PROCEEDINGS OF THE ROYAL SOCIETY A

FirstCite e-publishing

Proc. R. Soc. A
doi:10.1098/rspa.2009.0377
Published online

Travelling waves of a diffusive Kermack–McKendrick epidemic model with non-local delayed transmission

By Zhi-Cheng Wang[1,*] and Jianhong Wu[2,3]

[1]School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China
[2]Centre for Disease Modeling, York Institute for Health Research, and
[3]Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada

We obtain full information about the existence and non-existence of travelling wave solutions for a general class of diffusive Kermack–McKendrick SIR models with non-local and delayed disease transmission. We show that this information is determined by the basic reproduction number of the corresponding ordinary differential model, and the minimal wave speed is explicitly determined by the delay (such as the latent period) and non-locality in disease transmission, and the spatial movement pattern of the infected individuals. The difficulty is the lack of order-preserving property of the general system, and we obtain the threshold dynamics for spatial spread of the disease by constructing an invariant cone and applying Schauder's fixed point theorem.

Keywords: travelling wave solution; diffusive SIR model; non-local interaction; time delay

1. Introduction

The basic compartmental models to describe the transmission of communicable diseases are contained in a sequence of three papers by W. O. Kermack and A. G. McKendrick (1927, 1932, 1933) (Anderson 1991; Brauer 2008), where the SIR model

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(t)&=-\beta S(t)I(t),\\ \frac{\mathrm{d}}{\mathrm{d}t}I(t)&=\beta S(t)I(t)-\gamma I(t)\\ \text{and}\qquad \frac{\mathrm{d}}{\mathrm{d}t}R(t)&=\gamma I(t)\end{aligned}\right\}\tag{1.1}$$

was formulated and analysed, where $S(t)$, $I(t)$ and $R(t)$ denote the sizes of the susceptible, infected and removed individuals, respectively. The constant β is the transmission coefficient, and γ is the recovery rate. Let $S_0=S(0)$ be the density of the population at the beginning of the epidemic with everyone susceptible,

*Author for correspondence (wangzhch@lzu.edu.cn).

Received 17 July 2009
Accepted 7 September 2009

then it is well known that the so-called basic reproduction number $R_0 = \beta S_0/\gamma$ completely determines the transmission dynamics and epidemic potential: if $R_0 > 1$, the $I(t)$ first increases to its maximum and then decreases to zero and hence an epidemic occurs; if $R_0 < 1$, then $I(t)$ decreases to zero and epidemic does not happen.

Such a model and the aforementioned threshold result have been playing a pivotal role in subsequent developments in the mathematical modelling-assisted study of infectious disease transmission dynamics. This model is based on the assumption of a high degree of homogeneity in the population, including the mobility. In reality, however, individuals can be exposed to the infection from contact with infectives in different spatial locations. This consideration led to the Kendall non-local model in 1957 that involves space-dependent integro-differential equations

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}S(x,t) = -\beta S(x,t)\int_{-\infty}^{+\infty} I(y,t)K(x-y)\,\mathrm{d}y,\\
&\frac{\partial}{\partial t}I(x,t) = \beta S(x,t)\int_{-\infty}^{+\infty} I(y,t)K(x-y)\,\mathrm{d}y - \gamma I(x,t)\\
\text{and}\qquad &\frac{\partial}{\partial t}R(x,t) = \gamma I(x,t),
\end{aligned}\right\} \tag{1.2}$$

where the kernel $K(x-y) \geq 0$ weights the contributions of the infected individuals at location y to the infection of susceptible individuals at location x, with $\int_{-\infty}^{+\infty} K(y)\,\mathrm{d}y = 1$. Disease propagation in space is relevant to the so-called travelling waves, solutions of the form $(S(x+ct), I(x+ct), R(x+ct))$ for which c is called the wave speed. In the case where $R_0 > 1$, Kendall (1965) proved that there exists $c^* > 0$ such that equation (1.2) admits non-trivial travelling wave solutions for all speeds $c \geq c^*$ and no non-trivial travelling wave solution with speeds less than c^*. Aronson (1977) later showed that equation (1.2) can be reduced to a scalar integro-differential equation, and this reduction enabled him to formally link the wave speed to the asymptotic speed of propagation. Similar reductions can be done for more general situations, for example, the nonlinear (double) integral equation model

$$u(x,t) = \int_0^t \int_{\Omega} g(u(y,t-\theta))k(\theta,x,y)\,\mathrm{d}y\,\mathrm{d}\theta + f(x,t) \tag{1.3}$$

was used to include the incubation period in Diekmann (1978, 1979) and Thieme (1977*a*,*b*, 1979). When $\Omega = \mathbb{R}^N$, the existence of travelling wave solutions and the asymptotic speed of propagation were considered in Diekmann (1979) and Thieme (1979) for equation (1.3), see also Thieme and Zhao (2003), Ruan (2007) and references therein for other subsequent works in the subject area. An important feature of the reduced model is a certain order-preserving property that permits the applications of the powerful monotone dynamical systems and comparison arguments.

Random movement of individuals in space was further incorporated into the Kendall model by De Mottoni *et al.* (1979) by adding some diffusion terms as follows:

$$\left.\begin{aligned}\frac{\partial}{\partial t}S(x,t)&=\Delta S(x,t)+\mu-\sigma S(x,t)-S(x,t)\int_{\Omega}I(y,t)K(x,y)\,\mathrm{d}y\\ \text{and}\quad \frac{\partial}{\partial t}I(x,t)&=d\Delta I(x,t)+S(x,t)\int_{\Omega}I(y,t)K(x,y)\,\mathrm{d}y-\gamma I(x,t),\end{aligned}\right\}\tag{1.4}$$

subject to the Neumannn boundary condition. When the space is unbounded, $\mu=\sigma=0$ and $K(\cdot)$ is a constant (β) multiple of the delta function, system (1.4) reduces to the following reaction–diffusion model:

$$\left.\begin{aligned}\frac{\partial}{\partial t}S(x,t)&=\Delta S(x,t)-\beta S(x,t)I(x,t)\\ \text{and}\quad \frac{\partial}{\partial t}I(x,t)&=d\Delta I(x,t)+\beta S(x,t)I(x,t)-\gamma I(x,t).\end{aligned}\right\}\tag{1.5}$$

This equation was considered by Hosono and Ilyas (1994), where it is proved that if $S(0,x)=S_0$ is a constant and if $\gamma/\beta S_0<1$, then for each $c\geq c^*=2\sqrt{\beta S_0 d(1-\gamma/\beta S_0)}$ there exists a positive constant $\varepsilon<S_0$ such that system (1.5) has a travelling wave solution $(S(x+ct),I(x+ct))$ satisfying $S(+\infty)=S_0, S(-\infty)=\varepsilon, I(\pm\infty)=0$.

Extension of the above result for system (1.4) with a general kernel $K(\cdot)$, even if $\mu=\sigma=0$, is difficult due to the fundamental issue that the system of equations governing the wave solutions is no longer an ordinary differential system: it is a system of functional differential equations with both advanced and delayed arguments and it is a system without any obvious order-preserving property. In addition, if we wish to consider, as pointed out in Ruan (2007), the effect of spatial heterogeneity (geographical movement), non-local interaction and time delay such as latent period on the spread of the disease, we need to examine an even more general model of the following form:

$$\left.\begin{aligned}\frac{\partial}{\partial t}S(x,t)&=d_1\Delta S(x,t)-\beta S(x,t)\int_{-\infty}^{t}\int_{-\infty}^{+\infty}I(y,s)K(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s,\\ \frac{\partial}{\partial t}I(x,t)&=d_2\Delta I(x,t)\\ &\quad+\beta S(x,t)\int_{-\infty}^{t}\int_{-\infty}^{+\infty}I(y,s)K(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s-\gamma I(x,t)\\ \text{and}\quad \frac{\partial}{\partial t}R(x,t)&=d_3\Delta R(x,t)+\gamma I(x,t),\end{aligned}\right\}\tag{1.6}$$

where d_1, d_2 and d_3 are the diffusion rates for the susceptible, infective and removed individuals, respectively. The kernel $K(x-y,t-s)\geq 0$ describes the interaction between the infective and the susceptible individuals at location x

and the present t which occurred at location y and at earlier instance s, which throughout this paper is assumed to satisfy the following conditions:

(K1) K is non-negative and integrable, and satisfies

$$\int_0^\infty \int_{-\infty}^\infty K(x,t)\,\mathrm{d}x\,\mathrm{d}t = 1 \text{ and } K(x,t) = K(-x,t), (x,t) \in \mathbb{R} \times [0,\infty);$$

(K2) For every $c \geq 0$, there exists $\lambda_c \in (0, +\infty]$ such that $\int_0^\infty \int_{-\infty}^\infty K(x,t)\,\mathrm{e}^{-\lambda(x+ct)}\,\mathrm{d}x\,\mathrm{d}t < +\infty$ for any $\lambda \in [0, \lambda_c)$, and $\int_0^\infty \int_{-\infty}^\infty K(x,t)\,\mathrm{e}^{-\lambda(x+ct)}\,\mathrm{d}x\,\mathrm{d}t \to +\infty$ as $\lambda \to \lambda_c - 0$.

It is not difficult to verify that λ_c is non-decreasing on $c \geq 0$.

We focus here on the existence and non-existence of travelling wave solutions of system (1.6). We shall prove that if $R_0 = \beta S_0/\gamma > 1$, then there exists $c_* > 0$ such that for every $c > c_*$, system (1.6) admits a non-trivial travelling wave solution with wave speed c, and if $R_0 = \beta S_0/\gamma < 1$, then for any $c \geq 0$, system (1.6) admits no non-trivial travelling wave solutions with wave speed c. Therefore, the existence and non-existence of travelling wave solutions is determined completely by the basic reproduction number and the condition $R_0 = \beta S_0/\gamma = 1$ coincides with the threshold for the existence of wavefronts. Furthermore, when $\beta S_0/\gamma > 1$ we show that equation (1.6) admits no non-trivial travelling wave solutions with wave speed $c \in [0, c_*)$. Therefore, we confirm that $c_* > 0$ is indeed the minimal wave speed. We do anticipate that c_* is the asymptotic speed of propagation for equation (1.6), following the work described in Murray (1989), though verification of this requires some additional work. Our approach for the existence of travelling wave solutions is to construct a suitable invariant set and then apply Schauder's fixed point theorem, see also Li *et al.* (2006) and Ma (2001). Our construction of the invariant set is motivated by the work of Ducrot and Magal (2009). For $c \in [0, c_*)$, we conclude the non-existence of non-trivial travelling wave solutions by an argument applying the Laplace transform to the $I(x+ct)$ component, this argument was first introduced by Carr and Chmaj (2004) and further used by Wang *et al.* (2008, 2009).

We should point out that when $K(x,t) = \delta(t-\tau)1/(\sqrt{4\pi d_2 \tau})\mathrm{e}^{(-x^2)/(4d_2\tau)}$, the existence and non-existence of non-trivial travelling wave solutions of equation (1.6) were proved by Ducrot and Magal (2009) (see also Ducrot *et al.* 2009). These studies considered the following infection-age structured model with diffusion:

$$\left.\begin{aligned}
\frac{\partial}{\partial t}S(x,t) &= d_1\Delta_x S - S(x,t)\int_0^{a_+} \beta(a)i(x,a,t)\,\mathrm{d}a,\\
\frac{\partial}{\partial t}i(x,a,t) + \frac{\partial}{\partial a}i(x,a,t) &= d_2\Delta_x i - \gamma(a)i(x,a,t), \quad a \in (0,a_+)\\
i(x,0,t) &= S(x,t)\int_0^{a_+} \beta(a)i(x,a,t)\,\mathrm{d}a, \quad x \in \mathbb{R}, t \geq 0\\
\text{and} \qquad S(x,0) &= S_0(x), \quad i(x,a,0) = i_0(x,a),
\end{aligned}\right\} \tag{1.7}$$

with a being the time since the infection and $a_+ \in (0, +\infty]$ the maximum attainable age of infection. When the incubation is exactly equal to $\tau > 0$, then the function $\beta(a)$ takes the form $\beta(a) = \hat{\beta} 1_{[\tau,+\infty)}(a)$, $\quad a \geq 0$. So, if we further assume that $a_+ = \infty$ and $\gamma(a) \equiv \gamma$, then equation (1.7) reduces to the two former equations of (1.6) with $\beta = \hat{\beta}\, \mathrm{e}^{-\gamma\tau}$ and $K(x, t) = \delta(t - \tau) 1/(\sqrt{4\pi\, d_2 \tau}) \mathrm{e}^{(-x^2)/(4 d_2 \tau)}$. However, the aforementioned papers did not prove the existence of the minimal wave speed. We also refer to Faria *et al.* (2006), Gourley *et al.* (2004), Li & Zou (in press), Ou & Wu (2007), Wang *et al.* (2006, 2008) and references therein for some relevant progress on the existence of travelling wave solutions of reaction–diffusion equations with non-local interaction and time delay.

Our analytic study about the minimal speed permits discussion of the effect on the minimal wave speed c^* of (i) the diffusion rate d_2 of infective individuals, (ii) non-local interaction between the infective and the susceptible individuals and (iii) the latent period of disease. We confirm that the latent period of disease can slow down the speed of the disease, the non-local interaction between the infective and the susceptible individuals and the spatial movement of infective individuals can increase the speed of the spread of the disease.

2. Main results

In this section, we will state precisely and prove the main results of this paper. In the sequel, we always assume that the initial disease free equilibrium is $(S_0, 0, 0)$.

Because the first equation and the second equation of system (1.6) form a closed system, we consider only the following system:

$$\left.\begin{aligned} &\frac{\partial}{\partial t} S(x,t) = d_1 \Delta S(x,t) - \beta S(x,t) \int_{-\infty}^{t} \int_{-\infty}^{+\infty} I(y,s) K(x-y, t-s)\, \mathrm{d}y\, \mathrm{d}s \\ \text{and} \quad &\frac{\partial}{\partial t} I(x,t) = d_2 \Delta I(x,t) \\ &\qquad + \beta S(x,t) \int_{-\infty}^{t} \int_{-\infty}^{+\infty} I(y,s) K(x-y, t-s)\, \mathrm{d}y\, \mathrm{d}s - \gamma I(x,t). \end{aligned}\right\} \tag{2.1}$$

We look for the non-trivial travelling wave solutions $(S_c(x + ct), I_c(x + ct))$ of (2.1) satisfying the following conditions:

$$S_c(-\infty) = S_0, \quad S_c(+\infty) = S_c^{\infty}, \quad I(\pm\infty) = 0 \tag{2.2}$$

and

$$S(\cdot) \text{ is decreasing and } I(\cdot) \geq 0,\ S_0 > S_c^{\infty} \geq 0.$$

Let $\xi = x + ct$. Then the system describing travelling wave solutions is as follows:

$$\left.\begin{aligned} &c S_c' = d_1 S_c'' - \beta S_c(\xi) \int_0^{\infty} \int_{-\infty}^{+\infty} I_c(\xi - y - cs) K(y,s)\, \mathrm{d}y\, \mathrm{d}s \\ \text{and} \quad &c I_c' = d_2 I_c'' + \beta S_c(\xi) \int_0^{\infty} \int_{-\infty}^{+\infty} I_c(\xi - y - cs) K(y,s)\, \mathrm{d}y\, \mathrm{d}s - \gamma I_c(\xi). \end{aligned}\right\} \tag{2.3}$$

Linearizing the second equation of (2.3) at the initial disease free point $(S_0, 0)$, we have

$$cJ' = d_2 J'' + \beta S_0 \int_0^\infty \int_{-\infty}^{+\infty} J(\xi - y - cs) K(y, s)\,\mathrm{d}y\,\mathrm{d}s - \gamma J(\xi). \tag{2.4}$$

Let $J(\xi) = \mathrm{e}^{\lambda\xi}$, then we get a characteristic equation

$$\Theta(\lambda, c) := d_2\lambda^2 - c\lambda + \beta S_0 \int_0^\infty \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda y - c\lambda s} K(y, s)\,\mathrm{d}y\,\mathrm{d}s - \gamma. \tag{2.5}$$

For the sake of convenience, we set

$$G(\lambda, c) := \int_0^\infty \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda y - c\lambda s} K(y, s)\,\mathrm{d}y\,\mathrm{d}s.$$

It is easy to prove the following lemma, see also Li *et al.* (2007, lemma 3.27) and Wang *et al.* (2006, lemma 2.2).

Lemma 2.1. *Assume that $S_0 > \gamma/\beta$. Then there exists $c_* > 0$ and $\lambda^* > 0$ such that $\partial/\partial\lambda\Theta(\lambda, c)|_{(\lambda^*, c_*)} = 0$ and $\Theta(\lambda^*, c_*) = 0$. Furthermore,*

(i) *if $0 < c < c_*$, then $\Theta(\lambda, c) > 0$ for all $\lambda \in [0, \lambda_c)$;*
(ii) *if $c > c_*$, then the equation $\Theta(\lambda, c) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda^* < \lambda_2(c) < \lambda_c$ such that $\lambda_1'(c) < 0$, $\lambda_2'(c) > 0$ and*

$$\Theta(\lambda, c) \begin{cases} > 0 & \text{for } \lambda \in [0, \lambda_1(c)) \cup (\lambda_2(c), \lambda_c) \\ < 0 & \text{for } \lambda \in (\lambda_1(c)\lambda_2(c)). \end{cases}$$

In the following, we always assume that $S_0 > \gamma/\beta$. In addition, we fix $c > c_*$ and always denote $\lambda_i(c)$ by λ_i, $i = 1, 2$.

Lemma 2.2. *The function $I^+(\xi) = \mathrm{e}^{\lambda_1\xi}$ satisfies the following linear equation:*

$$cI' = d_2 I'' + \beta S_0 \int_0^\infty \int_{-\infty}^{+\infty} I(\xi - y - cs) K(y, s)\,\mathrm{d}y\,\mathrm{d}s - \gamma I(\xi). \tag{2.6}$$

Lemma 2.3. *For $\alpha > 0$ sufficiently small and $\sigma > S_0$ large enough, the function $S^-(\xi) := \max\{S_0 - \sigma\mathrm{e}^{\alpha\xi}, 0\}$ satisfies*

$$cS' \le d_1 S'' - \beta S(\xi)\,\mathrm{e}^{\lambda_1\xi} \int_0^\infty \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda_1 y - c\lambda_1 s} K(y, s)\,\mathrm{d}y\,\mathrm{d}s \tag{2.7}$$

for any $\xi \ne 1/\alpha \ln S_0/\sigma$.

Proof. When $\xi > (1/\alpha)\ln(S_0/\sigma)$, we have $S^-(\xi) = 0$, which implies that equation (2.7) holds.

When $\xi < (1/\alpha)\ln(S_0/\sigma)$, namely $S^-(\xi) = S_0 - \sigma\,\mathrm{e}^{\alpha\xi} > 0$, we need to prove that

$$-c\alpha\sigma + d_1\alpha^2\sigma + \beta(S_0 - \sigma\,\mathrm{e}^{\alpha\xi})\,\mathrm{e}^{(\lambda_1-\alpha)\xi}\int_0^{\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 y - c\lambda_1 s}K(y,s)\,\mathrm{d}y\,\mathrm{d}s \le 0. \tag{2.8}$$

Obviously, it is sufficient to ensure

$$-c\alpha\sigma + d_1\alpha^2\sigma + \beta S_0\,\mathrm{e}^{((\lambda_1-\alpha)/\alpha)\ln(S_0/\sigma)}\int_0^{\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 y - c\lambda_1 s}K(y,s)\,\mathrm{d}y\,\mathrm{d}s \le 0.$$

That is,

$$-c\alpha\sigma + d_1\alpha^2\sigma + \beta S_0\left(\frac{S_0}{\sigma}\right)^{(\lambda_1-\alpha)/\alpha}\int_0^{\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 y - c\lambda_1 s}K(y,s)\,\mathrm{d}y\,\mathrm{d}s \le 0. \tag{2.9}$$

Keeping $\alpha\sigma = 1$ and letting $\sigma \to \infty$, for some σ large enough we have that equation (2.9) holds. This completes the proof. ■

Lemma 2.4. *Let $\varepsilon > 0$ satisfy $\varepsilon < \alpha/2$ and $\varepsilon < \lambda_2 - \lambda_1$. Then for $M > 0$ sufficiently large, the function $I^-(\xi) := \mathrm{e}^{\lambda_1\xi}(1 - M\mathrm{e}^{\varepsilon\xi})$ satisfies*

$$cI \le d_2 I'' + \beta S^-(\xi)\int_0^{\infty}\int_{-\infty}^{+\infty} I(\xi - y - cs)K(y,s)\,\mathrm{d}y\,\mathrm{d}s - \gamma I(\xi).$$

Proof. If $S_0 - \sigma\mathrm{e}^{\alpha\xi} \le 0$, that is, $\xi \ge (1/\alpha)\ln(S_0/\sigma)$, it is needed to prove that

$$cI^{-\prime} \le d_2 I^{-\prime\prime} - \gamma I^-(\xi). \tag{2.10}$$

Namely

$$\begin{aligned} &c\lambda_1\,\mathrm{e}^{\lambda_1\xi} - cM(\lambda_1+\varepsilon)\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi} \\ &\quad\le d_2\lambda_1^2\,\mathrm{e}^{\lambda_1\xi} - d_2M(\lambda_1+\varepsilon)^2\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi} - \gamma\,\mathrm{e}^{\lambda_1\xi} + M\gamma\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi}. \end{aligned}$$

Consequently, we need to verify that

$$\beta S_0\int_0^{\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 y - c\lambda_1 s}K(y,s)\,\mathrm{d}y\,\mathrm{d}s \le M[c(\lambda_1+\varepsilon) - d_2(\lambda_1+\varepsilon)^2 + \gamma]\,\mathrm{e}^{\varepsilon\xi}.$$

Since $\xi \ge (1/\alpha)\ln(S_0/\sigma)$, it is sufficient to prove

$$\beta S_0\int_0^{\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 y - c\lambda_1 s}K(y,s)\,\mathrm{d}y\,\mathrm{d}s \le M[c(\lambda_1+\varepsilon) - d_2(\lambda_1+\varepsilon)^2 + \gamma]\,\mathrm{e}^{(\varepsilon/\alpha)\ln(S_0/\sigma)}.$$

Then, for sufficiently large $M > 0$ with

$$M \ge \frac{\beta S_0\int_0^{\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 y - c\lambda_1 s}K(y,s)\,\mathrm{d}y\,\mathrm{d}s}{M[c(\lambda_1+\varepsilon) - d_2(\lambda_1+\varepsilon)^2 + \gamma]\,\mathrm{e}^{(\varepsilon/\alpha)\ln(S_0/\sigma)}},$$

we have that equation (2.10) holds.

If $S_0 - \sigma\,\mathrm{e}^{\alpha\xi} > 0$, that is, $\xi < (1/\alpha)\ln(S_0/\sigma)$, it is sufficient to prove that

$$\begin{aligned} c\lambda_1\,\mathrm{e}^{\lambda_1\xi} - cM(\lambda_1+\varepsilon)\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi} &\\ \le d_2\lambda_1^2\,\mathrm{e}^{\lambda_1\xi} - d_2M(\lambda_1+\varepsilon)^2\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi} &\\ + \beta(S_0-\sigma\,\mathrm{e}^{\alpha\xi})(G(\lambda_1,c)\,\mathrm{e}^{\lambda_1\xi} - MG(\lambda_1+\varepsilon,c)\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi}) &\\ - \gamma\,\mathrm{e}^{\lambda_1\xi} + M\gamma\,\mathrm{e}^{(\lambda_1+\varepsilon)\xi}. & \end{aligned}$$

that is,

$$\begin{aligned} 0 &\le M[c(\lambda_1+\varepsilon) - d_2(\lambda_1+\varepsilon)^2 - \beta S_0 G(\lambda_1+\varepsilon,c)+\gamma]\,\mathrm{e}^{\varepsilon\xi} \\ &\quad - \sigma\beta G(\lambda_1,c)\,\mathrm{e}^{\alpha\xi} + M\sigma\beta G(\lambda_1+\varepsilon,c)\,\mathrm{e}^{(\alpha+\varepsilon)\xi} \\ &= -M\Theta(\lambda_1+\varepsilon,c)\,\mathrm{e}^{\varepsilon\xi} - \sigma\beta G(\lambda_1,c)\,\mathrm{e}^{\alpha\xi} + M\sigma\beta G(\lambda_1+\varepsilon,c)\,\mathrm{e}^{(\alpha+\varepsilon)\xi}. \end{aligned}$$

Consequently, we need only to show that

$$-M\Theta(\lambda_1+\varepsilon,c) - \sigma\beta G(\lambda_1,c)\,\mathrm{e}^{(\alpha-\varepsilon)\xi} \ge 0.$$

Since $\xi < (1/\alpha)\ln(S_0/\sigma) < 0$ (see lemma 2.3, $\sigma > S_0$), we have $\sigma\,\mathrm{e}^{(\alpha-\varepsilon)\xi} \le \sigma\,\mathrm{e}^{(\alpha/2)\xi} \le \sqrt{\sigma S_0}$. Therefore, we have

$$-M\Theta(\lambda_1+\varepsilon,c) - \sigma\beta G(\lambda_1,c)\,\mathrm{e}^{(\alpha-\varepsilon)\xi} \ge 0$$

if $M \ge -(\beta\sqrt{\sigma S_0}\,G(\lambda_1,c))/(\Theta(\lambda_1+\varepsilon,c))$. Thus, the proof is completed. ∎

Define

$$\Gamma = \left\{ (S(\cdot),I(\cdot)) \in C(\mathbb{R},\mathbb{R}^2) \;\middle|\; \begin{array}{l} S^-(x) \le S(x) \le S_0, \\ \max\{I^-(x),0\} \le I(x) \le I^+(x) \end{array} \right\}$$

and

$$\begin{aligned} \Lambda_{11} &= \frac{c-\sqrt{c^2+4d_1\beta\alpha_1}}{2d_1}, \quad \Lambda_{12} = \frac{c+\sqrt{c^2+4d_1\beta\alpha_1}}{2d_1}, \quad \rho_1 = d_1(\Lambda_{12}-\Lambda_{11}), \\ \Lambda_{21} &= \frac{c-\sqrt{c^2+4d_2\alpha_2}}{2d_2}, \quad \Lambda_{22} = \frac{c+\sqrt{c^2+4d_2\alpha_2}}{2d_2}, \quad \rho_2 = d_2(\Lambda_{22}-\Lambda_{21}), \end{aligned}$$

where $\alpha_1 \ge S_0$ and $\alpha_2 \ge \gamma$ satisfy $-\Lambda_{11} > 2\lambda_1$ and $-\Lambda_{21} > 2\lambda_1$. Furthermore, define an operator $F : \Gamma \to C(\mathbb{R},\mathbb{R}^2)$ by

$$F[S(\cdot),I(\cdot)](\xi) = \begin{pmatrix} F_1[S(\cdot),I(\cdot)](\xi) \\ F_2[S(\cdot),I(\cdot)](\xi) \end{pmatrix},$$

where

$$F_1[S(\cdot),I(\cdot)](\xi)=\frac{\beta}{\rho_1}\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}[\alpha_1 S(x)-S(x)(K*I)(x)]\,\mathrm{d}x$$
$$+\frac{\beta}{\rho_1}\int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}[\alpha_1 S(x)-S(x)(K*I)(x)]\,\mathrm{d}x,$$
$$F_2[S(\cdot),I(\cdot)](\xi)=\frac{\beta}{\rho_2}\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{21}(\xi-x)}\left[S(x)(K*I)(x)+\frac{\alpha_2-\gamma}{\beta}I(x)\right]\mathrm{d}x$$
$$+\frac{\beta}{\rho_2}\int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{22}(\xi-x)}\left[S(x)(K*I)(x)+\frac{\alpha_2-\gamma}{\beta}I(x)\right]\mathrm{d}x.$$

In the following, for $u\in\Gamma$ we denote

$$(K*u)(\xi):=\int_0^{+\infty}\int_{-\infty}^{+\infty}K(y,s)u(\xi-y-cs)\,\mathrm{d}y\,\mathrm{d}s,\quad\forall\xi\in\mathbb{R}. \tag{2.11}$$

Lemma 2.5. *The set Γ is closed and convex in $C(\mathbb{R},\mathbb{R}^2)$.*

The proof is very easy and we omit it.

Lemma 2.6. *The operator F maps Γ into Γ.*

Proof. Give $(S(\cdot),I(\cdot))\in\Gamma$. It is obvious that we only need to prove that

$$S^-(x)\le F_1[S(\cdot),I(\cdot)](x)\le S_0$$

and

$$\max\{I^-(x),0\}\le F_2[S(\cdot),I(\cdot)](x)\le I^+(x)$$

for all $x\in\mathbb{R}$.

First, we consider $F_1[S(\cdot),I(\cdot)](x)$. Since $\alpha_1 S(x)-S(x)(K*I)(x)\le\alpha_1 S_0$, for any $x\in\mathbb{R}$ we have

$$F_1[S(\cdot),I(\cdot)](\xi)=\frac{\beta}{\rho_1}\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}[\alpha_1 S(x)-S(x)(K*I)(x)]\,\mathrm{d}x$$
$$+\frac{\beta}{\rho_1}\int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}[\alpha_1 S(x)-S(x)(K*I)(x)]\,\mathrm{d}x$$
$$\le\frac{\beta\alpha_1 S_0}{\rho_1}\left[\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{d}x+\int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\,\mathrm{d}x\right]$$
$$=\frac{\beta\alpha_1 S_0}{\rho_1}\left(\frac{1}{\Lambda_{12}}-\frac{1}{\Lambda_{11}}\right)$$
$$=S_0.$$

Since $S^-(\xi) := \max\{S_0 - \sigma\, \mathrm{e}^{\alpha\xi}, 0\}$ satisfies

$$\begin{aligned} cS^{-\prime}(\xi) &\le d_1 S^{-\prime\prime}(\xi) - \beta S^-(\xi)\, \mathrm{e}^{\lambda_1 \xi} \int_0^\infty \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda_1 y - c\lambda s} K(y,s)\, \mathrm{d}y\, \mathrm{d}s \\ &\le d_1 S^{-\prime\prime}(\xi) - \beta S^-(\xi)(K * I)(\xi) \\ &= d_1 S^{-\prime\prime}(\xi) - \beta\alpha_1 S^-(\xi) + \beta S^-(\xi)[\alpha_1 - (K*I)(\xi)] \\ &\le d_1 S^{-\prime\prime}(\xi) - \beta\alpha_1 S^-(\xi) + \beta S(\xi)[\alpha_1 - (K*I)(\xi)] \end{aligned}$$

for $\xi \ne (1/\alpha)\ln(S_0/\sigma)$, we have

$$\beta S(\xi)[\alpha_1 - (K*I)(\xi)] \ge -d_1 S^{-\prime\prime}(\xi) + cS^{-\prime}(\xi) + \beta\alpha_1 S^-(\xi).$$

It follows that

$$\begin{aligned} F_1[S(\cdot), I(\cdot)](\xi) &= \frac{\beta}{\rho_1} \int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi - x)} [\alpha_1 S(x) - S(x)(K*I)(x)]\, \mathrm{d}x \\ &\quad + \frac{\beta}{\rho_1} \int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi - x)} [\alpha_1 S(x) - S(x)(K*I)(x)]\, \mathrm{d}x \\ &\ge \frac{1}{\rho_1} \int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi - x)} [-d_1 S^{-\prime\prime}(x) + cS^{-\prime}(x) + \beta\alpha_1 S^-(x)]\, \mathrm{d}x \\ &\quad + \frac{1}{\rho_1} \int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi - x)} [-d_1 S^{-\prime\prime}(x) + cS^{-\prime}(x) + \beta\alpha_1 S^-(x)]\, \mathrm{d}x. \end{aligned}$$

When $\xi \ge \xi_0 := (1/\alpha)\ln(S_0/\sigma)$, we have

$$\begin{aligned} F_1[S(\cdot), I(\cdot)](\xi) &\ge \frac{1}{\rho_1} \int_{\xi_0}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi - x)} [-d_1 S^{-\prime\prime}(x) + cS^{-\prime}(x) + \beta\alpha_1 S^-(x)]\, \mathrm{d}x \\ &\quad \times \frac{1}{\rho_1} \int_{-\infty}^{\xi_0} \mathrm{e}^{\Lambda_{11}(\xi - x)} [-d_1 S^{-\prime\prime}(x) + cS^{-\prime}(x) + \beta\alpha_1 S^-(x)]\, \mathrm{d}x \\ &\quad + \frac{1}{\rho_1} \int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi - x)} [-d_1 S^{-\prime\prime}(x) + cS^{-\prime}(x) + \beta\alpha_1 S^-(x)]\, \mathrm{d}x \\ &= S^-(\xi) + \frac{d_1}{\rho_1} \mathrm{e}^{\Lambda_{11}(\xi - \xi_0)} [S^{-\prime}(\xi_0 + 0) - S^{-\prime}(\xi_0 - 0)] \\ &\ge S^-(\xi). \end{aligned}$$

Similarly, when $\xi < \xi_0 := (1/\alpha)\ln(S_0/\sigma)$, we have

$$F_1[S(\cdot), I(\cdot)](\xi) \ge S^-(\xi).$$

Secondly, we consider $F_2[S(\cdot), I(\cdot)](x)$. Since $S(x)(K*I)(x) + ((\alpha_2 - \gamma)/\beta)I(x) \ge 0$ for all $x \in \mathbb{R}$, we have $F_2[(S(\cdot), I(\cdot))](x) \ge 0$ for all $x \in \mathbb{R}$. Because

$S(\xi) \geq S^-(\xi) = \max\{S_0 - \sigma\, \mathrm{e}^{\gamma\xi}, 0\}$ and $I(\xi) \geq \max\{I^-(\xi), 0\}$, we have that $I^-(\xi) := \mathrm{e}^{\lambda_1\xi}(1 - M\mathrm{e}^{\varepsilon\xi})$ satisfies

$$\begin{aligned} cI^{-\prime}(\xi) &\leq d_2 I^{-\prime\prime}(\xi) + \beta S^-(\xi)\int_0^\infty\int_{-\infty}^{+\infty} I^-(\xi - y - cs)K(y,s)\,\mathrm{d}y\,\mathrm{d}s - \gamma I^-(\xi) \\ &\leq d_2 I^{-\prime\prime}(\xi) - \alpha_2 I^-(\xi) \\ &\quad + \beta S(\xi)\int_0^\infty\int_{-\infty}^{+\infty} I(\xi - y - cs)K(y,s)\,\mathrm{d}y\,\mathrm{d}s + (\alpha_2 - \gamma)I(\xi). \end{aligned}$$

Hence, we have

$$\begin{aligned} F_2[S(\cdot), I(\cdot)](\xi) &= \frac{1}{\rho_2}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}(\xi - x)}[\beta S(x)(K * I)(x) + (\alpha_2 - \gamma)I(x)]\,\mathrm{d}x \\ &\quad + \frac{1}{\rho_2}\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{22}(\xi - x)}[\beta S(x)(K * I)(x) + (\alpha_2 - \gamma)I(x)]\,\mathrm{d}x \\ &\geq \frac{1}{\rho_2}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}(\xi - x)}[-d_2 I^{-\prime\prime}(x) + cI^{-\prime}(x) + \alpha_2 I^-(x)]\,\mathrm{d}x \\ &\quad + \frac{1}{\rho_2}\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{22}(\xi - x)}[-d_2 I^{-\prime\prime}(x) + cI^{-\prime}(x) + \alpha_2 I^-(x)]\,\mathrm{d}x \\ &= I^-(\xi). \end{aligned}$$

Because $I(\xi) \leq I^+(\xi)$ and $S(\xi) \leq S_0$, by equation (2.6) we have

$$\begin{aligned} cI^{+\prime}(\xi) &= d_2 I^{+\prime\prime}(\xi) + \beta S_0\int_0^\infty\int_{-\infty}^{+\infty} I^+(\xi - y - cs)K(y,s)\,\mathrm{d}y\,\mathrm{d}s - \gamma I^+(\xi) \\ &\geq d_2 I^{+\prime\prime}(\xi) - \alpha_2 I^+(\xi) \\ &\quad + \beta S(\xi)\int_0^\infty\int_{-\infty}^{+\infty} I(\xi - y - cs)K(y,s)\,\mathrm{d}y\,\mathrm{d}s + (\alpha_2 - \gamma)I(\xi). \end{aligned}$$

Consequently, we have

$$\begin{aligned} F_2[S(\cdot), I(\cdot)](\xi) &= \frac{1}{\rho_2}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}(\xi - x)}[\beta S(x)(K * I)(x) + (\alpha_2 - \gamma)I(x)]\,\mathrm{d}x \\ &\quad + \frac{1}{\rho_2}\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{22}(\xi - x)}[\beta S(x)(K * I)(x) + (\alpha_2 - \gamma)I(x)]\,\mathrm{d}x \\ &\leq \frac{1}{\rho_2}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}(\xi - x)}[-d_2 I^{+\prime\prime}(x) + cI^{+\prime}(x) + \alpha_2 I^+(x)]\,\mathrm{d}x \\ &\quad + \frac{1}{\rho_2}\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{22}(\xi - x)}[-d_2 I^{+\prime\prime}(x) + cI^{+\prime}(x) + \alpha_2 I^+(x)]\,\mathrm{d}x \\ &= I^+(\xi). \end{aligned}$$

This completes the proof. ■

Define

$$B_\mu(\mathbb{R},\mathbb{R}^2) := \left\{ \Phi = (\phi_1,\phi_2) \in C(\mathbb{R},\mathbb{R}^2) \,\middle|\, \begin{array}{l} \sup_{\xi\in\mathbb{R}} |\phi_1(\xi)|\,\mathrm{e}^{-\mu|\xi|} < +\infty, \\ \sup_{\xi\in\mathbb{R}} |\phi_2(\xi)|\,\mathrm{e}^{-\mu|\xi|} < +\infty \end{array} \right\}$$

with norm

$$|\Phi|_\mu = \max\left\{ \sup_{\xi\in\mathbb{R}} |\phi_1(\xi)|\,\mathrm{e}^{-\mu|\xi|}, \sup_{\xi\in\mathbb{R}} |\phi_2(\xi)|\,\mathrm{e}^{-\mu|\xi|} \right\},$$

where $\mu > 0$ is a constant and satisfies $2\lambda_1 < \mu < \min\{-\Lambda_{11}, -\Lambda_{21}\}$.

Lemma 2.7. *The map* $F : \Gamma \to C(\mathbb{R},\mathbb{R}^2)$ *is continuous with respect to the norm* $|\cdot|_\mu$ *in* $B_\mu(\mathbb{R},\mathbb{R}^2)$.

Proof. For any $(S_1(\cdot), I_1(\cdot)) \in \Gamma$ and $(S_2(\cdot), I_2(\cdot)) \in \Gamma$, we have

$$\begin{aligned} &|F_1[S_1(\cdot), I_1(\cdot)](\xi) - F_1[S_2(\cdot), I_2(\cdot)](\xi)|\,\mathrm{e}^{-\mu|\xi|} \\ &\quad\le \frac{\beta}{\rho_1}\,\mathrm{e}^{-\mu|\xi|}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)}|[\alpha_1 S_1(x) - S_1(x)(K*I_1)(x)] \\ &\qquad - [\alpha_1 S_2(x) - S_2(x)(K*I_2)(x)]|\,\mathrm{d}x \\ &\qquad + \frac{\beta}{\rho_1}\,\mathrm{e}^{-\mu|\xi|}\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}|[\alpha_1 S_1(x) - S_1(x)(K*I_1)(x)] \\ &\qquad - [\alpha_1 S_2(x) - S_2(x)(K*I_2)(x)]|\,\mathrm{d}x. \end{aligned}$$

Note that $(K*I)(x) \le G(\lambda_1, c)\,\mathrm{e}^{\lambda_1 x}$. When $\xi \ge 0$, we have

$$\begin{aligned} &|F_1[S_1(\cdot), I_1(\cdot)](\xi) - F_1[S_2(\cdot), I_2(\cdot)](\xi)|\,\mathrm{e}^{-\mu|\xi|} \\ &\quad\le \frac{\beta}{\rho_1}\,\mathrm{e}^{-\mu\xi}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)}[\alpha_1|S_1(x) - S_2(x)| \\ &\qquad + |S_1(x)(K*I_1)(x) - S_2(x)(K*I_2)(x)|]\,\mathrm{d}x \\ &\qquad + \frac{\beta}{\rho_1}\,\mathrm{e}^{-\mu\xi}\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}[\alpha_1|S_1(x) - S_2(x)| \\ &\qquad + |S_1(x)(K*I_1)(x) - S_2(x)(K*I_2)(x)|]\,\mathrm{d}x \\ &\quad\le \frac{\beta\alpha_1}{\rho_1}\,\mathrm{e}^{-\mu\xi}[|S_1(\cdot) - S_2(\cdot)|_\mu + |(K*I_1)(\cdot) - (K*I_2)(\cdot)|_\mu]\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{e}^{\mu|x|}\,\mathrm{d}x \\ &\qquad + \frac{\beta\alpha_1}{\rho_1}\,\mathrm{e}^{-\mu\xi}[|S_1(\cdot) - S_2(\cdot)|_\mu + |(K*I_1)(\cdot) - (K*I_2)(\cdot)|_\mu]\int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}\,\mathrm{e}^{\mu|x|}\,\mathrm{d}x \\ &\qquad + \frac{\beta\,G(\lambda_1, c)}{\rho_1}|S_1(\cdot) - S_2(\cdot)|_{\mu/2}\,\mathrm{e}^{-\mu\xi} \\ &\qquad \times \left[\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{e}^{\lambda_1 x+(\mu|x|)/2}\,\mathrm{d}x + \int_{\xi}^{\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}\,\mathrm{e}^{\lambda_1 x+(\mu|x|)/2}\,\mathrm{d}x\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\beta\alpha_1}{\rho_1}\,\mathrm{e}^{-\mu\xi}[|S_1(\cdot)-S_2(\cdot)|_\mu + |(K*I_1)(\cdot)-(K*I_2)(\cdot)|_\mu] \\
&\quad\times\left[\int_{-\infty}^{0}\mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{e}^{-\mu x}\,\mathrm{d}x + \int_0^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{e}^{\mu x}\,\mathrm{d}x + \int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\,\mathrm{e}^{\mu x}\,\mathrm{d}x\right] \\
&\quad+ \frac{\beta G(\lambda_1,c)}{\rho_1}|S_1(\cdot)-S_2(\cdot)|_{\mu/2}\,\mathrm{e}^{-\mu\xi} \\
&\quad\times\left[\int_{-\infty}^{0}\mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{e}^{(\lambda_1-\mu/2)x}\,\mathrm{d}x + \int_0^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}\,\mathrm{e}^{(\lambda_1+\mu/2)x}\,\mathrm{d}x\right. \\
&\qquad\left.+ \int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\,\mathrm{e}^{(\lambda_1+\mu/2)x}\,\mathrm{d}x\right] \\
&= \frac{\beta\alpha_1}{\rho_1}\,\mathrm{e}^{-\mu\xi}[|S_1(\cdot)-S_2(\cdot)|_\mu + |(K*I_1)(\cdot)-(K*I_2)(\cdot)|_\mu] \\
&\quad\times\left[\frac{\Lambda_{12}-\Lambda_{11}}{(\mu-\Lambda_{11})(\Lambda_{12}-\mu)}\,\mathrm{e}^{\mu\xi} + \frac{2\mu}{\Lambda_{11}^2-\mu^2}\,\mathrm{e}^{\Lambda_{11}\xi}\right] + \frac{\beta G(\lambda_1,c)}{\rho_1}|S_1(\cdot) \\
&\quad- S_2(\cdot)|_{\mu/2}\left[\frac{\mu\,\mathrm{e}^{(\Lambda_{11}-\mu)\xi}}{(\lambda_1-\Lambda_{11})^2-\mu^2/4} + \frac{(\Lambda_{12}-\Lambda_{11})\,\mathrm{e}^{(\lambda_1-\mu/2)\xi}}{(\lambda_1-\Lambda_{11}+\mu/2)(\Lambda_{12}-\lambda_1-\mu/2)}\right] \\
&\le \frac{\beta\alpha_1}{\rho_1}\left[\frac{\Lambda_{12}-\Lambda_{11}}{(\mu-\Lambda_{11})(\Lambda_{12}-\mu)} + \frac{2\mu}{\Lambda_{11}^2-\mu^2}\right] \\
&\quad\times[|S_1(\cdot)-S_2(\cdot)|_\mu + |(K*I_1)(\cdot)-(K*I_2)(\cdot)|_\mu] + \frac{\beta G(\lambda_1,c)}{\rho_1}|S_1(\cdot) \\
&\quad- S_2(\cdot)|_{\mu/2}\left[\frac{\mu}{(\lambda_1-\Lambda_{11})^2-\mu^2/4} + \frac{\Lambda_{12}-\Lambda_{11}}{(\lambda_1-\Lambda_{11}+\mu/2)(\Lambda_{12}-\lambda_1-\mu/2)}\right].
\end{aligned}
$$

Similarly, for $\xi < 0$ we have

$$
\begin{aligned}
&|F_1[S_1(\cdot),I_1(\cdot)](\xi) - F_1[S_2(\cdot),I_2(\cdot)](\xi)|\mathrm{e}^{-\mu|\xi|} \\
&\quad\le \frac{\beta\alpha_1}{\rho_1}\left[\frac{\Lambda_{11}-\Lambda_{12}}{(\Lambda_{11}+\mu)(\Lambda_{12}+\mu)} + \frac{2\mu}{\Lambda_{12}^2-\mu^2}\right] \\
&\qquad\times[|S_1(\cdot)-S_2(\cdot)|_\mu + |(K*I_1)(\cdot)-(K*I_2)(\cdot)|_\mu] + \frac{\beta G(\lambda_1,c)}{\rho_1}|S_1(\cdot) \\
&\qquad- S_2(\cdot)|_{\mu/2}\left[\frac{\mu}{(\Lambda_{11}-\lambda_1)^2-\mu^2/4} + \frac{\Lambda_{12}-\Lambda_{11}}{(\lambda_1-\Lambda_{11}-\mu/2)(\Lambda_{12}-\lambda_1+\mu/2)}\right].
\end{aligned}
$$

Then, it is sufficient to prove that $|S_1(\cdot)-S_2(\cdot)|_\mu \to 0$ and $|I_1(\cdot)-I_2(\cdot)|_\mu \to 0$ imply $|S_1(\cdot)-S_2(\cdot)|_{\mu/2} \to 0$ and $|(K*I_1)(\cdot)-(K*I_2)(\cdot)|_\mu \to 0$, respectively. Given $\epsilon > 0$ sufficiently small. Note that $|S_1(x)-S_2(x)| \le S_0$ for any $x \in \mathbb{R}$. Then there exists $N > 0$ such that $|S_1(x)-S_2(x)|\mathrm{e}^{-\mu|x|/2} \le S_0\mathrm{e}^{-\mu N/2} \le \epsilon$ for any $|x| \ge N$. Furthermore, when $|S_1(\cdot)-S_2(\cdot)|_\mu < \epsilon\mathrm{e}^{-\mu N/2}$, for $|x| < N$ we have

$$|S_1(x)-S_2(x)|\,\mathrm{e}^{-\mu|x|/2} \le |S_1(x)-S_2(x)|\,\mathrm{e}^{-\mu|x|}\mathrm{e}^{\mu N/2} \le \epsilon.$$

Thus, we conclude that $|S_1(\cdot) - S_2(\cdot)|_{\mu/2} \to 0$ as $|S_1(\cdot) - S_2(\cdot)|_\mu \to 0$. Consider

$$|(K * I_1)(\cdot) - (K * I_2)(\cdot)|_\mu.$$

Since $\mathrm{e}^{-\lambda_1(y+cs)}K(y,s) \in L(\mathbb{R}\times\mathbb{R}^+)$, then there exists $N^* > 0$ such that

$$\iint_{(\mathbb{R}\times\mathbb{R}^+)*([0,N^*]\times[-N^*,N^*])} \mathrm{e}^{-\lambda_1(y+cs)}K(y,s)\,\mathrm{d}y\,\mathrm{d}s < \epsilon.$$

Furthermore, when $|I_1(\cdot) - I_2(\cdot)|_\mu \le \epsilon\,\mathrm{e}^{-\mu(1+c)N^*}$, we have

$$\begin{aligned} &\mathrm{e}^{-\mu|x|}\int_0^{N^*}\int_{-N^*}^{N^*} |I_1(x-y-cs) - I_2(x-y-cs)|K(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad\le |I_1(\cdot) - I_2(\cdot)|_\mu\,\mathrm{e}^{-\mu|x|}\,\mathrm{e}^{\mu(|x|+N^*+cN^*)}\int_0^{N^*}\int_{-N^*}^{N^*} K(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad\le \epsilon. \end{aligned}$$

Combining the above arguments and the fact $|I_1(x) - I_2(x)| \le \mathrm{e}^{\lambda_1 x}$ for any $x \in \mathbb{R}$, we have that

$$|(K * I_1)(\cdot) - (K * I_2)(\cdot)|_\mu \to 0 \quad \text{as } |I_1(\cdot) - I_2(\cdot)|_\mu \to 0.$$

Thus, we conclude that $F_1 : \Gamma \to C(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R})$. Similarly, we can prove that $F_2 : \Gamma \to C(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R})$. This completes the proof. ■

Lemma 2.8. *The map $F : \Gamma \to \Gamma$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$.*

Proof. For any $(S, I) \in \Gamma$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}F_1[S(\cdot), I(\cdot)](\xi) = \frac{\beta\Lambda_{11}}{\rho_1}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)}[\alpha_1 S(x) - S(x)(K * I)(x)]\,\mathrm{d}x \\ + \frac{\beta\Lambda_{12}}{\rho_1}\int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}[\alpha_1 S(x) - S(x)(K * I)(x)]\,\mathrm{d}x. \end{aligned}$$

Therefore, for any $\xi \in \mathbb{R}$ we have

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}\xi}F_1[S(\cdot), I(\cdot)](\xi)\right| &\le \frac{\beta|\Lambda_{11}|S_0}{\rho_1}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)}[\alpha_1 + G(\lambda_1, c)\mathrm{e}^{\lambda_1 x}]\,\mathrm{d}x \\ &\quad + \frac{\beta\Lambda_{12}S_0}{\rho_1}\int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}[\alpha_1 + G(\lambda_1, c)\mathrm{e}^{\lambda_1 x}]\,\mathrm{d}x \\ &= \frac{2\beta S_0\alpha_1}{\rho_1} + \left[\frac{\beta\Lambda_{12}S_0 G(\lambda_1, c)}{\rho_1(\Lambda_{12} - \lambda_1)} - \frac{\beta\Lambda_{11}S_0 G(\lambda_1, c)}{\rho_1(\lambda_1 - \Lambda_{11})}\right]\mathrm{e}^{\lambda_1\xi}. \end{aligned}$$

Similarly, for any $\xi \in \mathbb{R}$ we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}\xi}F_2[S(\cdot), I(\cdot)](\xi)\right| \le \left[\frac{\Lambda_{22}}{\rho_2(\Lambda_{22} - \lambda_1)} - \frac{\Lambda_{21}}{\rho_2(\lambda_1 - \Lambda_{21})}\right][\beta S_0 G(\lambda_1, c) + \alpha_2 - \gamma]\mathrm{e}^{\lambda_1\xi}.$$

For each integer $n \in \mathbb{N}$, define an operator F^n by

$$F^n[S(\cdot), I(\cdot)](\xi) = \begin{cases} F[(S(\cdot), I(\cdot))](\xi), & \xi \in [-n, n], \\ F[(S(\cdot), I(\cdot))](-n), & \xi \in (-\infty, -n] \\ F[(S(\cdot), I(\cdot))](n), & \xi \in [n, +\infty). \end{cases}$$

By Ascoli–Arzela lemma, we have that $F^n : \Gamma \to \Gamma$ is compact with respect to supremum norm in $C(\mathbb{R}, \mathbb{R}^2)$ because $F^n[S(\cdot), I(\cdot)](\cdot)$ is also uniformly bounded and equicontinuous for $(S, I) \in \Gamma$. Consequently, we have that $F^n : \Gamma \to \Gamma$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$. Furthermore, since $\{F^n\}_0^\infty$ is a compact series and

$$\begin{aligned} &|F^n[S(\cdot), I(\cdot)](\cdot) - F[S(\cdot), I(\cdot)](\cdot)|_\mu \\ &\quad = \sup_{\xi \in \mathbb{R}} |F^n[S(\cdot), I(\cdot)](\xi) - F[S(\cdot), I(\cdot)](\xi)|\, \mathrm{e}^{-\mu|\xi|} \\ &\quad = \sup_{\xi \in (-\infty, -n] \cup [n, \infty)} |F^n[S(\cdot), I(\cdot)](\xi) - F[S(\cdot), I(\cdot)](\xi)|\, \mathrm{e}^{-\mu|\xi|} \\ &\quad \le \sup_{\xi \in (-\infty, -n] \cup [n, \infty)} \max\{S_0, \mathrm{e}^{-\lambda_1 n}, \mathrm{e}^{\lambda_1 \xi}\} \mathrm{e}^{-\mu|\xi|} \\ &\quad \le \max\{S_0 \mathrm{e}^{-\mu n}, \mathrm{e}^{-(\mu - \lambda_1)n}\} \to 0 \quad \text{as } n \to +\infty, \end{aligned}$$

by proposition 2.1 in Zeilder (1986) we have that $\{F^n\}_0^\infty$ converges to F in Γ with respect to the norm $|\cdot|_\mu$ and hence, $F : \Gamma \to \Gamma$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$. The proof is completed. ■

Theorem 2.9. *Assume that $S_0 > \gamma/\beta$. For every $c > c_*$, system (2.1) admits a travelling wave solution $(S_c(x + ct), I_c(x + ct))$ such that (2.2), $0 \le I_c(\xi) \le S_0 - S_c^\infty$ for any $\xi \in \mathbb{R}$ and $S_c(\cdot)$ is non-increasing in $\mathbb{R}$. In addition, we have*

$$\lim_{\xi \to -\infty} \mathrm{e}^{-\lambda_1 \xi} I(\xi) = 1 \quad \textit{and} \quad \int_{-\infty}^{\infty} I_c(x)\, \mathrm{d}x = \frac{c}{\gamma}[S_0 - S_c^\infty].$$

Proof. When $c > c_*$, Schauder's fixed point theorem implies that there exists a pair of $(S_c(\cdot), I_c(\cdot)) \in \Gamma$, which is a fixed point of the operator F. Consequently, the solution $(S_c(x + ct), I_c(x + ct))$ is a non-negative travelling wave solution of equation (2.1). It is obvious that $S_c(-\infty) = S_0$, $\lim_{\xi \to -\infty} \mathrm{e}^{-\lambda_1 \xi} I_c(\xi) = 1$, $0 \le S_c(\xi) \le S_0$ and $0 \le I_c(\xi) \le \mathrm{e}^{\lambda_1 \xi}$ for any $\xi \in \mathbb{R}$. In the following, we first prove that $S_c(\xi)$ is non-increasing and equation (2.2) holds.

Note that $(S_c(\cdot), I_c(\cdot)) \in \Gamma$ is a fixed point of the operator F. Applying the L'Hospital theorem to the maps F_1 and F_2, it is easy to show that

$$S_c'(-\infty) = 0 \quad \text{and} \quad I_c'(-\infty) = 0.$$

Consequently, it follows from equation (2.3) that $S_c''(-\infty) = 0$ and $I_c''(-\infty) = 0$. Integrating the two sides of

$$cS_c'(\xi) - d_1 S_c''(\xi) = -\beta S_c(\xi)(K * I_c)(\xi) \tag{2.12}$$

from $-\infty$ to ξ, we have

$$d_1 S_c'(\xi) = c[S_c(\xi) - S_0] + \beta \int_{-\infty}^{\xi} S_c(x)(K * I_c)(x)\,\mathrm{d}x.$$

Since $0 \le S_c(\xi) \le S_0$ for any $\xi \in \mathbb{R}$, we conclude that $\int_{-\infty}^{+\infty} S_c(x)(K * I_c)(x)\,\mathrm{d}x < +\infty$ and hence, $S_c'(\xi)$ is bounded on $x \in \mathbb{R}$. Otherwise, if $\int_{-\infty}^{+\infty} S_c(x)(K * I_c)(x)\,\mathrm{d}x = +\infty$, then there exists a constant $\delta_0 > 0$ such that $S_c'(\xi) \ge \delta_0$ for large $\xi > 0$, which contradicts the fact $0 \le S_c(x) \le S_0$. Therefore, $\int_{-\infty}^{+\infty} S_c(x)(K * I_c)(x)\,\mathrm{d}x < +\infty$ and $S_c'(\xi)$ is bounded on $x \in \mathbb{R}$. Multiplying $(1/d_1)\,\mathrm{e}^{-(c/d_1)\xi}$ for the two sides of the equality (2.12), we have

$$(S_c'(\xi)\mathrm{e}^{-(c/d_1)\xi})' = \beta/d_1\,\mathrm{e}^{-(c/d_1)\xi} S_c(\xi)(K * I_c)(\xi), \quad \forall \xi \in \mathbb{R}.$$

Integrating the last equality from ξ to $+\infty$, we have

$$S_c'(\xi) = -\frac{\beta}{d_1}\,\mathrm{e}^{(c/d_1)\xi} \int_{\xi}^{+\infty} \mathrm{e}^{-(c/d_1)x} S_c(x)(K * I_c)(x)\,\mathrm{d}x \le 0, \quad \forall \xi \in \mathbb{R},$$

which implies that $S_c(\xi)$ is non-increasing in $\xi \in \mathbb{R}$. Because $(S_c, I_c) \in \Gamma$, for $\xi < 0$ with $|\xi|$ sufficiently large, we have

$$\int_{\xi}^{+\infty} \mathrm{e}^{-(c/d_1)x} S_c(x)(K * I_c)(x)\,\mathrm{d}x > 0.$$

Hence, there exists $\xi^* < 0$ such that $\mathrm{d}/\mathrm{d}\xi(S_c(\xi)) < 0$ for $\xi < \xi^*$. Therefore, we have $0 \le S_c(+\infty) := S_c^{\infty} < S_0$.

Furthermore, since

$$cI_c'(\xi) - d_2 I_c''(\xi) = \beta S_c(\xi)(K * I_c)(\xi) - \gamma I_c(\xi), \quad \forall \xi \in \mathbb{R}, \tag{2.13}$$

we have

$$I_c(\xi) = \frac{\beta}{\rho_2'} \int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}'(\xi - x)} S_c(x)(K * I_c)(x)\,\mathrm{d}x + \frac{\beta}{\rho_2'} \int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{22}'(\xi - x)} S_c(x)(K * I_c)(x)\,\mathrm{d}x$$

for any $\xi \in \mathbb{R}$, where

$$\Lambda_{21}' = \frac{c - \sqrt{c^2 + 4d_2\gamma}}{2d_2}, \quad \Lambda_{22}' = \frac{c + \sqrt{c^2 + 4d_2\gamma}}{2d_2}, \quad \rho_2' = d_2(\Lambda_{22}' - \Lambda_{21}').$$

In view of $\int_{-\infty}^{+\infty} S_c(x)(K * I_c)(x)\,\mathrm{d}x := A_0 < +\infty$, we have that

$$\int_{-\infty}^{+\infty} I_c(x)\,\mathrm{d}x < \infty.$$

Since

$$\begin{aligned} I_c'(\xi) &= \frac{\Lambda_{21}'\beta}{\rho_2'} \int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}'(\xi-x)} S_c(x)(K * I_c)(x)\,\mathrm{d}x \\ &\quad + \frac{\Lambda_{22}'\beta}{\rho_2'} \int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{22}'(\xi-x)} S_c(x)(K * I_c)(x)\,\mathrm{d}x \\ &= \frac{\Lambda_{21}'\beta}{\rho_2'} \int_{0}^{+\infty} \mathrm{e}^{\Lambda_{21}'x} S_c(\xi-x)(K * I_c)(\xi-x)\,\mathrm{d}x \\ &\quad + \frac{\Lambda_{22}'\beta}{\rho_2'} \int_{-\infty}^{0} \mathrm{e}^{\Lambda_{22}'x} S_c(\xi-x)(K * I_c)(\xi-x)\,\mathrm{d}x, \end{aligned}$$

we have

$$\begin{aligned} I_c(\xi) &= \int_{-\infty}^{\xi} I_c'(x)\,\mathrm{d}x \le \frac{-\Lambda_{21}'\beta}{\rho_2'} \int_{0}^{+\infty} \mathrm{e}^{\Lambda_{21}'x} \int_{-\infty}^{\xi} S_c(y-x)(K * I_c)(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \frac{\Lambda_{22}'\beta}{\rho_2'} \int_{-\infty}^{0} \mathrm{e}^{\Lambda_{22}'x} \int_{-\infty}^{\xi} S_c(y-x)(K * I_c)(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{2\beta A_0}{\rho_2'}, \quad \forall \xi \in \mathbb{R}. \end{aligned}$$

Consequently, it follows that $|I_c'(\xi)| \le (4\beta^2 A_0 S_0/(\rho_2')^2)$ for any $\xi \in \mathbb{R}$. Thus, we have $\lim_{\xi\to+\infty} I_c(\xi) = 0$ because $I_c'(\xi)$ is bounded.

Now by lemma 2.3 in Wu & Zou (2001), we have

$$S_c'(\pm\infty) = 0, \quad S_c''(\pm\infty) = 0, \quad I_c'(\pm\infty) = 0, \quad I_c''(\pm\infty) = 0.$$

Consequently, integrating equation (2.12) on $\mathbb{R}$ yields

$$\beta \int_{-\infty}^{\infty} S_c(\xi)(K * I_c)(\xi) = c[S_0 - S_c^{\infty}].$$

Furthermore, by integrating equation (2.13) on $\mathbb{R}$, we obtain

$$\int_{-\infty}^{+\infty} I_c(x)\,\mathrm{d}x = \frac{c}{\gamma}[S_0 - S_c^{\infty}].$$

To prove that $0 \le I_c(\xi) \le S_0 - S_c^{\infty}$, we define a function $\widehat{R}_c(\xi) = \gamma/c \int_{-\infty}^{\xi} I_c(x)\mathrm{d}x + \gamma/c \int_{\xi}^{+\infty} \mathrm{e}^{(c/d_2)(\xi-x)} I_c(x)\,\mathrm{d}x$ for any $\xi \in \mathbb{R}$ which satisfies the following equation:

$$c\widehat{R}_c'(\xi) = d_2 \widehat{R}_c''(\xi) + \gamma I_c(\xi), \quad \forall \xi \in \mathbb{R}.$$

Obviously, $\widehat{R}_c(-\infty) = 0$, $\widehat{R}_c(+\infty) = S_0 - S_c^{\infty}$ and $\widehat{R}_c'(\pm\infty) = 0$. Furthermore, we can show that $N_c(\xi) := I_c(\xi) + \widehat{R}_c(\xi)$ is non-decreasing in $\mathbb{R}$. In fact, $N_c(\xi)$ satisfies

$$cN_c'(\xi) = d_2 N_c''(\xi) + \beta S_c(\xi)(K * I_c)(\xi), \quad \forall \xi \in \mathbb{R}.$$

Following this, we have that

$$N_c'(\xi) = \frac{\beta}{d_2} e^{(c/d_2)\xi} \int_{\xi}^{+\infty} e^{-(c/d_2)x} S_c(x)(K * I_c)(x)\, dx \geq 0, \quad \forall \xi \in \mathbb{R}.$$

In view of $N_c(+\infty) = S_0 - S_c^{\infty}$, we have that $0 \leq I_c(\xi) \leq S_0 - S_c^{\infty}$ for any $\xi \in \mathbb{R}$. This completes the proof. ■

Theorem 2.10. *Assume that* $S_0 > \gamma/\beta$. *For* $c' \in (0, c_*)$, *there exist no non-trivial travelling wave solutions* $(S_{c'}(x + c't), I_{c'}(x + c't))$ *of equation (2.1) such that equation (2.2) and* $0 \leq S_{c'}(\xi) \leq S_0$ *and* $0 \leq I_{c'}(\xi) \leq S_0$.

Proof. Now we consider the case $c \in (0, c_*)$. Fix $c \in (0, c_*)$. We prove the theorem by way of contradiction. Assume that there exists a non-trivial travelling wave solutions $(S_c(x + ct), I_c(x + ct))$ of (2.1) such that equation (2.2). Since $S(-\infty) = S_0$, there exists $\xi' < 0$ such that $S(\xi) > (\beta S_0 + \gamma)/2\beta$ for any $\xi \leq \xi'$. Therefore, we have

$$\begin{aligned} cI'(\xi) &= d_2 I''(\xi) + \beta S(\xi)(K * I)(\xi) - \gamma I(\xi) \\ &\geq d_2 I''(\xi) + \frac{\beta S_0 + \gamma}{2}[(K * I)(\xi) - I(\xi)] + \frac{\beta S_0 - \gamma}{2} I(\xi) \end{aligned} \tag{2.14}$$

for any $\xi \leq \xi'$. Let $J(\xi) = \int_{-\infty}^{\xi} I(\eta)\, d\eta$ for any $\xi \in \mathbb{R}$. It is not difficult to verify $\int_{-\infty}^{\xi} (K * I)(\eta) d\eta = (K * J)(\xi)$, see also Wang & Li (2009, theorem 3.5), where the convolution is defined by equation (2.11). Then, integrating two sides of inequality (2.14) from $-\infty$ to ξ with $\xi \leq \xi'$, we have

$$\frac{\beta S_0 - \gamma}{2} J(\xi) \leq cI(\xi) - d_2 I'(\xi) - \frac{\beta S_0 + \gamma}{2}[(K * J)(\xi) - J(\xi)]. \tag{2.15}$$

In view of

$$\begin{aligned} &\int_{-\infty}^{\xi} [(K * J)(\eta) - J(\eta)]\, d\eta \\ &= \lim_{z \to -\infty} \int_z^{\xi} [(K * J)(\eta) - J(\eta)]\, d\eta \\ &= \lim_{z \to -\infty} \int_z^{\xi} \int_0^{\infty} \int_{-\infty}^{\infty} K(y, s)[J(\eta - y - cs) - J(\eta)]\, dy\, ds\, d\eta \\ &= \lim_{z \to -\infty} -\int_z^{\xi} \int_0^{\infty} \int_{-\infty}^{\infty} (y + cs) K(y, s) \int_0^1 I(\eta - \theta(y + cs))\, d\theta\, dy\, ds\, d\eta \\ &= \lim_{z \to -\infty} -\int_0^{\infty} \int_{-\infty}^{\infty} (y + cs) K(y, s) \int_0^1 \int_z^{\xi} I(\eta - \theta(y + cs))\, d\eta\, d\theta\, dy\, ds \end{aligned}$$

$$= \lim_{z\to-\infty} -\int_0^\infty \int_{-\infty}^\infty (y+cs)K(y,s)\int_0^1 [J(\xi-\theta(y+cs)) - J(z-\theta(y+cs))]\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s$$

$$= -\int_0^\infty \int_{-\infty}^\infty (y+cs)K(y,s)\int_0^1 J(\xi-\theta(y+cs))\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s,$$

we have that $(K*J)(\xi)-J(\xi)$ is integrable on $(-\infty,\xi]$ for any $\xi\in\mathbb{R}$. Consequently, from equation (2.15) we have that $J(\xi)$ is integrable on $(-\infty,\xi]$ for any $\xi\in\mathbb{R}$. Now integrating the two sides of inequality (2.15) from $-\infty$ to ξ with $\xi\le\xi'$, we have

$$\frac{\beta S_0-\gamma}{2}\int_{-\infty}^{\xi} J(\eta)\,\mathrm{d}\eta + d_2 I(\xi) \le cJ(\xi) + \frac{\beta S_0+\gamma}{2}\int_0^\infty\int_{-\infty}^\infty (y+cs)K(y,s)\int_0^1 J(\xi-\theta(y+cs))\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s.$$

Since $(y+cs)J(\xi-\theta(y+cs))$ is non-increasing on $\theta\in[0,1]$, we have

$$\frac{\beta S_0-\gamma}{2}\int_{-\infty}^{\xi} J(\eta)\,\mathrm{d}\eta + d_2 I(\xi) \le \left[c+\frac{\beta S_0+\gamma}{2}\int_0^\infty\int_{-\infty}^\infty (y+cs)K(y,s)\,\mathrm{d}y\,\mathrm{d}s\right]J(\xi).$$

Let $K_1=(\beta S_0+\gamma)/2\int_0^\infty\int_{-\infty}^\infty sK(y,s)\,\mathrm{d}y\,\mathrm{d}s$. Since the kernel $K(y,s)$ is an even function of y, we have $\int_0^\infty\int_{-\infty}^\infty yK(y,s)\,\mathrm{d}y\,\mathrm{d}s=0$. Then, we have

$$\frac{\beta S_0-\gamma}{2}\int_{-\infty}^{\xi} J(\eta)\,\mathrm{d}\eta + d_2 I(\xi) \le c(1+K_1)J(\xi), \quad \forall \xi\le\xi'. \tag{2.16}$$

Therefore, for any $\xi\le\xi'$ we have

$$\frac{\beta S_0-\gamma}{2}\int_{-\infty}^{0} J(\xi+\eta)\,\mathrm{d}\eta \le c(1+K_1)J(\xi).$$

Since $J(\cdot)$ is increasing, then for any $\xi\le\xi'$ and any $\eta>0$ we have

$$\frac{\beta S_0-\gamma}{2}\eta J(\xi-\eta) \le c(1+K_1)J(\xi).$$

Thus, there exists $\eta_0>0$ sufficiently large and some $\theta_0\in(0,1)$ such that

$$J(\xi-\eta_0)\le\theta_0 J(\xi), \quad \forall\xi\le\xi'.$$

Let $p(x)=J(x)\,\mathrm{e}^{-\mu_0 x}$ with $\mu_0=(1/\eta_0)\ln(1/\theta_0)<\lambda_1$. Then,

$$p(\xi-\eta_0)=J(\xi-\eta_0)\,\mathrm{e}^{-\mu_0(\xi-\eta_0)}\le\theta_0 J(\xi)\,\mathrm{e}^{-\mu_0(\xi-\eta_0)}=p(\xi), \quad \forall\xi\le\xi'.$$

By virtue of $p(x)\to0$ as $x\to+\infty$, we have that there exists $p_0>0$ such that

$$p(x)\le p_0 \quad \text{for any } x\in\mathbb{R},$$

which implies that

$$J(x)\le p_0\mathrm{e}^{\mu_0 x} \quad \text{for any } x\in\mathbb{R}.$$

Consequently, there exists $q_0 > 0$ such that $\int_{-\infty}^{x} J(y)\,\mathrm{d}y \le q_0\,\mathrm{e}^{\mu_0 x}$ for any $x \in \mathbb{R}$. Furthermore, by equations (2.14)–(2.16), we have

$$\sup_{x\in\mathbb{R}}\{I(x)\,\mathrm{e}^{-\mu_0 x}\} < \infty, \quad \sup_{x\in\mathbb{R}}\{|I'(x)|\,\mathrm{e}^{-\mu_0 x}\} < \infty, \quad \sup_{x\in\mathbb{R}}\{|I''(x)|\,\mathrm{e}^{-\mu_0 x}\} < \infty.$$

Now consider $S_0 - S(\xi)$. By $cS'(\xi) = d_1 S''(\xi) - \beta S(\xi)(K * I)(\xi)$, integrating from $-\infty$ to ξ yields

$$c[S(\xi) - S_0] = d_1 S'(\xi) - \beta \int_{-\infty}^{\xi} S(\eta)(K * I)(\eta)\,\mathrm{d}\eta.$$

Let $f(\xi) = \beta \int_{-\infty}^{\xi} S(\eta)(K * I)(\eta)\,\mathrm{d}\eta$. It is obvious that $f(x) \le C_0\,\mathrm{e}^{\mu_0 x}$ for any $x \in \mathbb{R}$ and some constant $C_0 > 0$. Let $R(\xi) = S_0 - S(\xi) \ge 0$ for any $\xi \in \mathbb{R}$. Then, we have

$$d_1 R'(\xi) - cR(\xi) = -f(\xi).$$

Solving the last ODE yields

$$R(\xi) = \hat{C}_0\,\mathrm{e}^{(c/d_1)\xi} + \frac{1}{d_1}\,\mathrm{e}^{(c/d_1)\xi} \int_{\xi}^{0} \mathrm{e}^{-(c/d_1)\eta} f(\eta)\,\mathrm{d}\eta, \quad \forall \xi \in \mathbb{R},$$

where $\hat{C}_0 = R(0)$. Since $f(\xi) = O(\mathrm{e}^{\mu_0 \xi})$ as $\xi \to -\infty$, it is easy to see that $R(\xi) = O(\mathrm{e}^{\mu_0' \xi})$ as $\xi \to -\infty$, where $\mu_0' = \min\{\mu_0, c/d_1\}$. In view of $0 \le R(\xi) \le S_0$, we have

$$\sup_{x\in\mathbb{R}}\{R(x)\,\mathrm{e}^{-\mu_0' x}\} < \infty.$$

For $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \mu_0$, we can define a two-sided Laplace transform of I by

$$\mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda\xi} I(\xi)\,\mathrm{d}\xi.$$

Applying the property of Laplace transforms (see Widder 1941), we know that either there exists a real number $\alpha > 0$ such that $\mathcal{L}(\lambda)$ is analytic for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \alpha$ and $\lambda = \alpha$ is singular point of $\mathcal{L}(\lambda)$, or for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$, $\mathcal{L}(\lambda)$ is well defined. Now we use this property to conclude that for $c \in (0, c_*)$, equation (2.1) admits no travelling wave solutions $(S(x+ct), I(x+ct))$ satisfying equation (2.2).

By Fubini's theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} (K * I)(\xi)\,\mathrm{d}\xi \\
&\quad = \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s) I(\xi - y - cs)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\xi \\
&\quad = \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)\,\mathrm{e}^{-\lambda(y+cs)} \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda(\xi - y - cs)} I(\xi - y - cs)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = \mathcal{L}(\lambda) \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)\,\mathrm{e}^{-\lambda(y+cs)}\,dy\,ds \\
&\quad = \mathcal{L}(\lambda)\,G(\lambda, c),
\end{aligned}$$

where $G(\lambda, c) := \int_0^\infty \int_{-\infty}^{+\infty} e^{-\lambda y - c\lambda s} K(y, s)\,dy\,ds$. In view of

$$d_2 I''(\xi) - cI'(\xi) + \beta S_0 (K * I)(\xi) - \gamma I(\xi) = \beta[S_0 - S(\xi)](K * I)(\xi), \quad \forall \xi \in \mathbb{R},$$

we have

$$\Theta(\lambda, c)\mathcal{L}(\lambda) = \beta \int_{-\infty}^{\infty} e^{-\lambda\xi}[S_0 - S(\xi)](K * I)(\xi)\,d\xi \tag{2.17}$$

for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \mu_0$, where $\Theta(\lambda, c)$ is defined by equation (2.5). Note that the right-hand side of equation (2.17) is defined for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \mu_0 + \mu_0'$. For $c \in (0, c_*)$, since $\Theta(\lambda, c) > 0$ for $\lambda \in (0, \lambda_c)$, we have that $\mathcal{L}(\lambda)$ is defined for all $\lambda \in \mathbb{C}$ with $\lambda_c > \mathrm{Re}\lambda > 0$ and there are no singularity of $\mathcal{L}(\lambda)$ in $\lambda \in [0, \lambda_c)$. Because equation (2.17) can be re-written as

$$\int_{-\infty}^{\infty} e^{-\lambda\xi}[\Theta(\lambda, c)I(\xi) - \beta(S_0 - S(\xi))(K * I)(\xi)]\,d\xi = 0.$$

However, for $c \in (0, c_*)$, we have that $\Theta(\lambda, c) \to +\infty$ as $\lambda \to \lambda_c - 0$, which implies that the last equality is false. This is a contradiction. The proof is complete. ■

Theorem 2.11. *Assume that $S_0 < \gamma/\beta$. Then for any $c \geq 0$, there exists no travelling wave solutions $(S(x + ct), I(x + ct))$ satisfying*

$$S(-\infty) = S_0, \quad S(+\infty) = S_\infty < S_0, \quad I(\pm\infty) = 0. \tag{2.18}$$

Proof. Assume that there exists non-trivial travelling wave solution $(S(x + ct), I(x + ct))$ such that equation (2.18). Then, we have

$$\begin{aligned} I(\xi) &= \frac{\beta}{\rho_2'} \int_{-\infty}^{\xi} e^{\Lambda_{21}'(\xi - x)} S(x)(K * I)(x)\,dx + \frac{\beta}{\rho_2'} \int_{\xi}^{+\infty} e^{\Lambda_{22}'(\xi - x)} S(x)(K * I)(x)\,dx \\ &= \frac{\beta}{\rho_2'} \int_0^{+\infty} e^{\Lambda_{21}' x} S(\xi - x)(K * I)(\xi - x)\,dx \\ &\quad + \frac{\beta}{\rho_2'} \int_{-\infty}^{0} e^{\Lambda_{22}' x} S(\xi - x)(K * I)(\xi - x)\,dx. \end{aligned}$$

Integrating the two sides of the last equality, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} I(\xi)\,d\xi &= \frac{\beta}{\rho_2'} \int_0^{+\infty} e^{\Lambda_{21}' x} \int_{-\infty}^{+\infty} S(\xi - x)(K * I)(\xi - x)\,d\xi\,dx \\ &\quad + \frac{\beta}{\rho_2'} \int_{-\infty}^{0} e^{\Lambda_{22}' x} \int_{-\infty}^{+\infty} S(\xi - x)(K * I)(\xi - x)\,d\xi\,dx \\ &= \frac{\beta}{\rho_2'} \left[\int_0^{+\infty} e^{\Lambda_{21}' x}\,dx + \int_{-\infty}^{0} e^{\Lambda_{22}' x}\,dx\right] \int_{-\infty}^{+\infty} S(\xi)(K * I)(\xi)\,d\xi \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\beta S_0}{\rho_2'}\left(\frac{1}{\Lambda_{22}'}-\frac{1}{\Lambda_{21}'}\right)\int_{-\infty}^{+\infty}(K*I)(\xi)\,\mathrm{d}\xi \\
&= \frac{\beta S_0}{\gamma}\int_{-\infty}^{+\infty}(K*I)(\xi)\,\mathrm{d}\xi \\
&= \frac{\beta S_0}{\gamma}\int_{-\infty}^{+\infty}I(\xi)\,\mathrm{d}\xi \\
&< \int_{-\infty}^{+\infty}I(\xi)\,\mathrm{d}\xi.
\end{aligned}
$$

This is a contradiction. This completes the proof. ■

Remark 2.12. When $d_1 \cdot d_2 = 0$, the conclusions in theorems 2.9, 2.10 and 2.11 remain valid. In fact, if $d_1 = 0$, then it is sufficient to define F_1 by

$$F_1[S(\cdot), I(\cdot)] = \frac{\beta}{c}\int_{-\infty}^{\xi} \mathrm{e}^{-(\beta\alpha_1/c)(\xi-x)}[\alpha_1 S(x) - S(x)(K*I)(x)]\,\mathrm{d}x,$$

where $\alpha_1 \geq S_0$ satisfies $\beta\alpha_1/c > 2\lambda_1$. Similarly, if $d_2 = 0$, then we need only to redefine F_2 by

$$F_2[S(\cdot), I(\cdot)] = \frac{\beta}{c}\int_{-\infty}^{\xi} \mathrm{e}^{-(\alpha_2/c)(\xi-x)}\left[S(x)(K*I)(x) + \frac{\alpha_2-\gamma}{\beta}I(x)\right]\mathrm{d}x,$$

where $\alpha_2 \geq \gamma$ satisfies $\alpha_2/c > 2\lambda_1$.

3. Discussion

In this paper, we study the existence and non-existence of non-trivial travelling wave solutions for model (1.6). As the travelling wave solutions obtained or excluded in this work describe the transition from a disease-free equilibrium to an endemic equilibrium, the existence and non-existence of non-trivial travelling wave solutions indicates whether or not the disease can spread.

Theorems 2.9 and 2.11 combined provide a threshold condition for the existence of travelling wave solutions in terms of the basic reproduction number $\beta S_0/\gamma$ of the corresponding ODE system in the absence of non-local interaction, time delays and spatial diffusion. Therefore, whether disease spreads or not is independent of the non-local delayed interaction and spatial movement patterns of the population.

The speed at which the disease spreads (if it does), however, depends on the aforementioned factors. We have shown that if the basic reproduction number is larger than one, then system (1.6) admits a non-trivial travelling wave solution with wave speed $c > c_*$, where c_* is the minimal wave speed. As discussed in §1, this minimal wave speed c_* should be the asymptotic speed of propagation of the disease. This minimal wave speed c_* is defined by lemma 2.1, from which it is easy to see that c_* is dependent on the diffusion rate d_2 of the infected individuals, the pattern of non-local interaction between the infected and the susceptible individuals, and the latent period of disease.

More specifically, for $c>0$ and $\lambda\in(0,\lambda_c)$, direct calculations yield that

$$\left.\begin{aligned}\frac{\partial\Theta}{\partial c}&=-\lambda-\lambda\beta S_0\int_0^\infty\int_{-\infty}^\infty s\,\mathrm{e}^{-\lambda(y+cs)}K(y,s)\,\mathrm{d}y\,\mathrm{d}s<0,\\ \frac{\partial^2\Theta}{\partial\lambda^2}&=2d_2+\beta S_0\int_0^\infty\int_{-\infty}^\infty(y+cs)^2\,\mathrm{e}^{-\lambda(y+cs)}K(y,s)\,\mathrm{d}y\,\mathrm{d}s>0\\ \text{and}\qquad\Theta(0,c)&=\beta S_0-\gamma>0\quad\text{and}\quad\Theta(\lambda_c-0,c)=+\infty.\end{aligned}\right\}\tag{3.1}$$

In the case where $K(x,t)=\delta(t-\tau)\delta(x)$ with $\tau>0$, we have

$$\Theta(\lambda,c;\tau)=d_2\lambda^2-c\lambda+\beta S_0\,\mathrm{e}^{-\lambda c\tau}-\gamma,\quad\frac{\partial\Theta}{\partial\tau}=-\lambda c\beta S_0\,\mathrm{e}^{-\lambda c\tau}<0.$$

Therefore, from equation (3.1), we can conclude that $c_*=c_*(\tau)$ is a decreasing function of $\tau>0$.

In the case where $K(x,t)=\delta(t)(1/\sqrt{4\pi\rho})\mathrm{e}^{-x^2/4\rho}$, we have

$$\Theta(\lambda,c;\rho)=d_2\lambda^2-c\lambda+\beta S_0\,\mathrm{e}^{\rho\lambda^2}-\gamma,\quad\frac{\partial\Theta}{\partial\rho}=\lambda^2\beta S_0\,\mathrm{e}^{\rho\lambda^2}>0.$$

Consequently, $c_*=c_*(\rho)$ is an increasing function of $\rho>0$. Indeed, direct calculations also give

$$\frac{dc_*(\tau)}{d\tau}=-\frac{c_*\beta S_0\,\mathrm{e}^{-\lambda_*c_*\tau}}{1+\tau\beta S_0\,\mathrm{e}^{-\lambda_*c_*\tau}}<0\tag{3.2}$$

and

$$\frac{dc_*(\rho)}{d\rho}=\lambda_*\beta S_0\,\mathrm{e}^{\rho\lambda_*{}^2}>0.\tag{3.3}$$

Hence, we observed that the latent period can reduce the speed of the spread of the disease, and the non-locality of interaction can increase the speed of disease spread, an observation in coincidence with those reported by Li *et al.* (2007) and Wang *et al.* (2008).

Similar conclusions can be made for the case where $K(x,t)=\delta(t-\tau)(1/\sqrt{4\pi d_2\tau})\,\mathrm{e}^{x^2/4d_2\tau}$ with $\tau>0$. This case reduces to equation (1.7), and we have

$$\Theta(\lambda,c,d_2,\tau)=d_2\lambda^2-c\lambda+\beta S_0\,\mathrm{e}^{(d_2\lambda^2-\lambda c)\tau}-\gamma,\quad\frac{\partial\Theta}{\partial d_2}=\lambda^2+\tau\lambda^2\beta S_0\,\mathrm{e}^{(d_2\lambda^2-\lambda c)\tau}>0.$$

This implies that c_* is an increasing function of $d_2>0$ and hence, we know that the geographical movement of infected individuals can increase the speed of the spread of the disease. Now fix $d_2>0$, then for any $\tau_0>0$, there exists a unique pair of $\lambda_*(\tau_0)>0$ and $c_*(\tau_0)>0$ such that $\Theta(\lambda_*(\tau_0),c_*(\tau_0),d_2,\tau_0)=0$ and $\Theta(\lambda,c_*(\tau_0),d_2,\tau_0)\geq 0$ for any $\lambda\geq 0$. It is easy to see that $d_2\lambda_*^2(\tau_0)-\lambda_*(\tau_0)c_*(\tau_0)<0$. Then we have

$$\begin{aligned}&\frac{d}{d\epsilon}\Theta(\lambda_*(\tau_0),c_*(\tau_0),d_2,\tau_0+\epsilon)\\&\quad=(d_2\lambda_*^2(\tau_0)-\lambda_*(\tau_0)c_*(\tau_0))\beta S_0\,\mathrm{e}^{(d_2\lambda_*^2(\tau_0)-\lambda_*(\tau_0)c_*(\tau_0))(\tau_0+\epsilon)}<0,\end{aligned}$$

which implies that c_* is a decreasing function of $\tau>0$.

Z.-C.W. was supported by NSF of Gansu Province of China (0710RJZA020) and The Fundamental Research Fund for Physics and Mathematic of Lanzhou University (LZULL200807).

This work was partially supported by the Canada Research Chairs Program, by Natural Sciences and Engineering Research Council of Canada (NSERC), by Canadian Institute of Health Research (CIHR), by Mathematics for Information Technology and Complex Systems (MITACS) and by Geomatics for Informed Decisions (GEOIDE).

References

Anderson, R. M. 1991 Discussion: the Kermack–McKendrick epidemic threshold theorem. *Bull. Math. Biol.* **53**, 3–32. (doi:10.1016/S0092-8240(05)80039-4)

Aronson, D. G. 1977 The asymptotic speed if a propagation of a simple epidemic. In *Nonlinear diffusion* (eds W. E. Fitzgibbon & H. F. Walker). Research Notes in Mathematics, no. 14, pp. 1–23. London, UK: Pitman.

Brauer, F. 2008 Compartmental models in epidemiology. In *Mathematical epidemiology* (eds F. Frauer, P. van den Driessche & J. Wu). Lecture Notes in Mathematics, no. 1945, pp. 19–80. Berlin, Germany: Springer.

Carr, J. & Chmaj, A. 2004 Uniqueness of travelling waves for nonlocal monostable equations. *Proc. Amer. Math. Soc.* **132** 2433–2439. (doi:10.1090/S0002-9939-04-07432-5)

De Mottoni, P., Orlandi, E. & Tesei, A. 1979 Asymptotic behavior for a system describing epidemics with migration and spatial spread of infection. *Nonlinear Anal.* **3**, 663–675.

Diekmann, O. 1978 Thresholds and travelling waves for the geographical spread of infection. *J. Math. Biol.* **69**, 109–130. (doi:10.1007/BF02450783)

Diekmann, O. 1979 Run for your life, a note on the asymptotic speed of propagation of an epidemic. *J. Diff. Equat.* **33**, 58–73. (doi:10.1016/0022-0396(79)90080-9)

Ducrot, A. & Magal, P. 2009 Travelling wave solutions for an infection-age structured model with diffusion. *Proc. R. Soc. Edin. Ser. A Math.* **139**, 459–482.

Ducrot, A., Magal, P. & Ruan, S. 2009 Travelling wave solutions in multigroup age-structured epidemic models. *Arch. Rational Mech. Anal.* (doi:10.1007/s00205-008-0203-8)

Faria, T., Huang, W. & Wu, J. 2006 Travelling waves for delayed reaction–diffusion equations with global response. *Proc. R. Soc. A* **462A**, 229–261. (doi:10.1098/rspa.2005.1554)

Gourley, S. A., So, J. H. W. & Wu, J. 2004 Non-locality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics. *J. Math. Sci.* **124**, 5119–5153. (doi:10.1023/B:JOTH.0000047249.39572.6d)

Hosono, Y. & Ilyas, B. 1994 Travelling waves for a simple diffusive epidemic model. *Math. Model Meth. Appl. Sci.* **5**, 935–966. (doi:10.1142/S0218202595000504)

Kendall, D. G. 1959 Discussion of measles periodicity and community size. *J. Roy. Stat. Soc.* **120A**, 64–76.

Kendall, D. G. 1965 Mathematical models of the spread of injection. In *Mathematics and computer science in biology and medicine*, pp. 213–225. London, UK: Medical Research Council.

Kermack, W. O. & McKendrick, A. G. 1927 A contribution to the mathematical theory of epidemics, *Proc. R. Soc. Lond. B* **115**, 700–721. (doi:10.1098/rspa.1927.0118)

Kermack, W. O. & McKendrick, A. G. 1932 Contributions to the mathematical theory of epidemics, part II. *Proc. R. Soc. Lond. B* **138**, 55–83. (doi:10.1098/rspa.1932.0171)

Kermack, W. O. & McKendrick, A. G. 1933 Contributions to the mathematical theory of epidemics, part III. *Proc. R. Soc. Lond. B* **141**, 94–112. (doi:10.1098/rspa.1933.0106)

Li, W. T., Lin, G. & Ruan, S. 2006 Existence of travelling wave solutions in delayed reaction–diffusion systems with applications to diffusion–competition systems. *Nonlinearity* **19**, 1253–273. (doi:10.1088/0951-7715/19/6/003)

Li, W. T., Ruan, S. & Wang, Z. C. 2007 On the diffusive Nicholson's blowflies equation with nonlocal delays. *J. Nonlinear Sci.* **17**, 505–525. (doi:10.1007/s00332-007-9003-9)

Li, J. & Zou, X. In press. Modeling spatial spread of infectious diseases with a fixed latent period in a spatially continuous domain. *Bull. Math. Biol.*

Ma, S. 2001 Travelling wavefronts for delayed reaction-diffusion systems via a fixed point theorem. *J. Diff. Equat.* **171**, 294–314. (doi:10.1006/jdeq.2000.3846)

Murray, J. D. 1989 *Mathematical biology*, Berlin, Germany: Springer-Verlag.

Ou, C. & Wu. J. 2007 Persistence of wavefronts in delayed non-local reaction diffusion equations. *J. Diff. Equat.* **235**, 216–239.

Ruan, S. 2007 Spatial-temporal dynamics in nonlocal epidemiological models. In *Mathematics for life science and medicine*, vol. 2 (eds Y. Iwasa, K. Sato & Y. Takeuchi), pp. 97–122. Berlin, Germany: Springer-Verlag.

Thieme, H. R. 1977*a* A model for the spatial spread of an epidemic. *J. Math. Biol.* **4**, 337–351. (doi:10.1007/BF00275082)

Thieme, H. R. 1977*b* The asymptotic behavior of solutions of nonlinear integral equations. *Math. Zeitchrift* **157**, 141–154. (doi:10.1007/BF01215148)

Thieme, H. R. 1979 Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations. *J. Renie Angew. Math.* **306**, 94–121.

Thieme, H. R. & Zhao, X. Q. 2003 Asymptotic speeds of spread and travelling waves for integral equations and delayed reaction-diffusion models. *J. Diff. Equat.* **195**, 430–470. (doi:10.1016/S0022-0396(03)00175-X)

Wang, Z. C., Li, W. T. & Ruan, S. 2006 Travelling wave fronts of reaction–diffusion systems with spatio-temporal delays, *J. Diff. Equat.* **222**, 185–232. (doi:10.1016/j.jde.2005.08.010)

Wang, Z. C., Li, W. T. & Ruan, S. 2008 Travelling fronts in monostable equations with nonlocal delayed effects. *J. Dynam. Diff. Equat.* **20**, 573–607. (doi:10.1007/s10884-008-9103-8)

Wang, Z. C., Li, W. T. & Ruan, S. 2009 Entire solutions in bistable reaction–diffusion equations with nonlocal delayed nonlinearity. *Trans. Amer. Math. Soc.* **361**, 2047–2084. (doi:10.1090/S0002-9947-08-04694-1)

Widder, D. V. 1941 *The Laplace transform.* Princeton, NJ: Princeton University Press.

Wu, J. & Zou, X. 2001 Travelling wave fronts of reaction–diffusion systems with delay, *J. Dynam. Diff. Eqns.* **13**, 651–687. (doi:10.1023/A:1016690424892)

Zeilder, E. 1986 *Nonlinear functional analysis and its applications: I, fixed-point theorems.* New York, NY: Springer-Verlag.