Mathematical and Computer Modelling 51 (2010) 329–337

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Global dynamics – convergence to equilibria – of epidemic patch models with immigration[☆]

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ARTICLE INFO

Article history:
Received 31 July 2009
Received in revised form 26 December 2009
Accepted 28 December 2009

Keywords:
Epidemic model
Population dispersal
Threshold
Equilibrium
Global stability

ABSTRACT

We consider SIR and SIS epidemic models with bilinear incidence and migration between two patches, where infected individuals cannot migrate from one patch to another due to medical screening. We find the thresholds classifying the global dynamics of the models in terms of the model parameters, and we obtain the global asymptotical stability of the disease free and the disease endemic (in one patch or in both patches) equilibria. This global asymptotic stability of endemic equilibria is established by using a novel Lyapunov function.

1. Introduction

Some communicable diseases can be transmitted by person to person contacts within a single patch (region or country), and can be spread among different patches where the disease transmission characteristics such as the transmission and recovery rates are different from one patch to another. The aspect of spatial spread due to immigration and travel has been the focus of many recent studies, although results about the global asymptotic stability of the disease endemic (within a patch or in all patches) equilibria are rare and most such results are obtained under additional technical conditions.

In [1], an SIR epidemic model with population dispersal was proposed by adding an immigration term, where infective individuals enter the population at a constant rate. This work was later expanded in [2] to include SIR and SIRS epidemic models with immigration, where the general incidence and constant rates entering all the compartments are included. The work [3] then considered an SIS model with dispersal and bilinear incidence, where only susceptible individuals are allowed to migrate between two patches. Some SIS models with dispersal were further studied in [4–7], where the patch number n can be arbitrary and both susceptible and infectious individuals can migrate among the patches. The cases of bilinear incidence and general birth rate were discussed in [4], and the case of standard incidence was investigated in [5–7]. A 2-patch SIS model with a general birth rate and standard incidence was considered in [8]. Other heterogeneities involved in the spread of disease were also incorporated into epidemic models with spatial dispersal, such as the age-structure [9], the periodic seasonality [10], and time delay as the constant infection period [11].

In [12], an SEIRS epidemic model with population moving among an arbitrary number of patches was considered, and a frequency-dependent SIS model with two patches was discussed in details for some specific cases. An SEIR epidemic model

☆ This work was partially supported by the Nature Science Foundation of China (10971163) and the Nature Science Foundation of Shaanxi, China (SJ08-ZT13).

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0895-7177/$ – see front matter
doi:10.1016/j.mcm.2009.12.031

with spatial dynamics was considered for a population consisting of s species and occupying n spatial patches in [13]. The work on SEIR model in [13] was also extended to a variety of models with spatial dynamics in [14–16].

The aforementioned studies focused mainly on the calculation and analysis of the basic reproduction number and the persistence of the disease. The papers [1,2] addressed the issue of the global stability of equilibria under some technical conditions. The work [12] also examined the global stability of equilibria, but only for the specific case where susceptible and infectious individuals travel at the same rate. Some of these studies assumed no disease induced death so that the models considered can be simplified by using the theory of limiting systems.

The direct Lyapunov method is common to prove the global stability of biological systems, epidemic models, and so on. For many biological and epidemic models, Lyapunov functions are chosen in the form of either $(x - x^*)^2$ type or $x - x^* - x^* \ln(x/x^*)$ type, and the associated total derivative is required to be negative definite according to the linear combination of some perfect square expressions. In this paper, we will make use of a Lyapunov function in the ingenious linear combination form of $(x - x^*)^2$ and $x - x^* - x^* \ln(x/x^*)$ types, and show that the associated total derivative is of quadratic form.

In this paper, we consider SIR and SIS epidemic models with bilinear incidence and migration between two patches, where infected individuals cannot migrate from one patch to another due to medical screening. We find the thresholds that completely classify the global dynamics of the models in terms of the model parameters, and we show that the system can have a disease free equilibrium, an equilibrium where the disease can be maintained in one patch only, and a disease endemic equilibrium where the disease is maintained in both patches. We obtain the global asymptotical stability of the disease free and the disease endemic (in one patch or in both patches) equilibria. This global asymptotic stability of endemic equilibria is established by using a novel Lyapunov function.

2. Models and thresholds

We consider both the SIR model

$$\begin{aligned} S_1' &= A_1 - \mu_1 S_1 - \beta_1 S_1 I_1 - a_1 S_1 + a_2 S_2, \\ I_1' &= \beta_1 S_1 I_1 - (\mu_1 + \gamma_1 + \alpha_1) I_1, \\ R_1' &= \gamma_1 I_1 - \mu_1 R_1 - b_1 R_1 + b_2 R_2, \\ S_2' &= A_2 - \mu_2 S_2 - \beta_2 S_2 I_2 - a_2 S_2 + a_1 S_1, \\ I_2' &= \beta_2 S_2 I_2 - (\mu_2 + \gamma_2 + \alpha_2) I_2, \\ R_2' &= \gamma_2 I_2 - \mu_2 R_2 - b_2 R_2 + b_1 R_1, \end{aligned} \tag{1}$$

and the SIS model

$$\begin{aligned} S_1' &= A_1 - \mu_1 S_1 - \beta_1 S_1 I_1 - a_1 S_1 + a_2 S_2 + \gamma_1 I_1, \\ I_1' &= \beta_1 S_1 I_1 - (\mu_1 + \gamma_1 + \alpha_1) I_1, \\ S_2' &= A_2 - \mu_2 S_2 - \beta_2 S_2 I_2 - a_2 S_2 + a_1 S_1 + \gamma_2 I_2, \\ I_2' &= \beta_2 S_2 I_2 - (\mu_2 + \gamma_2 + \alpha_2) I_2. \end{aligned} \tag{2}$$

Here, $S_i = S_i(t)$, $I_i = I_i(t)$, and $R_i = R_i(t)$ represent the numbers of individuals in the susceptible, infected, and recovered compartments in patch i $(i = 1, 2)$, respectively. A_i denotes the recruitment of susceptible individuals in patch i, β_i the disease transmission coefficient, μ_i the per capita death rate, γ_i the recovery rate of an infected individual, α_i the per capita disease induced death rate, a_i the rate at which a susceptible individual migrates from patch i to the other patch (patch $3 - i$), b_i the rate at which a recovered individual migrates from patch i to the other patch. In these two models, we neglect the death and birth processes of individuals when they are dispersing and neglect the time that individuals take to move between patches.

For simplicity, we denote by $\varepsilon_1 = \mu_1 + \gamma_1 + \alpha_1$ and $\varepsilon_2 = \mu_2 + \gamma_2 + \alpha_2$. Since the variables R_i $(i = 1, 2)$ do not appear in the equations of S_i and I_i, we only need to consider the subsystem

$$\begin{aligned} S_1' &= A_1 - \mu_1 S_1 - \beta_1 S_1 I_1 - a_1 S_1 + a_2 S_2, \\ I_1' &= \beta_1 S_1 I_1 - \varepsilon_1 I_1, \\ S_2' &= A_2 - \mu_2 S_2 - \beta_2 S_2 I_2 - a_2 S_2 + a_1 S_1, \\ I_2' &= \beta_2 S_2 I_2 - \varepsilon_2 I_2, \end{aligned} \tag{3}$$

and

$$\begin{aligned} S_1' &= A_1 - \mu_1 S_1 - \beta_1 S_1 I_1 - a_1 S_1 + a_2 S_2 + \gamma_1 I_1, \\ I_1' &= \beta_1 S_1 I_1 - \varepsilon_1 I_1, \\ S_2' &= A_2 - \mu_2 S_2 - \beta_2 S_2 I_2 - a_2 S_2 + a_1 S_1 + \gamma_2 I_2, \\ I_2' &= \beta_2 S_2 I_2 - \varepsilon_2 I_2. \end{aligned} \tag{4}$$

We will focus on models (3) and (4). Let $N = S_1 + I_1 + S_2 + I_2$. Then as long as $I_i \geq 0$ $(i = 1, 2)$, we have

$$N' \leq A_1 + A_2 - \mu_1(S_1 + I_1) - \mu_2(S_2 + I_2) \leq (A_1 + A_2) - \min\{\mu_1, \mu_2\} N.$$

Therefore,

$$\limsup_{t\to\infty} N(t) \leq \frac{A_1 + A_2}{\min\{\mu_1, \mu_2\}},$$

and the region

$$\Omega = \left\{(S_1, I_1, S_2, I_2) \in R_+^4 : S_1 + I_1 + S_2 + I_2 \le \frac{A_1 + A_2}{\min\{\mu_1, \mu_2\}}\right\}$$

is a positively invariant attractive set for (3) and (4).

Let

$$S_1^{(0)} = \frac{A_1(\mu_2 + a_2) + a_2 A_2}{\mu_1\mu_2 + \mu_1 a_2 + \mu_2 a_1}, \qquad S_2^{(0)} = \frac{A_2(\mu_1 + a_1) + a_1 A_1}{\mu_1\mu_2 + \mu_1 a_2 + \mu_2 a_1},$$
$$S_1^{(1)} = \frac{\varepsilon_1}{\beta_1}, \qquad S_2^{(1)} = \frac{1}{\mu_2 + a_2}\left(A_2 + a_1\frac{\varepsilon_1}{\beta_1}\right),$$
$$S_1^{(2)} = \frac{1}{\mu_1 + a_1}\left(A_1 + a_2\frac{\varepsilon_2}{\beta_2}\right), \qquad S_2^{(2)} = \frac{\varepsilon_2}{\beta_2}.$$

And let

$$R_0^{(1)} = \frac{\beta_1 S_1^{(0)}}{\varepsilon_1} = \frac{\beta_1[A_1(\mu_2 + a_2) + a_2 A_2]}{\varepsilon_1(\mu_1\mu_2 + \mu_1 a_2 + \mu_2 a_1)},$$
$$R_0^{(2)} = \frac{\beta_2 S_2^{(0)}}{\varepsilon_2} = \frac{\beta_2[A_2(\mu_1 + a_1) + a_1 A_1]}{\varepsilon_2(\mu_1\mu_2 + \mu_1 a_2 + \mu_2 a_1)},$$
$$R_1^{(1)} = \frac{\beta_1 S_1^{(2)}}{\varepsilon_1} = \frac{\beta_1}{(\mu_1 + a_1)\varepsilon_1}\left(A_1 + a_2\frac{\varepsilon_2}{\beta_2}\right),$$
$$R_1^{(2)} = \frac{\beta_2 S_2^{(1)}}{\varepsilon_2} = \frac{\beta_2}{(\mu_2 + a_2)\varepsilon_2}\left(A_2 + a_1\frac{\varepsilon_1}{\beta_1}\right).$$

Remark 1. $(S_1^{(0)}, S_2^{(0)})$ is the solution of equations

$$\begin{aligned} &A_1 - (\mu_1 + a_1)S_1 + a_2 S_2 = 0, \\ &A_2 - (\mu_2 + a_2)S_2 + a_1 S_1 = 0. \end{aligned} \tag{5}$$

We also note that

Remark 2. $R_1^{(1)} > 1$ and $R_1^{(2)} > 1$ implies that $R_0^{(1)} > 1$ and $R_0^{(2)} > 1$.

In fact, $R_1^{(1)} > 1$ and $R_1^{(2)} > 1$ are equivalent to the inequalities

$$\Phi_1 = A_1 - (\mu_1 + a_1)\frac{\varepsilon_1}{\beta_1} + a_2\frac{\varepsilon_2}{\beta_2} > 0,$$

and

$$\Phi_2 = A_2 - (\mu_2 + a_2)\frac{\varepsilon_2}{\beta_2} + a_1\frac{\varepsilon_1}{\beta_1} > 0,$$

respectively. Then, it follows from $(\mu_2 + \alpha_2)\Phi_1 + a_2\Phi_2 > 0$ and $(\mu_1 + \alpha_1)\Phi_2 + a_1\Phi_1 > 0$ that $R_0^{(1)} > 1$ and $R_0^{(2)} > 1$, respectively.

3. Global convergence and full classification of dynamics

Direct calculations show that, for (3), there is always the disease free equilibrium $E_0(S_1^{(0)}, 0, S_2^{(0)}, 0)$. When $R_0^{(1)} > 1$, there exists the boundary equilibrium $E_1(S_1^{(1)}, I_1^{(1)}, S_2^{(1)}, 0)$. When $R_0^{(2)} > 1$, there exists the boundary equilibrium $E_2(S_1^{(2)}, 0, S_2^{(2)}, I_2^{(2)})$. When $R_1^{(1)} > 1$ and $R_1^{(2)} > 1$, there exists a unique endemic equilibrium $E^*(S_1^{(1)}, I_1^*, S_2^{(2)}, I_2^*)$, with

$$I_1^{(1)} = \frac{A_1(\mu_2 + a_2) + a_2 A_2}{\varepsilon_1(\mu_2 + a_2)}\left(1 - \frac{1}{R_0^{(1)}}\right),$$
$$I_2^{(2)} = \frac{A_2(\mu_1 + a_1) + a_1 A_1}{\varepsilon_2(\mu_1 + a_1)}\left(1 - \frac{1}{R_0^{(2)}}\right),$$
$$I_1^* = \frac{\mu_1 + a_1}{\beta_1}\left(R_1^{(1)} - 1\right),$$
$$I_2^* = \frac{\mu_2 + a_2}{\beta_2}\left(R_1^{(2)} - 1\right).$$

Direct calculations can also determine the local stability of the aforementioned equilibria. Namely, we have that, when $R_0^{(1)} < 1$ and $R_0^{(2)} < 1$, E_0 is locally asymptotically stable; when $R_0^{(1)} > 1$ and $R_1^{(2)} < 1$, E_1 is locally asymptotically stable; when $R_0^{(2)} > 1$ and $R_1^{(1)} < 1$, E_2 is locally asymptotically stable.

To have a full classification of the model dynamics and, in particular, to prove the global stability we will need the following technical lemma.

Lemma 3. *There are positive constants m_1 and m_2 such that the quadratic function $F(u, v) = m_1(\mu_1 + a_1)u^2 - (m_1a_2 + m_2a_1)uv + m_2(\mu_2 + a_2)v^2$ is positive definite.*

Proof. A direct calculation shows that $[2(\mu_1 + a_1)(\mu_2 + a_2) - a_1a_2]^2 - a_1^2a_2^2 > 0$, then there must be m_1 and m_2 such that

$$(m_1a_2 + m_2a_1)^2 - 4m_1m_2(\mu_1 + a_1)(\mu_2 + a_2) = m_1^2a_2^2 + m_2^2a_1^2 - 2m_1m_2[2(\mu_1 + a_1)(\mu_2 + a_2) - a_1a_2] < 0,$$

for otherwise the above quadratic in (m_1, m_2) is positive definite (note that $2(\mu_1 + a_1)(\mu_2 + a_2) - a_1a_2 > 0$). □

We can now state our main results for the global dynamics of (3).

Theorem 4. *We have the global dynamics for model (3).*

(i) *When $R_0^{(1)} \le 1$ and $R_0^{(2)} \le 1$, E_0 is globally stable on the set Ω;*
(ii) *When $R_0^{(1)} > 1$ and $R_1^{(2)} \le 1$, E_1 is globally stable in the set Ω;*
(iii) *When $R_0^{(2)} > 1$ and $R_1^{(1)} \le 1$, E_2 is globally stable in the set Ω;*
(iv) *When $R_1^{(1)} > 1$ and $R_1^{(2)} > 1$, E^* is globally stable in the set Ω.*

Proof. (i) Since $E_0(S_1^{(0)}, 0, S_2^{(0)}, 0)$ is the equilibrium of (3), we can rewrite (3) as

$$\begin{aligned}
S_1' &= -(\mu_1 + a_1)(S_1 - S_1^{(0)}) - \beta_1 I_1\left(S_1 - S_1^{(0)}\right) - \beta_1 S_1^{(0)} I_1 + a_2\left(S_2 - S_2^{(0)}\right),\\
I_1' &= I_1\left[\beta_1\left(S_1 - S_1^{(0)}\right) + \left(\beta_1 S_1^{(0)} - \varepsilon_1\right)\right],\\
S_2' &= -(\mu_2 + a_2)\left(S_2 - S_2^{(0)}\right) - \beta_2 I_2\left(S_2 - S_2^{(0)}\right) - \beta_2 S_2^{(0)} I_2 + a_1\left(S_1 - S_1^{(0)}\right),\\
I_2' &= I_2\left[\beta_2\left(S_2 - S_2^{(0)}\right) + \left(\beta_2 S_2^{(0)} - \varepsilon_2\right)\right].
\end{aligned} \tag{6}$$

Define the function

$$V_1 = m_1\left[\frac{\left(S_1 - S_1^{(0)}\right)^2}{2} + S_1^{(0)} I_1\right] + m_2\left[\frac{\left(S_2 - S_2^{(0)}\right)^2}{2} + S_2^{(0)} I_2\right].$$

Then the derivative of V_1 along the solution of (6) is given by

$$\begin{aligned}
V_1' = {} & m_1\left\{-(\mu_1 + a_1)\left(S_1 - S_1^{(0)}\right)^2 - \beta_1 I_1\left(S_1 - S_1^{(0)}\right)^2 + a_2\left(S_1 - S_1^{(0)}\right)\left(S_2 - S_2^{(0)}\right)\right\}\\
& + m_2\left\{-(\mu_2 + a_2)\left(S_2 - S_2^{(0)}\right)^2 - \beta_2 I_2\left(S_2 - S_2^{(0)}\right)^2 + a_1\left(S_1 - S_1^{(0)}\right)\left(S_2 - S_2^{(0)}\right)\right\}\\
& + m_1 S_1^{(0)}\left(\beta_1 S_1^{(0)} - \varepsilon_1\right) I_1 + m_2 S_2^{(0)}\left(\beta_2 S_2^{(0)} - \varepsilon_2\right) I_2\\
= {} & -F\left(S_1 - S_1^{(0)}, S_2 - S_2^{(0)}\right) + m_1\left[S_1^{(0)}\varepsilon_1\left(R_0^{(1)} - 1\right) - \beta_1\left(S_1 - S_1^{(0)}\right)^2\right] I_1\\
& + m_2\left[S_2^{(0)}\varepsilon_2\left(R_0^{(2)} - 1\right) - \beta_2\left(S_2 - S_2^{(0)}\right)^2\right] I_2.
\end{aligned}$$

According to Lemma 3, we can choose positive constants m_1 and m_2 such that $F(S_1 - S_1^{(0)}, S_2 - S_2^{(0)})$ is positive definite. Thus, $R_0^{(1)} \le 1$ and $R_0^{(2)} \le 1$ implies that $V_1' \le 0$ for the corresponding m_1 and m_2. Therefore, it follows from the LaSalle Invariance Principle [17] that E_0 is globally stable for $R_0^{(1)} \le 1$ and $R_0^{(2)} \le 1$.
(ii) To discuss the global stability of $E_1(S_1^{(1)}, I_1^{(1)}, S_2^{(1)}, 0)$, we rewrite (3) as

$$\begin{aligned}
S_1' &= -(\mu_1 + a_1)(S_1 - S_1^{(1)}) - \beta_1 I_1(S_1 - S_1^{(1)}) - \beta_1 S_1^{(1)}(I_1 - I_1^{(1)}) + a_2(S_2 - S_2^{(1)}),\\
I_1' &= \beta_1 I_1(S_1 - S_1^{(1)}),\\
S_2' &= -(\mu_2 + a_2)(S_2 - S_2^{(1)}) - \beta_2 I_2(S_2 - S_2^{(1)}) - \beta_2 S_2^{(1)} I_2 + a_1(S_1 - S_1^{(1)}),\\
I_2' &= \beta_2 I_2(S_2 - S_2^{(1)}) + (\beta_2 S_2^{(1)} - \varepsilon_2) I_2.
\end{aligned} \tag{7}$$

Define the function

$$V_2 = m_1\left[\frac{1}{2}(S_1 - S_1^{(1)})^2 + S_1^{(1)}\left(I_1 - I_1^{(1)} - I_1^{(1)}\ln\frac{I_1}{I_1^{(1)}}\right)\right] + m_2\left[\frac{1}{2}(S_2 - S_2^{(1)})^2 + S_2^{(1)} I_2\right],$$

then the derivative of V_2 along the solution of (7) is given by

$$\begin{aligned} V_2' &= -F\left(S_1 - S_1^{(1)}, S_2 - S_2^{(1)}\right) - \beta_1 m_1 I_1 (S_1 - S_1^{(1)})^2 - m_2 I_2\left[\beta_2 (S_2 - S_2^{(1)})^2 - S_2^{(1)}(\beta_2 S_2^{(1)} - \varepsilon_2)\right] \\ &= -F\left(S_1 - S_1^{(1)}, S_2 - S_2^{(1)}\right) - \beta_1 m_1 I_1 (S_1 - S_1^{(1)})^2 - m_2 I_2\left[\beta_2 (S_2 - S_2^{(1)})^2 - \varepsilon_2 S_2^{(1)}\left(R_1^{(2)} - 1\right)\right]. \end{aligned}$$

According to Lemma 3, we can choose positive constants m_1 and m_2 such that $F(S_1 - S_1^{(1)}, S_2 - S_2^{(1)})$ is positive definite. Thus, $R_1^{(2)} \le 1$ implies that $V_2' \le 0$ for the corresponding m_1 and m_2. Therefore, it follows from the LaSalle Invariance Principle [17] that E_1 is globally stable for $R_0^{(1)} > 1$ and $R_1^{(2)} \le 1$.
(iii) This can be proved in a similar way as for (ii).
(iv) From (3), $S_1^{(1)}, I_1^*, S_2^{(2)}$ and I_2^* satisfy

$$\begin{aligned} &\mu_1 + a_1 = \frac{A_1 - \beta_1 S_1 I_1 + a_2 S_2}{S_1}, \\ &\varepsilon_1 = \beta_1 S_1, \\ &\mu_2 + a_2 = \frac{A_2 - \beta_2 S_2 I_2 + a_1 S_1}{S_2}, \\ &\varepsilon_2 = \beta_2 S_2. \end{aligned} \tag{8}$$

Substituting (8) into (3) gives

$$\begin{aligned} &S_1' = S_1\left[A_1\left(\frac{1}{S_1} - \frac{1}{S_1^{(1)}}\right) - \beta_1(I_1 - I_1^*) + a_2\left(\frac{S_2}{S_1} - \frac{S_2^{(2)}}{S_1^{(1)}}\right)\right], \\ &I_1' = \beta_1 I_1 (S_1 - S_1^{(1)}), \\ &S_2' = S_2\left[A_2\left(\frac{1}{S_2} - \frac{1}{S_2^{(2)}}\right) - \beta_2(I_2 - I_2^*) + a_1\left(\frac{S_1}{S_2} - \frac{S_1^{(1)}}{S_2^{(2)}}\right)\right], \\ &I_2' = \beta_2 I_2 (S_2 - S_2^{(2)}). \end{aligned} \tag{9}$$

Define the function

$$\begin{aligned} V_4 = m_1&\left[\left(I_1 - I_1^* - I_1^*\ln\frac{I_1}{I_1^*}\right) + \left(S_1 - S_1^{(1)} - S_1^{(1)}\ln\frac{S_1}{S_1^{(1)}}\right)\right] \\ &+ m_2\left[\left(I_2 - I_2^* - I_2^*\ln\frac{I_2}{I_2^*}\right) + \left(S_2 - S_2^{(2)} - S_2^{(2)}\ln\frac{S_2}{S_2^{(2)}}\right)\right], \end{aligned}$$

then the derivative of V_4 along the solution of (9) is given by

$$\begin{aligned} V_4' = &-\frac{m_1 A_1 (S_1 - S_1^{(1)})^2}{S_1 S_1^{(1)}} - \frac{m_2 A_2 (S_2 - S_2^{(2)})^2}{S_2 S_2^{(2)}} + S_1\left(m_2 a_1 - m_1 a_2 \frac{S_2^{(2)}}{S_1^{(1)}}\right) + S_2\left(m_1 a_2 - m_2 a_1 \frac{S_1^{(1)}}{S_2^{(2)}}\right) \\ &+ \left(m_1 a_2 S_2^{(2)} + m_2 a_1 S_1^{(1)} - m_1 a_2 S_1^{(1)}\frac{S_2}{S_1} - m_2 a_1 S_2^{(2)}\frac{S_1}{S_2}\right). \end{aligned}$$

To make the coefficients of terms for S_1 and S_2 be zero, we choose $m_1 = a_1 S_1^{(1)}$ and $m_2 = a_2 S_2^{(2)}$. Then

$$\begin{aligned} V_4' &= -\frac{a_1 A_1 (S_1 - S_1^{(1)})^2}{S_1} - \frac{a_2 A_2 (S_2 - S_2^{(2)})^2}{S_2} + a_1 a_2 S_1^{(1)} S_2^{(2)}\left(2 - \frac{S_1^{(1)} S_2}{S_1 S_2^{(2)}} - \frac{S_1 S_2^{(2)}}{S_1^{(1)} S_2}\right) \\ &= -\frac{a_1 A_1 (S_1 - S_1^{(1)})^2}{S_1} - \frac{a_2 A_2 (S_2 - S_2^{(2)})^2}{S_2} - a_1 a_2 S_1^{(1)} S_2^{(2)}\left(\sqrt{\frac{S_1^{(1)} S_2}{S_1 S_2^{(2)}}} - \sqrt{\frac{S_1 S_2^{(2)}}{S_1^{(1)} S_2}}\right)^2 \le 0. \end{aligned}$$

It is easy to see that $V_4' = 0$ is equivalent to the fact that $S_1 = S_1^{(1)}$ and $S_2 = S_2^{(2)}$. And the largest invariant set of (8) on the set $\{(S_1, I_1, S_2, I_2) \in \Omega : S_1 = S_1^{(1)}, S_2 = S_2^{(2)}\}$ is the singleton set $\{E^*\}$. Therefore, it follows from the LaSalle Invariance Principle [17] that E^* is globally stable when it exists. □

We now state and derive parallel results for (4): There is always the disease free equilibrium $E_0(S_1^{(0)}, 0, S_2^{(0)}, 0)$. When $R_0^{(1)} > 1$, there exists the boundary equilibrium $E_1(S_1^{(1)}, I_1^{(1)}, S_2^{(1)}, 0)$, while when $R_0^{(2)} > 1$, there exists the boundary equilibrium $E_2(S_1^{(2)}, 0, S_2^{(2)}, I_2^{(2)})$. Furthermore, when $R_1^{(1)} > 1$ and $R_1^{(2)} > 1$, there exists a unique equilibrium $E^*(S_1^{(1)}, I_1^*, S_2^{(2)}, I_2^*)$. Here

$$I_1^{(1)} = \frac{A_1(\mu_2 + a_2) + a_2 A_2}{(\mu_1 + \alpha_1)(\mu_2 + a_2)} \left(1 - \frac{1}{R_0^{(1)}}\right),$$
$$I_2^{(2)} = \frac{A_2(\mu_1 + a_1) + a_1 A_1}{(\mu_2 + \alpha_2)(\mu_1 + a_1)} \left(1 - \frac{1}{R_0^{(2)}}\right),$$
$$I_1^* = \frac{\varepsilon_1(\mu_1 + a_1)}{\beta_1(\mu_1 + \alpha_1)} \left(R_1^{(1)} - 1\right),$$
$$I_2^* = \frac{\varepsilon_2(\mu_2 + a_2)}{\beta_2(\mu_2 + \alpha_2)} \left(R_1^{(2)} - 1\right).$$

As for the local stability of the boundary equilibria, direct calculations show that when $R_0^{(1)} < 1$ and $R_0^{(2)} < 1$, E_0 is locally asymptotically stable; when $R_0^{(1)} > 1$ and $R_1^{(2)} < 1$, E_1 is locally asymptotically stable; when $R_0^{(2)} > 1$ and $R_1^{(1)} < 1$, E_2 is locally asymptotically stable. Finally, we have the global threshold dynamics:

Theorem 5. *For (4), we have*

(i) *When $R_0^{(1)} \le 1$ and $R_0^{(2)} \le 1$, E_0 is globally stable on the set Ω;*
(ii) *When $R_0^{(1)} > 1$ and $R_1^{(2)} \le 1$, E_1 is globally stable in the set Ω;*
(iii) *When $R_0^{(2)} > 1$ and $R_1^{(1)} \le 1$, E_2 is globally stable in the set Ω;*
(iv) *When $R_1^{(1)} > 1$ and $R_1^{(2)} > 1$, E^* is globally stable in the set Ω.*

Proof. The proof is quite similar to that for Theorem 4, and hence we just give the Lyapunov function and its derivative for each of the case (i), (ii) and (iv).

For (i), since $E_0(S_1^{(0)}, 0, S_2^{(0)}, 0)$ is the equilibrium of (4), we can rewrite (4) as

$$\begin{aligned} S_1' &= -(\mu_1 + a_1)\left(S_1 - S_1^{(0)}\right) + a_2\left(S_2 - S_2^{(0)}\right) + I_1(\gamma_1 - \beta_1 S_1), \\ I_1' &= I_1(\beta_1 S_1 - \varepsilon_1), \\ S_2' &= -(\mu_2 + a_2)\left(S_2 - S_2^{(0)}\right) + a_1\left(S_1 - S_1^{(0)}\right) + I_2(\gamma_2 - \beta_2 S_2), \\ I_2' &= I_2(\beta_2 S_2 - \varepsilon_2). \end{aligned} \tag{10}$$

Therefore, for the function

$$V_1 = m_1\left[\frac{\left(S_1 - S_1^{(0)}\right)^2}{2} + \frac{2\varepsilon_1 - \beta_1 S_1^{(0)} - \gamma_1}{\beta_1} I_1\right] + m_2\left[\frac{\left(S_2 - S_2^{(0)}\right)^2}{2} + \frac{2\varepsilon_2 - \beta_2 S_2^{(0)} - \gamma_2}{\beta_2} I_2\right],$$

where $2\varepsilon_i - \beta_i S_i^{(0)} - \gamma_i = (\varepsilon_i - \gamma_i) + (\varepsilon_i - \beta_i S_i^{(0)}) = (\mu_i + \alpha_i) + \varepsilon_i(1 - R_0^{(i)}) > 0$ for $R_0^{(i)} \le 1$ $(i = 1, 2)$, we have

$$\begin{aligned} V_1' =\ & m_1\left\{-(\mu_1 + a_1)\left(S_1 - S_1^{(0)}\right)^2 + a_2\left(S_1 - S_1^{(0)}\right)\left(S_2 - S_2^{(0)}\right)\right\} \\ & + m_2\left\{-(\mu_2 + a_2)\left(S_2 - S_2^{(0)}\right)^2 + a_1\left(S_1 - S_1^{(0)}\right)\left(S_2 - S_2^{(0)}\right)\right\} \\ & + m_1 I_1\left[\left(S_1 - S_1^{(0)}\right)(\gamma_1 - \beta_1 S_1) + \frac{2\varepsilon_1 - \beta_1 S_1^{(0)} - \gamma_1}{\beta_1}(\beta_1 S_1 - \varepsilon_1)\right] \\ & + m_2 I_2\left[\left(S_2 - S_2^{(0)}\right)(\gamma_2 - \beta_2 S_2) + \frac{2\varepsilon_2 - \beta_2 S_2^{(0)} - \gamma_2}{\beta_2}(\beta_2 S_2 - \varepsilon_2)\right] \\ =\ & -F\left(S_1 - S_1^{(0)}, S_2 - S_2^{(0)}\right) - m_1\beta_1 I_1\left[\left(S_1 - \frac{\varepsilon_1}{\beta_1}\right)^2 + \frac{(\gamma_1 - \varepsilon_1)\left(\beta_1 S_1^{(0)} - \varepsilon_1\right)}{\beta_1^2}\right] \end{aligned}$$

$$- m_2\beta_2 I_2 \left[\left(S_2 - \frac{\varepsilon_2}{\beta_2} \right)^2 + \frac{(\gamma_2 - \varepsilon_2)\left(\beta_2 S_2^{(0)} - \varepsilon_2\right)}{\beta_2^2} \right]$$

$$= -F\left(S_1 - S_1^{(0)}, S_2 - S_2^{(0)}\right) - m_1\beta_1 I_1 \left[\left(S_1 - \frac{\varepsilon_1}{\beta_1} \right)^2 + \frac{\varepsilon_1(\mu_1+\alpha_1)\left(1 - R_0^{(1)}\right)}{\beta_1^2} \right]$$

$$- m_2\beta_2 I_2 \left[\left(S_2 - \frac{\varepsilon_2}{\beta_2} \right)^2 + \frac{\varepsilon_2(\mu_2+\alpha_2)\left(1 - R_0^{(2)}\right)}{\beta_2^2} \right].$$

As for (ii), we rewrite (4) as

$$\begin{aligned} S_1' &= -(\mu_1 + a_1)(S_1 - S_1^{(1)}) - \beta_1 I_1 (S_1 - S_1^{(1)}) - (\varepsilon_1 - \gamma_1)(I_1 - I_1^{(1)}) + a_2(S_2 - S_2^{(1)}), \\ I_1' &= \beta_1 I_1 (S_1 - S_1^{(1)}), \\ S_2' &= -(\mu_2 + a_2)(S_2 - S_2^{(1)}) - I_2(\beta_2 S_2 - \gamma_2) + a_1(S_1 - S_1^{(1)}), \\ I_2' &= I_2(\beta_2 S_2 - \varepsilon_2). \end{aligned} \tag{11}$$

Therefore, for the function

$$V_2 = m_1 \left[\frac{1}{2}(S_1 - S_1^{(1)})^2 + \frac{\varepsilon_1 - \gamma_1}{\beta_1}\left(I_1 - I_1^{(1)} - I_1^{(1)} \ln \frac{I_1}{I_1^{(1)}} \right) \right] + m_2 \left[\frac{1}{2}(S_2 - S_2^{(1)})^2 + \frac{2\varepsilon_2 - \beta_2 S_2^{(1)} - \gamma_2}{\beta_2} I_2 \right],$$

we get

$$V_2' = -F\left(S_1 - S_1^{(1)}, S_2 - S_2^{(1)}\right) - m_1\beta_1 I_1 (S_1 - S_1^{(1)})^2 - m_2\beta_2 I_2 \left[\left(S_2 - \frac{\varepsilon_2}{\beta_2} \right)^2 + \frac{(\varepsilon_2 - \gamma_2)(\varepsilon_2 - \beta_2 S_2^{(1)})}{\beta_2^2} \right]$$

$$= -F\left(S_1 - S_1^{(1)}, S_2 - S_2^{(1)}\right) - m_1\beta_1 I_1 (S_1 - S_1^{(1)})^2 - m_2\beta_2 I_2 \left[\left(S_2 - \frac{\varepsilon_2}{\beta_2} \right)^2 + \frac{\varepsilon_2(\mu_2+\alpha_2)\left(1 - R_1^{(2)}\right)}{\beta_2^2} \right].$$

Finally, for (iv), we rewrite (3) as

$$\begin{aligned} S_1' &= -(\mu_1 + a_1)(S_1 - S_1^{(1)}) - \beta_1 I_1 (S_1 - S_1^{(1)}) - (\varepsilon_1 - \gamma_1)(I_1 - I_1^*) + a_2(S_2 - S_2^{(2)}), \\ I_1' &= \beta_1 I_1 (S_1 - S_1^{(1)}), \\ S_2' &= -(\mu_2 + a_2)(S_2 - S_2^{(2)}) - \beta_2 I_2 (S_2 - S_2^{(2)}) - (\varepsilon_2 - \gamma_2)(I_2 - I_2^*) + a_1(S_1 - S_1^{(1)}), \\ I_2' &= \beta_2 I_2 (S_2 - S_2^{(2)}). \end{aligned} \tag{12}$$

Define the function

$$V_4 = m_1 \left[\frac{(S_1 - S_1^{(1)})^2}{2} + \frac{\varepsilon_1 - \gamma_1}{\beta_1}\left(I_1 - I_1^* - I_1^* \ln \frac{I_1}{I_1^*} \right) \right] + m_2 \left[\frac{(S_2 - S_2^{(2)})^2}{2} + \frac{\varepsilon_2 - \gamma_2}{\beta_2}\left(I_2 - I_2^* - I_2^* \ln \frac{I_2}{I_2^*} \right) \right],$$

then we have

$$V_4' = -F\left(S_1 - S_1^{(1)}, S_2 - S_2^{(2)}\right) - m_1\beta_1 I_1 (S_1 - S_1^{(1)})^2 - m_2\beta_2 I_2 (S_2 - S_2^{(2)})^2. \quad \square$$

4. Discussion

It is easy to see that $R_0^{(i)} (i = 1, 2)$ defined in Section 2 is the basic reproduction number in patch i in the case where spatial dispersal occurs between two patches and when the two patches are both in the disease free steady state. By the method of next generation matrix in [18,19], direct calculation shows that the basic reproduction number of (3) and (4) is $R_0 = \max\{R_0^{(1)}, R_0^{(2)}\}$, but, from the results in Theorems 4 and 5, R_0 cannot determine their dynamics completely. In fact, R_0 can only determine if the disease dies out in total population including two patches. To determine the transmission of disease in each patch, the thresholds $R_1^{(1)}$ and $R_1^{(2)}$ are also necessary according to Theorems 4 and 5.

It is obvious that $R_1^{(i)} (i = 1, 2)$ defined in Section 2 is the basic reproduction number in patch i with dispersal when the other patch is at the endemic steady state. By the expressions of $R_0^{(1)}$ and $R_0^{(2)}$, we have

$$R_1^{(1)} = \frac{R_0^{(1)}}{(\mu_1 + a_1) S_1^{(0)}} \left(A_1 + a_2 \cdot \frac{S_2^{(0)}}{R_0^{(2)}} \right), \qquad R_1^{(2)} = \frac{R_0^{(2)}}{(\mu_2 + a_2) S_2^{(0)}} \left(A_2 + a_1 \cdot \frac{S_1^{(0)}}{R_0^{(1)}} \right). \tag{13}$$

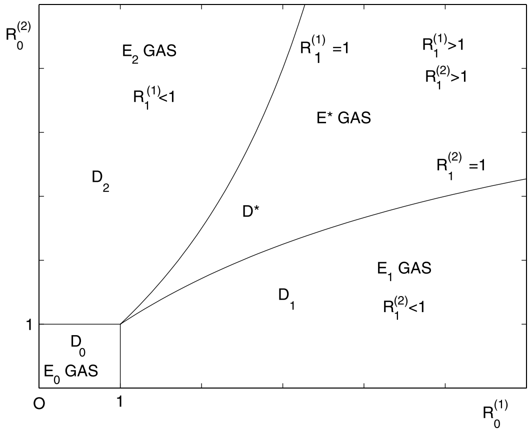

Fig. 1. The regions of parameters for the global stability of (3) and (4). Here, two curves correspond to equations, $R_1^{(1)} = 1$ and $R_1^{(2)} = 1$, respectively.

Therefore, the results on the dynamics of (3) and (4) may be summarized in Fig. 1, where the first quadrant of the $R_0^{(1)} - R_0^{(2)}$ plane is divided into four parts: D_i $(i = 0, 1, 2)$ and D^*. In the region $D_0 = \{(R_0^{(1)}, R_0^{(2)}) : R_0^{(1)} \le 1, R_0^{(2)} \le 1\}$, E_0 is globally stable; in the region $D_1 = \{(R_0^{(1)}, R_0^{(2)}) : R_0^{(1)} > 1, R_1^{(2)} \le 1\}$, E_1 is globally stable; in the region $D_2 = \{(R_0^{(1)}, R_0^{(2)}) : R_1^{(1)} \le 1, R_0^{(2)} > 1\}$, E_2 is globally stable; in the region $D^* = \{(R_0^{(1)}, R_0^{(2)}) : R_1^{(1)} > 1, R_1^{(2)} > 1\}$, E^* is globally stable.

Since $R_1^{(1)} = 1$ is equivalent to $R_0^{(2)} = a_2 S_2^{(0)} R_0^{(1)} / ((\mu_1 + a_1) S_1^{(0)} - A_1 R_0^{(1)})$, we note that $R_0^{(1)} = (\mu_1 + a_1) S_1^{(0)} / A_1$ is the asymptotic line of the curve $R_1^{(1)} = 1$, where $(\mu_1 + a_1) S_1^{(0)} / A_1 > 1$. Similarly, the curve $R_1^{(2)} = 1$ has the asymptotic line $R_0^{(2)} = (\mu_2 + a_2) S_2^{(0)} / A_2$. From Fig. 1, it is not difficult to see the following facts: when $1 < R_0^{(1)} < (\mu_1 + a_1) S_1^{(0)} / A_1$, the dynamical behavior of (3) and (4) can all change twice when $R_0^{(2)}$ varies from 0 to $+\infty$. E_1 is globally stable for $0 < R_0^{(2)} \le (\mu_2 + a_2) S_2^{(0)} R_0^{(1)} / (A_2 R_0^{(1)} + a_1 S_1^{(0)})$; E^* is globally stable for $(\mu_2 + a_2) S_2^{(0)} R_0^{(1)} / (A_2 R_0^{(1)} + a_1 S_1^{(0)}) < R_0^{(2)} < R_0^{(1)} (A_1 + a_2 S_2^{(0)} / R_0^{(2)}) / ((\mu_1 + a_1) S_1^{(0)})$; E_2 is globally stable for $R_0^{(2)} \ge a_2 S_2^{(0)} R_0^{(1)} / ((\mu_1 + a_1) S_1^{(0)} - A_1 R_0^{(1)})$. When $R_0^{(1)} < 1$ or $R_0^{(1)} > (\mu_1 + a_1) S_1^{(0)} / A_1$, the dynamical behavior of (3) and (4) can change only once when $R_0^{(2)}$ varies from 0 to $+\infty$. Similarly, when $1 < R_0^{(2)} < (\mu_2 + a_2) S_2^{(0)} / A_2$, the dynamical behavior of (3) and (4) can all change twice when $R_0^{(1)}$ varies from 0 to $+\infty$; when $R_0^{(2)} < 1$ or $R_0^{(2)} > (\mu_2 + a_2) S_2^{(0)} / A_2$, it can change only once.

Denote by $K_i = A_i / \mu_i$ $(i = 1, 2)$. Then it represents the size of population in path i at the equilibrium in the absence of disease and dispersal. Thus, it is easy to see that, in the absence of dispersal, the basic reproduction number in patch i is given by $R_{0i} = \beta_i K_i / \varepsilon_i = \beta_i A_i / (\varepsilon_i \mu_i)$. Direct calculation shows that

$$R_{0i} - R_0^{(i)} = \frac{\beta_i \mu_j}{\varepsilon_i (\mu_1 \mu_2 + \mu_1 a_2 + \mu_2 a_1)} (a_i K_i - a_j K_j), \quad (j = 3 - i).$$

Then $R_{0i} > R_0^{(i)}$ if and only if $a_i K_i > a_j K_j$. It implies that, under the condition that at the disease free steady state the number of migrating individuals from patch i to patch j is greater than that from patch j to patch i, the basic reproduction number in patch i in the absence of dispersal is greater than that in the presence of dispersal. So increasing the migration of susceptible individuals from patch i to patch j may be a helpful control strategy for the disease management in patch i.

Some of the outcomes described in this work resemble other competitive systems, though there is a delicate difference, since we assume that the same disease spreads in two patches, and that susceptible individuals are allowed to transit between these two patches. As such, there exists a cooperative relation between susceptible individuals in two patches. In our setup, the disease competes for mobile susceptible individuals but the disease can only invade the susceptible individuals that live in its domain (patch). In a sense, this looks like we had two distinct patch-specific diseases. The analog of this situation in a two-strain system seems to be that in two-strain systems differential susceptibility to each strain was assumed [20,21], except now that the susceptible individuals are mobile.

Acknowledgements

We are very grateful to referees for their valuable comments and suggestions which led to an improvement of our original manuscript.

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