the previous inequality, we have

$$\|A(\lambda^2 I + A^2)^{-1}\| \le \|(\lambda I + iA)^{-1}\| + |\lambda|\|(\lambda^2 I + A^2)^{-1}\| \le 2(\operatorname{Re}\lambda)^{-1}, \quad \operatorname{Re}\lambda > 0. \tag{2.26}$$

Using inequalities (2.25) and (2.26), we arrive at the inequality (valid for $\operatorname{Re}\lambda > \nu_0 > 0$)

$$\left\| \sum_{j=1}^{\infty} \exp(-\lambda h_j)[B_j A + \lambda D_j](\lambda^2 I + A^2) \right\| \le \text{const} \left(\sum_{j=1}^{\infty} \exp(\nu_0 h_j)(\|B_j\| + \|D_j\|)(\operatorname{Re}\lambda)^{-1} \right). \tag{2.27}$$

Using the relations

$$\|\hat{K}(\lambda)\| \to 0, \quad \|\hat{Q}(\lambda)\| \to 0 \quad (\operatorname{Re}\lambda \to +\infty),$$

which are well-known properties of the Laplace transform, and estimates (2.25) and (2.26), we obtain

$$\|[\hat{K}(\lambda)A + \lambda\hat{Q}(\lambda)](\lambda^2 I + A^2)^{-1}\| \to 0, \quad \operatorname{Re}\lambda \to +\infty. \tag{2.28}$$

Taking sufficiently large $\varkappa_+ > 0$ and using (2.27) and (2.28), we obtain the estimate

$$\|\mathcal{L}(\lambda)(\lambda^2 I + A^2)^{-1} - I\| < 1, \quad \operatorname{Re}\lambda > \varkappa^*. \tag{2.29}$$

Now, using (2.29), the representation

$$\mathcal{L}^{-1}(\lambda) = (\lambda^2 I + A^2)^{-1}(I + (\mathcal{L}(\lambda) - (\lambda^2 I + A^2))(\lambda^2 I + A^2)^{-1})^{-1},$$

and the estimates in (2.25) and (2.26), we obtain (1.10), which is the desired inequality. The proposition is proved. □

Theorem 1.2 is proved under the following additional assumption: $\varphi(t) = 0$, $t \le 0$, and $B_j = D_j = 0$, $j = 1, 2, \ldots, N$. As it was shown in the proof of Theorem 1.1, this leads to no loss of generality. Indeed, Problem (1.11),(1.12) is reduced to problem (2.30) and (2.31) (see below) exactly as in the proof of Theorem 1.1 (see also [23]).

Proof. We introduce the function $v(t) = \exp(-\gamma t)u(t)$. Then Eq. (1.11) for the function u is equivalent to the following equation for the function v:

$$v^{(2)}(t) + 2\gamma v^{(1)}(t) + (A^2 + \gamma^2 I)v(t) + \int_0^t e^{-\gamma(t-s)} K(t-s) A^2 v(s)\, ds$$
$$+ \int_0^t e^{-\gamma(t-s)} Q(t-s) v^{(1)}(s)\, ds + \gamma \int_0^t e^{-\gamma(t-s)} Q(t-s) v(s)\, ds = e^{-\gamma t} f(t), \quad t > 0, \tag{2.30}$$

with the initial condition

$$v(+0) = 0, \quad v^{(1)}(+0) = 0. \tag{2.31}$$

The solution of problem (2.30), (2.31) is sought in the form

$$v(t) = Lz(t) = \int_0^t \exp(-\gamma(t-s)) A^{-1} \sin(A(t-s)) z(s)\, ds, \tag{2.32}$$

where z is a new unknown function.

Substituting $v = Lz$ into Eq. (2.30), we obtain the following integral equation for the function z:

$$z(t) + \int_0^t e^{-\gamma(t-s)} K(t-s) A^2 \left(\int_0^s e^{-\gamma(s-\theta)} A^{-1} \sin(A(s-\theta)) z(\theta)\, d\theta \right) ds$$
$$+ \int_0^t e^{-\gamma(t-s)} Q(t-s) \left(\int_0^s e^{-\gamma(s-\theta)} \cos(A(s-\theta)) z(\theta)\, d\theta \right) ds = e^{-\gamma t} f(t), \quad t > 0. \quad (2.33)$$

We introduce the function $y(t) = Az(t)$. Then the function y satisfies the following integral equation:

$$y(t) + \int_0^t e^{-\gamma(t-s)} K(t-s) A \left(\int_0^s e^{-\gamma(s-\theta)} A \sin(A(s-\theta)) y(\theta) d\theta \right) ds$$
$$+ \int_0^t e^{-\gamma(t-s)} Q(t-s) \left(\int_0^s e^{-\gamma(s-\theta)} \cos(A(s-\theta)) y(\theta) d\theta \right) ds = e^{-\gamma t} F(t), \quad t > 0, \quad (2.34)$$

where $F(t) = Af(t)$.

By virtue of the conditions imposed on the kernel K, its Laplace transform

$$\hat{K}(\lambda) = \int_0^{+\infty} \exp(-\lambda t) K(t)\, dt, \quad \lambda = \mu + i\nu,$$

satisfies the estimate

$$|\hat{K}((\mu+\gamma) + i\nu)| \le \frac{\text{const}}{((\mu+\gamma)^2 + \nu^2)^{1/2}}, \quad \mu \ge 0. \quad (2.35)$$

On the other hand, the Laplace transform $\hat{Q}$ of the function Q satisfies the inequality

$$|\hat{Q}((\mu+\gamma) + i\nu)| \le \text{const}, \qquad \mu \ge 0. \quad (2.36)$$

Applying the Laplace transform to Eq. (2.34), we obtain

$$((I + \hat{K}(\lambda+\gamma) A^2 ((\lambda+\gamma)^2 I + A^2)^{-1} + (\lambda+\gamma)\hat{Q}(\lambda+\gamma)((\lambda+\gamma)^2 I + A^2)^{-1})\hat{y}(\lambda) = \hat{F}(\lambda+\gamma), \quad (2.37)$$

where $\lambda = \mu + i\nu$, $\operatorname{Re}\lambda > 0$, and $\hat{y}(\lambda)$ is the Laplace transform of the function y.

We set $\tau = \mu + \gamma$ and estimate the norm of the operator $\hat{K}(\tau + i\nu) A^2 ((\tau + i\nu)^2 I + A^2)^{-1}$. To do so, we consider the function

$$a^2 (|(\tau + i\nu)^2 + a^2|)^{-1} (\tau^2 + \nu^2)^{-1/2},$$

where $a \in [\alpha_0, +\infty)$, $\alpha_0 = \inf\limits_{\|u\|_H = 1} (Au, u) > 0$, and $\nu \in \mathbb{R}$. Let $d \in (0,1]$. We estimate the function $f(a, \nu, \tau) = (|(\tau + i\nu)^2 + a^2|)^2 (\tau^2 + \nu^2)$ from below:

$$f(a,\nu,\tau) = ((\tau^2 + a^2 - \nu^2)^2 + 4\tau^2\nu^2)(\tau^2 + \nu^2)$$
$$\ge \min\left[\min_{\nu^2 \in [0, da^2]} f(a,\nu,\tau), \min_{\nu^2 \in [da^2, +\infty]} f(a,\nu,\tau) \right] \ge \min[(\tau^2 + (1-d)a^2)^2 \tau^2, (\tau^2 + da^2)^2 4da^2\tau^2].$$

Using the latter estimate, inequality (2.35), and the theorem on spectral decomposition for the self-adjoint positive operator A, we obtain the following chain of inequalities:

$$\|\hat{K}(\tau+i\nu)A^2((\tau+i\nu)^2I+A^2)^{-1}\| \le \mathrm{const}\max\left[\frac{a^2}{(\tau^2+(1-d)a^2)\tau}, \frac{a^2}{2a\tau(d(\tau^2+da^2))^{1/2}}\right] \le$$

$$\le \mathrm{const}\max\left[\frac{1}{\left(\dfrac{\tau^2}{a^2}+(1-d)\right)\tau}, \frac{1}{2\tau\left(d\left(\dfrac{\tau^2}{a^2}+d\right)\right)^{1/2}}\right] \le \mathrm{const}\max\left[\frac{1}{(1-d)\tau}, \frac{1}{2\tau d}\right].$$

Assigning $d=\dfrac{1}{3}$, we arrive at the following estimate:

$$\|\hat{K}(\tau+i\nu)A^2((\tau+i\nu)^2I+A^2)^{-1}\| \le \mathrm{const}\cdot\frac{3}{2\tau} \le \frac{\mathrm{const}_1}{\mu+\gamma}. \tag{2.38}$$

Now we estimate the expression

$$\|\hat{Q}(\tau+i\nu)(\tau+i\nu)((\tau+i\nu)^2I+A^2)^{-1}\|.$$

To do so, consider the function

$$g(\tau,\nu,a) = \left|\frac{(\tau+i\nu)}{(\tau+i\nu)^2+a^2}\right|^2,$$

where $a\in[\alpha_0,+\infty)$, $\tau=\mu+\gamma$, $\mu\ge 0$, and $\nu\in\mathbb{R}$. It is easy to see that the following chain of inequalities holds:

$$g(\tau,\nu,a) = \frac{\tau^2+\nu^2}{(a^2-\nu^2)^2+\tau^4+2a^2\tau^2+2\tau^2\nu^2} \le \frac{(\tau^2+\nu^2)}{\tau^4+2a^2\nu^2+2\nu^2\tau^2}$$

$$\le \frac{1}{\tau^2}\left[\frac{\tau^2+\nu^2}{\tau^2+2\nu^2}\right] \le \frac{1}{\tau^2} = \frac{1}{(\mu+\gamma)^2}. \tag{2.39}$$

By virtue of (2.36) and (2.39), we obtain that

$$\|\hat{Q}(\tau+i\nu)(\tau+i\nu)((\tau+i\nu)^2I+A^2)^{-1}\| \le \frac{\mathrm{const}}{\mu+\gamma}. \tag{2.40}$$

Taking sufficiently large γ_0, we conclude from (2.38) and (2.40) that, for $\gamma\ge\gamma_0$, the operator-valued function

$$(I+\hat{K}(\lambda+\gamma)A^2((\lambda+\gamma)^2I+A^2)^{-1}+(\lambda+\gamma)\hat{Q}(\lambda+\gamma)((\lambda+\gamma)^2I+A^2)^{-1})^{-1} \tag{2.41}$$

is defined and uniformly bounded in the right-hand semi-plane ($\mathrm{Re}\,\lambda>0$). Hence, the vector-valued function $\hat{y}$ belongs to the Hardy space $H_2(\mathrm{Re}\,\lambda>0)$ in the right-hand semi-plane. By virtue of the Hardy theorem, this implies the unique solvability of Eq. (2.34), which is an integral equation in the space $L_2(\mathbb{R}_+,H)$. This yields the unique solvability of Eq. (2.33) in the space $L_2(\mathbb{R}_+,H_1)$. By virtue of Lemma 2.1, we obtain the unique solvability of problem (2.30),(2.31) in the space $W_2^2(\mathbb{R}_+,A^2)$. Hence, the original problem is uniquely solvable in the space $W_{2,\gamma}^2(\mathbb{R}_+,A^2)$. □

Note that the following estimate of the operator-valued function $\mathcal{L}_1^{-1}(\lambda)$ established during the proof of Theorem 1.2 has an independent interest as well.

Lemma 2.2. *Let $K(t)\in W_1^1(\mathbb{R}_+)$ and $Q(t)\in L_1(\mathbb{R}_+)$. Then there exists $\varkappa^*$ such that the operator-valued function $\mathcal{L}_1^{-1}(\lambda)$, where*

$$\mathcal{L}_1(\lambda) = \lambda^2I+A^2+\hat{K}(\lambda)A^2+\lambda Q(\lambda),$$

satisfies the following estimate in the semi-plane $\operatorname{Re}\lambda > \gamma^*$*:*

$$\|A\mathcal{L}_1^{-1}(\lambda)\| + \|\lambda\mathcal{L}_1^{-1}(\lambda)\| \le \frac{\text{const}}{\operatorname{Re}\lambda}. \tag{2.42}$$

The proof of Lemma 2.2 is based on the following representation for the operator-valued function $\mathcal{L}^{-1}(\lambda)$:

$$\mathcal{L}_1^{-1}(\lambda) = (\lambda^2 I + A^2)^{-1}(I + \hat{K}A^2(\lambda)(\lambda^2 I + A^2)^{-1} + Q(\lambda)\lambda(\lambda^2 I + A^2)^{-1})^{-1}. \tag{2.43}$$

It is known from the proof of Theorem 1.2 that the second factor on the right-hand side of (2.43) is a uniformly bounded operator-valued function for $\operatorname{Re}\lambda > \varkappa_*$. Therefore, the desired estimate (2.42) follows from inequalities (2.25) and (2.26) established in the proof of Proposition 2.1.

The proof of Theorem 1.3 is based on the following lemma, which is formulated here (as well as the proof of Theorem 1.3) under the assumption that the eigenvalues a_n of the operator A satisfy the strict inequalities $0 < a_1 < a_2 < \cdots < a_n < \ldots$.

Lemma 2.3. *The zeros of the function*

$$l_{n,N}(\lambda) = \lambda^2 + a_n^2\left(1 - \sum_{k=1}^{N}\frac{c_k}{\lambda + \alpha_k}\right) \tag{2.44}$$

form a real sequence $\left\{\mu_{n,k}\right\}_{n=1}^{\infty}$ *such that*

$$-a_k < \mu_{n,k} < y_k < -a_{k-1}, \mu_{n,k} = y_k + \overline{O}\left(\frac{1}{a_n^2}\right), \quad k = 1, 2, \ldots, N, \quad a_n \to +\infty, \tag{2.45}$$

where y_k *are the real zeros of the function*

$$g^0(\lambda) = 1 - \sum_{k=1}^{N}\frac{c_k}{\lambda + \alpha_k}, \tag{2.46}$$

and a pair of complex-conjugate roots $\mu_n^+ = \overline{\mu_n^-}$ *such that*

$$\mu_n^\pm = -\frac{1}{2}\sum_{k=1}^{N} c_k \pm i a_n + \underline{O}\left(\frac{1}{a_n^2}\right), \quad a_n \to +\infty. \tag{2.47}$$

The proof of the lemma is based on the Vièta theorem and is technical. It is given in the diploma thesis of Kabirova. In [9], close results are contained.

We pass to the proof of Theorem 1.3.

Proof. It follows from the convergence of the series $\sum_{k=1}^{\infty} c_k$ that, for any ε, there exists N such that

$$\sum_{k=N+1}^{\infty} c_k < \frac{\varepsilon}{10}. \tag{2.48}$$

Consider the circle of radius ε centered at the point μ_n^+:

$$D_\varepsilon(\mu_n^+) = \{\lambda\colon \lambda = \mu_n^+ + \varepsilon e^{i\varphi}, 0 \le \varphi < 2\pi\}$$

(the reasoning for the root μ_n^- is entirely the same).

We present the function $f_n(\lambda)$ as follows:

$$f_n(\lambda) = l_{n,N}(\lambda) + m_{n,N}(\lambda), \tag{2.49}$$

where

$$l_{n,N}(\lambda)=\lambda^2+\alpha_n^2\left(1-\sum_{k=1}^{N}\frac{c_k}{\lambda+a_k}\right),$$
$$m_{n,N}(\lambda)=-\alpha_n^2\left(\sum_{k=N+1}^{\infty}\frac{c_k}{\lambda+a_k}\right). \tag{2.50}$$

Consider sufficiently small positive ε such that the disk $B_\varepsilon(\mu_n^+)=\{\lambda:|\lambda-\mu_n^+|<\varepsilon\}$ contains no other zeros of the function $l_{n,N}(\lambda)$. We estimate the function $l_{n,N}(\lambda)$ on the circle $D_\varepsilon(\mu_n^+)$. To do so, we note that if $\lambda\in D_\varepsilon(\mu_n^+)$, then

$$|\lambda-\mu_{nk}|=\left(\left(-\frac{1}{2}\sum_{k=1}^{N}c_k-\mu_{n,k}+\varepsilon\cos\varphi\right)^2+(a_n+\varepsilon\sin\varphi)^2\right)^{1/2}\geq a_n-\varepsilon,$$

$$|\lambda-\mu_n^+|=\varepsilon,\qquad |\lambda-\mu_n^-|>2a_n-\varepsilon.$$

By virtue of the above inequalities, we obtain the following estimate from below:

$$|(\lambda-\mu_{n,1})\dots(\lambda-\mu_{n,N})(\lambda-\mu_n^+)(\lambda-\mu_n^-)|_{\lambda\in D_\varepsilon(\mu_n^+)}\geq(a_n-\varepsilon)^{N+1}\varepsilon. \tag{2.51}$$

Using the inequalities

$$|\lambda+\alpha_k|_{\lambda\in D_\varepsilon(\mu_n^+)}\leq\left|\alpha_k-\frac{1}{2}\sum_{j=1}^{N}c_j+\varepsilon\sin\varphi\right|+|a_n+\varepsilon\sin\varphi|$$
$$\leq\alpha_N+\frac{1}{2}\sum_{j=1}^{N}c_j+\varepsilon+(a_n+\varepsilon)\leq a_n+C+2\varepsilon,$$

where $C=\frac{1}{2}\sum\limits_{j=1}^{\infty}c_j+\alpha_N$, we obtain the estimate

$$|(\lambda+\alpha_1)\dots(\lambda+\alpha_N)|\leq(a_n+C+2\varepsilon)^N,\quad \lambda\in D_\varepsilon(\mu_n^+). \tag{2.52}$$

Therefore,

$$|l_{n,N}(\lambda)|=\Big|\frac{(\lambda-\mu_{n,1})\dots(\lambda-\mu_{n,N})(\lambda-\mu_n^+)(\lambda-\mu_n^-)}{(\lambda+\alpha_1)\dots(\lambda+\alpha_N)}\Big|\geq\frac{(a_n-\varepsilon)^{N+1}\varepsilon}{(a_n+2\varepsilon+C)^N}$$
$$=\varepsilon a_n\big(1-\frac{N(3\varepsilon+C)}{a_n}+\frac{N(N+1)}{a_n^2}(\varepsilon+C)+\overline{O}\left(\frac{1}{a_n^2}\right)\big),\quad a_n\to+\infty. \tag{2.53}$$

This implies that the following inequality holds for sufficiently large a_n:

$$|l_{n,N}(\lambda)|\geq\frac{\varepsilon}{2}a_n,\qquad a_n\to+\infty. \tag{2.54}$$

We estimate $|m_{n,N}(\lambda)|$ from above for sufficiently large a_n:

$$|m_{n,N}(\lambda)|\leq a_n^2\left(\sum_{k=N+1}^{\infty}\frac{c_k}{|\lambda+\alpha_k|}\right)\leq a_n^2\sum_{k=N+1}^{\infty}\frac{c_k}{a_n-\varepsilon}\leq\frac{\varepsilon}{5}a_n,\quad \lambda\in D_\varepsilon(\mu_n^+). \tag{2.55}$$

By virtue of inequalities (2.54) and (2.55), we obtain from the Rouché theorem that the function $f_n(\lambda)$ has one simple zero λ_n^+ in the disk $D_\varepsilon(\mu_n^+)$ if a_n is sufficiently large. □

We conclude this section by considering another form of the Gurtin–Pipkin equation.

Suppose that $Q(t) \equiv 0$ and $B_j \equiv D_j \equiv 0$, $j = 1, 2, \dots N$, and integrate Eq. (1.11) over $(0, t)$ with respect to the time variable:

$$u^{(1)}(t) + \int_0^t A^2 u(\theta)\, d\theta + \int_0^t \left(\int_0^s K(s-\theta) A^2 u(\theta)\, d\theta \right) ds = \int_0^t f(\theta) d\,\theta + u^{(1)}(+0), \tag{2.56}$$

$$u(+0) = \varphi_0. \tag{2.57}$$

Changing the order of integration in the third term of the left-hand side of Eq. (2.56), we arrive at the equation

$$u^{(1)}(t) + \int_0^t A^2 u(\theta)\, d\theta + \int_0^t Q(t-s) A^2 u(\theta)\, d\theta = f_1(t),$$
$$u(+0) = \varphi_0,$$

where

$$Q(t) = \int_0^t K(\xi)\, d\xi, \qquad f_1(t) = \int_0^t f(\theta)\, d\theta + \varphi_1.$$

The latter problem can be written as follows:

$$u^{(1)}(t) + \int_0^t G(t-\theta) A^2 u(\theta)\, d\theta = f_1(t), \tag{2.58}$$

$$u(+0) = \varphi_0, \tag{2.59}$$

where

$$G(t) = 1 + Q(t).$$

Note that the Gurtin–Pipkin equation was studied in the above form in [7, 17].

The Darcy law as well as other problems arising by averaging in strongly heterogeneous media can be represented in the form (2.58), (2.59) as well (see [9] and references therein for more detail).

3. Remarks and Comments

The distinctive property of our results is that we consider the case of variable coefficients $B_j(t)A$ and $D_j(t)$ and variable delays $g_j(t)$ (see Theorem 1.1), while the papers [2–4, 10, 11, 14, 15, 24, 25] deal with the case of constant delays and constant operator coefficients at delay terms.

The method of the proof of Theorem 1.1 substantially differs from the methods of proving the solvability used in [2–5, 10, 11, 14, 15, 17, 24, 25]. On the other hand, it is quite similar to the method of proving the well-posed solvability used in [21, 22]. Theorem 1.1 generalizes the corresponding result from [22]. Let us explain this in more detail. Similarly to [22], we reduce the original problem (1.1),(1.2) to Eq. (2.10) in the space $L_2(\mathbb{R}_+, H)$, which is a functional-integral equation equivalent (in the sense of solvability) to the original problem. The distinctive property of the specified functional-integral equation (cf. [22, Eq. (29)]) is the boundedness of its operator coefficients. This substantially facilitates its study, especially, obtaining estimates for the norms of the operators in the space $L_2(\mathbb{R}_+, H)$.

Note that the method of the proof of Theorem 1.2 on the well-posed solvability of problem (1.11), (1.12) substantially differs from the approach of [17] as well. To prove Theorem 1.2, we reduce problem (1.11), (1.12) to an equivalent (in the sense of solvability) convolution-type functional-integral equation in the space $L_2(\mathbb{R}_+, H)$ (in [21, 23], the proof of Theorem 1.2 is given under additional assumptions). Then, to prove the unique solvability of the latter equation, we estimate the Laplace images of the kernels of integral operators in Hardy spaces.

In [17], the Gurtin–Pipkin equation represented as (2.58), (2.59) is studied. Using the operator-valued functions $\sin(At)$ and $\cos(At)$, the author reduces the original problem to an integral equation (with respect to the unknown function u) equivalent (in the sense of solvability) to the original problem. The distinction from the present paper is that the solvability is investigated in function spaces defined on a finite interval $(0,T)$ of the time variable t, while, in the present paper,the solvability is investigated in the weighted Sobolev spaces $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ on the semi-axis $\mathbb{R}_+$.

When proving Theorem 1.2, we substantially use the Hilbert structure of the spaces $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ and $L_{2,\gamma}(\mathbb{R}_+, H)$ and the Hardy theorem, while the case of Banach function spaces on a finite interval $(0,T)$ of the time variable t is considered in [17].

As we have noted above, the results of the present paper generalize the results of papers [21, 23] and naturally continue papers [18–20] devoted to functional differential equations with unbounded coefficients such that the principal part of the equation is an abstract parabolic equation.

Finally, note that problems of propagation of oscillations in viscoelastic media with memory naturally lead to integrodifferential equations of the kind (1.11) with $B_j \equiv D_j \equiv 0$, $j = 1, 2, \dots, N$ (see [8, 16] and references therein).

Problems arising in the of propagation of theory oscillations in strongly heterogeneous media (the Darcy law) also lead to equations of the kind (1.11). Problems of that kind are described in [9] (see references therein as well).

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REFERENCES

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations* [in Russian], Nauka, Moscow (1991).
2. G. Di Blasio, "Parabolic Volterra equations of convolution type," *J. Integral Equations Appl.*, **6**, No. 4, 479–508 (1994).
3. G. Di Blasio, K. Kunisch, and E. Sinestari, "L^2-regularity for parabolic partial integrodifferential equations with delays in the highest order derivatives," *J. Math. Anal. Appl.*, **102**, 38–57 (1984).
4. G. Di Blasio, K. Kunisch, and E. Sinestari, "Stability for abstract linear functional differential equations," *Israel J. Math.*, **50**, No. 3, 231–263 (1985).
5. W. Desch and R. K. Miller, "Exponential stabilization of Volterra integrodifferential equations in Hilbert space," *J. Differential Equations*, **70**, 366–389 (1987).
6. M. E. Gurtin and A. C. Pipkin, "Theory of heat conduction with finite wave speed," *Arch. Ration. Mech. Anal.*, **31**, 113–126 (1968).
7. S. Ivanov and L. Pandolfi, "Heat equation with memory: lack of controllability to rest," *J. Math. Anal. Appl.*, **355**, No. 1, 1–11 (2009).
8. N. D. Kopachevsky and S. G. Krein, *Operator Approach to Linear Problems of Hydrodynamics. Vol. 2. Nonself-Adjoint Problems for Viscous Fluids. Operator Theory: Advances and Applications*, Birkhäuser, Basel (2003).
9. D. A. Kosmodem'yanskiĭ and A. S. Shamaev, "Spectral properties of some problems in the mechanics of strongly inhomogeneous media," *J. Math. Sci. (N. Y.)*, **149**, No. 6, 1679–1700 (2008).
10. K. Kunisch and M. Mastinsek, "Dual semigroups and structual operators for partial differential equations with unbounded operators acting on the delays," *Differential Integral Equations*, **3**, No. 3, 733–756 (1990).
11. K. Kunisch and W. Shappacher, "Necessary conditions for partial differential equations with delay to generate C_0-semigroup," *J. Differential Equations*, **50**, 49–79 (1983).
12. J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary-Value Problems and Their Applications* [Russian translation], Mir, Moscow (1971).

13. D. A. Medvedev, V. V. Vlasov, and J. Wu, "Solvability and structural properties of abstract neutral functional differential equations," *Funct. Differ. Equ.*, **15**, No. 3-4, 249–272 (2008).
14. R. K. Miller, "An integrodifferential equation for rigid heat conductors with memory," *J. Math. Anal. Appl.*, **66**, No. 2, 313–332 (1978).
15. R. K. Miller and R. L. Wheeler, "Well-posedness and stability of linear Volterra interodifferential equations in abstract spaces," *Funkcial. Ekvac.*, **21**, 279–305 (1978).
16. A. I. Miloslavskii, "Spectral properties of operator pencils arising in viscoelasticity," Deposited in Ukr. NIINTI, No. 1229-87 (1987).
17. L. Pandolfi, "The controllability of the Gurtin–Pipkin equations: a cosine operator approach," *Appl. Math. Optim.*, **52**, 143–165 (2005).
18. V. V. Vlasov, "On the solvability and properties of solutions of functional-differential equations in a Hilbert space," *Sb. Math.*, **186**, No. 8, 1147–1172 (1995).
19. V. V. Vlasov, "On the solvability and estimates for the solutions of functional-differential equations in Sobolev spaces," *Proc. Steklov Inst. Math.*, **227**, No. 4, 104–115 (1999).
20. V. V. Vlasov, "Well-defined solvability of abstract parabolic equations with aftereffect," *Dokl. Math.*, **76**, No. 1, 511–513 (2007).
21. V. V. Vlasov and D. A. Medvedev, "Functional-differential equations in Sobolev spaces and related problems in spectral theory," *Sovrem. Mat. Fundam. Napravl.*, **30**, 3–173 (2008).
22. V. V. Vlasov and K. I. Shmatov, "Correct solvability of hyperbolic-type equations with aftereffect in a Hilbert space," *Proc. Steklov Inst. Math.*, **243**, No. 4, 120–130 (2003).
23. V. V. Vlasov and J. Wu, "Solvability and spectral analysis of abstract hyperbolic equations with delay," *Funct. Differ. Equ.*, **16**, No. 1-2, 751–768 (2009).
24. J. Wu, "Semigroup and integral form of class of partial differential equations with infinite delay," *Differential Integral Equations*, **4**, No. 6, 1325–1351 (1991).
25. J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Springer, New York (1996).

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