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RESEARCH ARTICLE

Second-order differentiability with respect to parameters for differential equations with adaptive delays*

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Abstract In this paper, we study the second-order differentiability of solutions with respect to parameters in a class of delay differential equations, where the evolution of the delay is governed explicitly by a differential equation involving the state variable and the parameters. We introduce the notion of locally complete triple-normed linear space and obtain an extension of the well-known uniform contraction principle in such spaces. We then apply this extended principle and obtain the second-order differentiability of solutions with respect to parameters in the $W^{1,p}$-norm ($1 \leqslant p < \infty$).

Keywords Delay differential equation, adaptive delay, differentiability of solution, state-dependent delay, uniform contraction principle, locally complete triple-normed linear space
MSC 34K05

1 Introduction

Delay differential equations have been widely used in modeling evolution of a dynamical system for which the growth rate of the system is governed by the current and historical states of the system. The incorporation of time lag into a model is usually through the use of terms like $x(t-\tau)$ or $\int_{-\infty}^{t} k(t-s)x(s)\,\mathrm{d}s$, where $x(t)$ is the state variable of the system under consideration. In the

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first case, the constant τ is called a *discrete delay*; and in the second case, $k(\cdot)$ is a suitable kernel, and the delay is said to be of the *distributed* type.

However, complicated situations occur in which the time lag depends on the unknown state variable. We call such a time lag a *state-dependent delay*. Delay differential equations with state-dependent delay arise naturally in many structured population models (see, for example, Refs. [1,3,5–7,12,26, 30]) and other applications, and have received considerable attention recently. See Refs. [1–4,8–10,12–14,17,20,22–25,27–32] and a recent survey article [18].

The focus of this paper is a very classical problem, namely, the high-order differentiability of solutions with respect to parameters, including initial conditions. This problem was addressed in Refs. [11,16,19], and the importance of such a problem was appropriately described in Ref. [19] as "Differentiability results with respect to parameters, besides the obvious theoretical importance, have a natural application in the problem of identification of unknown parameters of the equation (such as the initial function, some coefficients in the equation, or for a constant delay equation, the delay itself). In this direction, it is important to know if the solution is differentiable with respect to the parameters in some sense, since many identification methods require the use of optimization techniques, in which the knowledge of the derivative of the solution with respect to the parameters is essential."

The references, Refs. [11,16,19], addressed the first-order differentiability issue, and already these studies have inspired new results in functional analysis and illustrated the great difficulty caused by the state dependence of the delay. Our goal is to obtain high-order differentiability as this is required in most optimization results.

For simplicity, we will focus on the delay differential equation with the so-called *adaptive delay*. More precisely, we consider the following system of state-dependent delay differential equations:

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-\tau(t)), \sigma), & t \in [0, \alpha], \\ \dot{\tau}(t) = g(x(t), \tau(t), \sigma), & t \in [0, \alpha], \\ x(t) = \varphi(t), & t \in [-r, 0], \\ \tau(0) = \tau_0, \end{cases} \tag{1.1}$$

where $0 \leqslant \tau_0 \leqslant r$ ($r > 0$ is a fixed constant throughout this paper), $\alpha = T$ (T is a constant) or ∞, $\sigma \in \Sigma \subset \mathbb{R}^m$ (a normed linear space with norm $|\cdot|_{\mathbb{R}^m}$), and $\varphi \in W^{2,\infty}$ (see Section 2 for details). As the evolution of the delay is governed by a differential equation involving the state variable $x(t)$ itself and the parameters, we call such a system a *differential system with an adaptive delay*. System (1.1) includes the following model considered in Refs. [4,25]:

$$\begin{cases} \dot{x}(t) = -f(x(t-\tau(t))), \\ \dot{\tau}(t) = g(x(t), \tau(t)), \end{cases} \tag{1.2}$$

which is used by Arino et al. [5] to describe the evolution of a fish population whose larvae share a limited resource.

Our study is inspired by that of Hale and Ladeira [15] and that of Hartung and Turi [19]. In Ref. [15], the authors studied the differentiability of solutions with respect to constant delays. While in Ref. [19], the authors considered the following state-dependent delay equation:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t - \tau(t, x_t, \sigma)), \theta), & t \in [0, T], \\ x(t) = \varphi(t), & t \in [-r, 0]. \end{cases}$$

They obtained some sufficient conditions for the first-order differentiability of solutions with respect to parameters. The approach developed in Refs. [15,19] is standard. One transforms the initial value problem into an equivalent integral equation and then reformulates the problem as a fixed-point problem in a certain functional space of an operator and then obtains differentiability of the fixed point with respect to parameters. We will follow the same approach. In order to achieve the second-order differentiability with respect to parameters, we use the state space $W^{2,\infty}$ equipped with norms $|\cdot|_{W^{2,p}}$ and $|\cdot|_{W^{1,p}}$ (whose precise definitions will be given in the coming section). Our presentation is parallel to that of Hartung and Turi [19], but we should quickly emphasize that in order to deal with the second-order differentiability with respect to parameters, quasi-Banach spaces (see Remark 2.1), which are the state spaces used in the study in Refs. [15,16], are no longer sufficient and appropriate. Instead, we introduce the notion of locally complete triple-normed linear spaces (Definition 2.1). More specifically, we define a new topology on the space $W^{2,\infty}$ and then extend the uniform contraction principle from quasi-Banach spaces to locally complete triple-normed linear spaces. As shown in Section 5, we can obtain the second-order differentiability with respect to parameters in these locally complete triple-normed linear spaces.

The rest of this paper is organized as follows. In Section 2, we first define a locally complete triple-normed linear space as well as the differentiability on such a space, and we then obtain the uniform contraction principle in such spaces. Section 3 gives some preliminary results that relate different norms. Then, in Section 4, we first transform system (1.1) into an equivalent integral form and then give some sufficient conditions guaranteeing the well-posedness of system (1.1). We then verify the conditions required in the uniform contraction principle for locally complete triple-normed linear spaces and obtain the second-order differentiability of solutions with respect to parameters in the $W^{1,p}_{\alpha}$ norm in Section 5.

2 Uniform contraction principles

To introduce the main tool of this paper, i.e., uniform contraction principle, we first present some notations.

Let $(X, |\cdot|)$ be a normed linear space. For $x_0 \in X$ and $R > 0$, $\mathscr{G}_{X,|\cdot|}(x_0; R)$ denotes the open ball in X with center x_0 and radius R, and $\overline{\mathscr{G}}_{X,|\cdot|}(x_0; R)$

denotes the corresponding closed ball. If the center is the origin, the corresponding open ball and closed ball are abbreviated as $\mathscr{G}_{X,|\cdot|}(R)$ and $\overline{\mathscr{G}}_{X,|\cdot|}(R)$, respectively. Moreover, if X is endowed with the norm $|\cdot|$ only, then we use $\mathscr{G}_X(R)$ and $\overline{\mathscr{G}}_X(R)$ to represent $\mathscr{G}_{X,|\cdot|}(R)$ and $\overline{\mathscr{G}}_{X,|\cdot|}(R)$, respectively. On the other hand, if there is no confusion about the involved space X, especially when the space X is endowed with different norms, $\mathscr{G}_{|\cdot|}(R)$ and $\overline{\mathscr{G}}_{|\cdot|}(R)$ stand for $\mathscr{G}_{X,|\cdot|}(R)$ and $\overline{\mathscr{G}}_{X,|\cdot|}(R)$, respectively.

In this paper, we use $\mathscr{L}(X,Y)$ to denote the set of all bounded linear operators $S\colon X\to Y$, where X, Y are normed linear spaces.

Definition 2.1 Let X be a linear space endowed with three norms $|\cdot|$, $|\cdot|_M$ and $|\cdot|_N$. We say that $(X,|\cdot|)$ is a *locally complete triple-normed linear space* with respect to the norm $|\cdot|_M$ and the norm $|\cdot|_N$ if every intersection of the closed balls $(\overline{\mathscr{G}}_{|\cdot|_M}(R)\cap\overline{\mathscr{G}}_{|\cdot|_N}(R_0),|\cdot|)$ is a complete space, i.e., for every $R>0$ and $R_0>0$, the set $\{x\in X\colon |x|_M\leqslant R,\, |x|_N\leqslant R_0\}$ is complete in the $|\cdot|$ norm.

In order to avoid confusion between these three norms, we will emphasize the norm $|\cdot|_M$ or $|\cdot|_N$ when referring to it.

Remark 2.1 In Ref. [15], Hale and Ladeira used the terminology *quasi-Banach* space to denote the space $(X,|\cdot|)$ for which every closed ball $\mathscr{G}_{X,|\cdot|_M}(R)$, $R>0$, is complete in the $|\cdot|$ norm. However, this terminology was also used for a complete linear space equipped with a quasi-norm (see Ref. [21]). To avoid this confusion, we refer the spaces used in Ref. [15] as locally complete double-normed linear spaces in the sequel.

Since every locally complete triple-normed linear space is equipped with three norms, we need to specify the norm in the definition of differentiability of a map defined over a subset of this space (as well as the continuity of derivatives).

Definition 2.2 Let $(X,|\cdot|_X)$ be a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$, and $(Y,|\cdot|_Y)$ be a normed linear space. $U=U_M\cap U_N$, where U_M and U_N are $|\cdot|_M$- and $|\cdot|_N$-open subsets of X, respectively. A function $F\colon (U\subset X)\to Y$ is said to be *differentiable in the norm* $|\cdot|_N$ if, for any $x\in U$, there exists $A(x)\in\mathscr{L}(X,Y)$ such that

$$\lim_{\substack{|h|_X\to 0\\ x+h\in U}}\frac{|F(x+h)-F(x)-A(x)h|_Y}{|h|_N}=0. \tag{2.1}$$

The map $A(x)$ is uniquely determined and is called the (Fréchet) derivative of F at x, denoted by $F'(x)$. Higher order Fréchet derivatives can be defined recursively.

Definition 2.3 Let $(X,|\cdot|_X)$ and $(Y,|\cdot|_Y)$ be normed linear spaces. Suppose $U\subset X$. The operator function $A\colon U\ni x\mapsto A(x)\in\mathscr{L}(X,Y)$ is said to be *continuous at x with respect to the norm* $|\cdot|_X$ if

$$|A(x+\Delta x)-A(x)|_{\mathscr{L}(X,Y)}\to 0\quad(|\Delta x|_X\to 0).$$

Definition 2.4 Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be normed linear spaces. Suppose $U \subset X$. The operator function $A\colon U \ni x \mapsto A(x) \in \mathscr{L}(X,Y)$ is said to be *pointwise continuous at x with respect to the norm* $|\cdot|_X$ if for any $h \in X$, $|(A(x+\Delta x) - A(x))h|_Y \to 0$ as $|\Delta x|_X \to 0$.

Remark 2.2 In this paper, we mainly deal with operators defined in the following way:

$$S\colon (X, |\cdot|_X) \times (Y, |\cdot|_Y) \to (X, |\cdot|_X),$$

where $(X, |\cdot|_X)$ is a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$, and $(Y, |\cdot|_Y)$ is a normed linear space. We make the following convention for the operators involved with a locally complete triple-normed linear space.

(a) We say that $S(x,y)$ is C_p^k $(k \geqslant 0)$ in the norm $|\cdot|_N$ if $S(x,y)$ has pointwise continuous Fréchet derivatives up to the k^{th} order in some neighborhood of (x,y) in $(X, |\cdot|_X) \times (Y, |\cdot|_Y)$. We strictly distinguish the continuity and pointwise continuity of the Fréchet derivatives throughout this paper.

(b) If there is no specification, the convergence topology of the locally complete triple-normed linear space $(X, |\cdot|_X)$ is generated by the norm $|\cdot|_X$ throughout this paper.

(c) We frequently use the phrase "in the norm $|\cdot|$" to refer that the topology of the range space of an operator is generated by the norm $|\cdot|$.

For the second-order differentiability of solutions with respect to parameters, we need to extend the Uniform Contraction Principle to locally complete triple-normed linear spaces. To achieve this, we first generalize some technical lemmas of Hale and Ladeira [15] to locally complete triple-normed linear spaces. For the sake of completeness and the convenience of readers, we present some details, though in many cases, we shall just indicate the modifications from those in Ref. [15]. If $(X, |\cdot|)$ is a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$, we define

$$\begin{aligned}\widetilde{L}(X) := \{T\colon X \to X \mid\ & T \text{ is linear, and there is a constant } K > 0\\ & \text{such that } |Tx| \leqslant K|x|,\ |Tx|_M \leqslant K|x|_M,\\ & \text{and } |Tx|_N \leqslant K|x|_N \text{ for } x \in X\}.\end{aligned}$$

As usual, we use the same notations for the induced operator norms on $\widetilde{L}(X)$.

Lemma 2.1 *If $(X, |\cdot|)$ is a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$, then $(\widetilde{L}(X), |\cdot|)$ is also a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$.*

Proof For $R > 0$ and $R_0 > 0$, let $\{T_n\} \subseteq \overline{\mathscr{G}}_{|\cdot|_M}(R) \cap \overline{\mathscr{G}}_{|\cdot|_N}(R_0)$ be a Cauchy sequence in the $|\cdot|$ norm. Given $x \in X$, it is easy to see that $\{T_n x\} \subseteq \overline{\mathscr{G}}_{|\cdot|_M}(R|x|_M) \cap \overline{\mathscr{G}}_{|\cdot|_N}(R_0|x|_N)$ is a Cauchy sequence in the $|\cdot|$-norm. Since

X is a locally complete triple-normed linear space, there exists a unique $\widetilde{x} \in \overline{\mathscr{G}}_{|\cdot|_M}(R|x|_M) \cap \overline{\mathscr{G}}_{|\cdot|_N}(R_0|x|_N)$ such that $\lim_{n\to\infty} |T_n x - \widetilde{x}| = 0$. Then, we define $T\colon X \to X$ by $Tx = \widetilde{x}$ for $x \in X$. Obviously, T is linear and $T \in \overline{\mathscr{G}}_{|\cdot|_M}(R) \cap \overline{\mathscr{G}}_{|\cdot|_N}(R_0)$. We are left to show $\lim_{n\to\infty} |T_n - T| = 0$. For any $\varepsilon > 0$, there exists an n_0 such that $|T_m - T_n| \leqslant \varepsilon$ for m, $n \geqslant n_0$. Then, for any $x \in X$, we have $|T_m x - T_n x| \leqslant \varepsilon |x|$ for $m, n \geqslant n_0$. Letting $m \to \infty$ gives us $|T_n x - Tx| \leqslant \varepsilon |x|$ for $n \geqslant n_0$, which implies that $|T_n - T| \leqslant \varepsilon$ for $n \geqslant n_0$. Since ε is arbitrary, we have $\lim_{n\to\infty} |T_n - T| = 0$. This completes the proof. □

The next lemma gives a Neumann series representation for inverses in locally complete triple-normed linear spaces.

Lemma 2.2 *Let $(X, |\cdot|)$ be a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$. Suppose $G \in \widetilde{L}(X)$ and there exists a $\theta \in [0, 1)$ such that $|G| \leqslant \theta$, $|G|_M \leqslant \theta$, and $|G|_N \leqslant \theta$. Then the following three statements hold.*

(i) *$I - G$ has an inverse which belongs to $\widetilde{L}(X)$ and satisfies*

$$|(I-G)^{-1}| \leqslant (1-\theta)^{-1}, \quad |(I-G)^{-1}|_M \leqslant (1-\theta)^{-1}, \quad |(I-G)^{-1}|_N \leqslant (1-\theta)^{-1}.$$

(ii) *The series $\sum_{n=0}^{\infty} G^n$ converges to $(I - G)^{-1}$ in all norms $|\cdot|$, $|\cdot|_M$ and $|\cdot|_N$.*

(iii)

$$|(I-G)^{-1} - I - G| = O(|G|^2) \quad (|G| \to 0),$$
$$|(I-G)^{-1} - I - G|_M = O(|G|_M^2) \quad (|G|_M \to 0),$$
$$|(I-G)^{-1} - I - G|_N = O(|G|_N^2) \quad (|G|_N \to 0).$$

Proof Let $S_n = \sum_{i=0}^{n} G^i$. It is easy to see that $\{S_n\} \subseteq \overline{\mathscr{G}}_{|\cdot|_M}((1-\theta)^{-1}) \cap \overline{\mathscr{G}}_{|\cdot|_N}((1-\theta)^{-1})$ is a Cauchy sequence in the $|\cdot|$ norm. By Lemma 2.1, there exists $S \in \widetilde{L}(X) \cap (\overline{\mathscr{G}}_{|\cdot|_M}((1-\theta)^{-1}) \cap \overline{\mathscr{G}}_{|\cdot|_N}((1-\theta)^{-1}))$ such that

$$\lim_{n\to\infty} |S_n - S| = 0, \tag{2.2}$$

that is, $\sum_{n=0}^{\infty} G^n = S$ in the norm $|\cdot|$. Taking limit in

$$(I-G)S_n = I - G^{n+1} = S_n(I-G),$$

we have $(I-G)S = I = S(I-G)$ since $\lim_{n\to\infty} |G^n| = 0$. This means that $(I-G)^{-1}$ exists and $(I-G)^{-1} = \sum_{n=0}^{\infty} G^n$ in the norm $|\cdot|$. It follows that

$$|(I-G)^{-1}| = \left|\sum_{n=0}^{\infty} G^n\right| \leqslant \sum_{n=0}^{\infty} |G|^n \leqslant \sum_{n=0}^{\infty} \theta^n = (1-\theta)^{-1}.$$

On the other hand, for any $x \in X$, we have

$$|(I-G)x|_M \geqslant |x|_M - |Gx|_M \geqslant (1-\theta)|x|_M,$$

and, similarly,

$$|(I-G)x|_N \geqslant (1-\theta)|x|_N.$$

Therefore, $|(I-G)^{-1}|_M \leqslant (1-\theta)^{-1}$ and $|(I-G)^{-1}|_N \leqslant (1-\theta)^{-1}$. This proves (i).

Next, we prove (ii). Note that

$$\lim_{n\to\infty} |I-(I-G)S_n|_M = \lim_{n\to\infty} |I-S_n(I-G)|_M = \lim_{n\to\infty} |G^{n+1}|_M = 0.$$

This, combined with the existence and boundedness of $(I-G)^{-1}$ in norm $|\cdot|_M$, gives us $\sum_{n=0}^{\infty} G^n = (I-G)^{-1}$ in the norm $|\cdot|_M$. Similar argument produces $\sum_{n=0}^{\infty} G^n = (I-G)^{-1}$ in the norm $|\cdot|_N$. This completes the proof of (ii).

Observe that

$$(I-G)^{-1} - I - G = G^2 \sum_{n=0}^{\infty} G^n = G^2(I-G)^{-1}.$$

Then part (iii) follows directly from this observation, and the proof is complete. □

Lemma 2.3 *Suppose that $(X, |\cdot|)$ is a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$, and T, $T^{-1} \in \widetilde{L}(X)$. Then, for any $H \in \widetilde{L}(X)$ such that*

$$|H| < |T^{-1}|^{-1}, \quad |H|_M < (|T^{-1}|_M)^{-1}, \quad |H|_N < (|T^{-1}|_N)^{-1},$$

$T+H$ is invertible. Moreover, one has

$$(T+H)^{-1} = \sum_{n=0}^{\infty} (-T^{-1}H)^n T^{-1}$$

in all norms, and

$$\begin{aligned} |(T+H)^{-1} - T^{-1} + T^{-1}HT^{-1}| &= O(|H|^2) \quad (|H| \to 0), \\ |(T+H)^{-1} - T^{-1} + T^{-1}HT^{-1}|_M &= O(|H|_M^2) \quad (|H|_M \to 0), \\ |(T+H)^{-1} - T^{-1} + T^{-1}HT^{-1}|_N &= O(|H|_N^2) \quad (|H|_N \to 0). \end{aligned}$$

Proof Note that $T+H = T(I+T^{-1}H)$. Moreover, it is easy to see that

$$|T^{-1}H| < 1, \quad |T^{-1}H|_M < 1, \quad |T^{-1}H|_N < 1.$$

Then, by Lemma 2.2, $I+T^{-1}H$ is invertible and

$$(I+T^{-1}H)^{-1} = \sum_{n=0}^{\infty} (-T^{-1}H)^n.$$

Hence, $T+H$ is invertible and

$$(T+H)^{-1} = (I+T^{-1}H)^{-1}T^{-1} = \sum_{n=0}^{\infty}(-T^{-1}H)^n T^{-1}.$$

The remaining conclusions follow directly from Lemma 2.2 (iii). □

Lemma 2.4 *Let $(X, |\cdot|)$ be a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$. Let U be a metric space. Let the operator function $G\colon U \to \widetilde{L}(X)$ satisfy $|G(y)| \leqslant \theta$, $|G(y)|_M \leqslant \theta$ and $|G(y)|_N \leqslant \theta$ for all $y \in U$ and some $0 \leqslant \theta < 1$. If $G(y)$ is (pointwise) continuous, then the map $\Phi\colon U \ni y \mapsto (I - G(y))^{-1} \in \widetilde{L}(X)$ is (pointwise) continuous.*

Proof By Lemma 2.2 (ii), we have $(I - G(y))^{-1} = \sum_{n=0}^{\infty}(G(y))^n$ in all three norms. Now, the conclusion follows from the fact that the convergence is uniform. □

Lemma 2.5 *Let $(X, |\cdot|)$ be a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$. Let Y be a normed space. Suppose that $U \subset Y$ is open. Let $G\colon U \to \widetilde{L}(X)$ satisfy $|G(y)| \leqslant \theta$, $|G(y)|_M \leqslant \theta$ and $|G(y)|_N \leqslant \theta$ for all $y \in U$ and some $0 \leqslant \theta < 1$. If $G\colon U \to \widetilde{L}(X)$ is C_p^k, then the map $\Phi\colon U \ni y \mapsto (I - G(y))^{-1} \in \widetilde{L}(X)$ is also C_p^k.*

Proof The proof is essentially the same as that of Corollary 2.5 of Ref. [15] except that, here, the norm in which the maps G and Φ are C_p^k is not specified, and here, the differentiability is in a weaker sense. We can choose any induced norm to finish the proof without loss of generality. □

Now, we are ready to present an extended version of the uniform contraction principle for locally complete triple-normed linear spaces.

Theorem 2.1 *Suppose that $(X, |\cdot|)$ is a locally complete triple-normed linear space with respect to the norms $|\cdot|_M$ and $|\cdot|_N$, where the norm $|\cdot|$ is stronger than the norm $|\cdot|_N$. $(Y, |\cdot|_Y)$ is a normed linear space. Let $U = U_M \cap U_N$, where U_M and U_N are $|\cdot|_M$- and $|\cdot|_N$-open subsets of X, respectively. Let W be a $|\cdot|$-closed subset of U and V be an open subset of Y. Assume that the function $S\colon U \times V \to U$ satisfies the following conditions.*

(i) *$S(W \times V) \subset W$.*

(ii) *S is a uniform $|\cdot|$, $|\cdot|_M$ and $|\cdot|_N$ contraction on $W \times V$, i.e., there exists $0 \leqslant c < 1$ such that, for x, $\overline{x} \in W$ and $y \in V$,*

$$|S(x,y) - S(\overline{x},y)| \leqslant c|x - \overline{x}|,$$
$$|S(x,y) - S(\overline{x},y)|_M \leqslant c|x - \overline{x}|_M,$$
$$|S(x,y) - S(\overline{x},y)|_N \leqslant c|x - \overline{x}|_N.$$

(iii) *For each $\rho > 0$, there exists $U_\rho \subset U$ such that*

$$S((U_\rho \cap W) \times (\mathscr{G}_Y(\rho) \cap V)) \subset (U_\rho \cap W).$$

(iv) *For all* $x \in W$, *the function* $S(x, \cdot)\colon (V \subset Y) \to X$ *is continuous in all the norms on* X.

Then for each $y \in V$, *there exists a unique fixed point* $g(y)$ *of* $S(\cdot, y)$ *in* W, *which depends continuously on* y *in all the norms on* X. *The following statements are true.*

(v) *If* $S\colon (X, |\cdot|) \times (Y, |\cdot|_Y) \to (X, |\cdot|)$ *is* C_p^k $(k \geqslant 1,\ k \in \mathbb{N})$, *and* $D_x S(x, y) \in \widetilde{L}(X)$, $|D_x S(x, y)| \leqslant c$, $|D_x S(x, y)|_M \leqslant c$, $|D_x S(x, y)|_N \leqslant c$, *then the fixed point* $g\colon (V \subset Y) \to (X, |\cdot|)$ *is* C_p^k.

(vi) *Let* $(\overline{X}, |\cdot|_N)$ *be a locally complete double-normed linear space with respect to the norm* $|\cdot|_{\overline{X}}$ *such that* $X \subseteq \overline{X}$. *The fixed point* $g\colon (V \subset Y) \to (\overline{X}, |\cdot|_N)$ *is* C_p^k $(k \geqslant 1)$ *if the following conditions are satisfied*:

(a) $S\colon (X, |\cdot|_N) \times (Y, |\cdot|_Y) \to (\overline{X}, |\cdot|_N)$ *is* C_p^k *and the partial derivatives* $D_y^{(i)} D_x^{(j-i)} S(x, y)$ $(i, j, k \in \mathbb{N},\ 0 \leqslant i < j \leqslant k)$ *are bounded* j*-linear operators in* $\mathscr{L}(\underbrace{\overline{X} \times \overline{X} \times \cdots \times \overline{X}}_{j-i} \times \underbrace{Y \times Y \times \cdots \times Y}_{i}, \overline{X})$ *for any* $(x, y) \in W \times V$.

(b) *There exists* $0 \leqslant \theta < 1$ *such that for any* $(x, y) \in W \times V$, $D_x S(x, y) \in \widetilde{L}(\overline{X})$, $|D_x S(x, y)|_{\overline{X}} \leqslant \theta$, $|D_x S(x, y)|_N \leqslant \theta$.

Proof The existence of $g(y)$ is shown as follows. Pick an $x \in W$. Define $x_1 = x$ and $x_{n+1} = S(x_n, y)$ for $n \geqslant 1$. Using assumptions (i) and (ii), one can easily see that $\{x_n\} \subset W \cap (\mathscr{G}_{X, |\cdot|_M}(R) \cap \mathscr{G}_{X, |\cdot|_N}(R))$ is a Cauchy sequence in the $|\cdot|$ norm, where

$$R = \max\left\{\frac{1}{1-c}\,|x_2 - x_1|_M + |x_1|_M,\ \frac{1}{1-c}\,|x_2 - x_1|_N + |x_1|_N\right\}.$$

Then there exists an $\widetilde{x} \in X$ such that $\lim_{n\to\infty} x_n = \widetilde{x}$ in the $|\cdot|$ norm. Since W is $|\cdot|$-closed, we also have $\widetilde{x} \in W$. By continuity, we know that $\widetilde{x}$ is a fixed point of $S(\cdot, y)$. The uniqueness of the fixed point of $S(\cdot, y)$ in W follows easily from assumption (ii).

Next, we show that $g(y)$ is continuous at y in all norms on X. It follows from the triangle inequality and assumption (ii) that

$$\begin{aligned}|g(y+h) - g(y)| &= |S(g(y+h), y+h) - S(g(y), y)| \\ &\leqslant c|g(y+h) - g(y)| + |S(g(y), y+h) - S(g(y), y)|.\end{aligned}$$

Then

$$|g(y+h) - g(y)| \leqslant (1-c)^{-1} |S(g(y), y+h) - S(g(y), y)|,$$

and hence $g(y)$ is continuous at y in the norm $|\cdot|$. Similarly, one can show that $g(y)$ is also continuous at y in the norms $|\cdot|_M$ and $|\cdot|_N$.

Before proving (v), we first show that $(I - D_x S(x, y))^{-1}$ exists in $\widetilde{L}(X)$ and is pointwise continuous at (x, y) in the norm $|\cdot|$. Indeed, since $D_x S(x, y) \in \widetilde{L}(X)$, $|D_x S(x, y)| \leqslant c$, $|D_x S(x, y)|_M \leqslant c$, $|D_x S(x, y)|_N \leqslant c$, it follows from Lemma 2.2 that $(I - D_x S(x, y))^{-1} \in \widetilde{L}(X)$ for $(x, y) \in W \times V$. Since

S is C_p^1, $D_xS(x,y)$ is pointwise continuous in the norm $|\cdot|$. Therefore, $(I-D_xS(x,y))^{-1}$ is also pointwise continuous in the norm $|\cdot|$.

Noting that we need only pointwise continuity here, the remaining proof for (v) will be essentially the same as that in Ref. [15] since only the norm $|\cdot|$ and the norm on V are involved. We omit the details here.

Finally, we turn to (vi) and prove that $g(y)$ is a C_p^k map in the norm $|\cdot|_N$. First, suppose $k=1$. By the assumption (vi)(b) and Lemma 2.2, we have $D_xS(x,y)\in\widetilde{L}(\overline{X})$ and $(I-D_xS(x,y))^{-1}\in\widetilde{L}(\overline{X})$ for $(x,y)\in W\times V$. Since S is C_p^1, $D_xS(x,y)$ is pointwise continuous in the norm $|\cdot|_N$. It follows that $(I-D_xS(x,y))^{-1}$ is also pointwise continuous.

Now, we are ready to prove that $g(y)$ is a C_p^1 map in the norm $|\cdot|_N$. Let

$$F(y)\colon\ V\ni y\mapsto (I-D_xS(g(y),y))^{-1}D_yS(g(y),y)\in\mathscr{L}(Y,\overline{X}). \tag{2.3}$$

We shall show that $|g(y+h)-g(y)-F(y)h|_N=o(|h|_Y)$ as $|h|_Y\to 0$. Denote $\gamma(h):=g(y+h)-g(y)$. Then

$$\begin{aligned}\gamma(h)&=S(g(y)+\gamma(h),y+h)-S(g(y),y)\\&=D_xS(g(y),y)\gamma(h)+D_yS(g(y),y)h+\Delta(h),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\Delta(h)=&\,S(g(y)+\gamma(h),y+h)-S(g(y),y)\\&-D_xS(g(y),y)\gamma(h)-D_yS(g(y),y)h.\end{aligned} \tag{2.5}$$

Since S is C_p^1 in the norm $|\cdot|_N$ and g is continuous in all the norms, it follows from Definition 2.2 and (2.5) that, for any $\varepsilon_2\in(0,1-\theta)$, there exists $\delta_2>0$ such that $|\Delta(h)|_N\leqslant\varepsilon_2(|\gamma(h)|_N+|h|_Y)$ if $|h|_Y<\delta_2$. From (2.4), we have

$$|\gamma(h)|_N\leqslant\frac{1}{1-\theta-\varepsilon_2}\,(|D_yS(g(y),y))|_Y+\varepsilon_2)|h|_Y=:k|h|_Y\quad\text{if } |h|_Y<\delta_2.$$

Therefore, $|\Delta(h)|_N\leqslant\varepsilon_2(1+k)|h|_Y$ if $|h|_Y<\delta_2$. It also follows from (2.4) that

$$(I-D_xS(g(y),y))(\gamma(h)-F(y)h)=\Delta(h).$$

Then

$$|\gamma(h)-F(y)h|_N<\frac{\varepsilon_2(1+k)}{1-\theta}\,|h|_Y\quad\text{if } |h|_Y<\delta_2.$$

This implies that $g(y)$ is differentiable and $Dg(y)h=F(y)h$. From the expression for $F(y)$ in (2.3), it is clear that Dg is pointwise continuous, i.e., g is C_p^1 in the norm $|\cdot|_N$.

Now, suppose that the result holds for $k-1$ $(k>1)$ and S is C_p^k in the norm $|\cdot|_N$. Then $g\colon\ (V\subset Y)\to(\overline{X},|\cdot|_N)$ is C_p^{k-1}.

Since the partial derivatives $D_y^{(i)}D_x^{(j-i)}S(x,y)$ $(0\leqslant i<j\leqslant k)$ are bounded j-linear operators in $\mathscr{L}(\underbrace{\overline{X}\times\overline{X}\times\cdots\times\overline{X}}_{j-i}\times\underbrace{Y\times Y\times\cdots\times Y}_{i},\overline{X})$ for any $(x,y)\in\overline{D}\times V$, one has $D_xS(g(y),y)$ is C_p^{k-1} with respect to y in the

norm $|\cdot|_N$. It follows from Lemma 2.5 that $(I - D_xS(g(y), y))^{-1}$ is C_p^{k-1}, and hence g is C_p^k in the norm $|\cdot|_N$. This completes the proof. □

Remark 2.3 If the $|\cdot|$ norm is identical to the $|\cdot|_N$ norm, then the locally complete triple-normed linear space $(X, |\cdot|)$ with respect to the norms $|\cdot|_M$ and $|\cdot|_N$ reduces to a locally complete double-normed linear space. In this case, Theorem 2.1 reduces to the uniform contraction principle for locally complete double-normed linear spaces with U being an $|\cdot|_M$-open subset of X.

Remark 2.4 In Theorem 2.1, we only assume that the Fréchet derivatives of $S(x, y)$ are pointwise continuous, and hence, the derivatives of the fixed point $g(y)$ are also pointwise continuous over its domain. In fact, following the proof of Corollary 2.5 of Ref. [15] will yield that Lemma 2.5 still holds if the differentiability is continuous Fréchet differentiability. Therefore, if the Fréchet derivatives of $S(x, y)$ are continuous as operator functions, then the derivatives of the fixed point $g(y)$ are uniformly continuous over its domain.

3 Preliminaries

In this section, we derive estimates of solutions of system (1.1) in different norms.

The following result can be easily deduced by elementary arguments and applications of the theory of ordinary differential equations. Similar results for the special case (1.2) were established by Arino et al. [5].

Lemma 3.1 *Assume that there exists a constant $L > 0$ such that $g(x, \tau, \sigma) < L/(L+1)$ for any x, τ, and σ. Let $z(t)$ be a continuous function defined on an interval $[t_0, \infty)$ and $\tau(t)$ be a function satisfying the second equation of system (1.1) with $x(t) = z(t)$ for $t \geqslant t_0$. Then the function $(t - \tau(t))$ is increasing on $[t_0, \infty)$. Moreover, assume that $g(x, \tau_1, \sigma) > 0$ and $g(x, \tau_2, \sigma) < 0$ for all x and σ. Then we have $\tau(t) \in [\tau_1, \tau_2]$ for all $t \geqslant t_0$, provided that $\tau(t_0) \in [\tau_1, \tau_2]$.*

The conclusion of Lemma 3.1 that $t - \tau(t)$ is increasing is a very natural restriction in the context of population dynamics, where it expresses the fact that no overlapping between generations arises from overcrowding. See Ref. [5].

As mentioned earlier, the main tool of this paper is the extended version of uniform contraction principle for locally complete triple-normed linear spaces, Theorem 2.1, combined with various inequalities relating different normed spaces. For the convenience of the readers, some of these results from Sections 2 and 3 of Hartung and Turi [19] are summarized below.

For differentiability of solutions with respect to parameters, we will work on the state space $W^{k,\infty}$, $k \in \mathbb{Z}^+$, equipped with different norms. For $1 \leqslant p \leqslant \infty$, $W_\alpha^{k,p}([-r, \alpha]; \mathbb{R}^n)$ denotes the space of all functions $\psi\colon [-r, \alpha] \to \mathbb{R}^n$ with absolutely continuous derivatives up to $(k-1)^{\text{st}}$ order and with finite norm defined by

$$|\psi|_{W_\alpha^{k,p}([-r,\alpha];\mathbb{R}^n)}=\begin{cases}\left(\int_{-r}^{\alpha}\left(\sum_{i=0}^{k}|\psi^{(i)}(t)|^p\right)\mathrm{d}t\right)^{1/p}, & 1\leqslant p<\infty,\\ \max\left\{\operatorname*{ess\,sup}_{-r\leqslant t\leqslant\alpha}|\psi^{(i)}(t)|,\ i=0,1,\ldots,k\right\}, & p=\infty.\end{cases}$$

The subspace of $W_\alpha^{k,\infty}$ defined by

$$\mathbb{Y}_\alpha^{k,p}:=\{y\in W_\alpha^{k,\infty}:\ y(t)=0 \text{ for } t\in[-r,0]\},\quad 1\leqslant p\leqslant\infty,$$

with the norm

$$|y|_{\mathbb{Y}_\alpha^{k,p}}=\begin{cases}\left(\int_{0}^{\alpha}\left(\sum_{i=1}^{k}|y^{(i)}(t)|^p\right)\mathrm{d}t\right)^{1/p}, & 1\leqslant p<\infty,\\ \max\left\{\operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}|y^{(i)}(t)|,\ i=1,2,\ldots,k\right\}, & p=\infty,\end{cases}$$

will be needed in the study of an integral operator, which will be defined soon in Section 4.

Since most of the functions we will encounter are defined on $[-r,0]$ or $[-r,\alpha]$, to keep the notations simple, we introduce the following abbreviations. For $1\leqslant p\leqslant\infty$,

$$W_\alpha^{k,p}:=W_\alpha^{k,p}([-r,\alpha];\mathbb{R}^n),\quad W^{k,p}:=W_0^{k,p};$$

$$L_\alpha^p:=L^p([-r,\alpha];\mathbb{R}^n),\quad L_{0,\alpha}^p:=L^p([0,\alpha];\mathbb{R}^n).$$

Lemma 3.2 (Lemma 3.6 in Ref. [19]) *Let $y\in\mathbb{Y}_\alpha^{1,p}$, $1\leqslant p\leqslant\infty$, and q be the conjugate to p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Then*

(i) $|y(t)|\leqslant\alpha^{1/q}|y|_{\mathbb{Y}_\alpha^{1,p}}$ *for $t\in[-r,\alpha]$ and $1\leqslant p\leqslant\infty$;*

(ii) $|y|_{\mathbb{Y}_\alpha^{1,p}}\leqslant\alpha^{1/p}|y|_{\mathbb{Y}_\alpha^{1,\infty}}$ *for $1\leqslant p<\infty$;*

(iii) $|y|_{\mathbb{Y}_\alpha^{1,p}}\leqslant|y|_{W_\alpha^{1,p}}\leqslant(a^p+1)^{1/p}|y|_{\mathbb{Y}_\alpha^{1,p}}$ *for $1\leqslant p<\infty$;*

(iv) $|y|_{\mathbb{Y}_\alpha^{1,\infty}}\leqslant|y|_{W_\alpha^{1,\infty}}\leqslant\max\{\alpha,1\}|y|_{\mathbb{Y}_\alpha^{1,\infty}}$;

(v) $|y|_{L_\alpha^p}\leqslant\alpha|y|_{\mathbb{Y}_\alpha^{1,p}}$ *for $1\leqslant p<\infty$.*

Lemma 3.3 *Let $y\in\mathbb{Y}_\alpha^{2,p}$, $1\leqslant p\leqslant\infty$, and q be the conjugate to p. Then*

(i) $|y(t)|\leqslant\alpha^{1/q}|y|_{\mathbb{Y}_\alpha^{2,p}}$ *for $t\in[-r,\alpha]$ and $1\leqslant p\leqslant\infty$;*

(ii) $|y|_{\mathbb{Y}_\alpha^{2,p}}\leqslant(2\alpha)^{1/p}|y|_{\mathbb{Y}_\alpha^{2,\infty}}$ *for $1\leqslant p<\infty$;*

(iii) $|y|_{\mathbb{Y}_\alpha^{2,p}}\leqslant|y|_{W_\alpha^{2,p}}\leqslant(a^p+1)^{1/p}|y|_{\mathbb{Y}_\alpha^{2,p}}$ *for $1\leqslant p<\infty$;*

(iv) $|y|_{\mathbb{Y}_\alpha^{2,\infty}}\leqslant|y|_{W_\alpha^{2,\infty}}\leqslant\max\{\alpha,1\}|y|_{\mathbb{Y}_\alpha^{2,\infty}}$;

(v) $|y|_{L_\alpha^p}\leqslant\alpha|y|_{\mathbb{Y}_\alpha^{2,p}}$ *for $1\leqslant p<\infty$.*

Proof Note that, for $1\leqslant p\leqslant\infty$, $\mathbb{Y}_\alpha^{2,p}\subseteq\mathbb{Y}_\alpha^{1,p}$ since $W_\alpha^{2,\infty}\subseteq W_\alpha^{1,\infty}$. By definition, $|y|_{\mathbb{Y}_\alpha^{1,p}}\leqslant|y|_{\mathbb{Y}_\alpha^{2,p}}$ for $y\in\mathbb{Y}_\alpha^{2,p}$ and $1\leqslant p\leqslant\infty$. Then (i) follows immediately from Lemma 3.2 (i).

By definition, we have

$$|y|^p_{\mathbb{Y}^{2,p}_\alpha} = \int_0^\alpha (|\dot{y}(t)|^p + |\ddot{y}(t)|^p)\mathrm{d}t \leqslant \int_0^\alpha 2|y|^p_{\mathbb{Y}^{2,\infty}_\alpha}\mathrm{d}t = 2\alpha|y|^p_{\mathbb{Y}^{2,\infty}_\alpha}.$$

This proves (ii). On the other hand,

$$\begin{aligned}
|y|_{\mathbb{Y}^{2,p}_\alpha} &\leqslant \left(\int_0^\alpha (|y(t)|^p + |\dot{y}(t)|^p + |\ddot{y}(t)|^p)\mathrm{d}t\right)^{1/p} \\
&= \left(\int_{-r}^\alpha (|y(t)|^p + |\dot{y}(t)|^p + |\ddot{y}(t)|^p)\mathrm{d}t\right)^{1/p} \\
&= |y|_{W^{2,\,p}_\alpha}.
\end{aligned}$$

Moreover, with the help of (i), it follows from

$$|y|_{W^{2,p}_\alpha} = \left(\int_0^\alpha |y(t)|^p\mathrm{d}t + \int_0^\alpha (|\dot{y}(t)|^p + |\ddot{y}(t)|^p)\mathrm{d}t\right)^{1/p}$$

that

$$|y|_{W^{2,p}_\alpha} \leqslant \left(\alpha^{\frac{p}{q}+1}|y|^p_{\mathbb{Y}^{2,p}_\alpha} + |y|^p_{\mathbb{Y}^{2,p}_\alpha}\right)^{1/p} = (\alpha^p+1)^{1/p}|y|_{\mathbb{Y}^{2,p}_\alpha}.$$

This proves (iii).

For (iv), by definition, we easily see that

$$\begin{aligned}
|y|_{\mathbb{Y}^{2,\infty}_\alpha} &= \max\Big\{\operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}|\dot{y}(t)|,\ \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}|\ddot{y}(t)|\Big\} \\
&\leqslant \max\Big\{\sup_{0\leqslant t\leqslant\alpha}|y(t)|,\ \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}|\dot{y}(t)|,\ \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}|\ddot{y}(t)|\Big\} \\
&= |y(t)|_{W^{2,\infty}_\alpha} \\
&\leqslant \max\Big\{\sup_{0\leqslant t\leqslant\alpha}|y(t)|,\ |y|_{\mathbb{Y}^{2,\infty}_\alpha}\Big\}.
\end{aligned}$$

This, combined with $\sup_{0\leqslant t\leqslant\alpha}|y(t)| \leqslant \alpha|y|_{\mathbb{Y}^{2,\infty}_\alpha}$ from (i), immediately produces (iv).

Item (v) can be proved similarly by using (i) as follows.

$$|y|^p_{L^p_\alpha} = \int_{-r}^\alpha |y(t)|^p\mathrm{d}t = \int_0^\alpha |y(t)|^p\mathrm{d}t \leqslant \int_0^\alpha \alpha^{p/q}|y|^p_{\mathbb{Y}^{2,p}_\alpha}\mathrm{d}s = \alpha^p|y|^p_{\mathbb{Y}^{2,p}_\alpha}.$$

This completes the proof. □

In the sequel, for the convenience of presentation, we introduce a notation. For $z\in\mathbb{Y}^{2,\infty}_\alpha$ and $\tau_0\in\mathbb{R}$, the function $z^{\tau_0}\colon [-r,\alpha]\to\mathbb{R}$ is defined as

$$z^{\tau_0}(t) = t - z(t) - \tau_0, \quad t\in[-r,\alpha].$$

Lemma 3.4 *Let $y,z\in\mathbb{Y}^{2,\infty}_\alpha$, $\tau_0\geqslant 0$ and there exist η_0^- and η_0^+ such that $\eta_0^-\leqslant\dot{z}\leqslant\eta_0^+<1$ and $\eta_0^-\alpha+\tau_0\geqslant 0$. Then, for $t\in[0,\,\alpha]$,*

$$(1-\eta_0^+)t-\tau_0 \leqslant z^{\tau_0}(t) \leqslant (1-\eta_0^-)t-\tau_0 \leqslant \alpha, \tag{3.1}$$

$$|y(z^{\tau_0}(t))| \leqslant |z^{\tau_0}(\alpha)|^{1/q} \cdot |y|_{\mathbb{Y}^{1,p}_\alpha} \leqslant \alpha^{1/q}|y|_{\mathbb{Y}^{1,p}_\alpha}, \tag{3.2}$$

$$|\dot{y}(z^{\tau_0}(t))| \leqslant |z^{\tau_0}(\alpha)|^{1/q} \cdot |y|_{\mathbb{Y}^{2,p}_\alpha} \leqslant \alpha^{1/q}|y|_{\mathbb{Y}^{2,p}_\alpha}, \tag{3.3}$$

where $1 \leqslant p \leqslant \infty$ *and* q *is the conjugate to* p.

Proof It follows from

$$\frac{\mathrm{d}z^{\tau_0}(t)}{\mathrm{d}t} = 1 - \dot{z}(t)$$

that

$$0 < 1 - \eta_0^+ \leqslant \frac{\mathrm{d}z^{\tau_0}(t)}{\mathrm{d}t} \leqslant 1 - \eta_0^-.$$

Note that $z^{\tau_0}(0) = -\tau_0$. Integrating the above inequalities from 0 to t immediately yields (3.1). It also follows that $z^{\tau_0}(t)$ is increasing on $[0,\alpha]$. To prove (3.2) and (3.3), first, it is easy to see from definitions that $\dot{y}(0) = 0$. Then, we distinct two cases. First, assume $z^{\tau_0}(t) \leqslant 0$. Then $y(z^{\tau_0}(t)) = \dot{y}(z^{\tau_0}(t)) = 0$, and (3.2) and (3.3) hold automatically. Second, assume $z^{\tau_0}(t) > 0$. Since $\eta_0^-\alpha + \tau_0 \geqslant 0$, one has $0 < z^{\tau_0}(t) \leqslant z^{\tau_0}(\alpha) \leqslant \alpha$ and

$$y(z^{\tau_0}(t)) = \int_0^{z^{\tau_0}(t)} \dot{y}(s)\mathrm{d}s, \quad \dot{y}(z^{\tau_0}(t)) = \int_0^{z^{\tau_0}(t)} \ddot{y}(s)\mathrm{d}s.$$

Then

$$|y(z^{\tau_0}(t))| \leqslant \int_0^{z^{\tau_0}(t)} |\dot{y}(s)|\mathrm{d}s, \quad |\dot{y}(z^{\tau_0}(t))| \leqslant \int_0^{z^{\tau_0}(t)} |\ddot{y}(s)|\mathrm{d}s.$$

It follows from Hölder's inequality that

$$|y(z^{\tau_0}(t))| \leqslant |z^{\tau_0}(t)|^{1/q}\, |y|_{\mathbb{Y}^{1,p}_\alpha} \leqslant |z^{\tau_0}(\alpha)|^{1/q}\, |y|_{\mathbb{Y}^{1,p}_\alpha} \leqslant \alpha^{1/q}|y|_{\mathbb{Y}^{1,p}_\alpha}$$

$$|\dot{y}(z^{\tau_0}(t))| \leqslant |z^{\tau_0}(t)|^{1/q}\, |y|_{\mathbb{Y}^{2,p}_\alpha} \leqslant |z^{\tau_0}(\alpha)|^{1/q}\, |y|_{\mathbb{Y}^{2,p}_\alpha} \leqslant \alpha^{1/q}|y|_{\mathbb{Y}^{2,p}_\alpha}.$$

This completes the proof. □

Lemma 3.5 *Let* $y, \overline{y} \in \mathbb{Y}^{2,\infty}_\alpha$, $\varphi, \overline{\varphi} \in W^{2,\infty}$, *and* $z, \overline{z} \in \mathbb{Y}^{2,\infty}_\alpha$. *Suppose that there exists* $\tau_0, \overline{\tau}_0 > 0$ *such that* $-r \leqslant z^{\tau_0}(t) \leqslant \alpha$ *and* $-r \leqslant \overline{z}^{\overline{\tau}_0}(t) \leqslant \alpha$ *for any* $t \in [-r,\alpha]$. *Then, for* $t \in [0,\alpha]$, $1 \leqslant p \leqslant \infty$,

$$|y(z^{\tau_0}(t)) - \overline{y}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |y|_{\mathbb{Y}^{1,\infty}_\alpha}(\alpha^{1/q}|z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha} + |\tau_0 - \overline{\tau}_0|) + \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha},$$

$$|\dot{y}(z^{\tau_0}(t)) - \dot{\overline{y}}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |y|_{\mathbb{Y}^{2,\infty}_\alpha}(\alpha^{1/q}|z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha} + |\tau_0 - \overline{\tau}_0|) + \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha},$$

$$|\widetilde{\varphi}(z^{\tau_0}(t)) - \widetilde{\overline{\varphi}}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |\varphi|_{W^{1,\infty}}(\alpha^{1/q}|z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha} + |\tau_0 - \overline{\tau}_0|) + |\varphi-\overline{\varphi}|_{W^{1,\infty}},$$

$$|\dot{\widetilde{\varphi}}(z^{\tau_0}(t)) - \dot{\widetilde{\overline{\varphi}}}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |\varphi|_{W^{2,\infty}}(\alpha^{1/q}|z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha} + |\tau_0 - \overline{\tau}_0|) + |\varphi-\overline{\varphi}|_{W^{2,\infty}}.$$

Proof We only prove the first inequality since the others can be proved in the same manner. Observe that

$$|y(z^{\tau_0}(t)) - \overline{y}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |y(z^{\tau_0}(t)) - y(\overline{z}^{\overline{\tau}_0}(t))| + |y(\overline{z}^{\overline{\tau}_0}(t)) - \overline{y}(\overline{z}^{\overline{\tau}_0}(t))|.$$

If $\overline{z}^{\overline{\tau}_0}(t) \leqslant 0$, then $|y(\overline{z}^{\overline{\tau}_0}(t)) - \overline{y}(\overline{z}^{\overline{\tau}_0}(t))| = 0$. On the other hand, if $\overline{z}^{\overline{\tau}_0}(t) > 0$, then by Lemma 3.4, we have

$$|y(\overline{z}^{\overline{\tau}_0}(t)) - \overline{y}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant (\overline{z}^{\overline{\tau}_0}(\alpha))^{1/q}|y - \overline{y}|_{\mathbb{Y}_\alpha^{1,p}} \leqslant \alpha^{1/q}|y - \overline{y}|_{\mathbb{Y}_\alpha^{1,p}}.$$

Therefore, in either case, we have

$$|y(z^{\tau_0}(t)) - \overline{y}(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |y(z^{\tau_0}(t)) - y(\overline{z}^{\overline{\tau}_0}(t))| + \alpha^{1/q}|y - \overline{y}|_{\mathbb{Y}_\alpha^{1,p}}.$$

It suffices to show

$$|y(z^{\tau_0}(t)) - y(\overline{z}^{\overline{\tau}_0}(t))| \leqslant |y|_{\mathbb{Y}_\alpha^{1,\infty}}(\alpha^{1/q}|z - \overline{z}|_{\mathbb{Y}_\alpha^{1,p}} + |\tau_0 - \overline{\tau}_0|).$$

If $z^{\tau_0}(t)\overline{z}^{\overline{\tau}_0}(t) \geqslant 0$, then the inequality follows from the Mean Value Theorem and Lemma 3.2. Now, suppose $z^{\tau_0}(t)\overline{z}^{\overline{\tau}_0}(t) < 0$. Without loss of generality, we assume $z^{\tau_0}(t) > 0$ and $\overline{z}^{\overline{\tau}_0}(t) < 0$. Then we have

$$\begin{aligned} 0 &< z^{\tau_0}(t) \\ &= t - \overline{z}(t) - \overline{\tau}_0 + \overline{z}(t) - z(t) + \overline{\tau}_0 - \tau_0 \\ &= \overline{z}^{\overline{\tau}_0}(t) + \overline{z}(t) - z(t) + \overline{\tau}_0 - \tau_0 \\ &< \overline{z}(t) - z(t) + \overline{\tau}_0 - \tau_0. \end{aligned}$$

An application of the Mean Value Theorem produces

$$\begin{aligned} |y(z^{\tau_0}(t)) - y(\overline{z}^{\overline{\tau}_0}(t))| &= |y(z^{\tau_0}(t)) - y(0)| \\ &\leqslant |z^{\tau_0}(t)| \cdot |y|_{\mathbb{Y}_\alpha^{1,\infty}} \\ &\leqslant |y|_{\mathbb{Y}_\alpha^{1,\infty}}(|z - \overline{z}|_C + |\overline{\tau}_0 - \tau_0|) \\ &\leqslant |y|_{\mathbb{Y}_\alpha^{1,\infty}}(\alpha^{1/q}|z - \overline{z}|_{\mathbb{Y}_\alpha^{1,p}} + |\overline{\tau}_0 - \tau_0|). \end{aligned}$$

This completes the proof. □

We also need the following results from Ref. [11].

Lemma 3.6 (Lemma 3.1 in Ref. [11]) *Let* $g \in L_\alpha^p$, $y \in Y_0 := \{y \in W^{1,\infty}([0,\alpha]; [-r,\alpha])\colon \operatorname{ess\,inf}_{t\in[0,\alpha]} \dot{y}(t) > 0\}$. *Let* $\varepsilon(y) = \operatorname{ess\,inf}_{t\in[0,\alpha]} \dot{y}(t)$. *Then*

$$\int_0^\alpha |g(y(s))|^p \mathrm{d}s \leqslant \frac{1}{\varepsilon(y)} |g|_{L_\alpha^p}^p.$$

Moreover, if $y_n \to y$ *in* Y_0 *in the norm of* $W^{1,\infty}$, *then*

$$\lim_{k\to\infty} \int_0^\alpha |g(y^k(s)) - g(y(s))|^p \mathrm{d}s = 0.$$

Now, in a similar manner as that in Ref. [19], we introduce a new norm on $W_\alpha^{k,\infty}$. First, we define two projection operators $\mathrm{Pr}_\varphi\colon W_\alpha^{k,\infty} \to W^{k,\infty}$ and $\mathrm{Pr}_y\colon W_\alpha^{k,\infty} \to \mathbb{Y}_\alpha^{k,p}$, respectively, by

$$(Pr_\varphi x)(t) = x(t), \quad t \in [-r, 0],\ x \in W_\alpha^{k,\infty},$$

$$(Pr_y x)(t) = \begin{cases} 0, & t \in [-r, 0], \\ x(t) - x(0), & t \in [0, \alpha], \end{cases} \quad x \in W_\alpha^{k,\infty}.$$

Then, we define a 'product' norm on the set $W_\alpha^{k,p}$ for $1 \leqslant p < \infty$ by

$$|x|_{\mathbb{X}_\alpha^{k,p}} = |\mathrm{Pr}_y x|_{\mathbb{Y}_\alpha^{k,p}} + |\mathrm{Pr}_\varphi x|_{W^{k,\infty}}$$

and denote the corresponding normed linear space by $\mathbb{X}_\alpha^{k,\,p} := (W_\alpha^{k,\infty}, |\cdot|_{\mathbb{X}_\alpha^{k,p}})$. Also, for $\varphi \in W^{k,\infty}$, we define its extension $\widetilde{\varphi}$ to $[-r, \alpha]$ by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in [-r, 0], \\ \varphi(0), & t \in [0, \alpha]. \end{cases}$$

Then, for $x \in W_\alpha^{k,\infty}$, we can decompose x as $x = \mathrm{Pr}_y x + \widetilde{\mathrm{Pr}_\varphi x}$.

Lemma 3.7 (Lemma 3.8 in Ref. [19]) *Let $1 \leqslant p < \infty$. Then there exist positive constants c_1 and c_2 such that, for any $x \in W_\alpha^{1,\infty}$,*

(i) $|x|_{W_\alpha^{1,p}} \leqslant c_1 |x|_{\mathbb{X}_\alpha^{1,p}}$;

(ii) $|x|_{\mathbb{X}_\alpha^{1,p}} \leqslant c_2 |x|_{W_\alpha^{1,\infty}}$.

Lemma 3.8 *Let $1 \leqslant p < \infty$. Then there exist positive constants c_1 and c_2 such that, for any $x \in W_\alpha^{2,\infty}$,*

(i) $|x|_{W_\alpha^{2,p}} \leqslant c_1 |x|_{\mathbb{X}_\alpha^{2,p}}$;

(ii) $|x|_{\mathbb{X}_\alpha^{2,p}} \leqslant c_2 |x|_{W_\alpha^{2,\infty}}$.

Proof Let $y = \mathrm{Pr}_y x \in \mathbb{Y}_\alpha^{2,p}$ and $\varphi = \mathrm{Pr}_\varphi x \in W^{2,\infty}$. Then $x = y + \widetilde{\varphi}$. Using the inequality $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$ and Lemma 3.3 (i), we obtain

$$\begin{aligned} |x|_{W_\alpha^{2,p}}^p &= \int_{-r}^0 (|\varphi(t)|^p + |\dot{\varphi}(t)|^p + |\ddot{\varphi}(t)|^p)\mathrm{d}t \\ &\quad + \int_0^\alpha (|y(t) + \varphi(0)|^p + |\dot{y}(t)|^p + |\ddot{y}(t)|^p)\mathrm{d}t \\ &= 3r|\varphi|_{W^{2,\infty}}^p + 2^{p-1}\int_0^\alpha |y(t)|^p \mathrm{d}t + \alpha \cdot 2^{p-1}|\varphi(0)|^p \\ &\quad + \int_0^\alpha |\dot{y}(t)|^p \mathrm{d}t + \int_0^\alpha |\ddot{y}(t)|^p \mathrm{d}t \\ &\leqslant (\alpha \cdot 2^{p-1} + 3r)|\varphi|_{W^{2,\infty}}^p + 2^{p-1} \cdot \alpha^{1+\frac{p}{q}} |y|_{\mathbb{Y}_\alpha^{2,p}}^p + |y|_{\mathbb{Y}_\alpha^{2,p}}^p \\ &\leqslant (2^{p-1}(\alpha + \alpha^p) + 3r + 1)|x|_{\mathbb{X}_\alpha^{2,p}}^p. \end{aligned}$$

This proves (i) with $c_1^p = 2^{p-1}(\alpha + \alpha^p) + 3r + 1$.

Now, by definition,

$$\begin{aligned} |x|_{\mathbb{X}_\alpha^{2,p}} &= \left(\int_0^\alpha (|\dot{y}(t)|^p + |\ddot{y}(t)|^p)\mathrm{d}t \right)^{1/p} + |\varphi|_{W^{2,\infty}} \\ &\leqslant \alpha^{1/p}(|\dot{y}|_{L_\alpha^\infty} + |\ddot{y}|_{L_\alpha^\infty}) + |\varphi|_{W^{2,\infty}} \\ &\leqslant (2\alpha^{1/p} + 1)|x|_{W_\alpha^{2,\infty}}. \end{aligned}$$

This proves (ii) with $c_2 = 2\alpha^{1/p} + 1$, and hence, the proof is complete. □

Remark 3.1 It follows from Lemmas 3.7 and 3.8 that the norm $|\cdot|_{\mathbb{X}_\alpha^{k,p}}$ is weaker than the norm $|\cdot|_{W_\alpha^{k,\infty}}$ but stronger than the norm $|\cdot|_{W_\alpha^{k,p}}$, $k=1,2$. So, for differentiability of solutions with respect to parameters in the norm $|\cdot|_{W_\alpha^{k,p}}$, it is sufficient to obtain differentiability in the norm $|\cdot|_{\mathbb{X}_\alpha^{k,p}}$. By definition of the norm $|\cdot|_{\mathbb{X}_\alpha^{k,p}}$, we know, for $y \in \mathbb{Y}_\alpha^{k,\infty}$,

$$|y|_{\mathbb{X}_\alpha^{k,p}} = |y|_{\mathbb{Y}_\alpha^{k,p}}.$$

As we will see in Section 4, the state space of the solutions will be transformed into $\mathbb{Y}_\alpha^{2,\infty}$. So one needs only to study the differentiability with respect to parameters in the norm $|\cdot|_{\mathbb{Y}_\alpha^{k,p}}$ ($k \leqslant 2$) to obtain the differentiability in the corresponding norm $|\cdot|_{\mathbb{X}_\alpha^{k,p}}$.

Lemma 3.9 *Suppose that* $\overline{y} \in \mathbb{Y}_\alpha^{2,\infty}$, $\delta_2 \geqslant \delta_1 > 0$, *and* $1 \leqslant p < \infty$. *Then the set* $\overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(\overline{y};\delta_2) \cap \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}}(\overline{y};\delta_1)$ *is a closed and complete subset of* $\mathbb{Y}_\alpha^{2,p}$.

Proof Let $\{y_n\} \subset \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(\overline{y};\delta_2) \cap \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}}(\overline{y};\delta_1)$ be a Cauchy sequence in the norm $|\cdot|_{\mathbb{Y}_\alpha^{2,p}}$. Since $|\cdot|_{\mathbb{Y}_\alpha^{2,p}}$ is equivalent to $|\cdot|_{W_\alpha^{2,p}}$, $\{y_n\}$ is also a Cauchy sequence in $W_\alpha^{2,p}$. We know that $W_\alpha^{2,p}$ is a Banach space; thus, there exists $y \in W_\alpha^{2,p}$ such that $|y_n - y|_{W_\alpha^{2,p}} \to 0$ and hence $|y_n - y|_{\mathbb{Y}_\alpha^{2,p}} \to 0$ as $n \to \infty$. Moreover, it follows from Lemma 3.3 (i) that

$$|y_k(t) - y_l(t)| \leqslant \alpha^{1/q} |y_k - y_l|_{\mathbb{Y}_\alpha^{2,p}}, \quad k,l \to \infty,$$

which implies that $\{y_n(t)\}$ is a Cauchy sequence in $\mathbb{R}^n$ for all $t \in [0,\alpha]$. Thus, $\{y_n(t)\}$ converges to $y(t)$ pointwisely. Now, it suffices to show $y \in \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(\overline{y};\delta_2) \cap \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}}(\overline{y};\delta_1)$. By way of contradiction, suppose $y \notin \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(\overline{y};\delta_2) \cap \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}}(\overline{y};\delta_1)$. Then there exists an $\varepsilon > 0$ such that either

$$\max\Big\{\operatorname*{ess\,sup}_{t\in[0,\alpha]} |\dot{y}(t) - \dot{\overline{y}}(t)|, \operatorname*{ess\,sup}_{t\in[0,\alpha]} |\ddot{y}(t) - \ddot{\overline{y}}(t)|\Big\} \geqslant \delta_2 + \varepsilon,$$

or

$$\operatorname*{ess\,sup}_{t\in[0,\alpha]} |\dot{y}(t) - \dot{\overline{y}}(t)| \geqslant \delta_1 + \varepsilon.$$

Thus, since $\delta_1 \leqslant \delta_2$, the set

$$A = \{t\colon\ |\dot{y}(t) - \dot{\overline{y}}(t)| > \delta_1 + \varepsilon\} \cup \{t\colon\ |\ddot{y}(t) - \ddot{\overline{y}}(t)| > \delta_2 + \varepsilon\}$$

has positive measure. On the other hand, $\{y_n\} \subset \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(\overline{y};\delta_2) \cap \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}}(\overline{y};\delta_1)$ implies that

$$\max\Big\{\operatorname*{ess\,sup}_{t\in[0,\alpha]} |\dot{y}_n(t) - \dot{\overline{y}}(t)|, \operatorname*{ess\,sup}_{t\in[0,\alpha]} |\ddot{y}_n(t) - \ddot{\overline{y}}(t)|\Big\} \leqslant \delta_2,$$

$$\operatorname*{ess\,sup}_{t\in[0,\alpha]} |\dot{y}_n(t) - \dot{\overline{y}}(t)| \leqslant \delta_1$$

for all $n\in\mathbb{N}$. Then the set $\{t\colon |\dot{y}_n(t)-\dot{\overline{y}}(t)|>\delta_1\}\cup\{t\colon |\ddot{y}_n(t)-\ddot{\overline{y}}(t)|>\delta_2\}$ has zero measure for all $n\in\mathbb{N}$. Thus, the set

$$\begin{aligned}B&=[0,\alpha]\setminus\bigcup_{n=1}^{\infty}(\{t\colon |\dot{y}_n(t)-\dot{\overline{y}}(t)|>\delta_1\}\cup\{t\colon |\ddot{y}_n(t)-\ddot{\overline{y}}(t)|>\delta_2\})\\&=\{t\colon |\dot{y}_n(t)-\dot{\overline{y}}(t)|\leqslant\delta_1,\ n\in\mathbb{N}\}\cap\{t\colon |\ddot{y}_n(t)-\ddot{\overline{y}}(t)|\leqslant\delta_2,\ n\in\mathbb{N}\}\end{aligned}$$

has measure α. Let

$$\begin{aligned}A_1&=\{t\colon |\dot{y}(t)-\dot{\overline{y}}(t)|<\delta_1+\varepsilon\}\cap\{t\colon |\ddot{y}(t)-\ddot{\overline{y}}(t)|>\delta_2+\varepsilon\},\\A_2&=\{t\colon |\dot{y}(t)-\dot{\overline{y}}(t)|>\delta_1+\varepsilon\}\cap\{t\colon |\ddot{y}(t)-\ddot{\overline{y}}(t)|<\delta_2+\varepsilon\},\\A_3&=\{t\colon |\dot{y}(t)-\dot{\overline{y}}(t)|>\delta_1+\varepsilon\}\cap\{t\colon |\ddot{y}(t)-\ddot{\overline{y}}(t)|>\delta_2+\varepsilon\}.\end{aligned}$$

Then $A_1,\ A_2$, and A_3 are disjoint to each other and $A=A_1\cup A_2\cup A_3$. Since A has positive measure, at least one of A_i's, $i=1,2,3$, has positive measure. Without loss of generality, we assume $m(A_1)>0$. Then

$$\begin{aligned}|y-y_n|_{\mathbb{Y}^{2,p}_\alpha}&\geqslant\Big(\int_{A\cap B}|\dot{y}(t)-\dot{\overline{y}}(t)+\dot{\overline{y}}(t)-\dot{y}_n(t)|^p\mathrm{d}t\\&\qquad+|\ddot{y}(t)-\ddot{\overline{y}}(t)+\ddot{\overline{y}}(t)-\ddot{y}_n(t)|^p\mathrm{d}t\Big)^{1/p}\\&\geqslant\Big(\int_{A_1\cap B}(|\ddot{y}(t)-\ddot{\overline{y}}(t)|-|\ddot{\overline{y}}(t)-\ddot{y}_n(t)|)^p\mathrm{d}t\Big)^{1/p}\\&\geqslant\Big(\int_{A_1\cap B}\varepsilon^p\mathrm{d}t\Big)^{1/p}\\&=\varepsilon\cdot(m(A_1\cap B))^{1/p}\\&>0,\end{aligned}$$

which contradicts with $|y-y_n|_{\mathbb{Y}^{2,p}_\alpha}\to0$ as $n\to0$. This completes the proof. □

Lemma 3.10 *Suppose* $\overline{y}\in\mathbb{Y}^{2,\infty}_\alpha,\ \delta>0,\ 1\leqslant p<\infty$. *Then* $\overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}^{2,\infty}_\alpha}}(\overline{y};\delta)$ *is a closed and complete subset of* $\mathbb{Y}^{2,p}_\alpha$.

Proof This follows from Lemma 3.9 by taking $\delta_2=\delta_1=\delta$ and noting that $\overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}^{2,\infty}_\alpha}}(\overline{y};\delta)\subseteq\overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}^{1,\infty}_\alpha}}(\overline{y};\delta)$. □

Note that $\mathbb{Y}^{2,\infty}_\alpha$ is a Banach space, since $\mathbb{Y}^{2,\infty}_\alpha$ is a closed subspace of $W^{2,\infty}_\alpha$. However, $\mathbb{Y}^{2,p}_\alpha$ is not a Banach space. Lemma 3.10 tells us that $\mathbb{Y}^{2,p}_\alpha$ is a locally complete double-normed linear space with respect to $|\cdot|_{\mathbb{Y}^{2,\infty}_\alpha}$ norm. It is also a locally complete triple-normed linear space with respect to the norms $|\cdot|_{\mathbb{Y}^{2,\infty}_\alpha}$ and $|\cdot|_{\mathbb{Y}^{1,p}_\alpha}$ from the following lemma. This locally complete triple-normed linear space is the main object of this paper. The reason we introduce the norm $|\cdot|_{\mathbb{Y}^{1,p}_\alpha}$ is that we need to study differentiability of some operators defined on $\mathbb{Y}^{2,\infty}_\alpha$ in the sense of the norm $|\cdot|_{\mathbb{Y}^{1,p}_\alpha}$.

Lemma 3.11 $\mathbb{Y}_\alpha^{2,p}$ *is a locally complete triple-normed linear space with respect to the norms* $|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}$ *and* $|\cdot|_{\mathbb{Y}_\alpha^{1,p}}$.

Proof For any $R > 0$ and $R_0 > 0$, let

$$\{y_n\} \subset \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(R_0) \cap \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,p}}}(R)$$

be a Cauchy sequence in the $|\cdot|_{\mathbb{Y}_\alpha^{2,p}}$ norm. Then $\{y_n\}$ is also a Cauchy sequence in $W_\alpha^{2,p}$, since these two norms $|\cdot|_{\mathbb{Y}_\alpha^{2,p}}$ and $|\cdot|_{W_\alpha^{2,p}}$ are equivalent. Thus, there exists $y \in W_\alpha^{2,p}$ such that $|y_n - y|_{W_\alpha^{2,p}} \to 0$ and hence $|y_n - y|_{\mathbb{Y}_\alpha^{2,p}} \to 0$ as $n \to \infty$. By Lemma 3.10, $\mathscr{G}_{|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}}(R_0)$ is complete in $\mathbb{Y}_\alpha^{2,p}$ norm, so $y \in \mathbb{Y}_\alpha^{2,\infty}$. Then we have

$$|y|_{\mathbb{Y}_\alpha^{1,p}} \leqslant |y_n|_{\mathbb{Y}_\alpha^{1,p}} + |y_n - y|_{\mathbb{Y}_\alpha^{1,p}} \leqslant |y_n|_{\mathbb{Y}_\alpha^{1,p}} + |y_n - y|_{\mathbb{Y}_\alpha^{2,p}} \leqslant R + |y_n - y|_{\mathbb{Y}_\alpha^{2,p}}.$$

Taking the limit gives us $|y|_{\mathbb{Y}_\alpha^{1,p}} \leqslant R$, i.e., $y \in \overline{\mathscr{G}}_{|\cdot|_{\mathbb{Y}_\alpha^{1,p}}}(R)$. This completes the proof. □

4 Well-posedness

In this section, we will give some assumptions on system (1.1) which guarantee the well-posedness as well as the first- and second-order differentiability of solutions with respect to parameters. Further, we will transform the system into an integral operator. By verifying the conditions not related to the differentiability of the operator in Theorem 2.1, we can obtain the existence and continuous dependence on parameters for solutions of system (1.1).

Let $\Omega_1 \times \Omega_2 \times \Omega_3$ be the domain of $f(\theta_1, \theta_2, \theta_3)$, and let $\Omega_1 \times \Omega_4 \times \Omega_3$ be the domain of $g(\gamma_1, \gamma_2, \gamma_3)$, where $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Sigma$, $\Omega_4 \subset \mathbb{R}$ and Ω_1, Ω_2, Ω_3 and Ω_4 are open in the corresponding spaces. We shall make the following assumptions.

(A1) f is C^2 over the domain $\Omega_1 \times \Omega_2 \times \Omega_3$.

(A2) (i) g is C^2 over the domain $\Omega_1 \times \Omega_4 \times \Omega_3$ and,

(ii) there exists a constant $L > 0$ such that $g(x, \tau, \sigma) < L/(L+1)$ for all x, τ, σ.

(A3) $\varphi \in W^{2,\infty}$.

Let $y(t) = x(t) - \widetilde{\varphi}(t)$, $z(t) = \widetilde{\tau}(t) - \tau_0$. Then $y(t) \in \mathbb{Y}_\alpha^{2,\infty}$, $z(t) \in \mathbb{Y}_\alpha^{2,\infty}$, and system (1.1) is transformed into

$$y(t) = \begin{cases} 0, & t \in [-r, 0], \\ \displaystyle\int_0^t f(y(s) + \widetilde{\varphi}(s), (y + \widetilde{\varphi})(z^{\tau_0}(s)), \sigma)\mathrm{d}s, & t \in [0, \alpha], \end{cases} \tag{4.1}$$

$$z(t) = \begin{cases} 0, & t \in [-r, 0], \\ \displaystyle\int_0^t g(y(s) + \widetilde{\varphi}(s), z(s) + \tau_0, \sigma)\mathrm{d}s, & t \in [0, \alpha]. \end{cases} \tag{4.2}$$

As a result, we define an integral operator as follows: for $(y,z)\in\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}$, $\varphi\in W^{2,\infty}$, $\tau_0\in\mathbb{R}^+$, $\sigma\in\Sigma$,

$$S(y,z,\varphi,\tau_0,\sigma)=\begin{cases}[0,0]^{\mathrm{T}}, & t\in[-r,0],\\ \Big[\displaystyle\int_0^t f((y+\widetilde{\varphi})(s),(y+\widetilde{\varphi})(z^{\tau_0}(s)),\sigma)\mathrm{d}s,\\ \qquad\displaystyle\int_0^t g((y+\widetilde{\varphi})(s),z(s)+\tau_0,\sigma)\mathrm{d}s\Big]^{\mathrm{T}}, & t\in[0,\alpha].\end{cases} \tag{4.3}$$

It is easy to show that the existence, continuity and differentiability with respect to parameters of solutions (x,τ) of system (1.1) are equivalent to those of the fixed point (y,z) of the integral operator (4.3).

In the following, we derive the domain of the integral operator S.

Lemma 4.1 *Assume that* (A1)–(A3) *hold and* $1\leqslant p<\infty$. *Also, assume that* $\varphi^*,\tau_0^*>0$, σ^* *satisfy* $\varphi^*(0)\in\Omega_1$, $\varphi^*(-\tau_0^*)\in\Omega_2$, $\sigma^*\in\Omega_3$, *and* $\tau_0^*\in\Omega_4$. *Then there exist positive constants* δ_1, δ_2, δ_3, α, *sets* M_1, M_2, M_3, M_4, M_5, U, *and* W *such that*

(i) $M_1\subset\Omega_1$, $M_2\subset\Omega_2$, $M_4\subset\Omega_4$, $M_5\subset\mathbb{R}^+$ *are compact subsets and* $M_3\subset\Omega_3$ *is a closed bounded subset of* Σ.

(ii) U *is the intersection of two* $|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}}$- *and* $|\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}$-*open subsets of* $\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}$, *and* $W\subseteq U$ *is a closed subset of* $\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}$.

(iii) *For* $t\in[0,\alpha]$, $(y,z)\in U$, $(\varphi,\tau_0,\sigma)\in V:=\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1)\times\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)\times\mathscr{G}_\Sigma(\sigma^*;\delta_3)$, *we have* $(y+\widetilde{\varphi})(t)\in M_1$, $(y+\widetilde{\varphi})(z^{\tau_0}(t))\in M_2$, $\mathscr{G}_\Sigma(\sigma^*;\delta_3)\in M_3$, $z(t)+\tau_0\in M_4$, *and* $\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)\in M_5$.

(iv) *The operator* $S\colon U\times V\subset\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}\times W^{2,\infty}\times\mathbb{R}^+\times\Sigma\to\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}$ *defined by* (4.3) *satisfies*

(a) $S(W\times V)\subset W$;

(b) S *is a uniform* $|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}}$, $|\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}$ *and* $|\cdot|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}}$ *contraction on* $W\times V$;

(c) *For all* $(y,z)\in U$, *the function* $S(y,z,\cdot,\cdot,\cdot)\colon V\to\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}$ *is continuous.*

Proof Choose $R_i>0$, $i=1,2,\dots,5$, such that

$$\begin{gathered}M_1=\overline{\mathscr{G}}_{\mathbb{R}^n}(\varphi^*(0);R_1)\subset\Omega_1\subset\mathbb{R}^n,\\ M_2=\overline{\mathscr{G}}_{\mathbb{R}^n}(\varphi^*(-\tau_0^*);R_2)\subset\Omega_2\subset\mathbb{R}^n,\\ M_3=\overline{\mathscr{G}}_\Sigma(\sigma^*;R_3)\subset\Omega_3\subset\Sigma,\\ M_4=\overline{\mathscr{G}}_{\mathbb{R}}(\tau_0^*;R_4)\subset\Omega_4\subset\mathbb{R},\\ M_5=\overline{\mathscr{G}}_{\mathbb{R}^+}(\tau_0^*;R_5)\subset\mathbb{R}^+.\end{gathered}$$

Then $f(\theta_1,\theta_2,\theta_3)$ and $\frac{\partial f}{\partial\theta_i}(\theta_1,\theta_2,\theta_3)$, $i=1,2,3$, are Lipschitzian continuous on $M_1\times M_2\times M_3$, and the Lipschitz constants are denoted by L_0 and L_i, respectively; $g(\gamma_1,\gamma_2,\gamma_3)$ and $\frac{\partial g}{\partial\gamma_i}(\gamma_1,\gamma_2,\gamma_3)$, $i=1,2,3$, are also Lipschitzian continuous on $M_1 \times M_4 \times M_3$, and the corresponding Lipschitz constants are denoted by H_0 and H_i, respectively. Moreover, by the C^2 smoothness of $f(\theta_1, \theta_2, \theta_3)$ and $g(\gamma_1, \gamma_2, \gamma_3)$, the following constants are finite:

$$\overline{\beta}_0 := \sup\{|f(\theta_1, \theta_2, \theta_3)|:\ \theta_1 \in M_1,\ \theta_2 \in M_2,\ \theta_3 \in M_3\},$$

$$\overline{\beta}_i := \sup\Big\{\Big\|\frac{\partial f}{\partial \theta_i}(\theta_1, \theta_2, \theta_3)\Big\|:\ \theta_1 \in M_1,\ \theta_2 \in M_2,\ \theta_3 \in M_3\Big\}, \quad i = 1, 2, 3,$$

$$\overline{\eta}_0 := \sup\{|g(\gamma_1, \gamma_2, \gamma_3)|:\ \gamma_1 \in M_1,\ \gamma_2 \in M_4,\ \gamma_3 \in M_3\},$$

$$\eta_0^+ := \sup\{g(\gamma_1, \gamma_2, \gamma_3):\ \gamma_1 \in M_1,\ \gamma_2 \in M_4,\ \gamma_3 \in M_3\},$$

$$\eta_0^- := \inf\{g(\gamma_1, \gamma_2, \gamma_3):\ \gamma_1 \in M_1,\ \gamma_2 \in M_4,\ \gamma_3 \in M_3\},$$

$$\overline{\eta}_i := \sup\Big\{\Big\|\frac{\partial g}{\partial \gamma_i}(\gamma_1, \gamma_2, \gamma_3)\Big\|:\ \gamma_1 \in M_1,\ \gamma_2 \in M_4,\ \gamma_3 \in M_3\Big\}, \quad i = 1, 2, 3.$$

It is easy to see that $\overline{\eta}_0 = \max\{\eta_0^+, -\eta_0^-\} \geqslant 0$ and $\eta_0^+ < 1$.

Choose $\beta_0 > 0$, $\eta^+ > 0$, and $\eta^- < 0$ such that $\beta_0 > \overline{\beta}_0$, $1 > \eta^+ > \eta_0^+ > \eta_0^- > \eta^-$. Then $\eta_0 := \max\{\eta^+, -\eta^-\} > \overline{\eta}_0 \geqslant 0$. Let $\varepsilon > 0$ be a small positive constant such that the following numbers are positive:

$$\delta_1 = \min\{R_1/2,\ R_2/3\},$$

$$\delta_2 = \min\Big\{\tau_0^* - \varepsilon,\ \frac{R_4}{2},\ \frac{R_2}{3|\varphi^*|_{W^{1,\infty}}},\ R_5\Big\},$$

$$\delta_3 = R_3,$$

$$\zeta_1 = \max\{\overline{\beta}_0,\ \overline{\beta}_0\overline{\beta}_1 + \overline{\beta}_2(1 - \eta_0^-)\max\{\overline{\beta}_0, \delta_1 + |\varphi^*|_{W^{1,\infty}}\}\},$$

$$\xi_1 = \max\{\overline{\eta}_0,\ \overline{\beta}_0\overline{\eta}_1 + \overline{\eta}_0\overline{\eta}_2\},$$

$$\Theta_1 = \overline{\beta}_1 + 2\overline{\beta}_0 L_1 + \overline{\beta}_2(1 + \overline{\eta}_0) + 2L_2(1 + \overline{\eta}_0)(\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}}),$$

$$\begin{aligned}\Theta_2 = {} & L_1\overline{\beta}_0(\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}}) + \overline{\beta}_2(2 + \overline{\eta}_0)(\zeta_1 + \delta_1 + |\varphi^*|_{W^{2,\infty}}) \\ & + L_2(1 + \overline{\eta}_0)(\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}})^2\end{aligned}$$

$$\begin{aligned}N_0 = {} & L_0 \max\{2, \overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}}\} + \max\{\Theta_1, \Theta_2\} + H_0 \\ & + \max\{\overline{\eta}_1 + \overline{\beta}_0 H_1 + \overline{\eta}_0 H_2,\ \overline{\eta}_2 + \overline{\beta}_0 H_1 + \overline{\eta}_0 H_2\},\end{aligned}$$

$$\alpha = \min\Big\{T,\ \frac{R_1}{2\beta_0},\ \frac{R_2}{3(\beta_0 + (1 - \eta^-)|\varphi^*|_{W^{1,\infty}})},\ \frac{R_4}{2\eta_0},\ \frac{1}{N_0 + \varepsilon},\ \frac{\tau_0^* - \delta_2}{\max\{-\eta^-, 0\}}\Big\}.$$

Let $y \in \mathscr{G}_{\mathbb{Y}_\alpha^{1,\infty}}(\beta_0)$ and $\varphi \in \mathscr{G}_{W^{2,\infty}}(\varphi^*; \delta_1)$. Then, for $t \in [0, \alpha]$, by Lemma 3.3 (i), we have

$$\begin{aligned}|(y + \widetilde{\varphi})(t) - \varphi^*(0)| &\leqslant |y(t)| + |\widetilde{\varphi}(t) - \varphi^*(0)| \\ &= |y(t)| + |\varphi(0) - \varphi^*(0)| \\ &\leqslant \alpha|y|_{\mathbb{Y}_\alpha^{1,\infty}} + |\varphi - \varphi^*|_{W^{1,\infty}} \\ &\leqslant \alpha\beta_0 + \delta_1 \\ &\leqslant \frac{R_1}{2} + \frac{R_1}{2} \\ &= R_1,\end{aligned}$$

That is, $(y+\widetilde{\varphi})(t)\in M_1$.

Let $z\in \mathscr{G}_{\mathbb{Y}^{1,\infty}_\alpha}(\eta^-,\eta^+):=\{z\in \mathbb{Y}^{1,\infty}_\alpha\colon\ \eta^-\leqslant \dot{z}(t)\leqslant \eta^+,\ t\in[0,\alpha]\}$ and $\tau_0\in \mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)$. Then, for $t\in[0,\alpha]$, we have

$$|z(t)+\tau_0-\tau_0^*|\leqslant |z(t)|+|\tau_0-\tau_0^*|\leqslant \alpha\eta_0+\delta_2\leqslant \frac{R_4}{2}+\frac{R_4}{2}\leqslant R_4,$$

i.e., $z(t)+\tau_0\in M_4$.

For $y\in \mathscr{G}_{\mathbb{Y}^{1,\infty}_\alpha}(\beta_0),\ z\in \mathscr{G}_{\mathbb{Y}^{1,\infty}_\alpha}(\eta^-,\eta^+),\ \varphi\in \mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1),\ \tau_0\in \mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)$, and $t\in[0,\alpha]$, we have

$$\begin{aligned}|(y+\widetilde{\varphi})(z^{\tau_0}(t))-\varphi^*(-\tau_0^*)|\leqslant{}& |y(z^{\tau_0}(t))|+|\widetilde{\varphi}(z^{\tau_0}(t))-\widetilde{\varphi}^*(z^{\tau_0}(t))|\\ &+|\widetilde{\varphi}^*(z^{\tau_0}(t))-\widetilde{\varphi}^*(z^{\tau_0^*}(t))|\\ &+|\widetilde{\varphi}^*(z^{\tau_0^*}(t))-\widetilde{\varphi}^*(-\tau_0^*)|. \end{aligned}\tag{4.4}$$

To show $(y+\widetilde{\varphi})(z^{\tau_0}(t))\in M_2$, we estimate the right-hand side of (4.4) term by term. First, we show

$$|y(z^{\tau_0}(t))|\leqslant \alpha|y|_{\mathbb{Y}^{1,\infty}_\alpha}\leqslant \alpha\beta_0.\tag{4.5}$$

The fact that $z(0)=0$, combined with $\eta^-\leqslant \dot{z}(t)\leqslant \eta^+<1$, implies $\eta^-t\leqslant z(t)\leqslant \eta^+t\leqslant t$ for $t\in[0,\alpha]$. It follows from the definition of α that $\alpha\leqslant (\tau_0^*-\delta_2)/\max\{-\eta^-,\,0\}$. Then

$$z^{\tau_0}(t)\leqslant (1-\eta^-)t-\tau_0\leqslant (1-\eta^-)\alpha-\tau_0^*+\delta_2\leqslant \alpha,\quad t\in[0,\alpha].$$

On the other hand,

$$z^{\tau_0}(t)\geqslant (1-\eta^+)t-\tau_0\geqslant (1-\eta^+)t-\tau_0^*-\delta_2\geqslant -\tau_0^*-\delta_2.$$

Thus,

$$-\tau_0^*-\delta_2\leqslant z^{\tau_0}(t)\leqslant \alpha.$$

In particular,

$$-\tau_0^*-\delta_2\leqslant z^{\tau_0}(\alpha)\leqslant \alpha.$$

If $z^{\tau_0}(\alpha)\leqslant 0$, then $|y(z^{\tau_0}(t))|=0$ since $z^{\tau_0}(\alpha)$ is the maximum of $z^{\tau_0}(t)$. If $z^{\tau_0}(\alpha)\geqslant 0$, then, by Lemma 3.4,

$$|y(z^{\tau_0}(t))|\leqslant |z^{\tau_0}(\alpha)|\,|y|_{\mathbb{Y}^{1,\infty}_\alpha}\leqslant \alpha\beta_0.$$

This proves (4.5). Second, by Lemma 3.5, we have

$$|\widetilde{\varphi}^*(z^{\tau_0}(t))-\widetilde{\varphi}^*(z^{\tau_0^*}(t))|\leqslant \delta_2|\varphi^*|_{W^{1,\infty}}.\tag{4.6}$$

Third, applying the Mean Value Theorem gives us

$$|\widetilde{\varphi}^*(z^{\tau_0^*}(t))-\widetilde{\varphi}^*(-\tau_0^*)|\leqslant |t-z(t)|\,|\varphi^*|_{W^{1,\infty}}\leqslant (1-\eta^-)\alpha|\varphi^*|_{W^{1,\infty}}.\tag{4.7}$$

Now, combining (4.5), (4.6) and (4.7), we have

$$\begin{aligned}|(y+\widetilde{\varphi})(z^{\tau_0}(t))-\varphi^*(-\tau_0^*)| &\leqslant \alpha\beta_0+\delta_1+\delta_2|\varphi^*|_{W^{1,\infty}}+(1-\eta^-)\alpha|\varphi^*|_{W^{1,\infty}}\\ &\leqslant \alpha(\beta_0+(1-\eta^-)|\varphi^*|_{W^{1,\infty}})+\delta_1+\delta_2|\varphi^*|_{W^{1,\infty}}\\ &\leqslant \frac{R_2}{3}+\frac{R_2}{3}+\frac{R_2}{3}\\ &\leqslant R_2.\end{aligned}$$

This proves $(y+\widetilde{\varphi})(z^{\tau_0}(t))\in M_2$.

By the definitions of δ_2 and δ_3, it follows that $\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)\subseteq M_5$ and $\mathscr{G}_{\Sigma}(\sigma^*;\delta_3)\subseteq M_3$. So far, we have proved parts (i) and (iii) of the lemma, which also imply that $S(y,z,\varphi,\tau_0,\sigma)$ is well defined on its domain.

Now, we turn to the proof of parts (ii) and (iv) of the lemma. Choose $\beta_1>\zeta_1,\ \eta_1>\xi_1$, and define

$$U=(\mathscr{G}_{\mathbb{Y}_\alpha^{2,\infty}}(\beta_1)\cap\mathscr{G}_{\mathbb{Y}_\alpha^{1,\infty}}(\beta_0))\times(\mathscr{G}_{\mathbb{Y}_\alpha^{2,\infty}}(\eta_1)\cap\mathscr{G}_{\mathbb{Y}_\alpha^{1,\infty}}(\eta^-,\eta^+)),$$

$$W=(\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{2,\infty}}(\zeta_1)\cap\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{1,\infty}}(\overline{\beta}_0))\times(\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{2,\infty}}(\xi_1)\cap\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{1,\infty}}(\eta_0^-,\eta_0^+)).$$

It is easy to check that U and W are convex sets, $W\subset U$, U is an open subset of $\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}$, and

$$U=U\cap(\mathscr{G}_{\mathbb{Y}_\alpha^{2,\infty},|\cdot|_{\mathbb{Y}_\alpha^{1,p}}}(\alpha^{1/p}\beta_0)\times\mathscr{G}_{\mathbb{Y}_\alpha^{2,\infty},|\cdot|_{\mathbb{Y}_\alpha^{1,p}}}(\alpha^{1/p}\eta_0)).$$

By Lemma 3.9, W is a closed subset of $\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}$. This proves part (ii) of the lemma.

Let $(y,z)\in W,\ \varphi\in\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1),\ \tau_0\in\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)$, and $\sigma\in\mathscr{G}_{\Sigma}(\sigma^*;\delta_3)$. For the simplicity of notations, for $t\in[0,\alpha]$, we introduce

$$\hat{f}(t;y,z,\varphi,\tau_0,\sigma)=f((y+\widetilde{\varphi})(t),(y+\widetilde{\varphi})(z^{\tau_0}(t)),\sigma),$$

$$\hat{g}(t;y,z,\varphi,\tau_0,\sigma)=g((y+\widetilde{\varphi})(t),z(t)+\tau_0,\sigma).$$

Noting that

$$(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))=\begin{cases}\dot{y}(z^{\tau_0}(t)), & z^{\tau_0}(t)>0,\\ \dot{\widetilde{\varphi}}(z^{\tau_0}(t)), & z^{\tau_0}(t)\leqslant 0,\end{cases}$$

we have

$$\Big|\int_0^t f((y+\widetilde{\varphi})(s),(y+\widetilde{\varphi})(z^{\tau_0}(s)),\sigma)\mathrm{d}s\Big|_{\mathbb{Y}_\alpha^{1,\infty}}\leqslant\overline{\beta}_0,$$

and

$$\begin{aligned}&\Big|\int_0^t f((y+\widetilde{\varphi})(s),(y+\widetilde{\varphi})(z^{\tau_0}(s)),\sigma)\mathrm{d}s\Big|_{\mathbb{Y}_\alpha^{2,\infty}}\\ &=\ \max\Big\{\operatorname*{ess\,sup}_{t\in[0,\alpha]}|\hat{f}(t;y,z,\varphi,\tau_0,\sigma)|,\ \operatorname*{ess\,sup}_{t\in[0,\alpha]}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(t;y,z,\varphi,\tau_0,\sigma)\Big|\Big\}\end{aligned}$$

$$\begin{aligned}
&\leqslant \max\Big\{\overline{\beta}_0,\ \operatorname*{ess\,sup}_{t\in[0,\alpha]}\Big|\frac{\partial f}{\partial\theta_1}(t)((\dot y+\dot{\widetilde{\varphi}})(t))\\
&\qquad\qquad + (1-\dot z(t))\frac{\partial f}{\partial\theta_2}(t)((\dot y+\dot{\widetilde{\varphi}})(z^{\tau_0}(t)))\Big|\Big\}\\
&\leqslant \max\Big\{\overline{\beta}_0,\ \operatorname*{ess\,sup}_{t\in[0,\alpha]}\Big(\Big\|\frac{\partial f}{\partial\theta_1}(t)\Big\|\,|\dot y(t)|\\
&\qquad\qquad + \Big\|\frac{\partial f}{\partial\theta_2}(t)\Big\|\,|(\dot y+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))|\,|(1-\dot z(t))|\Big)\Big\}\\
&\leqslant \max\{\overline{\beta}_0,\ \overline{\beta}_1\overline{\beta}_0+\overline{\beta}_2\max\{\overline{\beta}_0,\delta_1+|\varphi^*|_{W^{1,\infty}}\}(1-\eta_0^-)\}\\
&=\zeta_1.
\end{aligned}$$

Here and in the sequel, $\frac{\partial f}{\partial\theta_i}(\cdot)$ and $\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}(\cdot)$, $i,j=1,2,3$, are evaluated at $((y+\widetilde{\varphi})(\cdot),(y+\widetilde{\varphi})(z^{\tau_0}(\cdot)),\sigma)$, respectively. Thus, we have proved that the first component of the integral operator $S(y,z,\varphi,\tau_0,\sigma)$ is in $\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{2,\infty}}(\zeta_1)\cap\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{1,\infty}}(\overline{\beta}_0)$.

Similarly, noting $\overline{\eta}_0=\max\{\eta_0^+,-\eta_0^-\}$, we have

$$\Big|\int_0^t g((y+\widetilde{\varphi})(s),z(s)+\tau_0,\sigma)\mathrm{d}s\Big|_{\mathbb{Y}_\alpha^{1,\infty}}\leqslant\overline{\eta}_0,$$

and

$$\begin{aligned}
&\Big|\int_0^t g((y+\widetilde{\varphi})(s),z(s)+\tau_0,\sigma)\mathrm{d}s\Big|_{\mathbb{Y}_\alpha^{2,\infty}}\\
&=\max\Big\{\operatorname*{ess\,sup}_{t\in[0,\alpha]}|\hat g(t;y,z,\varphi,\tau_0,\sigma)|,\ \operatorname*{ess\,sup}_{t\in[0,\alpha]}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\hat g(t;y,z,\varphi,\tau_0,\sigma)\Big|\Big\}\\
&\leqslant\max\Big\{\overline{\eta}_0,\ \operatorname*{ess\,sup}_{t\in[0,\alpha]}\Big|\frac{\partial g}{\partial\gamma_1}(t)\dot y(t)+\frac{\partial g}{\partial\gamma_2}(t)\dot z(t)\Big|\Big\}\\
&\leqslant\max\Big\{\overline{\eta}_0,\ \operatorname*{ess\,sup}_{t\in[0,\alpha]}\Big(\Big\|\frac{\partial g}{\partial\gamma_1}(t)\Big\|\,|\dot y(t)|+\Big\|\frac{\partial g}{\partial\gamma_2}(t)\Big\|\,|\dot z|\Big)\Big\}\\
&\leqslant\max\{\overline{\eta}_0,\ \overline{\beta}_0\overline{\eta}_1+\overline{\eta}_0\overline{\eta}_2\}\\
&=\xi_1.
\end{aligned}$$

Here and in the sequel, $\frac{\partial g}{\partial\gamma_i}(\cdot)$ and $\frac{\partial^2 g}{\partial\gamma_i\partial\gamma_j}(\cdot)$, $i,j=1,2,3$, are evaluated at $((y+\widetilde{\varphi})(\cdot),z(\cdot)+\tau_0,\sigma)$, respectively. Thus, we have proved that the second component of the integral operator $S(y,z,\varphi,\tau_0,\sigma)$ is in $\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{2,\infty}}(\xi_1)\cap\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{1,\infty}}(\overline{\eta}_0)$. Since for $s\in[0,\alpha]$, $(y+\widetilde{\varphi})(s)\in M_1$, $(y+\widetilde{\varphi})(z^{\tau_0}(s))\in M_2$, and $z(s)+\tau_0\in M_4$, we have $S(y,z,\varphi,\tau_0,\sigma)\in W$. This proves part (iv) (a) of the lemma.

Let $(y,z)\in W$, $(\overline{y},\overline{z})\in W$, $\varphi\in\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1)$, $\tau_0\in\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)$ and $\sigma\in\mathscr{G}_\Sigma(\sigma^*;\delta_3)$. Then, by (A1)–(A3), we have

$$|S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)|_{\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}}=\max\{A_1,A_2\}+\max\{A_3,A_4\},$$

where

$$A_1 = \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha} |\hat{f}(t;y,z,\varphi,\tau_0,\sigma) - \hat{f}(t;\overline{y},\overline{z},\varphi,\tau_0,\sigma)|,$$

$$A_2 = \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(t;y,z,\varphi,\tau_0,\sigma) - \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{f}(t;\overline{y},\overline{z},\varphi,\tau_0,\sigma)\Big|,$$

$$A_3 = \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha} |\hat{g}(t;y,z,\varphi,\tau_0,\sigma) - \hat{g}(t;\overline{y},\overline{z},\varphi,\tau_0,\sigma)|,$$

$$A_4 = \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\hat{g}(t;y,z,\varphi,\tau_0,\sigma) - \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{g}(t;\overline{y},\overline{z},\varphi,\tau_0,\sigma)\Big|.$$

In the following, we estimate A_1, A_2, A_3, and A_4 in terms of $y-\overline{y}$ and $z-\overline{z}$ in the norms $|\cdot|_{\mathbb{Y}^{2,\infty}_\alpha}$, $|\cdot|_{\mathbb{Y}^{2,p}_\alpha}$ and $|\cdot|_{\mathbb{Y}^{1,p}_\alpha}$, respectively. Before doing this, we specify that if ψ is a continuous function from $[-r,\alpha]$ (or $[0,\alpha]$) to $\mathbb{R}^n$, then $|\psi|_C$ is defined as $\max_{t\in[-r,\alpha]}|\psi(t)|$ (or $\max_{t\in[0,\alpha]}|\psi(t)|$). Note also that related inequalities from Lemma 3.2 to Lemma 3.5 will be used in the long technical presentations below.

Let us first look at A_1 and A_3. By Lemmas 3.2 and 3.3,

$$|y(t)-\overline{y}(t)| \leqslant \alpha|y-\overline{y}|_{\mathbb{Y}^{1,\infty}_\alpha} \leqslant \alpha|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha},$$

$$|y(t)-\overline{y}(t)| \leqslant \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} \leqslant \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha},$$

$$|y(z^{\tau_0}(t))-\overline{y}(z^{\tau_0}(t))| \leqslant \alpha|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha},$$

$$|y(z^{\tau_0}(t))-\overline{y}(z^{\tau_0}(t))| \leqslant \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha}.$$

By Lemma 3.5, we have

$$|\overline{y}(z^{\tau_0}(t))-\overline{y}(\overline{z}^{\tau_0}(t))| \leqslant |\overline{y}|_{\mathbb{Y}^{1,\infty}_\alpha}\,|z-\overline{z}|_C \leqslant \alpha\overline{\beta}_0|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha},$$

and

$$|\overline{y}(z^{\tau_0}(t))-\overline{y}(\overline{z}^{\tau_0}(t))| \leqslant |\overline{y}|_{\mathbb{Y}^{1,\infty}_\alpha}\,|z-\overline{z}|_C \leqslant \alpha^{1/q}\overline{\beta}_0|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}.$$

Also, we have

$$|\widetilde{\varphi}(z^{\tau_0}(t))-\widetilde{\varphi}(\overline{z}^{\tau_0}(t))| \leqslant |\varphi|_{W^{1,\infty}}\,|z-\overline{z}|_C \leqslant \alpha(\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha},$$

and

$$|\widetilde{\varphi}(z^{\tau_0}(t))-\widetilde{\varphi}(\overline{z}^{\tau_0}(t))| \leqslant |\varphi|_{W^{1,\infty}}\,|z-\overline{z}|_C \leqslant \alpha^{1/q}(\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}.$$

So, by assumption (A1) and the inequalities above, we have

$$\begin{aligned} A_1 &\leqslant L_0 \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}(|y(t)-\overline{y}(t)| + |y(z^{\tau_0}(t))-\overline{y}(\overline{z}^{\tau_0}(t))| \\ &\quad + |\widetilde{\varphi}(z^{\tau_0}(t))-\widetilde{\varphi}(\overline{z}^{\tau_0}(t))|) \\ &\leqslant L_0 \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}(|y(t)-\overline{y}(t)| + |y(z^{\tau_0}(t))-\overline{y}(z^{\tau_0}(t))| \\ &\quad + |\overline{y}(z^{\tau_0}(t))-\overline{y}(\overline{z}^{\tau_0}(t))| + |\widetilde{\varphi}(z^{\tau_0}(t))-\widetilde{\varphi}(\overline{z}^{\tau_0}(t))|) \end{aligned}$$

$$\leqslant 2\alpha L_0|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + L_0\alpha(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}. \tag{4.8}$$

That is,

$$A_1 \leqslant 2\alpha L_0|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + L_0\alpha(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}. \tag{4.9}$$

To estimate A_1 in terms of the differences of y, $\overline{y}$ and z, $\overline{z}$ in all the norms, we combine inequality (4.8) with Lemmas 3.2 and 3.5 to obtain

$$\begin{aligned} A_1 &\leqslant L_0(\alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} + \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} \\ &\quad + |\overline{y}|_{\mathbb{Y}^{1,\infty}_\alpha}|z-\overline{z}|_C + |\varphi|_{W^{1,\infty}}|z-\overline{z}|_C) \\ &\leqslant 2\alpha^{1/q}L_0|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} + L_0\alpha^{1/q}(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha}. \end{aligned}$$

That is,

$$A_1 \leqslant 2\alpha^{1/q}L_0|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} + L_0\alpha^{1/q}(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha}, \tag{4.10}$$

and hence,

$$A_1 \leqslant 2\alpha^{1/q}L_0|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + L_0\alpha^{1/q}(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}. \tag{4.11}$$

Similarly, by assumption (A2) (i), Lemmas 3.2 and 3.3, we have

$$\begin{aligned} A_3 &\leqslant H_0 \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}(|y(t)-\overline{y}(t)| + |z(t)-\overline{z}(t)|) \\ &\leqslant H_0(|y-\overline{y}|_C + |z-\overline{z}|_C) \\ &\leqslant H_0\alpha(|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + |z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}), \end{aligned}$$

and

$$A_3 \leqslant H_0\alpha^{1/q}(|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} + |z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha}).$$

Therefore,

$$A_3 \leqslant H_0\alpha(|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + |z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}), \tag{4.12}$$

$$A_3 \leqslant H_0\alpha^{1/q}(|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + |z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}), \tag{4.13}$$

$$A_3 \leqslant H_0\alpha^{1/q}(|y-\overline{y}|_{\mathbb{Y}^{1,p}_\alpha} + |z-\overline{z}|_{\mathbb{Y}^{1,p}_\alpha}). \tag{4.14}$$

Next, we consider A_2.

$$\begin{aligned} A_2 &= \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial f}{\partial\theta_1}(t)\dot{y}(t) + \frac{\partial f}{\partial\theta_2}(t)((\dot{y}+\dot{\varphi})(z^{\tau_0}(t)))(1-\dot{z}(t)) - \frac{\partial\overline{f}}{\partial\theta_1}(t)\dot{\overline{y}}(t) \\ &\quad - \frac{\partial\overline{f}}{\partial\theta_2}(t)((\dot{\overline{y}}+\dot{\overline{\varphi}})(\overline{z}^{\tau_0}(t)))(1-\dot{\overline{z}}(t))\Big| \\ &= \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial f}{\partial\theta_1}(t)(\dot{y}(t)-\dot{\overline{y}}(t)) + \Big(\frac{\partial f}{\partial\theta_1}(t) - \frac{\partial\overline{f}}{\partial\theta_1}(t)\Big)\dot{\overline{y}}(t) \\ &\quad + \frac{\partial f}{\partial\theta_2}(t)[((\dot{y}+\dot{\varphi})(z^{\tau_0}(t)))(1-\dot{z}(t)) - ((\dot{\overline{y}}+\dot{\overline{\varphi}})(\overline{z}^{\tau_0}(t)))(1-\dot{\overline{z}}(t))] \\ &\quad + \Big(\frac{\partial f}{\partial\theta_2}(t) - \frac{\partial\overline{f}}{\partial\theta_2}(t)\Big)((\dot{\overline{y}}+\dot{\overline{\varphi}})(\overline{z}^{\tau_0}(t)))(1-\dot{\overline{z}}(t))\Big|. \end{aligned}$$

Here and in the sequel, $\frac{\partial \overline{f}}{\partial \theta_i}(\cdot)$ and $\frac{\partial^2 \overline{f}}{\partial \theta_i \partial \theta_j}(\cdot)$, $i, j = 1, 2, 3$, are evaluated at $((\overline{y}+\widetilde{\varphi})(\cdot), (\overline{y}+\widetilde{\varphi})(\overline{z}^{\tau_0}(\cdot)), \sigma)$, respectively. To get the estimation in terms of $|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}$ and $|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}$, we estimate the above expression term by term. By Lemma 3.4,

$$\Big|\frac{\partial f}{\partial \theta_1}(t)(\dot{y}(t)-\dot{\overline{y}}(t))\Big| \leqslant \alpha \sup_{0\leqslant t\leqslant \alpha}\Big\|\frac{\partial f}{\partial \theta_1}(t)\Big\|\, |y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} \leqslant \alpha\overline{\beta}_1 |y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}, \quad (4.15)$$

and

$$\begin{aligned}\Big|\frac{\partial f}{\partial \theta_1}(t)(\dot{y}(t)-\dot{\overline{y}}(t))\Big| &\leqslant \alpha^{1/q} \sup_{0\leqslant t\leqslant \alpha}\Big\|\frac{\partial f}{\partial \theta_1}(t)\Big\|\, |y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha}\\ &\leqslant \alpha^{1/q}\overline{\beta}_1 |y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha}. \end{aligned} \quad (4.16)$$

By (A1) and Lemma 3.5, we have

$$\begin{aligned}\Big\|\frac{\partial f}{\partial \theta_1}(t)-\frac{\partial \overline{f}}{\partial \theta_1}(t)\Big\| &\leqslant L_1(|y-\overline{y}|_C + |(\overline{y}+\widetilde{\varphi})(\overline{z}^{\tau_0}(t)) - (y+\widetilde{\varphi})(z^{\tau_0}(t))|)\\ &\leqslant L_1(|y-\overline{y}|_C + |y|_{\mathbb{Y}^{1,\infty}_\alpha}|z-\overline{z}|_C + \alpha|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}\\ &\quad + |\widetilde{\varphi}|_{W^{1,\infty}}|z-\overline{z}|_C)\\ &\leqslant 2\alpha L_1|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}\\ &\quad + L_1\alpha(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}, \end{aligned} \quad (4.17)$$

and

$$\begin{aligned}\Big\|\frac{\partial f}{\partial \theta_1}(t)-\frac{\partial \overline{f}}{\partial \theta_1}(t)\Big\| &\leqslant L_1(|y-\overline{y}|_C + |\overline{y}|_{\mathbb{Y}^{1,\infty}_\alpha}|z-\overline{z}|_C\\ &\quad + \alpha^{1/q}|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + |\widetilde{\varphi}|_{W^{1,\infty}}|z-\overline{z}|_C)\\ &\leqslant 2\alpha^{1/q} L_1|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha}\\ &\quad + L_1\alpha^{1/q}(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}. \end{aligned} \quad (4.18)$$

Similarly, we have

$$\Big\|\frac{\partial f}{\partial \theta_2}(t)-\frac{\partial \overline{f}}{\partial \theta_2}(t)\Big\| \leqslant 2\alpha L_2|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + L_2\alpha(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}, \quad (4.19)$$

$$\Big\|\frac{\partial f}{\partial \theta_2}(t)-\frac{\partial \overline{f}}{\partial \theta_2}(t)\Big\| \leqslant 2\alpha^{1/q} L_2|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + L_2\alpha^{1/q}(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}. \quad (4.20)$$

Using the triangle inequality and Lemma 3.5, we have the following estimations ((4.21)–(4.28)):

$$\begin{aligned}|\dot{y}(z^{\tau_0}(t)) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))| &\leqslant |\dot{y}(z^{\tau_0}(t)) - \dot{y}(\overline{z}^{\tau_0}(t))| + |\dot{y}(\overline{z}^{\tau_0}(t)) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))|\\ &\leqslant |y|_{\mathbb{Y}^{2,\infty}_\alpha}\,|z-\overline{z}|_C + \alpha|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}\\ &\leqslant \alpha\zeta_1|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha} + \alpha|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}, \end{aligned} \quad (4.21)$$

$$\begin{aligned}
&|\dot{y}(z^{\tau_0}(t))\dot{z}(t) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |\dot{y}(z^{\tau_0}(t))\dot{z}(t) - \dot{y}(\overline{z}^{\tau_0}(t))\dot{z}(t)| + |\dot{y}(\overline{z}^{\tau_0}(t))\dot{z}(t) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{z}(t)| \\
&\qquad + |\dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{z}(t) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |y|_{\mathbb{Y}^{2,\infty}_\alpha} |\dot{z}(t)|\, |z - \overline{z}|_C + \alpha|\dot{z}(t)|\, |y - \overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + |\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} |\dot{z} - \dot{\overline{z}}|_C \\
&\quad \leqslant \alpha\zeta_1(\overline{\eta}_0 + 1)|z - \overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha} + \alpha\overline{\eta}_0|y - \overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}, \qquad (4.22)
\end{aligned}$$

$$\begin{aligned}
&|\dot{\widetilde{\varphi}}(z^{\tau_0}(t))\dot{z}(t) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |\dot{\widetilde{\varphi}}(z^{\tau_0}(t))\dot{z}(t) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{z}(t)| + |\dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{z}(t) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |\widetilde{\varphi}|_{W^{2,\infty}} |\dot{z}(t)|\, |z - \overline{z}|_C + |\widetilde{\varphi}|_{W^{2,\infty}} |\dot{z} - \dot{\overline{z}}|_C \\
&\quad \leqslant \alpha(\delta_1 + |\varphi^*|_{W^{2,\infty}})(1 + \overline{\eta}_0)|z - \overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}, \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
|\dot{\widetilde{\varphi}}(z^{\tau_0}(t)) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))| &\leqslant |\widetilde{\varphi}|_{W^{2,\infty}} |z - \overline{z}|_C \\
&\leqslant \alpha(\delta_1 + |\varphi^*|_{W^{2,\infty}})|z - \overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}, \qquad (4.24)
\end{aligned}$$

$$\begin{aligned}
|\dot{y}(z^{\tau_0}(t)) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))| &\leqslant |\dot{y}(z^{\tau_0}(t)) - \dot{y}(\overline{z}^{\tau_0}(t))| + |\dot{y}(\overline{z}^{\tau_0}(t)) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))| \\
&\leqslant \alpha^{1/q}|y - \overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + |y|_{\mathbb{Y}^{2,\infty}_\alpha} |z - \overline{z}|_C \\
&\leqslant \alpha^{1/q}|y - \overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + \alpha^{1/q}\zeta_1|z - \overline{z}|_{\mathbb{Y}^{2,p}_\alpha}, \qquad (4.25)
\end{aligned}$$

$$\begin{aligned}
&|\dot{y}(z^{\tau_0}(t))\dot{z}(t) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |\dot{y}(z^{\tau_0}(t))\dot{z}(t) - \dot{y}(\overline{z}^{\tau_0}(t))\dot{z}(t)| + |\dot{y}(\overline{z}^{\tau_0}(t))\dot{z}(t) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{z}(t)| \\
&\qquad + |\dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{z}(t) - \dot{\overline{y}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |y|_{\mathbb{Y}^{2,\infty}_\alpha} |\dot{z}(t)|\, |z - \overline{z}|_C + \alpha^{1/q}|\dot{z}(t)|\, |y - \overline{y}|_{\mathbb{Y}^{2,p}_\alpha} + |\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} |\dot{z} - \dot{\overline{z}}|_C \\
&\quad \leqslant \alpha^{1/q}\zeta_1(\overline{\eta}_0 + 1)|z - \overline{z}|_{\mathbb{Y}^{2,p}_\alpha} + \alpha^{1/q}\overline{\eta}_0|y - \overline{y}|_{\mathbb{Y}^{2,p}_\alpha}, \qquad (4.26)
\end{aligned}$$

$$\begin{aligned}
&|\dot{\widetilde{\varphi}}(z^{\tau_0}(t))\dot{z}(t) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |\dot{\widetilde{\varphi}}(z^{\tau_0}(t))\dot{z}(t) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{z}(t)| + |\dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{z}(t) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))\dot{\overline{z}}(t)| \\
&\quad \leqslant |\widetilde{\varphi}|_{W^{2,\infty}} |\dot{z}(t)|\, |z - \overline{z}|_C + |\widetilde{\varphi}|_{W^{2,\infty}} |\dot{z} - \dot{\overline{z}}|_C \\
&\quad \leqslant \alpha^{1/q}(\delta_1 + |\varphi^*|_{W^{2,\infty}})(1 + \overline{\eta}_0)|z - \overline{z}|_{\mathbb{Y}^{2,p}_\alpha}, \qquad (4.27)
\end{aligned}$$

$$\begin{aligned}
|\dot{\widetilde{\varphi}}(z^{\tau_0}(t)) - \dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t))| &\leqslant |\widetilde{\varphi}|_{W^{2,\infty}} |z - \overline{z}|_C \\
&\leqslant \alpha^{1/q}(\delta_1 + |\varphi^*|_{W^{2,\infty}})|z - \overline{z}|_{\mathbb{Y}^{2,p}_\alpha}. \qquad (4.28)
\end{aligned}$$

Using inequalities (4.15), (4.17), (4.19) and (4.21)–(4.24), we have

$$\begin{aligned}
A_2 \leqslant\ & \alpha\overline{\beta}_1\, |y - \overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} + 2\alpha\overline{\beta}_0 L_1|y - \overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} \\
&+ \overline{\beta}_0 L_1\alpha(\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}})|z - \overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha} + \overline{\beta}_2(1 + \overline{\eta}_0)\alpha|y - \overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} \\
&+ \overline{\beta}_2[\alpha\zeta_1(2 + \overline{\eta}_0) + \alpha(\delta_1 + |\varphi^*|_{W^{2,\infty}})(1 + \overline{\eta}_0) + \alpha(\delta_1 + |\varphi^*|_{W^{2,\infty}})] \\
&\times |z - \overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha} + 2\alpha L_2(1 + \overline{\eta}_0)(\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}})|y - \overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha} \\
&+ \alpha L_2(1 + \overline{\eta}_0)(\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}})^2|z - \overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha},
\end{aligned}$$

or

$$A_2 \leqslant \alpha\Theta_1|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}+\alpha\Theta_2|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}, \tag{4.29}$$

where

$$\begin{aligned}\Theta_1 &= \overline{\beta}_1+2\overline{\beta}_0L_1+\overline{\beta}_2(1+\overline{\eta}_0)+2L_2(1+\overline{\eta}_0)(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}}),\\ \Theta_2 &= L_1\overline{\beta}_0(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})+\overline{\beta}_2(2+\overline{\eta}_0)(\zeta_1+\delta_1+|\varphi^*|_{W^{2,\infty}})\\ &\quad +L_2(1+\overline{\eta}_0)(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})^2.\end{aligned}$$

Similarly, it follows from inequalities (4.16), (4.18), (4.20) and (4.25)–(4.28) that

$$A_2 \leqslant \alpha^{1/q}\Theta_1|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha}+\alpha^{1/q}\Theta_2|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}. \tag{4.30}$$

Finally, we consider A_4.

$$\begin{aligned}A_4 &= \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial g}{\partial\gamma_1}(t)\dot y(t)+\frac{\partial g}{\partial\gamma_2}(t)\dot z(t)-\frac{\partial \overline{g}}{\partial\gamma_1}(t)\dot{\overline{y}}(t)-\frac{\partial \overline{g}}{\partial\gamma_2}(t)\dot{\overline{z}}(t)\Big|\\ &\leqslant \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big\{\Big|\frac{\partial g}{\partial\gamma_1}(t)\dot y(t)-\frac{\partial g}{\partial\gamma_1}(t)\dot{\overline{y}}(t)\Big|+\Big|\frac{\partial g}{\partial\gamma_1}(t)\dot{\overline{y}}(t)-\frac{\partial \overline{g}}{\partial\gamma_1}(t)\dot{\overline{y}}(t)\Big|\\ &\quad +\Big|\frac{\partial g}{\partial\gamma_2}(t)\dot z(t)-\frac{\partial g}{\partial\gamma_2}(t)\dot{\overline{z}}(t)\Big|+\Big|\frac{\partial g}{\partial\gamma_2}(t)\dot{\overline{z}}(t)-\frac{\partial \overline{g}}{\partial\gamma_2}(t)\dot{\overline{z}}(t)\Big|\Big\}\\ &\leqslant \operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big\{\Big\|\frac{\partial g}{\partial\gamma_1}(t)\Big\|\,|\dot y-\dot{\overline{y}}|_C+\Big\|\frac{\partial g}{\partial\gamma_1}(t)-\frac{\partial \overline{g}}{\partial\gamma_1}(t)\Big\|\,|\dot{\overline{y}}|_C\\ &\quad +\Big\|\frac{\partial g}{\partial\gamma_2}(t)\Big\|\,|\dot z-\dot{\overline{z}}|_C+\Big\|\frac{\partial g}{\partial\gamma_2}(t)-\frac{\partial \overline{g}}{\partial\gamma_2}(t)\Big\|\,|\dot{\overline{z}}|_C\Big\}\\ &\leqslant \overline{\eta}_1|\dot y-\dot{\overline{y}}|_C+\overline{\beta}_0H_1(|y-\overline{y}|_C+|z-\overline{z}|_C)+\overline{\eta}_2|\dot z-\dot{\overline{z}}|_C\\ &\quad +\overline{\eta}_0H_2(|y-\overline{y}|_C+|z-\overline{z}|_C).\end{aligned}$$

Here and in the sequel, $\frac{\partial\overline{g}}{\partial\gamma_i}(\cdot)$ and $\frac{\partial^2\overline{g}}{\partial\gamma_i\partial\gamma_j}(\cdot)$, $i,j=1,2,3$, are evaluated at $((\overline{y}+\widetilde{\varphi})(\cdot),\overline{z}(\cdot)+\tau_0,\sigma)$, respectively. So by Lemmas 3.3 and 3.4, we have

$$\begin{aligned}A_4 &\leqslant \alpha(\overline{\eta}_1+\overline{\beta}_0H_1+\overline{\eta}_0H_2)|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}\\ &\quad +\alpha(\overline{\eta}_2+\overline{\beta}_0H_1+\overline{\eta}_0H_2)|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha},\end{aligned}\tag{4.31}$$

$$\begin{aligned}A_4 &\leqslant \alpha^{1/q}(\overline{\eta}_1+\overline{\beta}_0H_1+\overline{\eta}_0H_2)|y-\overline{y}|_{\mathbb{Y}^{2,p}_\alpha}\\ &\quad +\alpha^{1/q}(\overline{\eta}_2+\overline{\beta}_0H_1+\overline{\eta}_0H_2)|z-\overline{z}|_{\mathbb{Y}^{2,p}_\alpha}.\end{aligned}\tag{4.32}$$

Now, we are able to estimate $S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)$ in $\mathbb{Y}^{2,\infty}_\alpha\times\mathbb{Y}^{2,\infty}_\alpha$ norm, $\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha$ norm and $\mathbb{Y}^{2,p}_\alpha\times\mathbb{Y}^{2,p}_\alpha$ norm by means of A_1, A_2, A_3, and A_4. Using (4.9), (4.12), (4.29), and (4.31), we have

$$\begin{aligned}&|S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)|_{\mathbb{Y}^{2,\infty}_\alpha\times\mathbb{Y}^{2,\infty}_\alpha}\\ &\leqslant \max\{2\alpha L_0|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}+L_0\alpha(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha},\\ &\quad \alpha\Theta_1|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}+\alpha\Theta_2|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}\}\\ &\quad +\max\{H_0\alpha(|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}+|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}),\\ &\quad \alpha(\overline{\eta}_1+\overline{\beta}_0H_1+\overline{\eta}_0H_2)|y-\overline{y}|_{\mathbb{Y}^{2,\infty}_\alpha}+\alpha(\overline{\eta}_2+\overline{\beta}_0H_1+\overline{\eta}_0H_2)|z-\overline{z}|_{\mathbb{Y}^{2,\infty}_\alpha}\}.\end{aligned}$$

Hence,

$$|S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)|_{\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}} \leqslant \alpha N_1(|y-\overline{y}|_{\mathbb{Y}_\alpha^{2,\infty}}+|z-\overline{z}|_{\mathbb{Y}_\alpha^{2,\infty}}), \tag{4.33}$$

where

$$\begin{aligned} N_1 = &\max\{L_0\max\{2,\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}}\},\max\{\Theta_1,\Theta_2\}\} \\ &+\max\{H_0,\max\{\overline{\eta}_1+\overline{\beta}_0H_1+\overline{\eta}_0H_2,\overline{\eta}_2+\overline{\beta}_0H_1+\overline{\eta}_0H_2\}\} \\ \leqslant &N_0. \end{aligned}$$

Therefore, the operator S is a uniform $|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}}$ contraction over $W\times V$ since $\alpha N_1 \leqslant \alpha N_0 = c < 1$. Similarly, by assumptions (A1)–(A3), and inequalities (4.11), (4.13), (4.30), and (4.32), we have

$$\begin{aligned} &|S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \\ &=\Big(\int_0^\alpha|\hat{f}(t;y,z,\varphi,\tau_0,\sigma)-\hat{f}(t;\overline{y},\overline{z},\varphi,\tau_0,\sigma)|^p \\ &\quad+\Big|\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(t;y,z,\varphi,\tau_0,\sigma)-\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(t;\overline{y},\bar{z},\varphi,\tau_0,\sigma)\Big|^p\mathrm{d}t\Big)^{1/p} \\ &\quad+\Big(\int_0^\alpha|\hat{g}(t;y,z,\varphi,\tau_0,\sigma)-\hat{g}(t;\overline{y},\overline{z},\varphi,\tau_0,\sigma)|^p \\ &\quad+\Big|\frac{\mathrm{d}}{\mathrm{d}t}\hat{g}(t;y,z,\varphi,\tau_0,\sigma)-\frac{\mathrm{d}}{\mathrm{d}t}\hat{g}(t;\overline{y},\bar{z},\varphi,\tau_0,\sigma)\Big|^p\mathrm{d}t\Big)^{1/p} \\ &\leqslant\Big(\int_0^\alpha(A_1^p+A_2^p)\mathrm{d}t\Big)^{1/p}+\Big(\int_0^\alpha(A_3^p+A_4^p)\mathrm{d}t\Big)^{1/p} \\ &\leqslant\alpha^{1/p}(A_1+A_2+A_3+A_4) \\ &\leqslant\alpha N_0(|y-\overline{y}|_{\mathbb{Y}_\alpha^{2,p}}+|z-\overline{z}|_{\mathbb{Y}_\alpha^{2,p}}) \\ &=c(|y-\overline{y}|_{\mathbb{Y}_\alpha^{2,p}}+|z-\overline{z}|_{\mathbb{Y}_\alpha^{2,p}}). \end{aligned}$$

i.e., S is a uniform $|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,p}}$ contraction over $W\times V$. It also follows from inequalities (4.10) and (4.14) that

$$\begin{aligned} |S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}} &\leqslant\Big(\int_0^\alpha(A_1^p+A_3^p)\mathrm{d}t\Big)^{1/p} \\ &\leqslant\alpha^{1/p}(A_1+A_3) \\ &\leqslant\alpha N_2(|y-\overline{y}|_{\mathbb{Y}_\alpha^{1,p}}+|z-\overline{z}|_{\mathbb{Y}_\alpha^{1,p}}), \end{aligned}$$

where

$$N_2=L_0\max\{2,\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}}\}+H_0\leqslant N_0.$$

Thus,

$$|S(y,z,\varphi,\tau_0,\sigma)-S(\overline{y},\overline{z},\varphi,\tau_0,\sigma)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\leqslant c(|y-\overline{y}|_{\mathbb{Y}_\alpha^{1,p}}+|z-\overline{z}|_{\mathbb{Y}_\alpha^{1,p}}),$$

i.e., S is a uniform $|\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}$ contraction over $W\times V$. This proves part (iv) (b) of the lemma.

Part (iv) (c) of the lemma follows easily from the continuity of f, g and the variables y, z, and the Lebesgue Dominated Convergence Theorem. This completes the proof. □

Combining Lemmas 3.11, 4.1 and Theorem 2.1, we can obtain the existence, uniqueness, and continuous dependence on parameters for solutions of system (1.1), which is formulated in the following theorem.

Theorem 4.1 *Assume that* (A1)–(A3) *hold,* $1\leqslant p<\infty$, *and* $\varphi^*,\tau_0^*>0$, σ^* *satisfy* $\varphi^*(0)\in\Omega_1$, $\varphi^*(-\tau_0^*)\in\Omega_2$, $\tau_0^*\in\Omega_4$ *and* $\sigma^*\in\Omega_3$. *Then there exist* $\alpha>0$ *and a neighborhood of the parameters, where system* (1.1) *has a unique solution* $(x(\varphi,\tau_0,\sigma),\tau(\varphi,\tau_0,\sigma))(\cdot)$ *on* $[0,\alpha]$, *which depends continuously on the parameters* φ, σ *and* τ_0 *in the norms* $|\cdot|_{\mathbb{X}_\alpha^{2,\infty}\times\mathbb{X}_\alpha^{2,\infty}}$, $|\cdot|_{\mathbb{X}_\alpha^{2,p}\times\mathbb{X}_\alpha^{2,p}}$ *and* $|\cdot|_{\mathbb{X}_\alpha^{1,p}\times\mathbb{X}_\alpha^{1,p}}$, *and hence in the norms* $|\cdot|_{W_\alpha^{2,\infty}\times W_\alpha^{2,\infty}}$, $|\cdot|_{W_\alpha^{2,p}\times W_\alpha^{2,p}}$ *and* $|\cdot|_{W_\alpha^{1,p}\times W_\alpha^{1,p}}$, *respectively.*

Theorem 4.1 establishes the existence, uniqueness, and continuous dependence on parameters for solutions in $(\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty},|\cdot|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}})$. We can also obtain the first-order differentiability in $(\mathbb{Y}_\alpha^{2,\infty},|\cdot|_{\mathbb{Y}_\alpha^{2,p}})$. This gives first-order differentiability in a stronger norm in a subspace of the functional space that Hartung and Turi [19] worked on, where they obtained the first-order differentiability of solutions with respect to parameters in the locally complete double-normed linear space $(\mathbb{Y}_\alpha^{1,\infty},|\cdot|_{\mathbb{Y}_\alpha^{1,p}})$ with respect to the norm $|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}$. We state this result below. The proof is similar to that in Hartung and Turi [19] and can be made available to readers upon request.

Theorem 4.2 *Assume that* (A1)–(A3) *hold,* $1\leqslant p<\infty$, *and* φ^*, τ_0^*, σ^* *satisfy* $\varphi^*(0)\in\Omega_1$, $\varphi^*(-\tau_0^*)\in\Omega_2$, $\tau_0^*\in\Omega_4$ *and* $\sigma^*\in\Omega_3$. *Then there exist* $\alpha>0$, $\delta_1,\delta_2,\delta_3>0$ *such that system* (1.1) *has a unique solution* $(x(\varphi,\tau_0,\sigma),\tau(\varphi,\tau_0,\sigma))(\cdot)$ *on* $[0,\alpha]$ *corresponding to* $(\varphi,\tau_0,\sigma)\in\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1)$ $\times\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)\times\mathscr{G}_\Sigma(\sigma^*;\delta_3)$. *Moreover, the function*

$$
\begin{aligned}
V\subset W^{2,\infty}\times\mathbb{R}^+\times\Sigma &\to \mathbb{X}_\alpha^{2,p}\times\mathbb{X}_\alpha^{2,p},\\
(\varphi,\tau_0,\sigma) &\mapsto (x(\varphi,\tau_0,\sigma)(\cdot),\tau(\varphi,\tau_0,\sigma)(\cdot))
\end{aligned}
$$

is C^1 *with respect to* φ, σ, *and* τ_0 *on its domain in the norm* $\mathbb{X}_\alpha^{2,p}\times\mathbb{X}_\alpha^{2,p}$ *and hence in the norm* $W_\alpha^{2,p}\times W_\alpha^{2,p}$.

5 Second-order differentiability with respect to parameters in $W_\alpha^{1,p}$-norm

We now focus on the second-order differentiability with respect to parameters using the uniform contraction principle for locally complete double-normed linear spaces. To achieve this, we would have to obtain the existence and continuity of the second-order differentials of the integral operator $S(y,z,\varphi,\tau_0,\sigma)$. However, the range spaces of the second-order derivatives of the integral operator are no longer in $\mathbb{Y}_\alpha^{2,p} \times \mathbb{Y}_\alpha^{2,p}$, but in $\mathbb{Y}_\alpha^{1,p} \times \mathbb{Y}_\alpha^{1,p}$ only. This means that, in general, we cannot obtain the second-order differentiability with respect to parameters in the norm $|\cdot|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}}$ by means of Theorem 2.1.

Fortunately, we know that $(\mathbb{Y}_\alpha^{2,\infty}, |\cdot|_{\mathbb{Y}_\alpha^{2,p}})$ is a locally complete triple-normed linear space with respect to the norms $|\cdot|_{\mathbb{Y}_\alpha^{2,\infty}}$ and $|\cdot|_{\mathbb{Y}_\alpha^{1,p}}$. Also, as mentioned earlier, Hartung and Turi have proved (see Lemma 3.7 in Ref. [19]) that $(\mathbb{Y}_\alpha^{1,\infty}, |\cdot|_{\mathbb{Y}_\alpha^{1,p}})$ is a locally complete double-normed linear space with respect to the norm $|\cdot|_{\mathbb{Y}_\alpha^{1,\infty}}$, and $S(u,v)$ is C^1 in the norm $|\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}$, where $u=(y,z)$, $v=(\varphi,\sigma,\tau_0)$. We can check the conditions in (vi) of Theorem 2.1 to obtain the second-order differentiability of solutions in the norm $|\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}$. In Lemma 5.3 below, we will show that $S(u,v)$ is C_p^2 in the norm $|\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}$, and its second-order partial derivatives are bounded bilinear operators.

We also remark that Lemma 5.2 below will be needed to verify condition (vi) (b) of Theorem 2.1.

Given $\varphi^*, \tau_0^* > 0, \sigma^*$ satisfying the assumptions in Lemma 4.1, let M_i $(i = 1,2,\dots,5)$ be the sets in the statement of Lemma 4.1, and let δ_1, L_0, H_0, β_0, β_1 and the others be the numbers chosen in the proof of Lemma 4.1. For convenience in the sequel, we introduce the following constants:

$$d_1 := \beta_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}}, \quad d_2 := \beta_1 + \delta_1 + |\varphi^*|_{W^{2,\infty}},$$

$$L_{m1} := \max\Big\{\Big\|\frac{\partial f}{\partial\theta_i}(\theta_1,\theta_2,\theta_3)\Big\|:\ \theta_1\in M_1,\ \theta_2\in M_2,\ \theta_3\in M_3,\ i=1,2,3\Big\},$$

$$H_{m1} := \max\Big\{\Big\|\frac{\partial g}{\partial\gamma_i}(\gamma_1,\gamma_2,\gamma_3)\Big\|:\ \gamma_1\in M_1,\ \gamma_2\in M_4,\ \gamma_3\in M_3,\ i=1,2,3\Big\},$$

$$L_{m2} := \max\Big\{\Big\|\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}(\theta_1,\theta_2,\theta_3)\Big\|: \theta_1\in M_1,\theta_2\in M_2,\theta_3\in M_3,i,j=1,2,3\Big\},$$

$$H_{m2} := \max\Big\{\Big\|\frac{\partial^2 g}{\partial\gamma_i\partial\gamma_j}(\gamma_1,\gamma_2,\gamma_3)\Big\|: \gamma_1\in M_1,\gamma_2\in M_4,\gamma_3\in M_3,i,j=1,2,3\Big\}.$$

In our discussion, we need the differentiability of the following composition operator:

$$\omega\colon\ \mathscr{G}_{W_\alpha^{2,\infty}}(y^*;\overline{\delta}_1)\times\mathscr{G}_{W_\alpha^{2,\infty}}(z^*;\overline{\delta}_2)\times\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\overline{\delta}_3)\subset\mathbb{X}_\alpha^{2,p}\times\mathbb{X}_\alpha^{2,p}\times\mathbb{R}^+\to L_{0,\alpha}^p,$$

$$\omega(y,z,\tau_0) := y(z^{\tau_0}(t)), \tag{5.1}$$

where $\overline{\delta}_1$, $\overline{\delta}_2$, $\bar{\delta}_3$ are given positive constants.

Lemma 5.1 *The composite operator $\omega(y,z,\tau_0)$ defined by* (5.1) *has continuous partial derivatives*

$$\frac{\partial^2\omega(y,z,\tau_0)}{\partial y^2}(l,m) = 0, \quad l,m\in\mathbb{X}_\alpha^{2,p}, \tag{5.2}$$

$$\Big(\frac{\partial^2\omega(y,z,\tau_0)}{\partial z^2}(l,m)\Big)(t)=\ddot{y}(z^{\tau_0}(t))l(t)m(t),\quad l,m\in\mathbb{X}_\alpha^{2,p},\tag{5.3}$$

$$\Big(\frac{\partial^2\omega(y,z,\tau_0)}{\partial\tau_0^2}\Big)(t)=\ddot{y}(z^{\tau_0}(t)).$$

Proof Noting $\dot{y}\in W^{1,\infty}$, we can easily use Lemma 5.4 in Ref. [19] for the case of the first derivative to obtain the conclusions. □

Lemma 5.2 *Assume that* (A1)–(A3) *hold, and let* $1\leqslant p<\infty$, $\varphi^*,\tau_0^*>0$, σ^*, α, δ_1, δ_2, δ_3, M_1, M_2, M_3, M_4, M_5, U, V, W *be from Lemma* 4.1. *Then the first derivative* $\mathscr{S}_u(u,v)$ *of the operator* $S(u,v)$,

$$S\colon\ U\times V\subset\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}\times W^{2,\infty}\times\mathbb{R}^+\times\Sigma\to\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p},$$

defined by (4.3) *satisfies that, for any* $(u,v)\in W\times V$, *the linear operator* $\mathscr{S}_u(u,v)\in\mathscr{L}(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p},\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p})$, *and there exists* $0\leqslant\theta<1$ *such that* $|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\leqslant\theta$ *and* $|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}}\leqslant\theta$.

Proof We know that the first differential of the operator S is given by

$$(\mathscr{S}_u(u,v)h)(t)=\begin{cases}[0,0]^{\mathrm{T}}, & t\in[-r,0],\\ \Big[\displaystyle\int_0^t\Big(\frac{\partial f}{\partial\theta_1}(s)h_1(s)+\frac{\partial f}{\partial\theta_2}(s)(h_1(z^{\tau_0}(s)) \\ \quad -(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))\Big)\mathrm{d}s, \\ \quad\displaystyle\int_0^t\Big(\frac{\partial g}{\partial\gamma_1}(s)h_1(s)+\frac{\partial g}{\partial\gamma_2}(s)h_2(s)\Big)\mathrm{d}s\Big]^{\mathrm{T}}, & t\in[0,\alpha],\end{cases}$$

for $h=(h_1,h_2)\in\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}$. From this expression, it is easy to see that $\mathscr{S}_u(u,v)$ is linear in h, and if $h=(h_1,h_2)\in\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$, then $\mathscr{S}_u(u,v)h\in\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$. This shows that $\mathscr{S}_u(u,v)$ is a linear operator from $\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$ to $\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$. Now, we turn to estimate the norm of the operator $\mathscr{S}_u(u,v)$. Let

$$S_1=\{h\in\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}\colon\ |h|_{\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}}=1\}.$$

We have

$$\begin{aligned}&|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}}\\ &\ =\sup_{h\in S_1}|\mathscr{S}_u(u,v)h|_{\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}}\\ &\ =\sup_{h\in S_1}\Big(\operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial f}{\partial\theta_1}(t)h_1(t)+\frac{\partial f}{\partial\theta_2}(t)(h_1(z^{\tau_0}(t))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))h_2(t))\Big|\\ &\qquad+\operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial g}{\partial\gamma_1}(t)h_1(t)+\frac{\partial g}{\partial\gamma_2}(t)h_2(t)\Big|\Big).\end{aligned}$$

It follows from Lemma 4.1 that

$$W=(\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{2,\infty}}(\zeta_1)\cap\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{1,\infty}}(\overline{\beta}_0))\times(\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{2,\infty}}(\xi_1)\cap\overline{\mathscr{G}}_{\mathbb{Y}_\alpha^{1,\infty}}(\eta_0^-,\eta_0^+)),$$

$$\begin{aligned}V&=\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1)\times\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2)\times\mathscr{G}_\Sigma(\sigma^*;\delta_3),\\N_0&=L_0\max\{2,\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}}\}+\max\{\Theta_1,\Theta_2\}+H_0\\&\quad+\max\{\overline{\eta}_1+\overline{\beta}_0H_1+\overline{\eta}_0H_2,\overline{\eta}_2+\overline{\beta}_0H_1+\overline{\eta}_0H_2\}.\end{aligned}$$

Using Lemma 3.2, we have

$$\begin{aligned}&|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}}\\&=\sup_{h\in S_1}\Big(\operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial f}{\partial\theta_1}(t)h_1(t)+\frac{\partial f}{\partial\theta_2}(t)(h_1(z^{\tau_0}(t))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))h_2(t))\Big|\\&\quad+\operatorname*{ess\,sup}_{0\leqslant t\leqslant\alpha}\Big|\frac{\partial g}{\partial\gamma_1}(t)h_1(t)+\frac{\partial g}{\partial\gamma_2}(t)h_2(t)\Big|\Big)\\&\leqslant\sup_{h\in S_1}\Big(\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial f}{\partial\theta_1}(t)\Big\|\,|h_1(t)|_C+\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial f}{\partial\theta_2}(t)\Big\|\\&\quad\times|h_1(z^{\tau_0}(t))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))h_2(t)|_C\\&\quad+\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial g}{\partial\gamma_1}(t)\Big\|\,|h_1(t)|_C+\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial g}{\partial\gamma_2}(t)\Big\|\,|h_2(t)|_C\Big)\\&\leqslant\sup_{h\in S_1}(L_0\alpha|h_1|_{\mathbb{Y}_\alpha^{1,\infty}}+L_0(\alpha|h_1|_{\mathbb{Y}_\alpha^{1,\infty}}+(\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}})\alpha|h_2|_{\mathbb{Y}_\alpha^{1,\infty}})\\&\quad+H_0\alpha|h_1|_{\mathbb{Y}_\alpha^{1,\infty}}+H_0\alpha|h_2|_{\mathbb{Y}_\alpha^{1,\infty}})\\&\leqslant\alpha L_0\max\{2,\overline{\beta}_0+\delta_1+|\varphi^*|_{W^{1,\infty}}\}+\alpha H_0\\&\leqslant\alpha N_0\\&<1.\end{aligned}$$

Again, let $\theta=\alpha N_0$. Then $\theta\in[0,1)$ and $|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}}\leqslant\theta$.

Similarly, let

$$S_2=\{h\in\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}\colon\ |h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}=1\}.$$

We have

$$\begin{aligned}|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}&=\sup_{h\in S_2}\bigg(\bigg(\int_0^\alpha\Big|\frac{\partial f}{\partial\theta_1}(t)h_1(t)+\frac{\partial f}{\partial\theta_2}(t)(h_1(z^{\tau_0}(t))\\&\quad-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))\times h_2(t))\Big|\mathrm{d}t\bigg)^{1/p}\\&\quad+\bigg(\int_0^\alpha\Big|\frac{\partial g}{\partial\gamma_1}(t)h_1(t)+\frac{\partial g}{\partial\gamma_2}(t)h_2(t)\Big|\mathrm{d}t\bigg)^{1/p}\bigg)\\&\leqslant\alpha^{1/p}\sup_{h\in S_2}\Big(\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial f}{\partial\theta_1}(t)\Big\|\,|h_1(t)|_C+\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial f}{\partial\theta_2}(t)\Big\|\\&\quad\times|h_1(z^{\tau_0}(t))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))h_2(t)|_C\\&\quad+\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial g}{\partial\gamma_1}(t)\Big\|\,|h_1(t)|_C+\max_{0\leqslant t\leqslant\alpha}\Big\|\frac{\partial g}{\partial\gamma_2}(t)\Big\|\,|h_2(t)|_C\Big)\\&\leqslant\alpha^{1/p}\sup_{h\in S_2}(L_0\alpha^{1/q}|h_1|_{\mathbb{Y}_\alpha^{1,p}}+L_0(\alpha^{1/q}|h_1|_{\mathbb{Y}_\alpha^{1,p}}\end{aligned}$$

$$
\begin{aligned}
&+ (\overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}})\alpha^{1/q}|h_2|_{\mathbb{Y}_\alpha^{1,p}}) \\
&+ H_0\alpha^{1/q}|h_1|_{\mathbb{Y}_\alpha^{1,p}} + H_0\alpha^{1/q}|h_2|_{\mathbb{Y}_\alpha^{1,p}}) \\
\leqslant\ & \alpha L_0 \max\{2, \overline{\beta}_0 + \delta_1 + |\varphi^*|_{W^{1,\infty}}\} + \alpha H_0 \\
\leqslant\ & \theta,
\end{aligned}
$$

i.e.,

$$|\mathscr{S}_u(u,v)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}} \leqslant \theta.$$

This completes the proof. □

Lemma 5.3 *Assume* (A1)–(A3) *and let* $1 \leqslant p < \infty$, φ^*, $\tau_0^* > 0$, σ^*, α, δ_1, δ_2, δ_3, M_1, M_2, M_3, M_4, M_5, U, V, W *be from Lemma* 4.1. *Then the operator*

$$
\begin{aligned}
S\colon\ U\times V \subset\ & (\mathbb{Y}_\alpha^{2,\infty}\times\mathbb{Y}_\alpha^{2,\infty}, |\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}) \times W^{2,\infty}\times\mathbb{R}^+\times\Sigma \\
&\to (\mathbb{Y}_\alpha^{1,\infty}\times\mathbb{Y}_\alpha^{1,\infty}, |\cdot|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}})
\end{aligned}
$$

defined by (4.3) *has pointwise continuous second-order partial derivatives with respect to* $u = (y,z)$ *and with respect to* $v = (\varphi,\tau_0,\sigma)$. *Moreover,*

$$
\begin{aligned}
\frac{\partial^2 S}{\partial u^2}(u,v) &= \mathscr{S}_{uu}(u,v) \in \mathscr{L}^2(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}, \mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}),\\
\frac{\partial^2 S}{\partial v^2}(u,v) &= \mathscr{S}_{vv}(u,v) \in \mathscr{L}^2(W^{2,\infty}\times\mathbb{R}^+\times\Sigma, \mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}),\\
\frac{\partial^2 S}{\partial u\partial v}(u,v) &= \mathscr{S}_{uv}(u,v) \in \mathscr{L}((W^{2,\infty}\times\mathbb{R}^+\times\Sigma)\times(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}), \mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}),
\end{aligned}
$$

where

$$
\begin{aligned}
&(\mathscr{S}_{uu}(u,v)(h,k))(t)\\
&= \begin{cases}
[0,0]^{\mathrm{T}}, & t\in[-r,0],\\
\Big[\displaystyle\int_0^t \Big(\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[(h_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s))k_1(s) & \\
\quad +(k_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))k_2(s))h_1(s)] + \dfrac{\partial^2 f}{\partial\theta_1^2}(s)h_1(s)k_1(s) & \\
\quad +\dfrac{\partial^2 f}{\partial\theta_2^2}(s)[h_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)] & \\
\quad \times[k_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))k_2(s)] - \dfrac{\partial f}{\partial\theta_2}(s)[\dot k_1(z^{\tau_0}(s))h_2(s) & \\
\quad +\dot h_1(z^{\tau_0}(s))k_2(s) - (\ddot y + \ddot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)k_2(s)]\Big)\mathrm{d}s, & \\
\quad \displaystyle\int_0^t \Big(\frac{\partial^2 g}{\partial\gamma_1^2}(s)h_1(s)k_1(s) + \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)(h_2(s)k_1(s) + k_2(s)h_1(s)) & \\
\quad +\dfrac{\partial^2 g}{\partial\gamma_2^2}(s)h_2(s)k_2(s)\Big)\mathrm{d}s\Big]^{\mathrm{T}}, & t\in[0,\alpha],
\end{cases}
\end{aligned}
$$

with

$$h = (h_1,h_2),\ \ k = (k_1,k_2) \in \mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p};$$

$$(\mathscr{S}_{uv}(u,v)(h,l))(t)$$

$$
= \begin{cases} [0,0]^{\mathrm{T}}, & t\in[-r,0], \\ \Big[\int_0^t \Big(\frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0)h_1(s) + \frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[(\widetilde{l}_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2) \\ \times h_1(s) + (h_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s))l_1(0)] + \frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_3h_1(s) \\ +\Big[\frac{\partial^2 f}{\partial\theta_2^2}(s)(\widetilde{l}_1(s) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2) + \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)l_3\Big][h_1(z^{\tau_0}(s)) \\ -(\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)] + \frac{\partial f}{\partial\theta_2}(s)[-\dot h_1(z^{\tau_0}(s))l_2 \\ +(\ddot y + \ddot{\tilde\varphi})(z^{\tau_0}(s))l_2h_2(s) - \dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s)]\Big)\mathrm{d}s, \\ \int_0^t \Big(\Big[\frac{\partial^2 g}{\partial\gamma_1^2}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_3\Big]h_1(s) \\ +\Big[\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_2^2}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_3\Big]h_2(s)\Big)\mathrm{d}s\Big]^{\mathrm{T}}, & t\in[0,\alpha], \end{cases}
$$

with $h=(h_1,h_2)\in\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p},\ l=(l_1,l_2,l_3)\in W^{2,\infty}\times\mathbb{R}^+\times\Sigma$;

$$
(\mathscr{S}_{vv}(u,v)(l,m))(t)
$$

$$
= \begin{cases} [0,0]^{\mathrm{T}}, & t\in[-r,0], \\ \Big[\int_0^t \Big(\Big[\frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0) + \frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)(\widetilde{l}_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2) \\ +\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_3\Big]m_1(0) + \Big[\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)l_1(0) + \frac{\partial^2 f}{\partial\theta_2^2}(s)(\widetilde{l}_1(s) \\ -(\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2) + \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)l_3\Big](\widetilde{m}_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))m_2) \\ +\Big[\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_1(0) + \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)\big(\widetilde{l}_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2\big) \\ +\frac{\partial^2 f}{\partial\theta_3^2}(s)l_3\Big]m_3 \\ +\frac{\partial f}{\partial\theta_2}(s)[-\dot{\widetilde{m}}_1(z^{\tau_0}(s))l_2 + (\ddot y + \ddot{\tilde\varphi})(z^{\tau_0}(s))l_2m_2 - \dot{\widetilde{l}}_1(z^{\tau_0}(s))m_2]\Big)\mathrm{d}s, \\ \int_0^t \Big(\Big[\frac{\partial^2 g}{\partial\gamma_1^2}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_3\Big]m_1(0) \\ +\Big[\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_2^2}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_3\Big]m_2 \\ +\Big[\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_3^2}(s)l_3\Big]m_3\Big)\mathrm{d}s\Big]^{\mathrm{T}}, & t\in[0,\alpha], \end{cases}
$$

with $l=(l_1,l_2,l_3),\ m=(m_1,m_2,m_3)\in W^{2,\infty}\times\mathbb{R}^+\times\Sigma$.

Proof We divide the proof into three parts.

Part 1 The second-order partial derivative of $S(u,v)$ with respect to u.

First of all, by the C^2 smoothness of $f(\theta_1,\theta_2,\theta_3)$ and $g(\gamma_1,\gamma_2,\gamma_3)$ and the definitions of y, z, h, k, φ, τ_0, σ, we know from the expression of $\mathscr{S}_{uu}(u,v)$

that $\mathscr{S}_{uu}(u,v)$ is a bilinear operator from $(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p})\times(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p})$ to $\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$. In the following, we verify that $\mathscr{S}_{uu}(u,v)$ is bounded on $(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p})\times(\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p})$ and is the continuous second derivative of $S(u,v)$.

We check the boundedness of $\mathscr{S}_{uu}(u,v)$ as follows. Suppose $h=(h_1,h_2)$, $k=(k_1,k_2)\in\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$. One has

$$
\begin{aligned}
&|\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\
&=\Big[\int_0^\alpha\Big|\frac{\partial^2 f}{\partial\theta_1^2}(t)h_1(t)k_1(t)+\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(t)[(h_1(z^{\tau_0}(t))\\
&\quad-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))h_2(t))k_1(t)+(k_1(z^{\tau_0}(t))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))k_2(t))h_1(t)]\\
&\quad+\frac{\partial^2 f}{\partial\theta_2^2}(t)[h_1(z^{\tau_0}(t))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))h_2(t)]\\
&\quad\times[k_1(z^{\tau_0}(t))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))k_2(t)]+\frac{\partial f}{\partial\theta_2}(t)\\
&\quad\times[(\ddot y+\ddot{\tilde\varphi})(z^{\tau_0}(t))h_2(t)k_2(t)-\dot k_1(z^{\tau_0}(t))h_2(t)-\dot h_1(z^{\tau_0}(t))k_2(t)]\Big|^p\mathrm{d}t\Big]^{1/p}\\
&\quad+\Big[\int_0^\alpha\Big|\frac{\partial^2 g}{\partial\gamma_1^2}(t)h_1(t)k_1(t)+\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(t)(h_2(t)k_1(t)+k_2(t)h_1(t))\\
&\quad+\frac{\partial^2 g}{\partial\gamma_1^2}(t)h_2(t)k_2(t)\Big|^p\mathrm{d}t\Big]^{1/p}.
\end{aligned}
$$

Note that

$$
\Big\|\frac{\partial f}{\partial\theta_i}(t)\Big\|\leqslant L_{m1},\quad\Big\|\frac{\partial g}{\partial\gamma_i}(t)\Big\|\leqslant H_{m1},
$$
$$
\Big\|\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}(t)\Big\|\leqslant L_{m2},\quad\Big\|\frac{\partial^2 g}{\partial\gamma_i\partial\gamma_j}(t)\Big\|\leqslant H_{m2},
$$

$i,j=1,2,3$. By Lemma 3.6, we have

$$
|\dot y(z^{\tau_0}(t))|_{L^p_{0,\alpha}}=\Big(\int_0^\alpha|\dot y(z^{\tau_0}(t))|^p\mathrm{d}t\Big)^{1/p}\leqslant\frac{1}{1-\eta^+}|\dot y|_{L^p_\alpha}\leqslant\frac{1}{1-\eta^+}|y|_{\mathbb{Y}_\alpha^{1,p}}
$$

for $(y,z)\in U$. By (A1) and Minkowski's Inequality, we get

$$
\begin{aligned}
&|\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\
&\leqslant L_{m2}|h_1|_C\,|k_1|_{L^p_{0,\alpha}}+L_{m2}|h_1(z^{\tau_0}(t))|_C\,|k_1|_{L^p_{0,\alpha}}\\
&\quad+L_{m2}|(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}}\,|k_1|_C\\
&\quad+L_{m2}|k_1(z^{\tau_0}(t))|_C\,|h_1|_{L^p_{0,\alpha}}+L_{m2}|h_1|_C\,|(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))k_2(t)|_{L^p_{0,\alpha}}\\
&\quad+L_{m2}|h_1(z^{\tau_0}(t))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))h_2(t)|_C\\
&\quad\times|k_1(z^{\tau_0}(t))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(t))k_2(t)|_{L^p_{0,\alpha}}\\
&\quad+L_{m1}|(\ddot y+\ddot{\tilde\varphi})(z^{\tau_0}(t))h_2(t)k_2(t)|_{L^p_{0,\alpha}}
\end{aligned}
$$

$$+ L_{m1}|\dot{k}_1(z^{\tau_0}(t))h_2(t) + \dot{h}_1(z^{\tau_0}(t))k_2(t)|_{L^p_{0,\alpha}}$$
$$+ H_{m2}(|h_1|_C\,|k_1|_{L^p_{0,\alpha}} + |h_1|_C\,|k_2|_{L^p_{0,\alpha}} + |h_2|_C\,|k_1|_{L^p_{0,\alpha}} + |h_2|_C\,|k_2|_{L^p_{0,\alpha}}).$$

Also, by Lemmas 3.2 and 3.4, we have

$$\begin{aligned}
&|\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\\
&\leqslant \alpha^{1+\frac{1}{q}}(4L_{m2}+H_{m2})|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|k_1|_{\mathbb{Y}^{1,p}_\alpha}\\
&\quad + \big(2\alpha^{1+\frac{1}{q}}L_{m2}d_1 + L_{m1}\alpha^{1/q}(1-\eta^+)^{-1} + H_{m2}\alpha^{1+\frac{1}{q}}\big)|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|k_2|_{\mathbb{Y}^{1,p}_\alpha}\\
&\quad + \big(2\alpha^{1+\frac{1}{q}}L_{m2}c_1L_{m1}\alpha^{1/q}(1-\eta^+)^{-1} + H_{m2}\alpha^{1+\frac{1}{q}}\big)|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|k_1|_{\mathbb{Y}^{1,p}_\alpha}\\
&\quad + \alpha^{1+\frac{1}{q}}\big(L_{m2}c_1^2 + L_{m1}d_2 + H_{m2}\big)|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|k_2|_{\mathbb{Y}^{1,p}_\alpha}.
\end{aligned}$$

Let

$$\begin{aligned}
\widetilde{L} = \alpha^{1+\frac{1}{q}}\max\{&4L_{m2}+H_{m2},\, 2L_{m2}c_1 + L_{m1}\alpha^{-1}(1-\eta^+)^{-1} + H_{m2},\\
&L_{m2}d_1^2 + L_{m1}d_2 + H_{m2}\}.
\end{aligned}$$

Then one has

$$|\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha} \leqslant \widetilde{L}|h|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\cdot|k|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha},$$

that is, $\mathscr{S}_{uu}(u,v)$ is a bounded bilinear operator in $\mathscr{L}^2(\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha, \mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha)$.

Next, we show that $\mathscr{S}_{uu}(u,v)$ is the second-order partial derivative of the operator $S(u,v)$ with respect to $u=(y,z)$. Let $u\in U$. Choose $h=(h_1,h_2)$, $k=(k_1,k_2)\in\mathbb{Y}^{2,p}_\alpha\times\mathbb{Y}^{2,p}_\alpha$ such that $h+u,\ k+u\in U$. Again, denote $\overline{u}=(\overline{y},\overline{z}):=(y+h_1,z+h_2)$. Then

$$\begin{aligned}
&|\mathscr{S}_u(\overline{u},v)k - \mathscr{S}_u(u,v)k - \mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\\
&\leqslant \bigg(\int_0^\alpha \Big|\frac{\partial\overline{f}}{\partial\theta_1}(t)k_1(t) + \frac{\partial\overline{f}}{\partial\theta_2}(t)(k_1(\overline{z}^{\tau_0}(t)) - (\dot{\overline{y}}+\dot{\widetilde{\varphi}})(\overline{z}^{\tau_0}(t))k_2(t))\\
&\quad - \frac{\partial f}{\partial\theta_1}(t)k_1(t) - \frac{\partial f}{\partial\theta_2}(t)(k_1(z^{\tau_0}(t)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))k_2(t))\\
&\quad - \frac{\partial^2 f}{\partial\theta_1^2}(t)h_1(t)k_1(t)\\
&\quad - \frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(t)[(h_1(z^{\tau_0}(t)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t))k_1(t) + (k_1(z^{\tau_0}(t))\\
&\quad - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))k_2(t))h_1(t)] - \frac{\partial^2 f}{\partial\theta_2^2}(t)(h_1(z^{\tau_0}(t)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t))\\
&\quad \times(k_1(z^{\tau_0}(t)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))k_2(t)) - \frac{\partial f}{\partial\theta_2}(-\dot{h}_1(z^{\tau_0}(t))k_2(t)\\
&\quad - \dot{k}_1(z^{\tau_0}(t))h_2(t) + (\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t)k_2(t))\Big|^p\mathrm{d}t\bigg)^{1/p}
\end{aligned}$$

$$+\bigg(\int_0^{\alpha}\Big|\frac{\partial\overline{g}}{\partial\gamma_1}(t)k_1(t)+\frac{\partial\overline{g}}{\partial\gamma_2}(t)k_2(t)-\frac{\partial g}{\partial\gamma_1}(t)k_1(t)-\frac{\partial g}{\partial\gamma_2}(t)k_2(t)$$
$$-\frac{\partial^2 g}{\partial\gamma_1^2}(t)h_1(t)k_1(t)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(t)(h_2(t)k_1(t)+k_2(t)h_1(t))$$
$$-\frac{\partial^2 g}{\partial\gamma_2^2}(t)h_2(t)k_2(t)\Big|^p\mathrm{d}t\bigg)^{1/p},$$

or

$$\begin{aligned}&|\mathscr{S}_u(\overline{u},v)k-\mathscr{S}_u(u,v)k-\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\&\quad\leqslant\big|\omega^{11}(t)k_1(t)+\omega^{12}(t)(k_1(z^{\tau_0}(t))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))k_2(t))+R_1(t)\big|_{L^p_{0,\alpha}}\\&\qquad+|\omega^{13}(t)k_1(t)+\omega^{14}(t)k_2(t)|_{L^p_{0,\alpha}},\end{aligned}\tag{5.4}$$

where

$$\begin{aligned}\omega^{11}(t)&=\frac{\partial\overline{f}}{\partial\theta_1}(t)-\frac{\partial f}{\partial\theta_1}(t)-\frac{\partial^2 f}{\partial\theta_1^2}(t)h_1(t)\\&\quad-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(t)((\overline{y}+\widetilde{\varphi})(\overline{z}^{\tau_0}(t))-(y+\widetilde{\varphi})(z^{\tau_0}(t))),\end{aligned}$$

$$\begin{aligned}\omega^{12}(t)&=\frac{\partial\overline{f}}{\partial\theta_2}(t)-\frac{\partial f}{\partial\theta_2}(t)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(t)h_1(t)\\&\quad-\frac{\partial^2 f}{\partial\theta_2^2}(t)((\overline{y}+\widetilde{\varphi})(\overline{z}^{\tau_0}(t))-(y+\widetilde{\varphi})(z^{\tau_0}(t))),\end{aligned}$$

$$\omega^{13}(t)=\frac{\partial\overline{g}}{\partial\gamma_1}(t)-\frac{\partial g}{\partial\gamma_1}(t)-\frac{\partial^2 g}{\partial\gamma_1^2}(t)h_1(t)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(t)h_2(t),$$

$$\omega^{14}(t)=\frac{\partial\overline{g}}{\partial\gamma_2}(t)-\frac{\partial g}{\partial\gamma_2}(t)-\frac{\partial^2 g}{\partial\gamma_2^2}(t)h_1(t)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(t)h_2(t),$$

$$\begin{aligned}R_1(t)&=\frac{\partial\overline{f}}{\partial\theta_2}(t)(k_1(\overline{z}^{\tau_0}(t))-(\dot{\overline{y}}+\dot{\widetilde{\varphi}})(\overline{z}^{\tau_0}(t))k_2(t)-k_1(z^{\tau_0}(t))\\&\quad-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))k_2(t)-\dot{h}_1(z^{\tau_0}(t))k_2(t)-\dot{k}_1(z^{\tau_0}(t))h_2(t)\\&\quad+(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t)k_2(t))+\Big(\frac{\partial\check{f}}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big)\\&\quad\times[\dot{h}_1(z^{\tau_0}(t))k_2(t)+\dot{k}_1(z^{\tau_0}(t))h_2(t)-(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t)k_2(t)]\\&\quad+\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(t)((\overline{y}+\widetilde{\varphi})(\overline{z}^{\tau_0}(t))-(y+\widetilde{\varphi})(z^{\tau_0}(t))-h_1(z^{\tau_0}(t))\\&\quad+(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t))k_1(t)\\&\quad+\frac{\partial^2 f}{\partial\theta_2^2}(t)((\overline{y}+\widetilde{\varphi})(\overline{z}^{\tau_0}(t))-(y+\widetilde{\varphi})(z^{\tau_0}(t))-h_1(z^{\tau_0}(t))\\&\quad+(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))h_2(t))[k_1(z^{\tau_0}(t))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(t))k_2(t)].\end{aligned}$$

We will decompose the integrands of (5.4) and use triangle inequality to give an estimation of each item in the $L^p_{0,\alpha}$ norm as follows.

Denote

$$\Delta_{11}(t) = |h_1(t)| + |(\overline{y} + \widetilde{\varphi})(\overline{z}^{\tau_0}(t)) - (y + \widetilde{\varphi})(z^{\tau_0}(t))|, \tag{5.5}$$

$$\Delta_{12}(t) = |h_1(t)| + |h_2(t)|. \tag{5.6}$$

Then, by Lemmas 3.3 and 3.5, one has

$$\begin{aligned}\Delta_{11}(t) &\leqslant (|y|_{\mathbb{Y}^{1,\infty}_\alpha} + |\varphi|_{W^{1,\infty}})|h_2|_C + 2|h_1|_C \\ &\leqslant \alpha^{1/q} \max\{d_1,\, 2\} \cdot |h|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha} \\ &\to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0),\end{aligned}$$

and

$$\Delta_{12}(t) \leqslant \alpha^{1/q} |h|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha} \to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0).$$

Define

$$\begin{aligned}\Omega^{11}(\overline{\theta}_1, \overline{\theta}_2; \theta_1, \theta_2) = {}& \frac{\partial f(\theta_1, \theta_2, \theta_3)}{\partial \theta_1} - \frac{\partial f(\overline{\theta}_1, \overline{\theta}_2, \theta_3)}{\partial \theta_1} \\ &- \frac{\partial^2 f(\overline{\theta}_1, \overline{\theta}_2,\, \theta_3)}{\partial \theta_1^2}(\theta_1 - \overline{\theta}_1) - \frac{\partial^2 f(\overline{\theta}_1, \overline{\theta}_2, \theta_3)}{\partial \theta_1 \partial \theta_2}(\theta_2 - \overline{\theta}_2).\end{aligned}$$

Since f is C^2, one has

$$\frac{\|\Omega^{11}(\overline{\theta}_1, \overline{\theta}_2; \theta_1, \theta_2)\|_{\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)}}{|\theta_1 - \overline{\theta}_1| + |\theta_2 - \overline{\theta}_2|} \to 0 \quad (|\theta_1 - \overline{\theta}_1| \to 0,\ |\theta_2 - \overline{\theta}_2| \to 0).$$

Then it follows from (5.5) and (5.6) that

$$\frac{\|\omega^{11}(t)\|}{\Delta_{11}(t)} \to 0 \quad \text{uniformly as } |h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0. \tag{5.7}$$

Noting that y, z, h, φ are continuous in t and f is C^2 in $(\theta_1, \theta_2, \theta_3)$, one gets that $\|\omega^{11}(t)\|/\Delta_{11}(t)$ is continuous in $t \in [0, \alpha]$ except for discontinuities such that $\Delta_{11}(t) = 0$. Moreover, if $\Delta_{11}(t_0) = 0$ for some $t_0 \in [0,\, \alpha]$, then by the C^2-smoothness of f, we have $\lim_{t \to t_0} |\|\omega^{11}(t)\|/\Delta_{11}(t)| = 0$. So we can rewrite (5.7) as

$$\Big|\frac{\|\omega^{11}(t)\|}{\Delta_{11}(t)}\Big|_C \to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0). \tag{5.8}$$

In a similar argument yielding (5.8), one can get

$$\Big|\frac{\|\omega^{12}(t)\|}{\Delta_{11}(t)}\Big|_C \to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0), \tag{5.9}$$

$$\Big|\frac{\|\omega^{13}(t)\|}{\Delta_{12}(t)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{14}(t)\|}{\Delta_{12}(t)}\Big|_C \to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0), \tag{5.10}$$

and

$$\Big|\frac{\partial \overline{f}}{\partial \theta_2}(s)-\frac{\partial f}{\partial \theta_2}(s)\Big| \leqslant \alpha^{1/q}\max\{d_1,2\}\cdot |h|_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}. \tag{5.11}$$

Therefore,

$$\begin{aligned}\Big(\int_0^\alpha \frac{|\omega^{11}(t)k_1(t)|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p} &\leqslant \alpha^{1/q}|k_1|_{\mathbb{Y}_\alpha^{1,p}}\Big(\int_0^\alpha \frac{\|\omega^{11}(t)\|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p}\\ &\leqslant \alpha^{1/q}|k_1|_{\mathbb{Y}_\alpha^{1,p}}\Big(\int_0^\alpha \Big|\frac{\|\omega^{11}(t)\|}{\Delta_{11}(t)}\Big|^p\frac{|\Delta_{11}(t)|^p\mathrm{d}s}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\Big)^{1/p}\\ &\leqslant \alpha^{1+\frac{1}{q}}\max\{d_1,2\}|k_1|_{\mathbb{Y}_\alpha^{1,p}}\Big|\frac{\|\omega^{11}(t)\|}{\Delta_{11}(t)}\Big|_C\\ &\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times \mathbb{Y}_\alpha^{2,p}}\to 0),\end{aligned} \tag{5.12}$$

$$\begin{aligned}&\Big(\int_0^\alpha \frac{|\omega^{12}(t)(k_1(z^{\tau_0}(t))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))k_2(t))|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p}\\ &\leqslant \alpha\max\{d_1,1\}|k|_{\mathbb{Y}_\alpha^{1,p}}\Big|\frac{\|\omega^{11}(t)\|}{\Delta_{11}(t)}\Big|_C\\ &\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times \mathbb{Y}_\alpha^{2,p}}\to 0),\end{aligned} \tag{5.13}$$

$$\begin{aligned}\Big(\int_0^\alpha \frac{|\omega^{13}(t)k_1(t)|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p} &\leqslant \alpha^{1/q}|k_1|_{\mathbb{Y}_\alpha^{1,p}}\Big(\int_0^\alpha \frac{\|\omega^{13}(t)\|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p}\\ &\leqslant \alpha^{1/q}|k_1|_{\mathbb{Y}_\alpha^{1,p}}\Big(\int_0^\alpha \Big|\frac{\|\omega^{13}(t)\|}{\Delta_{12}(t)}\Big|^p\frac{|\Delta_{12}(t)|^p\mathrm{d}t}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\Big)^{1/p}\\ &\leqslant \alpha^{1+\frac{1}{q}}|k_1|_{\mathbb{Y}_\alpha^{1,p}}\Big|\frac{\|\omega^{13}(t)\|}{\Delta_{12}(t)}\Big|_C\\ &\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times \mathbb{Y}_\alpha^{2,p}}\to 0),\end{aligned} \tag{5.14}$$

$$\begin{aligned}\Big(\int_0^\alpha \frac{|\omega^{14}(t)k_2(t)|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p} &\leqslant \alpha^{1/q}|k_2|_{\mathbb{Y}_\alpha^{1,p}}\Big(\int_0^\alpha \frac{\|\omega^{14}(t)\|^p}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\,\mathrm{d}t\Big)^{1/p}\\ &\leqslant \alpha^{1/q}|k_2|_{\mathbb{Y}_\alpha^{1,p}}\Big(\int_0^\alpha \Big|\frac{\|\omega^{14}(t)\|}{\Delta_{12}(t)}\Big|^p\frac{|\Delta_{12}(t)|^p\mathrm{d}t}{|h|^p_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}}}\Big)^{1/p}\\ &\leqslant \alpha^{1+\frac{1}{q}}|k_2|_{\mathbb{Y}_\alpha^{1,p}}\Big|\frac{\|\omega^{14}(t)\|}{\Delta_{12}(t)}\Big|_C\\ &\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times \mathbb{Y}_\alpha^{2,p}}\to 0).\end{aligned} \tag{5.15}$$

$$
\begin{aligned}
T_{10} := {} & \frac{1}{|h|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha}} \Big| \Big(\frac{\partial \overline{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s) \Big) \\
& \times [\dot{h}_1(z^{\tau_0}(t)) k_2(t) + \dot{k}_1(z^{\tau_0}(t)) h_2(t) - (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(t)) h_2(t) k_2(t)] \Big|_{L^p_{0,\alpha}} \\
\leqslant {} & \alpha^{1/q} \max\{d_1, 2\} L_{m2} |\dot{h}_1(z^{\tau_0}(t)) k_2(t) + \dot{k}_1(z^{\tau_0}(t)) h_2(t) \\
& - (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(t)) h_2(t) k_2(t)|_{L^p_{0,\alpha}} \\
\leqslant {} & \alpha^{1+\frac{1}{q}} \max\{d_1, 2\} L_{m2} [(1-\eta^+)^{-1} (|h_1|_{\mathbb{Y}^{1,p}_\alpha} |k_2|_{\mathbb{Y}^{1,p}_\alpha} + |k_1|_{\mathbb{Y}^{1,p}_\alpha} |h_2|_{\mathbb{Y}^{1,p}_\alpha}) \\
& + d_2 |h_2|_{\mathbb{Y}^{1,p}_\alpha} |k_2|_{\mathbb{Y}^{1,p}_\alpha}] \\
\leqslant {} & \alpha^{1+\frac{1}{q}} \max\{d_1, 2\} \max\{d_2, (1-\eta^+)^{-1}\} L_{m2} |k|_{\mathbb{Y}^{1,p}_\alpha} |h|_{\mathbb{Y}^{1,p}_\alpha} \\
\to {} & 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0). \qquad (5.16)
\end{aligned}
$$

On the other hand, by the triangle inequality, we have

$$
\begin{aligned}
|R_1|_{L^p_{0,\alpha}} \leqslant {} & L_{m1} |k_1(\overline{z}^{\tau_0}(t)) - (\dot{\overline{y}} + \dot{\widetilde{\varphi}})(\overline{z}^{\tau_0}(t)) k_2(t) - k_1(z^{\tau_0}(t)) \\
& - (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(t)) k_2(t) - \dot{h}_1(z^{\tau_0}(t)) k_2(t) - \dot{k}_1(z^{\tau_0}(t)) h_2(t) \\
& + (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(t)) h_2(t) k_2(t)|_{L^p_{0,\alpha}} + \Big| \Big(\frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s) \Big) \\
& \cdot [\dot{h}_1(z^{\tau_0}(t)) k_2(t) + \dot{k}_1(z^{\tau_0}(t)) h_2(t) - (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(t)) h_2(t) k_2(t)] \Big|_{L^p_{0,\alpha}} \\
& + H_{m2} |(\overline{y} + \widetilde{\varphi})(\overline{z}^{\tau_0}(t)) - (y + \widetilde{\varphi})(z^{\tau_0}(t)) - h_1(z^{\tau_0}(t)) \\
& + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(t)) h_2(t)|_{L^p_{0,\alpha}} |k_1|_C \\
& + H_{m2} |(\overline{y} + \widetilde{\varphi})(\overline{z}^{\tau_0}(t)) - (y + \widetilde{\varphi})(z^{\tau_0}(t)) - h_1(z^{\tau_0}(t)) \\
& + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(t)) h_2(t)|_{L^p_{0,\alpha}} |k_1(z^{\tau_0}(t)) - (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(t)) k_2(t)|_C.
\end{aligned}
$$

It follows from Lemmas 3.3–3.6 and 5.1 that

$$
\begin{aligned}
T_{11} &= \frac{1}{|h|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha}} |k_1(\overline{z}^{\tau_0}(t)) - k_1(z^{\tau_0}(t)) - \dot{k}_1(z^{\tau_0}(t)) h_2(t)|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}^{1,p}_\alpha}} |k_1(\overline{z}^{\tau_0}(t)) - k_1(z^{\tau_0}(t)) - \dot{k}_1(z^{\tau_0}(t)) h_2(t)|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0),
\end{aligned}
$$

$$
\begin{aligned}
T_{12} &= \frac{1}{|h|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha}} |y(\overline{z}^{\tau_0}(t)) - y(z^{\tau_0}(t)) - \dot{y}(z^{\tau_0}(t)) h_2(t)|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}^{1,p}_\alpha}} |y(\overline{z}^{\tau_0}(t)) - y(z^{\tau_0}(t)) - \dot{y}(z^{\tau_0}(t)) h_2(t)|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|h|_{\mathbb{Y}^{2,p}_\alpha \times \mathbb{Y}^{2,p}_\alpha} \to 0),
\end{aligned}
$$

$$
\begin{aligned}
T_{13} &= \frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}} |\widetilde{\varphi}(\overline{z}^{\tau_0}(t)) - \widetilde{\varphi}(z^{\tau_0}(t)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}_\alpha^{1,p}}} |\widetilde{\varphi}(\overline{z}^{\tau_0}(t)) - \widetilde{\varphi}(z^{\tau_0}(t)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \to 0), \\
T_{14} &= \frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}} |h_1(\overline{z}^{\tau_0}(t)) - h_1(z^{\tau_0}(t))|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}_\alpha^{1,p}}} |\dot{h}_1(t - z(t) - \lambda_{14}(t)h_2(t) - \tau_0)|_{L^p_{0,\alpha}} \alpha^{1/q} |h_2|_{\mathbb{Y}_\alpha^{1,p}} \\
&\quad \text{(where } \lambda_{14}(t) \in [0,1] \text{ for } t \in [0,\alpha]) \\
&\leqslant \frac{\alpha^{1/q}}{1-\eta^+} |h_1|_{\mathbb{Y}_\alpha^{1,p}} \\
&\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \to 0), \\
T_{15} &= \frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}} |\dot{h}_1(\overline{z}^{\tau_0}(t)) - \dot{h}_1(z^{\tau_0}(t))|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}_\alpha^{1,p}}} |\ddot{h}_1(z^{\tau_0}(t) - \lambda_{15}(t)h_2(t))|_C \cdot \alpha |h_2|_{\mathbb{Y}_\alpha^{1,p}} \\
&\quad \text{(where } \lambda_{15}(t) \in [0,1] \text{ for } t \in [0,\alpha]) \\
&\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \to 0), \\
T_{16} &= \frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}} |\dot{y}(\overline{z}^{\tau_0}(t)) - \dot{y}(z^{\tau_0}(t)) - \ddot{y}(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}_\alpha^{1,p}}} |\dot{y}(\overline{z}^{\tau_0}(t)) - \dot{y}(z^{\tau_0}(t)) - \ddot{y}(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \to 0), \\
T_{17} &= \frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}} |\dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(t)) - \ddot{\widetilde{\varphi}}(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|h_2|_{\mathbb{Y}_\alpha^{1,p}}} |\dot{\widetilde{\varphi}}(\overline{z}^{\tau_0}(t)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(t)) - \ddot{\widetilde{\varphi}}(z^{\tau_0}(t))h_2(t)|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \to 0).
\end{aligned}
$$

Using the above estimations, we get

$$
\begin{aligned}
\frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}} |R_1|_{L^p_{0,\alpha}} &\leqslant T_{10} + L_{m1}(T_{11} + (T_{15} + T_{16} + T_{17})|k_2|_C) \\
&\quad + H_{m2}(T_{12} + T_{13} + T_{14})|k_1|_C \\
&\quad + H_{m2}(T_{12} + T_{13} + T_{14})(|k_1|_C + d_1|k_2|_C) \\
&\leqslant T_{10} + L_{m1}(T_{11} + \alpha(T_{15} + T_{16} + T_{17})|k_2|_{\mathbb{Y}_\alpha^{1,p}}) \\
&\quad + \alpha H_{m2}(T_{12} + T_{13} + T_{14})(2|k_1|_{\mathbb{Y}_\alpha^{1,p}} + d_1|k_2|_{\mathbb{Y}_\alpha^{1,p}}) \\
&\to 0 \quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}} \to 0).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}}\,|\mathscr{S}_u(\overline{u},v)k-\mathscr{S}_u(u,v)k-\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\
&\quad\leqslant \frac{1}{|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}}(|\omega^{11}(t)(k_1(z^{\tau_0}(t))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(t))k_2(t)|_{L^p_{0,\alpha}}\\
&\qquad+|\omega^{12}(t)k_2(t)|_{L^p_{0,\alpha}}+|R_1|_{L^p_{0,\alpha}}+|\omega^{13}(t)k_1(t)|_{L^p_{0,\alpha}}+|\omega^{14}(t)k_2(t)|_{L^p_{0,\alpha}})\\
&\quad\to 0\quad (|h|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}}\to 0),
\end{aligned}
$$

which means that $\mathscr{S}_{uu}(u,v)$ is the second-order partial derivative of $S(u,v)$ with respect to u.

Next, we show that $\frac{\partial^2 S}{\partial u^2}(u,v)$ is pointwise continuous on its domain. Select sequences $u^n=(y^n,z^n)\in U$, $v^n=(\varphi^n,\tau_0^n,\sigma^n)\in V$ such that $|u^n-u|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}}\to 0$ and $|v^n-v|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0$ as $n\to\infty$. Choose $h=(h_1,h_2)$, $k=(k_1,k_2)\in\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}$ such that $u+h, u^n+h, u+k, u^n+k\in U$. We need to show

$$
\lim_{n\to\infty}|\mathscr{S}_{uu}(u^n,v^n)(h,k)-\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\to 0.
$$

Note that

$$
\begin{aligned}
&|\mathscr{S}_{uu}(u^n,v^n)(h,k)-\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\
&=\Big[\int_0^\alpha\Big|\Big(\frac{\partial^2 f^n}{\partial\theta_1^2}(s)-\frac{\partial^2 f}{\partial\theta_1^2}(s)\Big)h_1(s)k_1(s)+\Big(\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_2}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)\Big)\\
&\quad\times[(h_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))k_1(s)+(k_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))k_2(s))h_1(s)]+\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[(h_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))k_1(s)+(k_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))k_2(s))h_1(s)-((h_1(z^{\tau_0}(s))\\
&\quad-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))k_1(s)+(k_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))k_2(s))h_1(s))]\\
&\quad+\Big(\frac{\partial^2 f^n}{\partial\theta_2^2}(s)-\frac{\partial^2 f}{\partial\theta_2^2}(s)\Big)(h_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))\\
&\quad\times(k_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))k_2(s))+\frac{\partial^2 f}{\partial\theta_2^2}(s)[(h_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))(k_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))k_2(s))\\
&\quad-(h_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))(k_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))k_2(s))]\\
&\quad+\Big(\frac{\partial f^n}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big)((\ddot{y}^n+\ddot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)k_2(s)-\dot{k}_1((z^n)^{\tau_0^n}(s))\\
&\quad\times h_2(s)-\dot{h}_1((z^n)^{\tau_0^n}(s))k_2(s))-\frac{\partial f}{\partial\theta_2}(s)[(\ddot{y}^n+\ddot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)k_2(s)
\end{aligned}
$$

$$
\begin{aligned}
&- \dot{k}_1((z^n)^{\tau_0^n}(s))h_2(s) - \dot{h}_1((z^n)^{\tau_0^n}(s))k_2(s) - (\ddot{y} + \ddot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s)k_2(s) \\
&+ \dot{k}_1(z^{\tau_0}(s))h_2(s) + \dot{h}_1(z^{\tau_0}(s))k_2(s)]\Big|^p \mathrm{d}s\Big]^{1/p} \\
&+ \Big[\int_0^\alpha \Big|\Big(\frac{\partial^2 g^n}{\partial \gamma_1^2}(s) - \frac{\partial^2 g}{\partial \gamma_1^2}(s)\Big)h_1(s)k_1(s) + \Big(\frac{\partial^2 g^n}{\partial \gamma_1 \partial \gamma_2}(s) - \frac{\partial^2 g}{\partial \gamma_1 \partial \gamma_2}(s)\Big) \\
&\times (h_2(s)k_1(s) + k_2(s)h_1(s)) + \Big(\frac{\partial^2 g^n}{\partial \gamma_2^2}(s) - \frac{\partial^2 g}{\partial \gamma_2^2}(s)\Big)h_2(s)k_2(s)\Big|^p \mathrm{d}s\Big]^{1/p}.
\end{aligned}
$$

Now, we use Lemmas 3.3–3.5 and 5.1 to estimate the above expression term by term.

$$
\begin{aligned}
C_{11} &= \Big|\Big(\frac{\partial^2 f^n}{\partial \theta_1^2}(s) - \frac{\partial^2 f}{\partial \theta_1^2}(s)\Big)h_1(s)k_1(s)\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant \alpha}\Big\|\frac{\partial^2 f^n}{\partial \theta_1^2}(s) - \frac{\partial^2 f}{\partial \theta_1^2}(s)\Big\| \, |h_1|_C \, |k_1|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant \alpha}\Big\|\frac{\partial^2 f^n}{\partial \theta_1^2}(s) - \frac{\partial^2 f}{\partial \theta_1^2}(s)\Big\|\alpha^{1+\frac{1}{q}}|h_1|_{\mathbb{Y}^{1,p}_\alpha}\, |k_1|_{\mathbb{Y}^{1,p}_\alpha},
\end{aligned}
$$

$$
\begin{aligned}
C_{12} = &\Big|\Big(\frac{\partial^2 f^n}{\partial \theta_1 \partial \theta_2}(s) - \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)\Big) \\
&\times ([h_1((z^n)^{\tau_0^n}(s)) - (\dot{y^n} + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)]k_1(s) \\
&+ (k_1((z^n)^{\tau_0^n}(s)) - (\dot{y^n} + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))k_2(s))h_1(s))\Big|_{L^p_{0,\alpha}} \\
\leqslant &\sup_{0\leqslant s\leqslant \alpha}\Big\|\frac{\partial^2 f^n}{\partial \theta_1 \partial \theta_2}(s) - \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)\Big\|(|h_1((z^n)^{\tau_0^n}(s))k_1(s) + k_1((z^n)^{\tau_0^n}(s)) \\
&\times h_1(s)|_{L^p_{0,\alpha}} + |(\dot{y^n} + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))(h_2(s)k_1(s) + k_2(s)h_1(s))|_{L^p_{0,\alpha}}) \\
\leqslant &\sup_{0\leqslant s\leqslant \alpha}\Big\|\frac{\partial^2 f^n}{\partial \theta_1 \partial \theta_2}(s) - \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)\Big\|\alpha \max\{1, d_1\}|h|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha}\, |k|_{\mathbb{Y}^{1,p}_\alpha \times \mathbb{Y}^{1,p}_\alpha},
\end{aligned}
$$

$$
\begin{aligned}
C_{13} = &\Big|\frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)((h_1((z^n)^{\tau_0^n}(s)) - (\dot{y^n} + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))k_1(s) \\
&+ (k_1((z^n)^{\tau_0^n}(s)) - (\dot{y^n} + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))k_2(s))h_1(s) \\
&- (h_1(z^{\tau_0}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))k_1(s) \\
&- (k_1(z^{\tau_0}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s))k_2(s))h_1(s))\Big|_{L^p_{0,\alpha}} \\
= &\Big|\frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)\{[h_1((z^n)^{\tau_0^n}(s)) - h_1(z^{\tau_0}(s)) + ((\dot{y^n} + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s)) \\
&- (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s)))h_2(s)]k_1(s) + [k_1((z^n)^{\tau_0^n}(s)) - k_1(z^{\tau_0}(s)) \\
&+ ((\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s)))k_2(s)]h_1(s)\}\Big|_{L^p_{0,\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant L_{m2}\{\alpha^{1+\frac{1}{p}}|k_1|_{\mathbb{Y}_\alpha^{1,\infty}}[(|h_1|_{\mathbb{Y}_\alpha^{1,\infty}}+\alpha|h_2|_{\mathbb{Y}_\alpha^{1,\infty}}(|y|_{\mathbb{Y}_\alpha^{2,\infty}}+|\varphi|_{W^{2,\infty}}))\\
&\quad\times(\alpha^{1/q}|z^n-z|_{\mathbb{Y}_\alpha^{1,p}}+|\tau_0^n-\tau_0|)\\
&\quad+\alpha|h_2|_{\mathbb{Y}_\alpha^{1,\infty}}(\alpha^{1/p}|y-y^n|_{\mathbb{Y}_\alpha^{2,p}}+|\varphi-\varphi^n|_{W^{2,\infty}})]\\
&\quad+\alpha^{1+\frac{1}{p}}|h_1|_{\mathbb{Y}_\alpha^{1,\infty}}[(|k_1|_{\mathbb{Y}_\alpha^{1,\infty}}+\alpha|k_2|_{\mathbb{Y}_\alpha^{1,\infty}}(|y|_{\mathbb{Y}_\alpha^{2,\infty}}+|\varphi|_{W^{2,\infty}}))\\
&\quad\times(\alpha^{1/q}|z^n-z|_{\mathbb{Y}_\alpha^{1,p}}+|\tau_0^n-\tau_0|)\\
&\quad+\alpha|k_2|_{\mathbb{Y}_\alpha^{1,\infty}}(\alpha^{1/p}|y-y^n|_{\mathbb{Y}_\alpha^{2,p}}+|\varphi-\varphi^n|_{W^{2,\infty}})]\},
\end{aligned}
$$

$$
\begin{aligned}
C_{14}&=\Big|\Big(\frac{\partial^2 f^n}{\partial\theta_2^2}(s)-\frac{\partial^2 f}{\partial\theta_2^2}(s)\Big)(h_1((z^n)^{\tau_0^n}(s))-(\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))h_2(s))\\
&\quad\times(k_1((z^n)^{\tau_0^n}(s))-(\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))k_2(s))\Big|_{L^p_{0,\alpha}}\\
&\leqslant\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_2^2}(s)-\frac{\partial^2 f}{\partial\theta_2^2}(s)\Big\|\max\{1,d_1^2\}\alpha^{1+\frac{1}{q}}|h|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}|k|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}},
\end{aligned}
$$

$$
\begin{aligned}
C_{15}&=\Big|\frac{\partial^2 f}{\partial\theta_2^2}(s)\{[h_1((z^n)^{\tau_0^n}(s))-(\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))h_2(s)][k_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))k_2(s)]-[h_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)]\\
&\quad\times[k_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))k_2(s)]\}\Big|_{L^p_{0,\alpha}}\\
&=\Big|\frac{\partial^2 f}{\partial\theta_2^2}(s)\{(h_1((z^n)^{\tau_0^n}(s))-h_1(z^{\tau_0}(s))+((\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))\\
&\quad-(\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s)))h_2(s))[k_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))k_2(s)]\\
&\quad-[h_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)](k_1(z^{\tau_0}(s))-k_1((z^n)^{\tau_0^n}(s))\\
&\quad+((\dot y^n+\dot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s)))k_2(s))\}\Big|_{L^p_{0,\alpha}}\\
&\leqslant L_{m2}\{[(|h_1|_{\mathbb{Y}_\alpha^{1,\infty}}+\alpha|h_2|_{\mathbb{Y}_\alpha^{1,\infty}}(|y|_{\mathbb{Y}_\alpha^{2,\infty}}+|\varphi|_{W^{2,\infty}}))\\
&\quad\times(\alpha^{1/q}|z^n-z|_{\mathbb{Y}_\alpha^{1,p}}+|\tau_0^n-\tau_0|)+\alpha|h_2|_{\mathbb{Y}_\alpha^{1,\infty}}(\alpha^{1/q}|y-y^n|_{\mathbb{Y}_\alpha^{2,p}}\\
&\quad+|\varphi-\varphi^n|_{W^{2,\infty}})]\max\{1,d_1\}\alpha^{1+\frac{1}{p}}|k|_{\mathbb{Y}_\alpha^{1,\infty}}+[(|k_1|_{\mathbb{Y}_\alpha^{1,\infty}}+\alpha|k_2|_{\mathbb{Y}_\alpha^{1,\infty}}\\
&\quad\times(|y|_{\mathbb{Y}_\alpha^{2,\infty}}+|\varphi|_{W^{2,\infty}}))(\alpha^{1/q}|z^n-z|_{\mathbb{Y}_\alpha^{1,p}}+|\tau_0^n-\tau_0|)+\alpha|k_2|_{\mathbb{Y}_\alpha^{1,\infty}}\\
&\quad\times(\alpha^{1/q}|y-y^n|_{\mathbb{Y}_\alpha^{2,p}}+|\varphi-\varphi^n|_{W^{2,\infty}})]\max\{1,d_1\}\alpha^{1+\frac{1}{p}}|h|_{\mathbb{Y}_\alpha^{1,\infty}}\},
\end{aligned}
$$

$$
\begin{aligned}
C_{16}&=\Big|\Big(\frac{\partial f^n}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big)[(\ddot y^n+\ddot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))h_2(s)k_2(s)\\
&\quad-\dot k_1((z^n)^{\tau_0^n}(s))h_2(s)-\dot h_1((z^n)^{\tau_0^n}(s))k_2(s)]\Big|_{L^p_{0,\alpha}}\\
&\leqslant\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial f^n}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big\|(\alpha^{1+\frac{1}{q}}d_2|h_2|_{\mathbb{Y}_\alpha^{1,p}}|k_2|_{\mathbb{Y}_\alpha^{1,p}}\\
&\quad+(1-\eta^+)^{-1}\alpha^{1/q}(|k_1|_{\mathbb{Y}_\alpha^{1,p}}|h_2|_{\mathbb{Y}_\alpha^{1,p}}+|h_1|_{\mathbb{Y}_\alpha^{1,p}}|k_2|_{\mathbb{Y}_\alpha^{1,p}})),
\end{aligned}
$$

$$\begin{aligned}
C_{17} &= \Big|\frac{\partial f}{\partial\theta_2}(s)[(\ddot y^n+\ddot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))h_2(s)k_2(s)-\dot k_1((z^n)^{\tau_0^n}(s))h_2(s)\\
&\quad-\dot h_1((z^n)^{\tau_0^n}(s))k_2(s)-(\ddot y+\ddot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)k_2(s)\\
&\quad+\dot k_1(z^{\tau_0}(s))h_2(s)+\dot h_1(z^{\tau_0}(s))k_2(s)]\Big|_{L^p_{0,\alpha}}\\
&= \Big|\frac{\partial f}{\partial\theta_2}(s)[((\ddot y^n+\ddot{\tilde\varphi}^n)((z^n)^{\tau_0^n}(s))-(\ddot y+\ddot{\tilde\varphi})((z^n)^{\tau_0^n}(s))\\
&\quad+(\ddot y+\ddot{\tilde\varphi})((z^n)^{\tau_0^n}(s))-(\ddot y+\ddot{\tilde\varphi})(z^{\tau_0}(s)))h_2(s)k_2(s)-(\dot k_1((z^n)^{\tau_0^n}(s))\\
&\quad-\dot k_1(z^{\tau_0}(s)))h_2(s)-(\dot h_1((z^n)^{\tau_0^n}(s))-\dot h_1(z^{\tau_0}(s)))k_2(s)]\Big|_{L^p_{0,\alpha}}\\
&\leqslant L_{m2}\Big\{\alpha^2|h_2|_{\mathbb{Y}^{1,\infty}_\alpha}|k_2|_{\mathbb{Y}^{1,\infty}_\alpha}\Big[\frac{1}{1-\eta^+}|y-y^n|_{\mathbb{Y}^{2,p}_\alpha}+\alpha^{1/p}|\varphi-\varphi^n|_{W^{2,\infty}}\\
&\quad+|(\ddot y+\ddot{\tilde\varphi})((z^n)^{\tau_0^n}(s))-(\ddot y+\ddot{\tilde\varphi})(z^{\tau_0}(s))|_{L^p_{0,\alpha}}\Big]\\
&\quad+\alpha|k_1|_{\mathbb{Y}^{2,\infty}_\alpha}|h_2|_{\mathbb{Y}^{1,\infty}_\alpha}(\alpha^{1/q}|z-z^n|_{\mathbb{Y}^{1,p}_\alpha}+|\tau_0-\tau_0^n|)\\
&\quad+\alpha|k_2|_{\mathbb{Y}^{1,\infty}_\alpha}|h_1|_{\mathbb{Y}^{2,\infty}_\alpha}(\alpha^{1/q}|z-z^n|_{\mathbb{Y}^{1,p}_\alpha}+|\tau_0-\tau_0^n|)\Big\},
\end{aligned}$$

$$\begin{aligned}
C_{18} &= \Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_1^2}(s)\Big)h_1(s)k_1(s)\Big|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_1^2}(s)\Big\|\,|h_1|_C\,|k_1|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_1^2}(s)\Big\|\alpha^{1+\frac1q}|h_1|_{\mathbb{Y}^{1,p}_\alpha}|k_1|_{\mathbb{Y}^{1,p}_\alpha},
\end{aligned}$$

$$\begin{aligned}
C_{19} &= \Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_2}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\Big)(h_2(s)k_1(s)+k_2(s)h_1(s))\Big|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_2}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\Big\|\\
&\quad\times\alpha^{1+\frac1q}(|h_2|_{\mathbb{Y}^{1,p}_\alpha}|k_1|_{\mathbb{Y}^{1,p}_\alpha}+|h_1|_{\mathbb{Y}^{1,p}_\alpha}|k_2|_{\mathbb{Y}^{1,p}_\alpha}),
\end{aligned}$$

$$\begin{aligned}
C_{20} &= \Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_2^2}(s)-\frac{\partial^2 g}{\partial\gamma_2^2}(s)\Big)h_2(s)k_2(s)\Big|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_2^2}(s)\Big\|\,|h_2|_C\,|k_2|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_2^2}(s)\Big\|\alpha^{1+\frac1q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}|k_2|_{\mathbb{Y}^{1,p}_\alpha}.
\end{aligned}$$

We note that $\overline{M}_3 := \{\sigma^k:\ k\in\mathbb{N}\}\cup\{\sigma\}$ is compact and that $\frac{\partial f^n}{\partial\theta_1}$, $\frac{\partial f^n}{\partial\theta_2}$, $\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}$ and $\frac{\partial g^n}{\partial\gamma_1}$, $\frac{\partial g^n}{\partial\gamma_2}$, $\frac{\partial^2 g}{\partial\theta_i\partial\theta_j}$ $(i,j=1,2,3)$ are uniformly continuous on the compact sets $M_1\times M_2\times\overline{M}_3$ and $M_1\times M_4\times\overline{M}_3$, respectively. Since, by Lemma 3.3 (i), we know that $|u^n-u|_{\mathbb{Y}^{2,p}_\alpha\times\mathbb{Y}^{2,p}_\alpha}\to 0$ and $|v^n-v|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0$ as $n\to\infty$

imply that

$$\begin{aligned}\Delta_{01}(s) &= |(y^n+\widetilde{\varphi}^n-y-\widetilde{\varphi})(s)|+|\sigma^n-\sigma|\\ &\quad +|(y^n+\widetilde{\varphi}^n)((z^n)^{\tau_0^n}(s))-(y+\widetilde{\varphi})(z^{\tau_0}(s))|\\ &\leqslant (|y|_{\mathbb{Y}_\alpha^{2,\infty}}+|\varphi|_{W^{2,\infty}})(\alpha^{1/q}|z-z^n|_{\mathbb{Y}_\alpha^{1,p}}+|\tau_0-\tau_0^n|)\\ &\quad +|y-y^n|_{\mathbb{Y}_\alpha^{1,p}}+|\varphi-\varphi^n|_{W^{1,\infty}}+|\sigma^n-\sigma|\\ &\to 0 \quad (n\to\infty),\end{aligned}$$

$$\begin{aligned}\Delta_{02}(s) &= |(y^n+\widetilde{\varphi}^n-y-\widetilde{\varphi}(s)|+|z^n(s)+\tau_0^n-z(s)-\tau_0|+|\sigma^n-\sigma|\\ &\leqslant \alpha^{1/q}(|y-y^n|_{\mathbb{Y}_\alpha^{1,p}}+|z-z^n|_{\mathbb{Y}_\alpha^{1,p}})+|\varphi-\varphi^n|_{W^{1,\infty}}\\ &\quad +|\tau_0-\tau_0^n|+|\sigma^n-\sigma|\\ &\to 0 \quad (n\to\infty),\end{aligned}$$

we have

$$\lim_{n\to\infty}\sup_{0\leqslant s\leqslant\alpha}\left\|\frac{\partial^2 f^n}{\partial\theta_i\partial\theta_j}(s)-\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}(s)\right\|=0$$

and

$$\lim_{n\to\infty}\sup_{0\leqslant s\leqslant\alpha}\left\|\frac{\partial^2 g^n}{\partial\gamma_i\partial\gamma_j}(s)-\frac{\partial^2 g}{\partial\gamma_i\partial\gamma_j}(s)\right\|=0.$$

Also, for the term $|(\ddot{y}+\ddot{\widetilde{\varphi}})((z^n)^{\tau_0^n}(s))-(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))|_{L^p_{0,\alpha}}$ in C_{17}, we know that $(\ddot{y}+\ddot{\widetilde{\varphi}})((z^n)^{\tau_0^n}(s))\to(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))$ pointwisely a.e. in $[0,\alpha]$ as $n\to\infty$ and $|(\ddot{y}+\ddot{\widetilde{\varphi}})((z^n)^{\tau_0^n}(s))|\leqslant\beta_1+\delta_1+|\varphi^*|_{W^{2,\infty}}$ for any n. It follows from the Dominated Convergence Theorem that

$$|(\ddot{y}+\ddot{\widetilde{\varphi}})((z^n)^{\tau_0^n}(s))-(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))|_{L^p_{0,\alpha}}\to 0 \quad (n\to\infty).$$

Now, we are ready to conclude that $C_i\to 0$ as $n\to\infty$, $i=11,12,\dots,20$. Hence,

$$|\mathscr{S}_{uu}(u^n,v^n)(h,k)-\mathscr{S}_{uu}(u,v)(h,k)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\to 0 \quad (n\to\infty).$$

This completes the proof of the pointwise continuity of $\frac{\partial^2 S}{\partial u^2}(u,v)$.

Part 2 The second-order partial derivative of $S(u,v)$ with respect to v.

It is easy to see that $\mathscr{S}_{vv}(u,v)$ is a bilinear operator from $(W^{2,\infty}\times\mathbb{R}^+\times\Sigma)\times(W^{2,\infty}\times\mathbb{R}^+\times\Sigma)$ to $\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}$. We use Minkowski's Inequality to check the boundedness of $\mathscr{S}_{vv}(u,v)$ as follows.

$$\begin{aligned}&|\mathscr{S}_{vv}(u,v)(l,m)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\ &=\Big(\int_0^\alpha\Big|\Big[\frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0)+\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)(\widetilde{l}_1(z^{\tau_0}(s))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2)\\ &\quad+\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_3\Big]m_1(0)+\Big[\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)l_1(0)+\frac{\partial^2 f}{\partial\theta_2^2}(s)(\widetilde{l}_1(s)\end{aligned}$$

$$
\begin{aligned}
&- (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2) + \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)l_3\Big](\widetilde m_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))m_2)\\
&+ \Big[\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_1(0) + \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)(\widetilde l_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2)\\
&+ \frac{\partial^2 f}{\partial\theta_3^2}(s)l_3\Big]m_3 + \frac{\partial f}{\partial\theta_2}(s)[-\dot{\tilde m}_1(z^{\tau_0}(s))l_2 + (\ddot y + \ddot{\tilde\varphi})(z^{\tau_0}(s))l_2m_2\\
&- \dot{\tilde l}_1(z^{\tau_0}(s))m_2]\Big|^p\mathrm{d}s\Big)^{1/p} + \Big(\int_0^\alpha\Big|\Big(\frac{\partial^2 g}{\partial\gamma_1^2}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_2\\
&+ \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_3\Big)m_1(0) + \Big(\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_2^2}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_3\Big)m_2\\
&+ \Big(\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_1(0) + \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_2 + \frac{\partial^2 g}{\partial\gamma_3^2}(s)l_3\Big)m_3\Big|^p\mathrm{d}s\Big)^{1/p}\\
\leqslant{}& L_{m2}(|l_1(0)| + |\widetilde l_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2|_C + |l_3|)\\
&\times(\alpha^{1/p}|m_1(0)| + |\widetilde m_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))m_2|_{L^p_{0,\alpha}} + \alpha^{1/p}|m_3|)\\
&+ L_{m1}\big|\dot{\tilde m}_1(z^{\tau_0}(s))l_2 + \dot{\tilde l}_1(z^{\tau_0}(s))m_2 - (\ddot y + \ddot{\tilde\varphi})(z^{\tau_0}(s))l_2m_2\big|_{L^p_{0,\alpha}}\\
&+ \alpha^{1/p}H_{m2}(|l_1|_{W^{2,\infty}} + |l_2|_{\mathbb{R}^+} + |l_3|_\Sigma)(|m_1|_{W^{2,\infty}} + |m_2|_{\mathbb{R}^+} + |m_3|_\Sigma)\\
\leqslant{}& \alpha^{1/p}L_{m2}\big(2|l_1|_{W^{2,\infty}} + d_1|l_2|_{\mathbb{R}^+} + |l_3|_\Sigma\big)\big(2|m_1|_{W^{2,\infty}} + d_1|m_2|_{\mathbb{R}^+} + |m_3|_\Sigma)\\
&+ \alpha^{1/p}L_{m1}(|m_1|_{W^{2,\infty}}|l_2|_{\mathbb{R}^+} + |l_1|_{W^{2,\infty}}|m_2|_{\mathbb{R}^+} + d_2|l_2|_{\mathbb{R}^+}|m_2|_{\mathbb{R}^+})\\
&+ \alpha^{1/p}H_{m2}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}|m|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\\
\leqslant{}& \alpha^{1/p}(H_{m2} + \max\{d_1^2, 4\}L_{m2} + \max\{d_2, 1\}L_{m1})\\
&\times|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}|m|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}.
\end{aligned}
$$

Next, we show that $\mathscr{S}_{vv}$ is the second-order partial derivative of the operator $S(u,v)$ with respect to v. Let $v = (\varphi,\tau_0,\sigma) \in V$. Choose $l = (l_1,l_2,l_3)$, $m = (m_1,m_2,m_3) \in W^{2,\infty}\times\mathbb{R}^+\times\Sigma$ such that $\overline{v} = (\overline{\varphi},\overline{\tau}_0,\overline{\sigma}) := v + l \in V$ and $v + m \in V$. Then

$$
\begin{aligned}
&|\mathscr{S}_v(u,\overline{v})m - \mathscr{S}_v(u,v)m - \mathscr{S}_{vv}(u,v)(l,m)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\\
={}& \Big(\int_0^\alpha\Big|\frac{\partial\check f}{\partial\theta_1}(s)m_1(0) + \frac{\partial\check f}{\partial\theta_2}(s)(\widetilde m_1(z^{\overline{\tau}_0}(s)) - (\dot y + \dot{\tilde{\overline{\varphi}}})(z^{\overline{\tau}_0}(s))m_2)\\
&+ \frac{\partial\check f}{\partial\theta_3}(s)m_3 - \frac{\partial f}{\partial\theta_1}(s)m_1(0) - \frac{\partial f}{\partial\theta_2}(s)(\widetilde m_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))m_2)\\
&- \frac{\partial f}{\partial\theta_3}(s)m_3 - \frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0)m_1(0) - \frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[(\widetilde m_1(z^{\tau_0}(s))\\
&- (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))m_2)l_1(0) + (\widetilde l_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2)m_1(0)]\\
&- \frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)(m_1(0)l_3 + l_1(0)m_3) - \frac{\partial^2 f}{\partial\theta_2^2}(s)(\widetilde m_1(z^{\tau_0}(s))\\
&- (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))m_2)(\widetilde l_1(z^{\tau_0}(s)) - (\dot y + \dot{\tilde\varphi})(z^{\tau_0}(s))l_2)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)[l_3(\widetilde{m}_1(z^{\tau_0}(s)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))m_2) + m_3(\widetilde{l}_1(z^{\tau_0}(s)) \\
&- (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2)] - \frac{\partial^2 f}{\partial\theta_3^2}(s)l_3m_3 - \frac{\partial f}{\partial\theta_2}(s)(-\dot{\widetilde{m}}_1(z^{\tau_0}(s))l_2 \\
&- \dot{\widetilde{l}}_1(z^{\tau_0}(s))m_2 + (\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2m_2)\Big|^p \mathrm{d}s\Big)^{1/p} \\
&+ \Big(\int_0^\alpha \Big|\frac{\partial\breve{g}}{\partial\gamma_1}(s)m_1(0) + \frac{\partial\breve{g}}{\partial\gamma_2}(s)m_2 + \frac{\partial\breve{g}}{\partial\gamma_3}(s)m_3 - \frac{\partial g}{\partial\gamma_1}(s)m_1(0) \\
&- \frac{\partial g}{\partial\gamma_2}(s)m_2 - \frac{\partial g}{\partial\gamma_3}(s)m_3 - \frac{\partial^2 g}{\partial\gamma_1^2}(s)l_1(0)m_1(0) - \frac{\partial^2 g}{\partial\gamma_2^2}(s)l_2m_2 \\
&- \frac{\partial^2 g}{\partial\gamma_3^2}(s)l_3m_3 - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)(l_2m_1(0)+l_1(0)m_2) \\
&- \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)(l_3m_1(0)+l_1(0)m_3) - \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)(l_2m_3+l_3m_2)\Big|^p \mathrm{d}s\Big)^{1/p},
\end{aligned}
$$

or

$$
\begin{aligned}
&|\mathscr{S}_v(u,\overline{v})m - \mathscr{S}_v(u,v)m - \mathscr{S}_{vv}(u,v)(l,m)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}} \\
&\quad = |\omega^{21}(s)m_1(0) + \omega^{22}(s)(\widetilde{m}_1(z^{\tau_0}(s)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))m_2) + \omega^{23}(s)m_3 \\
&\qquad + R_2(s)|_{L^p_{0,\alpha}} + |\omega^{24}(s)m_1(0) + \omega^{25}(s)m_3 + \omega^{26}(s)m_2|_{L^p_{0,\alpha}},
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
\omega^{21}(s) &= \frac{\partial\breve{f}}{\partial\theta_1}(s) - \frac{\partial f}{\partial\theta_1}(s) - \frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0) - \frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_3 \\
&\quad - \frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)((y+\widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y+\widetilde{\varphi})(z^{\tau_0}(s))),
\end{aligned}
$$

$$
\begin{aligned}
\omega^{22}(s) &= \frac{\partial\breve{f}}{\partial\theta_2}(s) - \frac{\partial f}{\partial\theta_2}(s) - \frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)l_1(0) - \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)l_3 \\
&\quad - \frac{\partial^2 f}{\partial\theta_2^2}(s)((y+\widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y+\widetilde{\varphi})(z^{\tau_0}(s))),
\end{aligned}
$$

$$
\begin{aligned}
\omega^{23}(s) &= \frac{\partial\breve{f}}{\partial\theta_3}(s) - \frac{\partial f}{\partial\theta_3}(s) - \frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_1(0) - \frac{\partial^2 f}{\partial\theta_3^2}(s)l_3 \\
&\quad - \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)((y+\widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y+\widetilde{\varphi})(z^{\tau_0}(s))),
\end{aligned}
$$

$$
\omega^{24}(s) = \frac{\partial\breve{g}}{\partial\gamma_1}(s) - \frac{\partial g}{\partial\gamma_1}(s) - \frac{\partial^2 g}{\partial\gamma_1^2}(s)l_1(0) - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_2 - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_3,
$$

$$
\omega^{25}(s) = \frac{\partial\breve{g}}{\partial\gamma_2}(s) - \frac{\partial g}{\partial\gamma_2}(s) - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\widetilde{l}_1(z^{\tau_0}(s)) - \frac{\partial^2 g}{\partial\gamma_2^2}(s)l_2 - \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_3,
$$

$$
\omega^{26}(s) = \frac{\partial\breve{g}}{\partial\gamma_3}(s) - \frac{\partial g}{\partial\gamma_3}(s) - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)\widetilde{l}_1(z^{\tau_0}(s)) - \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_2 - \frac{\partial^2 g}{\partial\gamma_3^2}(s)l_3,
$$

$$
\begin{aligned}
R_2(s) = {} & \frac{\partial \check{f}}{\partial \theta_2}(s)(\widetilde{m}_1(z^{\overline{\tau}_0}(s)) - (\dot{y} + \dot{\widetilde{\varphi}})(z^{\overline{\tau}_0}(s))m_2 - \widetilde{m}_1(z^{\tau_0}(s)) \\
& + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))m_2 + \dot{\widetilde{m}}_1(z^{\tau_0}(s))l_2 + \dot{\widetilde{l}}_1(z^{\tau_0}(s))m_2 \\
& - (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2 m_2) + \Big(\frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s)\Big) \\
& \times (-\dot{\widetilde{m}}_1(z^{\tau_0}(s))l_2 - \dot{\widetilde{l}}_1(z^{\tau_0}(s))m_2 + (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2 m_2) \\
& + \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)((y + \widetilde{\varphi})(z^{\overline{\tau}_0}(s)) - (y + \widetilde{\varphi})(z^{\tau_0}(s)) - \widetilde{l}_1(z^{\tau_0}(s)) \\
& + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2)m_1(0) + \frac{\partial^2 f}{\partial \theta_2^2}(s)((y + \widetilde{\varphi})(z^{\overline{\tau}_0}(s)) \\
& - (y + \widetilde{\varphi})(z^{\tau_0}(s)) - \widetilde{l}_1(z^{\tau_0}(s)) + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2)(\widetilde{m}_1(z^{\tau_0}(s)) \\
& - (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))m_2) + \frac{\partial^2 f}{\partial \theta_2 \partial \theta_3}(s)((y + \widetilde{\varphi})(z^{\overline{\tau}_0}(s)) \\
& - (y + \widetilde{\varphi})(z^{\tau_0}(s)) - \widetilde{l}_1(z^{\tau_0}(s)) + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2)m_3.
\end{aligned}
$$

Again, we will use the triangle inequality to decompose the integrands of (5.17) and estimate each term in the $L^p_{0,\alpha}$ norm as follows. It follows from

$$
\begin{aligned}
(\Delta_2 l)(s) &:= |l_1(0)| + |(y + \widetilde{\varphi})(z^{\overline{\tau}_0}(s)) - (y + \widetilde{\varphi})(z^{\tau_0}(s))| + |l_3|_\Sigma \\
&\leqslant 2|l_1|_{W^{2,\infty}} + (|y|_{\mathbb{Y}^{1,\infty}_\alpha} + |\widetilde{\varphi}|_{W^{1,\infty}})|l_2|_{\mathbb{R}^+} + |l_3|_\Sigma \\
&\leqslant \max\{d_1, 2\}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}
\end{aligned}
$$

and

$$(\Delta_3 l)(s) := |l_1(0)| + |l_2| + |l_3| \leqslant |l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}$$

that

$$(\Delta_2 l)(s) \to 0, \quad (\Delta_3 l)(s) \to 0 \quad (|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0),$$

and

$$\Big\|\frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s)\Big\| \leqslant \max\{d_1, 2\}L_{m2}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}.$$

Similar to obtaining (5.8), one has

$$
\begin{aligned}
&\Big|\frac{\|\omega^{21}(s)\|}{(\Delta_2 l)(s)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{22}(s)\|}{(\Delta_2 l)(s)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{23}(s)\|}{(\Delta_2 l)(s)}\Big|_C \to 0, \\
&\Big|\frac{\|\omega^{24}(s)\|}{(\Delta_3 l)(s)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{25}(s)\|}{(\Delta_3 l)(s)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{26}(s)\|}{(\Delta_3 l)(s)}\Big|_C \to 0,
\end{aligned}
$$

as $|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0$. Thus, as $|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0$, we have the following estimations:

$$\frac{|\omega^{21}(s)m_1(0)|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \leqslant \max\{d_1, 2\}|m_1(0)|\Big|\frac{\|\omega^{21}(s)\|}{(\Delta_2 l)(s)}\Big|_C \to 0, \tag{5.18}$$

$$\begin{aligned}
&\frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\omega^{22}(s)(\widetilde{m}_1(z^{\tau_0}(s))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))m_2)|_{L^p_{0,\alpha}}\\
&\leqslant \max\{d_1,2\}|(\widetilde{m}_1(z^{\tau_0}(s))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))m_2)|_C\,\Big|\frac{\|\omega^{21}(s)\|}{(\Delta_2 l)(s)}\Big|_C\\
&\to 0, \qquad (5.19)
\end{aligned}$$

$$\frac{|\omega^{23}(s)m_3(s)|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\leqslant \alpha^{1/p}\max\{d_1,2\}|m_3|\,\Big|\frac{\|\omega^{23}(s)\|}{(\Delta_2 l)(s)}\Big|_C\to 0, \qquad (5.20)$$

$$\begin{aligned}
\frac{|\omega^{24}(s)m_1(0)|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}
&=\Big(\int_0^\alpha\Big(\frac{\|\omega^{24}(s)\|\,|m_1(0)|}{(\Delta_3 l)(s)}\,\frac{(\Delta_3 l)(s)}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big)^p\mathrm{d}s\Big)^{1/p}\\
&\leqslant \alpha^{1/p}|m_1(0)|\,\Big|\frac{\|\omega^{24}(s)\|}{(\Delta_3 l)(s)}\Big|_C\\
&\to 0, \qquad (5.21)
\end{aligned}$$

$$\begin{aligned}
\frac{|\omega^{25}(s)m_2|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}
&=\Big(\int_0^\alpha\Big(\frac{\|\omega^{25}(s)\|\,|m_2|}{(\Delta_3 l)(s)}\,\frac{(\Delta_3 l)(s)}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big)^p\mathrm{d}s\Big)^{1/p}\\
&\leqslant \alpha^{1/p}|m_2|\,\Big|\frac{\|\omega^{25}(s)\|}{(\Delta_3 l)(s)}\Big|_C\\
&\to 0, \qquad (5.22)
\end{aligned}$$

$$\begin{aligned}
\frac{|\omega^{26}(s)m_3|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}
&=\Big(\int_0^\alpha\Big(\frac{\|\omega^{26}(s)\|\,|m_3|}{(\Delta_3 l(s)}\,\frac{(\Delta_3 l)(s)}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big)^p\mathrm{d}s\Big)^{1/p}\\
&\leqslant \alpha^{1/p}|m_3|\,\Big|\frac{\|\omega^{26}(s)\|}{(\Delta_3 l)(s)}\Big|_C\\
&\to 0. \qquad (5.23)
\end{aligned}$$

$$\begin{aligned}
T_{20}=&\ \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,\Big|\Big(\frac{\partial\check{f}}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big)\\
&\quad\times(-\dot{\widetilde{m}}_1(z^{\tau_0}(s))l_2-\dot{\widetilde{l}}_1(z^{\tau_0}(s))m_2+(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2m_2)\Big|_{L^p_{0,\alpha}}\\
\leqslant&\ \max\{d_1,2\}L_{m2}|(\dot{\widetilde{m}}_1(z^{\tau_0}(s))l_2+\dot{\widetilde{l}}_1(z^{\tau_0}(s))m_2\\
&\quad-(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2m_2|_{L^p_{0,\alpha}}\\
\leqslant&\ \alpha^{1/p}\max\{d_1,2\}L_{m2}(|m_2|_{\mathbb{R}^+}|l_1|_{W^{2,\infty}}\\
&\quad+|l_2|_{\mathbb{R}^+}|m_1|_{W^{2,\infty}}+d_2|l_2|_{\mathbb{R}^+}|m_2|_{\mathbb{R}^+})\\
\leqslant&\ \alpha^{1/p}\max\{d_1,2\}\max\{d_2,1\}L_{m2}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}|m|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\\
\to&\ 0. \qquad (5.24)
\end{aligned}$$

On the other hand, it follows from Lemma 5.1 that $y(z^{\tau_0}(s))$, $\dot{y}(z^{\tau_0}(s))$, $\varphi(z^{\tau_0}(s))$, $\dot{\varphi}(z^{\tau_0}(s))$, $\widetilde{l}_1(z^{\tau_0}(s))$, $\dot{\widetilde{l}}_1(z^{\tau_0}(s))$, $\widetilde{m}_1(z^{\tau_0}(s))$, $\dot{\widetilde{m}}_1(z^{\tau_0}(s))$ all are continuously differentiable with respect to τ_0 in the $L^p_{0,\alpha}$ norm. This, combined with Lemmas 3.3–3.5, implies that, as $|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0$,

$$\begin{aligned}
T_{21} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |m_1(z^{\overline{\tau}_0}(s)) - m_1(z^{\tau_0}(s)) - \dot{m}_1(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}} \, |m_1(z^{\overline{\tau}_0}(s)) - m_1(z^{\tau_0}(s)) - \dot{m}_1(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
T_{22} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |y(z^{\overline{\tau}_0}(s)) - y(z^{\tau_0}(s)) - \dot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}} \, |y(z^{\overline{\tau}_0}(s)) - y(z^{\tau_0}(s)) - \dot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
T_{23} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\widetilde{\varphi}(z^{\overline{\tau}_0}(s)) - \widetilde{\varphi}(z^{\tau_0}(s)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}} \, |\widetilde{\varphi}(z^{\overline{\tau}_0}(s)) - \widetilde{\varphi}(z^{\tau_0}(s)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
T_{24} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\widetilde{l}_1(z^{\overline{\tau}_0}(s)) - \widetilde{l}_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\dot{\widetilde{l}}_1(z^{\tau_0}(s) - \lambda_{24}(s)l_2)|_{L^p_{0,\alpha}} \, |l_2|_{\mathbb{R}^+} \\
&\quad \text{(where } \lambda_{24}(s) \in [0,1] \text{ for } s \in [0,\alpha]) \\
&\leqslant \alpha^{1/p} |l_1|_{W^{2,\infty}} \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
T_{25} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\dot{\widetilde{l}}_1(z^{\overline{\tau}_0}(s)) - \dot{\widetilde{l}}_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\ddot{\widetilde{l}}_1(z^{\tau_0}(s) - \lambda_{25}(s)l_2)|_{L^p_{0,\alpha}} \, |l_2|_{\mathbb{R}^+} \\
&\quad \text{(where } \lambda_{25}(s) \in [0,1] \text{ for } s \in [0,\alpha]) \\
&\leqslant \alpha^{1/p} |l_1|_{W^{2,\infty}} \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
T_{26} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\dot{y}(z^{\overline{\tau}_0}(s)) - \dot{y}(z^{\tau_0}(s)) - \ddot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}} \, |\dot{y}(z^{\overline{\tau}_0}(s)) - \dot{y}(z^{\tau_0}(s)) - \ddot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
T_{27} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \, |\dot{\widetilde{\varphi}}(z^{\overline{\tau}_0}(s)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(s)) - \ddot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}} \, |\dot{\widetilde{\varphi}}(z^{\overline{\tau}_0}(s)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(s)) - \ddot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}
\end{aligned}$$

$$\to\ 0,$$

and

$$\begin{aligned}\frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}|R_2|_{L^p_{0,\alpha}} &\leqslant \overline{\beta}_2(T_{21}+(T_{25}+T_{26}+T_{27})|m_2|_{\mathbb{R}^+})\\ &\quad + T_{20}+L_{m2}(T_{22}+T_{23}+T_{24})\\ &\quad \times(|m_1|_{W^{2,\infty}}+|(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))|_C\,|m_2|_{\mathbb{R}^+})\\ &\to\ 0. \end{aligned}\tag{5.25}$$

Therefore, it follows from (5.19)–(5.25) that

$$\begin{aligned}&\frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\mathscr{S}_v(u,\overline{v})m-\mathscr{S}_v(u,v)m-\mathscr{S}_{vv}(u,v)(l,m)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\\ &\leqslant \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,(|\omega^{21}(s)m_1(0)|_{L^p_{0,\alpha}}+|\omega^{22}(s)(\widetilde{m}_1(z^{\tau_0}(s))\\ &\quad -(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s)))|_{L^p_{0,\alpha}}+|\omega^{23}(s)m_3|_{L^p_{0,\alpha}}+|\omega^{24}(s)m_1(0)|_{L^p_{0,\alpha}}\\ &\quad +|\omega^{25}(s)m_3|_{L^p_{0,\alpha}}+|\omega^{26}(s)m_2|_{L^p_{0,\alpha}}+|R_2|_{L^p_{0,\alpha}})\\ &\to 0\quad (|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0).\end{aligned}$$

That is, $\mathscr{S}_{vv}(u,v)$ is the second-order partial derivative of $S(u,v)$ with respect to v. The proof of the continuity of $\mathscr{S}_{vv}(u,v)$ on its domain is essentially the same as that of the continuity of $\mathscr{S}_{uu}(u,v)$, and hence is omitted.

Part 3 The mixed second-order partial derivative of $S(u,v)$.

Using the same argument for $\mathscr{S}_{uu}(u,v)$ in Part 1, we can show that $\mathscr{S}_{uv}(u,v)$ is a bilinear operator from $(W^{2,\infty}\times\mathbb{R}^+\times\Sigma)\times(\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha)$ to $\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha$. Note that the expression of $\mathscr{S}_{uv}(u,v)$ is the same as that of $\mathscr{S}_{vu}(u,v)$. We consider $\mathscr{S}_{uv}(u,v)$ only to save space.

Again, we first check the boundedness of $\mathscr{S}_{uv}(u,v)$ as follows. Let $(l,h)\in(W^{2,\infty}\times\mathbb{R}^+\times\Sigma)\times(\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha)$.

$$\begin{aligned}&|\mathscr{S}_{uv}(u,v)(h,l)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\\ &=\Big(\int_0^\alpha\Big|\frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0)h_1(s)+\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[h_1(s)(\widetilde{l}_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))l_2)\\ &\quad +l_1(0)(h_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))]+\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_3h_1(s)\\ &\quad +\frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)l_3(h_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))+\frac{\partial^2 f}{\partial\theta_2^2}(s)(\widetilde{l}_1(z^{\tau_0}(s))\\ &\quad -(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))l_2)(h_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))+\frac{\partial f}{\partial\theta_2}(s)\\ &\quad \times[-\dot{h}_1(z^{\tau_0}(s))l_2+(\ddot{y}+\ddot{\tilde{\varphi}})(z^{\tau_0}(s))l_2h_2(s)-\dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s)]\Big|^p\,\mathrm{d}s\Big)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_0^\alpha\Big|\Big(\frac{\partial^2 g}{\partial\gamma_1^2}(s)l_1(0)+\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)l_3+\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_2\Big)h_1(s)\\
&+\Big(\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)l_1(0)+\frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)l_3+\frac{\partial^2 g}{\partial\gamma_2^2}(s)l_2\Big)h_2(s)\Big|^p\mathrm{d}s\Big)^{1/p}\\
\leqslant{}& 2L_{m2}|l_1|_{W^{2,\infty}}\,|h_1|_{L^p_{0,\alpha}}+L_{m2}|(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))|_C\,|l_2|_{\mathbb{R}^+}\,|h_1|_{L^p_{0,\alpha}}\\
&+L_{m2}|l_1|_{W^{2,\infty}}\,|h_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}}+L_{m2}|l_1|_{W^{2,\infty}}\,|(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))|_C\,|h_2|_{L^p_{0,\alpha}}\\
&+L_{m2}|l_3|_\Sigma\,|h_1|_{L^p_{0,\alpha}}+L_{m2}|l_3|_\Sigma\,|h_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}}\\
&+L_{m2}|l_3|_\Sigma\,|(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))|_C\,|h_2|_{L^p_{0,\alpha}}\\
&+L_{m2}(|l_1|_{W^{2,\infty}}+d_1|l_2|_\Sigma)(|h_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}}+|(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))|_C\,|h_2|_{L^p_{0,\alpha}})\\
&+L_{m1}|\dot h_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}}\,|l_2|_{\mathbb{R}^+}+L_{m1}|(\ddot y+\ddot{\tilde\varphi})(z^{\tau_0}(s))|_C\,|h_2|_{L^p_{0,\alpha}}\,|l_2|_{\mathbb{R}^+}\\
&+L_{m1}|l_1|_{W^{2,\infty}}\,|h_2|_{L^p_{0,\alpha}}+H_{m2}(|l_1|_{W^{2,\infty}}\,|h_1|_{L^p_{0,\alpha}}+|l_3|_\Sigma\,|h_1|_{L^p_{0,\alpha}}\\
&+|l_2|_{\mathbb{R}^+}\,|h_1|_{L^p_{0,\alpha}}+|l_1|_{W^{2,\infty}}\,|h_2|_{L^p_{0,\alpha}}+|l_3|_\Sigma\,|h_2|_{L^p_{0,\alpha}}+|l_2|_{\mathbb{R}^+}\,|h_2|_{L^p_{0,\alpha}})\\
\leqslant{}&\alpha((4L_{m2}+H_{m2})|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|l_1|_{W^{2,\infty}}+(2L_{m2}+H_{m2})|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|l_3|_\Sigma+(2L_{m2}c_1\\
&+L_{m1}+H_{m2})|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|l_2|_{\mathbb{R}^+}+(2L_{m2}c_1+L_{m1}+H_{m2})|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|l_1|_{W^{2,\infty}}\\
&+(L_{m2}c_1+H_{m2})|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|l_3|_\Sigma+(L_{m2}c_1+L_{m1}d_2+H_{m2})|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|l_2|_\Sigma)\\
\leqslant{}&\alpha(C_0+H_{m2})|h|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\,|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma},
\end{aligned}
$$

where

$$C_0=\max\{4L_{m2},\,2L_{m2}c_1+L_{m1},\,L_{m2}c_1+L_{m1}d_2\}.$$

This completes the proof of the boundedness of $\mathscr{S}_{uv}(u,v)$ as a bilinear operator from $(W^{2,\infty}\times\mathbb{R}^+\times\Sigma)\times(\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha)$ to $\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha$. From the proof, it is easy to see that it is also bounded as a bilinear operator from $(W^{2,\infty}\times\mathbb{R}^+\times\Sigma)\times(\mathbb{Y}^{2,p}_\alpha\times\mathbb{Y}^{2,p}_\alpha)$ to $\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha$.

Next, we prove that $\mathscr{S}_{uv}(u,v)$ is the mixed second-order partial derivative of $S(u,v)$. Again, let $\overline{v}=(\overline{\varphi},\overline{\tau}_0,\overline{\sigma}):=v+l$. We have

$$
\begin{aligned}
&|\mathscr{S}_u(u,\overline{v})h-\mathscr{S}_u(u,v)h-\mathscr{S}_{uv}(u,v)(h,l)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}\\
={}&\Big(\int_0^\alpha\Big|\frac{\partial\check f}{\partial\theta_1}(s)h_1(s)+\frac{\partial\check f}{\partial\theta_2}(s)(h_1(z^{\overline{\tau}_0}(s))-(\dot y+\dot{\tilde{\overline{\varphi}}})(z^{\overline{\tau}_0}(s))h_2(s))\\
&-\frac{\partial f}{\partial\theta_2}(s)(h_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s))-\frac{\partial f}{\partial\theta_1}(s)h_1(s)\\
&-\frac{\partial^2 f}{\partial\theta_1^2}(s)l_1(0)h_1(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[(\widetilde{l}_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))l_2)h_1(s)\\
&+(h_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s))l_1(0)]-\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)l_3h_1(s)-\frac{\partial^2 f}{\partial\theta_2^2}(s)\\
&\times[\widetilde{l}_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))l_2][h_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s)]\\
&-\frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)[l_3(h_1(z^{\tau_0}(s))-(\dot y+\dot{\tilde\varphi})(z^{\tau_0}(s))h_2(s))]-\frac{\partial f}{\partial\theta_2}(s)
\end{aligned}
$$

$$
\begin{aligned}
&\times (-\dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s) - \dot{\widetilde{h}}_1(z^{\tau_0}(s))l_2 + (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s)l_2)\Big|^p \mathrm{d}s\Big)^{1/p} \\
&+ \Big(\int_0^\alpha \Big|\frac{\partial \check{g}}{\partial \gamma_1}(s)h_1(s) + \frac{\partial \check{g}}{\partial \gamma_2}(s)h_2(s) - \frac{\partial g}{\partial \gamma_1}(s)h_1(s) - \frac{\partial g}{\partial \gamma_2}(s)h_2(s) \\
&- \frac{\partial^2 g}{\partial \gamma_1^2}(s)h_1(s)l_1(0) - \frac{\partial^2 g}{\partial \gamma_2^2}(s)h_2(s)l_2 - \frac{\partial^2 g}{\partial \gamma_1 \partial \gamma_2}(s)(l_2h_1(s) + l_1(0)h_2(s)) \\
&- \frac{\partial^2 g}{\partial \gamma_1 \partial \gamma_3}(s)l_3h_1(s) - \frac{\partial^2 g}{\partial \gamma_2 \partial \gamma_3}(s)l_3h_2(s)\Big|^p \mathrm{d}s\Big)^{1/p},
\end{aligned}
$$

or

$$
\begin{aligned}
&|\mathscr{S}_u(u,\overline{v})h - \mathscr{S}_u(u,v)h - \mathscr{S}_{uv}(u,v)(h,l)|_{\mathbb{Y}_\alpha^{1,p}\times \mathbb{Y}_\alpha^{1,p}} \\
&= |\omega^{31}(s)h_1(s) + \omega^{32}(s)(h_1(z^{\tau_0}(s)) - (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s)) + R_3(s)|_{L^p_{0,\alpha}} \\
&\quad + |\omega^{33}(s)h_1(s) + \omega^{34}(s)h_2(s)|_{L^p_{0,\alpha}},
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\omega^{31}(s) &= \frac{\partial \check{f}}{\partial \theta_1}(s) - \frac{\partial f}{\partial \theta_1}(s) - \frac{\partial^2 f}{\partial \theta_1^2}(s)l_1(0) - \frac{\partial^2 f}{\partial \theta_1 \partial \theta_3}(s)l_3 \\
&\quad - \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)((y + \widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y + \widetilde{\varphi})(z^{\tau_0}(s))),
\end{aligned}
$$

$$
\begin{aligned}
\omega^{32}(s) &= \frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s) - \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s)l_1(0) - \frac{\partial^2 f}{\partial \theta_2 \partial \theta_3}(s)l_3 \\
&\quad - \frac{\partial^2 f}{\partial \theta_2^2}(s)((y + \widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y + \widetilde{\varphi})(z^{\tau_0}(s))),
\end{aligned}
$$

$$
\omega^{33}(s) = \frac{\partial \check{g}}{\partial \gamma_1}(s) - \frac{\partial g}{\partial \gamma_1}(s) - \frac{\partial^2 g}{\partial \gamma_1^2}(s)l_1(0) - \frac{\partial^2 g}{\partial \gamma_1 \partial \gamma_2}(s)l_2 - \frac{\partial^2 g}{\partial \gamma_1 \partial \gamma_3}(s)l_3,
$$

$$
\omega^{34}(s) = \frac{\partial \check{g}}{\partial \gamma_2}(s) - \frac{\partial g}{\partial \gamma_2}(s) - \frac{\partial^2 g}{\partial \gamma_1 \partial \gamma_2}(s)l_1(0) - \frac{\partial^2 g}{\partial \gamma_2^2}(s)l_2 - \frac{\partial^2 g}{\partial \gamma_2 \partial \gamma_3}(s)l_3,
$$

$$
\begin{aligned}
R_3(s) = {}& \frac{\partial \check{f}}{\partial \theta_2}(s)[h_1(z^{\overline{\tau}_0}(s)) - (\dot{y} + \dot{\widetilde{\overline{\varphi}}})(z^{\overline{\tau}_0}(s))h_2(s) - h_1(z^{\tau_0}(s)) \\
&+ (\dot{y} + \dot{\widetilde{\overline{\varphi}}})(z^{\overline{\tau}_0}(s))h_2(s) - \dot{h}_1(z^{\tau_0}(s))l_2 - \dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s) \\
&+ (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2h_2(s)] - \Big(\frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s)\Big)[\dot{h}_1(z^{\tau_0}(s))l_2 \\
&+ \dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s) - (\ddot{y} + \ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2h_2(s)] + \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}(s) \\
&\times ((y + \widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y + \widetilde{\varphi})(z^{\tau_0}(s)) - \widetilde{l}_1(z^{\tau_0}(s)) + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s)) \\
&\times l_2)h_1(s) + \frac{\partial^2 f}{\partial \theta_2^2}(s)[(y + \widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y + \widetilde{\overline{\varphi}})(z^{\tau_0}(s)) \\
&- \widetilde{l}_1(z^{\tau_0}(s)) + (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2][h_1(z^{\tau_0}(s)) - (\dot{y} + \dot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s)].
\end{aligned}
$$

As before, we show that the left-hand side of (5.26) goes to 0 as $|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0$ by estimating each term in the right-hand side of (5.26). Noting that

$$\begin{aligned}(\Delta_4 l)(s) &:= |l_1(0)| + |(y+\widetilde{\overline{\varphi}})(z^{\overline{\tau}_0}(s)) - (y+\widetilde{\varphi})(z^{\tau_0}(s))| + |l_3| \\ &\leqslant 2|l_1|_{W^{2,\infty}} + (|y|_{\mathbb{Y}^{1,\infty}_\alpha} + |\widetilde{\varphi}|_{W^{1,\infty}})|l_2|_{\mathbb{R}^+} + |l_3|_\Sigma \\ &\leqslant \max\{2, d_1\}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\end{aligned}$$

and

$$(\Delta_5 l)(s) := |l_1(0)| + |l_3| + |l_2| \leqslant |l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma},$$

we know that

$$(\Delta_4 l)(s) \to 0, \quad (\Delta_5 l)(s) \to 0 \quad \text{uniformly as } |l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0,$$

and

$$\Big\|\frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s)\Big\| \leqslant \max\{d_1, 2\}L_{m2}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}.$$

Then similar to obtaining (5.8), we have, as $|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0$,

$$\Big|\frac{\|\omega^{31}(s)\|}{(\Delta_4 l)(s)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{32}(s)\|}{(\Delta_4 l)(s)}\Big|_C \to 0,$$

$$\Big|\frac{\|(\omega^{33})(s)\|}{(\Delta_5 l)(s)}\Big|_C \to 0, \quad \Big|\frac{\|\omega^{34}(s)\|}{(\Delta_5 l)(s)}\Big|_C \to 0,$$

and

$$\Big\|\frac{\partial \check{f}}{\partial \theta_2}(s) - \frac{\partial f}{\partial \theta_2}(s)\Big\|_C \to 0.$$

Therefore, as $|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma} \to 0$, we have

$$\frac{|\omega^{31}(s)h_1(s)|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} \leqslant \max\{2, d_1\}|h_1|_C\Big|\frac{\|\omega^{31}(s)\|}{(\Delta_4 l)(s)}\Big|_C \to 0, \tag{5.27}$$

$$\begin{aligned}&\frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\omega^{32}(s)(h_1(z^{\tau_0}(s)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s))|_{L^p_{0,\alpha}} \\ &\leqslant \max\{2, d_1\}|h_1(z^{\tau_0}(s)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s)|_C\Big|\frac{\|\omega^{32}(s)\|}{(\Delta_4 l)(s)}\Big|_C \\ &\to 0,\end{aligned} \tag{5.28}$$

$$\begin{aligned}\frac{|\omega^{33}(s)h_1(s)|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} &= \Big(\int_0^\alpha \Big(\frac{\|\omega^{33}(s)\|\,|h_1(s)|}{(\Delta_5 l)(s)}\frac{(\Delta_5 l)(s)}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big)^p \mathrm{d}s\Big)^{1/p} \\ &\leqslant \alpha^{1/p}|h_1|_C\Big|\frac{\|\omega^{33}(s)\|}{(\Delta_5 l)(s)}\Big|_C \\ &\to 0,\end{aligned} \tag{5.29}$$

$$\begin{aligned}\frac{|\omega^{34}(s)h_2(s)|_{L^p_{0,\alpha}}}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}} &= \Big(\int_0^\alpha \Big(\frac{\|\omega^{34}(s)\|\,|h_2(s)|}{(\Delta_5 l)(s)}\frac{(\Delta_5 l)(s)}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big)^p \mathrm{d}s\Big)^{1/p}\\ &\leqslant \alpha^{1/p}|h_2|_C\,\Big|\frac{\|\omega^{34}(s)\|}{(\Delta_5 l)(s)}\Big|_C\\ &\to\ 0. \end{aligned} \tag{5.30}$$

$$\begin{aligned} T_{30} :=\ & \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big|\Big(\frac{\partial\check{f}}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big)\\ &\times[\dot{h}_1(z^{\tau_0}(s))l_2+\dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s)-(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2h_2(s)]\Big|_{L^p_{0,\alpha}}\\ \leqslant\ & \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\Big\|\frac{\partial\check{f}}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big\|_C\\ &\times|\dot{h}_1(z^{\tau_0}(s))l_2+\dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s)-(\ddot{y}+\ddot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2h_2(s)|_{L^p_{0,\alpha}}\\ \leqslant\ & \max\{d_1,2\}L_{m2}\Big(\frac{1}{1-\eta^+}|h_1|_{\mathbb{Y}^{1,p}_\alpha}|l_2|_{\mathbb{R}^+}+\alpha|l_1|_{W^{2,\infty}}|h_2|_{\mathbb{Y}^{1,p}_\alpha}\\ &+\alpha(\beta_1+\delta_1+|\varphi^*|_{W^{2,\infty}})|h_2|_{\mathbb{Y}^{1,p}_\alpha}|l_2|_{\mathbb{R}^+}\Big)\\ \leqslant\ & \max\{d_1,2\}\max\Big\{\alpha d_1,\frac{1}{1-\eta^+}\Big\}L_{m2}|h|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\\ \to\ & 0. \end{aligned} \tag{5.31}$$

On the other hand, it follows from Lemma 5.1 that $y(z^{\tau_0}(s))$, $\dot{y}(z^{\tau_0}(s))$, $\varphi(z^{\tau_0}(s))$, $\dot{\varphi}(z^{\tau_0}(s))$, $\widetilde{l}_1(z^{\tau_0}(s))$, $\dot{\widetilde{l}}_1(z^{\tau_0}(s))$ all are continuously differentiable with respect to τ_0 in $L^p_{0,\alpha}$ norm. This, combined with Lemmas 3.3–3.5, gives us

$$\begin{aligned} T_{31} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}|h_1(z^{\overline{\tau}_0}(s))-h_1(z^{\tau_0}(s))-\dot{h}_1(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}\\ &\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}}|h_1(z^{\overline{\tau}_0}(s))-h_1(z^{\tau_0}(s))-\dot{h}_1(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}\\ &\to 0\quad(|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0), \end{aligned}$$

$$\begin{aligned} T_{32} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}|y(z^{\overline{\tau}_0}(s))-y(z^{\tau_0}(s))-\dot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}\\ &\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}}|y(z^{\overline{\tau}_0}(s))-y(z^{\tau_0}(s))-\dot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}\\ &\to 0\quad(|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0), \end{aligned}$$

$$\begin{aligned} T_{33} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}|\widetilde{\varphi}(z^{\overline{\tau}_0}(s))-\widetilde{\varphi}(z^{\tau_0}(s))-\dot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}\\ &\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}}|\widetilde{\varphi}(z^{\overline{\tau}_0}(s))-\widetilde{\varphi}(z^{\tau_0}(s))-\dot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}}\\ &\to 0\quad(|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0), \end{aligned}$$

$$\begin{aligned}
T_{34} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\widetilde{l}_1(z^{\overline{\tau}_0}(s)) - \widetilde{l}_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\dot{\widetilde{l}}_1(z^{\tau_0}(s) - \lambda_{34}(s)l_2)|_{L^p_{0,\alpha}}\,|l_2|_{\mathbb{R}^+} \\
&\qquad (\text{where } \lambda_{34}(s)\in[0,1] \text{ for } s\in[0,\alpha]) \\
&\leqslant \alpha^{1/p}|l_1|_{W^{2,\infty}} \\
&\to 0 \quad (|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0), \\
T_{35} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\dot{\widetilde{l}}_1(z^{\overline{\tau}_0}(s)) - \dot{\widetilde{l}}_1(z^{\tau_0}(s))|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\ddot{\widetilde{l}}_1(z^{\tau_0}(s) - \lambda_{35}(s)l_2)|_{L^p_{0,\alpha}}\,|l_2|_{\mathbb{R}^+} \\
&\qquad (\text{where } \lambda_{35}(s)\in[0,1] \text{ for } s\in[0,\alpha]) \\
&\leqslant \alpha^{1/p}|l_1|_{W^{2,\infty}} \\
&\to 0 \quad (|l|_{W^{2,\infty}\times\Sigma\times\mathbb{R}^+}\to 0), \\
T_{36} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\dot{y}(z^{\overline{\tau}_0}(s)) - \dot{y}(z^{\tau_0}(s)) - \ddot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}}\,|\dot{y}(z^{\overline{\tau}_0}(s)) - \dot{y}(z^{\tau_0}(s)) - \ddot{y}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0), \\
T_{37} &= \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\dot{\widetilde{\varphi}}(z^{\overline{\tau}_0}(s)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(s)) - \ddot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\leqslant \frac{1}{|l_2|_{\mathbb{R}^+}}\,|\dot{\widetilde{\varphi}}(z^{\overline{\tau}_0}(s)) - \dot{\widetilde{\varphi}}(z^{\tau_0}(s)) - \ddot{\widetilde{\varphi}}(z^{\tau_0}(s))l_2|_{L^p_{0,\alpha}} \\
&\to 0 \quad (|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0).
\end{aligned}$$

Then, by the Minkowsky Inequality, we have

$$\begin{aligned}
\frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}|R_3|_{L^p_{0,\alpha}} &\leqslant \overline{\beta}_2(T_{31} + (T_{35}+T_{36}+T_{37})|h_2|_C) \\
&\quad + T_{30} + L_{m2}(T_{32}+T_{33}+T_{34}) \\
&\quad \times (|h_1|_C + |(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))|_C\,|h_2|_C) \\
&\to 0 \quad (|l|_{W^{2,\infty}\times\Sigma\times\mathbb{R}^+}\to 0). \qquad (5.32)
\end{aligned}$$

Therefore, it follows from (5.27)–(5.32) that

$$\begin{aligned}
&\frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,|\mathscr{S}_u(u,\overline{v})h - \mathscr{S}_u(u,v)h - \mathscr{S}_{uv}(u,v)(h,l)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha} \\
&\leqslant \frac{1}{|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}}\,[|\omega^{32}(s)(h_1(z^{\tau_0}(s)) - (\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s))|_{L^p_{0,\alpha}} \\
&\quad + |\omega^{31}(s)h_1(s)|_{L^p_{0,\alpha}} + |\omega^{33}(s)h_2(s)|_{L^p_{0,\alpha}} + |\omega^{34}(s)h_1(s)|_{L^p_{0,\alpha}} + |R_3|_{L^p_{0,\alpha}}] \\
&\to 0 \quad (|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0).
\end{aligned}$$

This shows that $\mathscr{S}_{uv}(u,v)$ is the mixed second-order partial derivative of $S(u,v)$.

The remaining proof is the continuity of $\mathscr{S}_{uv}(u,v)$. Let $(u,v)\in U\times V$. Select sequences $u^n=(y^n,z^n)\in U$ and $v^n=(\varphi^n,\tau_0^n,\sigma^n)\in V$ such that $|u^n-u|_{\mathbb{Y}_\alpha^{2,p}\times\mathbb{Y}_\alpha^{2,p}}\to 0$ and $|v^n-v|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}\to 0$ as $n\to\infty$. For $h=(h_1,h_2)\in U$, $l=(l_1,l_2,l_3)\in V$, we have

$$
\begin{aligned}
&|\mathscr{S}_{uv}(u^n,v^n)(h,l)-\mathscr{S}_{uv}(u,v)(h,l)|_{\mathbb{Y}_\alpha^{1,p}\times\mathbb{Y}_\alpha^{1,p}}\\
&=\Big(\int_0^\alpha\Big|\Big(\frac{\partial^2 f^n}{\partial\theta_1^2}(s)-\frac{\partial^2 f}{\partial\theta_1^2}(s)\Big)l_1(0)h_1(s)+\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_2}(s)[h_1(s)(\widetilde{l}_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))l_2)+l_1(0)(h_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))]-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)[h_1(s)(\widetilde{l}_1(z^{\tau_0}(s))\\
&\quad-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))l_2)+l_1(0)(h_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))]\\
&\quad+\Big(\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_3}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)\Big)l_3h_1(s)+\frac{\partial^2 f^n}{\partial\theta_2\partial\theta_3}(s)l_3(h_1((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))-\frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)l_3(h_1(z^{\tau_0}(s))\\
&\quad-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s))+\frac{\partial^2 f^n}{\partial\theta_2^2}(s)(\widetilde{l}_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))\\
&\quad\times l_2)(h_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2)-\frac{\partial^2 f}{\partial\theta_2^2}(s)(\widetilde{l}_1(z^{\tau_0}(s))\\
&\quad-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))l_2)(h_1(z^{\tau_0}(s))-(\dot{y}+\dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2)\\
&\quad+\frac{\partial f^n}{\partial\theta_2}(s)[-\dot{h}_1((z^n)\tau_0^n(s))l_2+(\ddot{y}^n+\ddot{\tilde{\varphi}}^n)((z^n)\tau_0^n(s))l_2h_2(s)\\
&\quad-\dot{\widetilde{l}}_1((z^n)\tau_0^n(s))h_2(s)]-\frac{\partial f}{\partial\theta_2}(s)[-\dot{h}_1(z^{\tau_0}(s))l_2+(\ddot{y}+\ddot{\tilde{\varphi}})(z^{\tau_0}(s))l_2h_2(s)\\
&\quad-\dot{\widetilde{l}}_1(z^{\tau_0}(s))h_2(s)]\Big|^p\mathrm{d}s\Big)^{1/p}+\Big(\int_0^\alpha\Big|\Big[\Big(\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_1^2}(s)\Big)l_1(0)\\
&\quad+\Big(\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_2}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\Big)l_2+\Big(\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_3}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)\Big)l_3\Big]h_1(s)\\
&\quad+\Big[\Big(\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_2}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\Big)l_1(0)+\Big(\frac{\partial^2 g^n}{\partial\gamma_2^2}(s)-\frac{\partial^2 g}{\partial\gamma_2^2}(s)\Big)l_2\\
&\quad+\Big(\frac{\partial^2 g^n}{\partial\gamma_2\partial\gamma_3}(s)-\frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)\Big)l_3\Big]h_2(s)\Big|^p\mathrm{d}s\Big)^{1/p}.
\end{aligned}
$$

Using the Minkowsky Inequality, we can estimate the above expression in the sum of the following terms. Again, the following estimations can be obtained by using Lemmas 3.3–3.6 and 5.1:

$$C_{31}=\Big|\Big(\frac{\partial^2 f^n}{\partial\theta_1^2}(s)-\frac{\partial^2 f}{\partial\theta_1^2}(s)\Big)l_1(0)h_1(s)\Big|_{L^p_{0,\alpha}}$$

$$\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_1^2}(s)-\frac{\partial^2 f}{\partial\theta_1^2}(s)\Big\|\,|l_1(0)|\,|h_1|_{L^p_{0,\alpha}}$$

$$\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_1^2}(s)-\frac{\partial^2 f}{\partial\theta_1^2}(s)\Big\|\alpha^{1/q}|l_1|_{W^{2,\infty}}\,|h_1|_{\mathbb{Y}^{1,p}_\alpha},$$

$$\begin{aligned}
C_{32} &= \Big|\Big(\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_2}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)\Big)([\widetilde{l}_1((z^n)^{\tau_0^n}(s))-(\dot{y^n}+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))l_2]\\
&\quad\times h_1(s)+[h_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)]l_1(0))\Big|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_2}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)\Big\|(|\widetilde{l}_1((z^n)^{\tau_0^n}(s))h_1(s)+h_1((z^n)^{\tau_0^n}(s))\\
&\quad\times l_1(0)|_{L^p_{0,\alpha}}+|(\dot{y^n}+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))(l_2h_1(s)+h_2(s)l_1(0))|_{L^p_{0,\alpha}})\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_2}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)\Big\|\\
&\quad\times\alpha\cdot\max\{1,d_1\}|h|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma},
\end{aligned}$$

$$\begin{aligned}
C_{33} &= \Big|\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)((\widetilde{l}_1((z^n)^{\tau_0^n}(s))-(\dot{y^n}+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))l_2)h_1(s)\\
&\quad+(h_1((z^n)^{\tau_0^n}(s))-(\dot{y}^n+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s))l_1(0)\\
&\quad-(\widetilde{l}_1(z^{\tau_0}(s))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))l_2)h_1(s)-(h_1(z^{\tau_0}(s))\\
&\quad-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s))h_2(s))l_1(0))\Big|_{L^p_{0,\alpha}}\\
&= \Big|\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)([\widetilde{l}_1((z^n)^{\tau_0^n}(s))-\widetilde{l}_1(z^{\tau_0}(s))+((\dot{y^n}+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))\\
&\quad-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s)))l_2]h_1(s)+[h_1((z^n)^{\tau_0^n}(s))-h_1(z^{\tau_0}(s))\\
&\quad+((\dot{y}^n+\dot{\widetilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))-(\dot{y}+\dot{\widetilde{\varphi}})(z^{\tau_0}(s)))h_2(s)]\,l_1(0))\Big|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}(s)\Big\|(\alpha|h_1|_{\mathbb{Y}^{1,p}_\alpha}[(|l_1|_{W^{1,\infty}}+|l_2|_{\mathbb{R}^+}(|y|_{\mathbb{Y}^{2,\infty}_\alpha}+|\varphi|_{W^{2,\infty}}))\\
&\quad\times(\alpha^{1/q}|z^n-z|_{\mathbb{Y}^{1,p}_\alpha}+|\tau_0^n-\tau_0|)\\
&\quad+|l_2|_{\mathbb{R}^+}(\alpha^{1/q}|y-y^n|_{\mathbb{Y}^{2,p}_\alpha}+|\varphi-\varphi^n|_{W^{2,\infty}})]\\
&\quad+\alpha^{1/p}|l_1|_{W^{2,\infty}}[(|h_1|_{\mathbb{Y}^{1,\infty}_\alpha}+\alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}(|y|_{\mathbb{Y}^{2,\infty}_\alpha}+|\varphi|_{W^{2,\infty}}))\\
&\quad\times(\alpha^{1/q}|z^n-z|_{\mathbb{Y}^{1,p}_\alpha}+|\tau_0^n-\tau_0|)\\
&\quad+\alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}(\alpha^{1/q}|y-y^n|_{\mathbb{Y}^{2,p}_\alpha}+|\varphi-\varphi^n|_{W^{2,\infty}})]),
\end{aligned}$$

$$\begin{aligned}
C_{34} &= \Big|\Big(\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_3}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)\Big)l_3h_1(s)\Big|_{L^p_{0,\alpha}}\\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_1\partial\theta_3}(s)-\frac{\partial^2 f}{\partial\theta_1\partial\theta_3}(s)\Big\|\alpha|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|l_3|_\Sigma,
\end{aligned}$$

$$\begin{aligned}
C_{35} &= \Big|\Big(\frac{\partial^2 f^n}{\partial\theta_2\partial\theta_3}(s) - \frac{\partial^2 f^n}{\partial\theta_2\partial\theta_3}(s)\Big) l_3[h_1((z^n)^{\tau_0^n}(s)) \\
&\quad - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)]\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_2\partial\theta_3}(s) - \frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)\Big\|\alpha\cdot\max\{1,d_1\}|h|_{\mathbb{Y}^{1,p}_\alpha}|l_3|_\Sigma,
\end{aligned}$$

$$\begin{aligned}
C_{36} &= \Big|\frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s) l_3(h_1((z^n)^{\tau_0^n}(s)) - h_1(z^{\tau_0}(s)) \\
&\quad - ((\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s)))h_2(s))\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f}{\partial\theta_2\partial\theta_3}(s)\Big\|\alpha^{1/p}|l_3|_\Sigma((|h_1|_{\mathbb{Y}^{1,\infty}_\alpha} \\
&\quad + \alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}(|y|_{\mathbb{Y}^{2,\infty}_\alpha} + |\varphi|_{W^{2,\infty}}))(\alpha^{1/q}|z - z^n|_{\mathbb{Y}^{1,p}_\alpha} + |\tau_0 - \tau_0^n|) \\
&\quad + \alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}(\alpha^{1/q}|y - y^n|_{\mathbb{Y}^{2,p}_\alpha} + |\varphi - \varphi^n|_{W^{2,\infty}})),
\end{aligned}$$

$$\begin{aligned}
C_{37} &= \Big|\Big(\frac{\partial^2 f^n}{\partial\theta_2^2}(s) - \frac{\partial^2 f}{\partial\theta_2^2}(s)\Big)([\widetilde{l}_1((z^n)^{\tau_0^n}(s)) - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))l_2] \\
&\quad \times [h_1((z^n)^{\tau_0^n}(s)) - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)])\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_2^2}(s) - \frac{\partial^2 f}{\partial\theta_2^2}(s)\Big\|\,|\widetilde{l}_1((z^n)^{\tau_0^n}(s)) - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))l_2|_C \\
&\quad \times |h_1((z^n)^{\tau_0^n}(s)) - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f^n}{\partial\theta_2^2}(s) - \frac{\partial^2 f}{\partial\theta_2^2}(s)\Big\|\alpha\cdot\max\{1,d_1^2\}|h|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma},
\end{aligned}$$

$$\begin{aligned}
C_{38} &= \Big|\frac{\partial^2 f}{\partial\theta_2^2}(s)([\widetilde{l}_1((z^n)^{\tau_0^n}(s)) - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))l_2][h_1((z^n)^{\tau_0^n}(s)) \\
&\quad - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)] - [\widetilde{l}_1(z^{\tau_0}(s)) \\
&\quad - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s))l_2][h_1(z^{\tau_0}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s)])\Big|_{L^p_{0,\alpha}} \\
&= \Big|\frac{\partial^2 f}{\partial\theta_2^2}(s)([\widetilde{l}_1((z^n)^{\tau_0^n}(s)) - \widetilde{l}_1(z^{\tau_0}(s)) - ((\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s)) \\
&\quad - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s)))l_2][h_1((z^n)^{\tau_0^n}(s)) - (\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)] \\
&\quad - [\widetilde{l}_1(z^{\tau_0}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s))l_2][h_1(z^{\tau_0}(s)) - h_1((z^n)^{\tau_0^n}(s)) \\
&\quad + ((\dot{y}^n + \dot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s)) - (\dot{y} + \dot{\tilde{\varphi}})(z^{\tau_0}(s)))h_2(s)])\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 f}{\partial\theta_2^2}(s)\Big\|\max\{1,d_1\}\alpha|h|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha}((|l_1|_{W^{1,\infty}} + |l_2|_{\mathbb{R}^+}(|y|_{\mathbb{Y}^{2,\infty}_\alpha} \\
&\quad + |\varphi|_{W^{2,\infty}}))(\alpha^{1/q}|z - z^n|_{\mathbb{Y}^{1,p}_\alpha} + |\tau_0 - \tau_0^n|) + |l_2|_{\mathbb{R}^+}(\alpha^{1/q}|y - y^n|_{\mathbb{Y}^{2,p}_\alpha}
\end{aligned}$$

$$+|\varphi-\varphi^n|_{W^{2,\infty}}))+\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial f^2}{\partial\theta_2^2}(s)\Big\|\max\{1,d_1\}\alpha|l|_{W^{2,\infty}\times\mathbb{R}^+\times\Sigma}$$
$$\times(((|h_1|_{\mathbb{Y}^{1,\infty}_\alpha}+\alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}(|y|_{\mathbb{Y}^{2,\infty}_\alpha}+|\varphi|_{W^{2,\infty}}))(\alpha^{1/q}|z-z^n|_{\mathbb{Y}^{1,p}_\alpha}$$
$$+|\tau_0-\tau_0^n|)+\alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}(\alpha^{1/q}|y-y^n|_{\mathbb{Y}^{2,p}_\alpha}+|\varphi-\varphi^n|_{W^{2,\infty}})),$$

$$\begin{aligned}C_{39}=&\Big|\Big(\frac{\partial f^n}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big)((\ddot{y}^n+\ddot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)l_2\\&-\dot{\tilde{l}}_1((z^n)^{\tau_0^n}(s))h_2(s)-\dot{h}_1((z^n)^{\tau_0^n}(s))l_2)\Big|_{L^p_{0,\alpha}}\\\leqslant&\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial f^n}{\partial\theta_2}(s)-\frac{\partial f}{\partial\theta_2}(s)\Big\|(\alpha d_2|h_2|_{\mathbb{Y}^{1,p}_\alpha}|l_2|_{\mathbb{R}^+}\\&+(\alpha|l_1|_{W^{1,\infty}}|h_2|_{\mathbb{Y}^{1,p}_\alpha}+(1-\eta^+)^{-1}|h_1|_{\mathbb{Y}^{1,p}_\alpha}|l_2|_{\mathbb{R}^+})),\end{aligned}$$

$$\begin{aligned}C_{40}=&\Big|\frac{\partial f}{\partial\theta_2}(s)[(\ddot{y}^n+\ddot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))h_2(s)l_2-\dot{\tilde{l}}_1((z^n)^{\tau_0^n}(s))h_2(s)\\&-\dot{h}_1((z^n)^{\tau_0^n}(s))l_2-(\ddot{y}+\ddot{\tilde{\varphi}})(z^{\tau_0}(s))h_2(s)l_2\\&+\dot{\tilde{l}}_1(z^{\tau_0}(s))h_2(s)+\dot{h}_1(z^{\tau_0}(s))l_2]\Big|_{L^p_{0,\alpha}}\\=&\Big|\frac{\partial f}{\partial\theta_2}(s)[((\ddot{y}^n+\ddot{\tilde{\varphi}}^n)((z^n)^{\tau_0^n}(s))-(\ddot{y}+\ddot{\tilde{\varphi}})((z^n)^{\tau_0^n}(s))\\&+(\ddot{y}+\ddot{\tilde{\varphi}})((z^n)^{\tau_0^n}(s))-(\ddot{y}+\ddot{\tilde{\varphi}})(z^{\tau_0}(s)))h_2(s)l_2-(\dot{\tilde{l}}_1((z^n)^{\tau_0^n}(s))\\&-\dot{\tilde{l}}_1(z^{\tau_0}(s)))h_2(s)-(\dot{h}_1((z^n)^{\tau_0^n}(s))-\dot{h}_1(z^{\tau_0}(s)))l_2]\Big|_{L^p_{0,\alpha}}\\\leqslant&\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial f}{\partial\theta_2}(s)\Big\|([(1-\eta^+)^{-1}|y-y^n|_{\mathbb{Y}^{2,p}_\alpha}+\alpha^{1/p}|\varphi-\varphi^n|_{W^{2,\infty}}\\&+|(\ddot{y}+\ddot{\tilde{\varphi}})((z^n)^{\tau_0^n}(s))-(\ddot{y}+\ddot{\tilde{\varphi}})(z^{\tau_0}(s))|_{L^p_{0,\alpha}}]\alpha^{1/q}|h_2|_{\mathbb{Y}^{1,p}_\alpha}|l_2|_{\mathbb{R}^+}\\&+\alpha|h_2|_{\mathbb{Y}^{1,p}_\alpha}|l_1|_{W^{2,\infty}}(\alpha^{1/q}|z-z^n|_{\mathbb{Y}^{1,p}_\alpha}+|\tau_0-\tau_0^n|)\\&+\alpha^{1/p}|h_1|_{\mathbb{Y}^{2,\infty}_\alpha}|l_2|_{\mathbb{R}^+}(\alpha^{1/q}|z-z^n|_{\mathbb{Y}^{1,p}_\alpha}+|\tau_0-\tau_0^n|)),\end{aligned}$$

$$\begin{aligned}C_{41}=&\Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_1^2}(s)\Big)h_1(s)l_1(0)\Big|_{L^p_{0,\alpha}}\\\leqslant&\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1^2}(s)-\frac{\partial^2 g}{\partial\gamma_1^2}(s)\Big\|\alpha|h_1|_{\mathbb{Y}^{1,p}_\alpha}|l_1|_{W^{2,\infty}},\end{aligned}$$

$$\begin{aligned}C_{42}=&\Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_2}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\Big)(h_2(s)l_1(0)+l_2h_1(s))\Big|_{L^p_{0,\alpha}}\\\leqslant&\sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_2}(s)-\frac{\partial^2 g}{\partial\gamma_1\partial\gamma_2}(s)\Big\|\alpha(|h_2|_{\mathbb{Y}^{1,p}_\alpha}|l_1|_{W^{2,\infty}}+|h_1|_{\mathbb{Y}^{1,p}_\alpha}|l_2|_{\mathbb{R}^+}),\end{aligned}$$

$$
\begin{aligned}
C_{43} &= \Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_2^2}(s) - \frac{\partial^2 g}{\partial\gamma_2^2}(s)\Big)h_2(s)l_2\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1^2}(s) - \frac{\partial^2 g}{\partial\gamma_2^2}(s)\Big\|\alpha|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|l_2|_{\mathbb{R}^+}, \\
C_{44} &= \Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_3}(s) - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)\Big)l_3h_1\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_1\partial\gamma_3}(s) - \frac{\partial^2 g}{\partial\gamma_1\partial\gamma_3}(s)\Big\|\alpha|h_1|_{\mathbb{Y}^{1,p}_\alpha}\,|l_3|_\Sigma, \\
C_{45} &= \Big|\Big(\frac{\partial^2 g^n}{\partial\gamma_2\partial\gamma_3}(s) - \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)\Big)l_3h_2\Big|_{L^p_{0,\alpha}} \\
&\leqslant \sup_{0\leqslant s\leqslant\alpha}\Big\|\frac{\partial^2 g^n}{\partial\gamma_2\partial\gamma_3}(s) - \frac{\partial^2 g}{\partial\gamma_2\partial\gamma_3}(s)\Big\|\alpha|h_2|_{\mathbb{Y}^{1,p}_\alpha}\,|l_3|_\Sigma.
\end{aligned}
$$

Similar arguments as those for the pointwise continuity of $\mathscr{S}_{uu}(u,v)$ will produce $C_i \to 0$ as $n\to\infty,\ i=31,32,\dots,45$. Thus,

$$|\mathscr{S}_{uv}(u^n,v^n)(h,l) - \mathscr{S}_{uv}(u,v)(h,l)|_{\mathbb{Y}^{1,p}_\alpha\times\mathbb{Y}^{1,p}_\alpha} \to 0 \quad (n\to\infty).$$

This completes the proof of the pointwise continuity of $\mathscr{S}_{uv}(u,v)$. □

According to Theorem 2.1, Lemmas 3.11, 4.1, 5.2, and 5.3, we then have verified all the conditions for the second-order differentiability of the fixed point in the norm $|\cdot|_{\mathbb{X}^{1,p}_\alpha\times\mathbb{X}^{1,p}_\alpha}$. Recall also Remark 3.1. We have established the main result of this paper, which is formulated below.

Theorem 5.1 *Assume that* (A1)–(A3) *hold,* $1\leqslant p<\infty$ *and* φ^*, σ^*, τ_0^* *satisfy* $\varphi^*(0)\in\Omega_1$, $\varphi^*(-\tau_0^*)\in\Omega_2$, $\tau_0^*\in\Omega_4$ *and* $\sigma^*\in\Omega_3$. *Then there exist positive constants* α, δ_1^*, δ_2^*, δ_3^* *such that system* (1.1) *has a unique solution*

$$(x(\varphi,\tau_0,\sigma),\tau(\varphi,\tau_0,\sigma))(t) \quad \textit{for } t\in[0,\alpha]$$

corresponding to $\varphi\in\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1^*)$, $\tau\in\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2^*)$ *and* $\sigma\in\mathscr{G}_\Sigma(\sigma^*;\delta_3^*)$. *Moreover, the function*

$$
\begin{aligned}
&\mathscr{G}_{W^{2,\infty}}(\varphi^*;\delta_1^*)\times\mathscr{G}_{\mathbb{R}^+}(\tau_0^*;\delta_2^*)\times\mathscr{G}_\Sigma(\sigma^*;\delta_3^*)\to\mathbb{X}^{2,p}_\alpha\times\mathbb{X}^{2,p}_\alpha, \\
&\qquad\qquad (\varphi,\tau_0,\sigma)\mapsto(x(\varphi,\tau_0,\sigma)(t),\tau(\varphi,\tau_0,\sigma)(t)),
\end{aligned}
$$

is C_p^2 *with respect to* φ, τ_0 *and* σ *on its domain in the norm* $|\cdot|_{\mathbb{X}^{1,p}_\alpha\times\mathbb{X}^{1,p}_\alpha}$ *and hence in the norm* $|\cdot|_{W^{1,p}_\alpha\times W^{1,p}_\alpha}$.

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