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SOLVABILITY AND SPECTRAL ANALYSIS OF ABSTRACT HYPERBOLIC EQUATIONS WITH DELAY*

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Dedicated to Professor M. E. Drakhlin

Abstract. In this paper, we study the solvability of abstract hyperbolic equations with variable delay and integral Volterra terms. We consider several spectral problems in autonomous cases by considering the operator-valued functions as the symbols of the equations under investigation. We also present some applications of our results to integrodifferential equations of Gurtin-Pipkin type arising from the theory of heat equations with memory.

Key Words. Functional differential equations, Integrodifferential equations, spectral analysis, Sobolev space, Gurtin-Pipkin heat equation.

AMS(MOS) subject classification. 34D05, 34C23

1. Introduction.

A considerable number of studies have recently been devoted to the solvability and asymptotic behaviors of solutions to functional differential equations and integrodifferential equations in Banach (in particular, Hilbert) spaces. See [1]–[10], [14], [15], [17]–[20] and references therein.

In this article, we study functional differential and integrodifferential equations with unbounded operator coefficients in a Hilbert space. The main

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part of the equation under consideration is an abstract hyperbolic-type equation, disturbed by terms with delay arguments and terms involving Volterra operators. Our work, in comparison with most existing studies in the subject area, focuses on the case of variable delays and variable operator coefficients. We establish the solvability of the corresponding classical initial value problem in weighted Sobolev spaces on semiaxis. We study several spectral problems in autonomous cases, by considering the operator-valued functions as the symbols of the equations under investigation. We also present some applications of our results to the integrodifferential equation of Gurtin-Pipkin type arising from the theory of heat equations with memory.

2. The Problems and Results. Let H be a given separable Hilbert space and suppose that A is a positive selfadjoint operator that acts in H and has a bounded inverse. Let $H_\beta = Dom(A^\beta \cdot)$, the domain of A^β with $\beta > 0$, be equipped with norm $\|\cdot\|_\beta = \|A^\beta \cdot \|$.

For a Hilbert space X and real number $\gamma \geq 0$ let $L_{2,\gamma}((a,b),X)$, $-\infty \leq a < b \leq +\infty$, denote the space of X-valued measurable functions, equipped with the norm

$$\|f\|_{L_{2,\gamma}((a,b),X)} \equiv \Big(\int_a^b e^{-2\gamma t} \|f(t)\|_X^2 dt \Big)^{1/2}.$$

Let $W_{2,\gamma}^m((a,b),A^m)$, $m \in \mathbb{N}$, denote the space of H-valued functions such that $A^{pm}u^{((1-p)m)} \in L_{2,\gamma}((a,b),H)$ $(p = 0,1)$ with the norm

$$\|u\|_{W_{2,\gamma}^m((a,b),A^m)} = \left(\|u^{(m)}\|_{L_{2,\gamma}((a,b),H)}^2 + \|A^m u\|_{L_{2,\gamma}((a,b),H)}^2 \right)^{1/2}.$$

A more detailed description of these spaces can be found in [16], Chapter I. In what follows, we will omit corresponding index if $\gamma = 0$.

Consider the following initial-value problem for functional differential equation

$$\begin{aligned} &\tfrac{d^2u}{dt^2}(t) + A^2u(t) + \textstyle\sum_{j=1}^{\infty} \left(B_j(t)Au\left(g_j(t)\right) + D_j(t)\tfrac{du}{dt}\left(g_j(t)\right)\right) \\ &\quad + \int_{-\infty}^{t} K(t-s)Au(s)ds + \int_{-\infty}^{t} Q(t-s)u^{(1)}(s)ds = f(t), \quad t > 0, \end{aligned} \tag{1}$$

subject to

$$u(t) = \varphi(t), \quad t \in \mathbb{R}_- = (-\infty, 0]. \tag{2}$$

Here $B_j(t)$ and $D_j(t)$, $j = 1, 2, \ldots$, are operator-valued functions taking values in the ring of bounded operators in H; $K(t)$ and $Q(t)$ are operator-valued functions taking values in the ring of bounded operators in H, Bochner integrable on the semiaxis $\mathbb{R}_+$ with weight $e^{-\varkappa t}$:

$$(3) \qquad \int_0^{+\infty} e^{-\varkappa t}\|K(t)\|dt < +\infty, \quad \int_0^{+\infty} e^{-\varkappa t}\|Q(t)\|dt < +\infty;$$

$g_j(t)$, $j = 1,2,\dots$, are real valued continuously differentiable functions on $\mathbb{R}_+$, such that $g_j(t) \le t$, $g_j^{(1)}(t) > 0$ for $t \in \mathbb{R}_+$, g_j^{-1} are the inverse functions of g_j. In addition we assume $f \in L_{2,\gamma_0}(\mathbb{R}_+, H_1)$, $\varphi \in W^2_{2,\gamma_1}(\mathbb{R}_-, A^2)$ for certain γ_0 and $\gamma_1 \in \mathbb{R}$.

Definition 1. *A function u is called a strong solution to problem* (1)-(2) *if it belongs to the space $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$, with certain $\gamma \ge 0$, satisfies equation* (1) *almost everywhere on $\mathbb{R}_+$ and initial condition* (2).

The following theorem establishes the solvability of the initial value problem (1)-(2).

Theorem 1. *Suppose that the operator-valued functions $B_j(t)$, $D_j(t)$, $\widetilde{B}_j(t) = AB_j(t)A^{-1}$, $\widetilde{D}_j(t) = AD_j(t)A^{-1}$, take values in the ring of bounded operators over H, are strongly continuous and satisfy the inequalities:*

$$(4) \quad \begin{gathered} \sum_{j\ge1}\sup_{t\in\mathbb{R}_+} e^{-\delta(t-g_j(t))}\left[\sup_{t\in\mathbb{R}_+}\left[\|B_j(t)\|^2\frac{1}{g_j^{(1)}(t)}\right]^{1/2} + \sup_{t\in\mathbb{R}_+}\left[\|D_j(t)\|^2\frac{1}{g_j^{(1)}(t)}\right]^{1/2}\right] < \infty, \\ \sum_{j\ge1}\sup_{t\in\mathbb{R}_+} e^{-\delta(t-g_j(t))}\left[\sup_{t\in\mathbb{R}_+}\left[\|\widetilde{B}_j(t)\|^2\frac{1}{g_j^{(1)}(t)}\right]^{1/2} + \sup_{t\in\mathbb{R}_+}\left[\|\widetilde{D}_j(t)\|^2\frac{1}{g_j^{(1)}(t)}\right]^{1/2}\right] < \infty, \end{gathered}$$

for a certain $\delta \ge 0$; functions $g_j(t)$ are such that $t - g_j(t) \ge \alpha_0 > 0$, $t \ge 0$, $j = 1,2,\dots$, $\alpha_0 = const > 0$. Suppose also that operator-valued functions $K(t)$, $Q(t)$, $\widetilde{K}(t) = AK(t)A^{-1}$, $\widetilde{Q}(t) = AQ(t)A^{-1}$ take values in the ring of bounded operators over H and satisfy the conditions (3) *and the conditions:*

$$(5) \qquad \int_0^{+\infty} e^{-\varkappa_1 t}\|\widetilde{K}(t)\| < +\infty, \quad \int_0^{+\infty} e^{-\varkappa_1 t}\|\widetilde{Q}(t)\| < +\infty$$

for certain $\varkappa_1 \ge 0$. Then for any given functions $\varphi \in W^2_{2,\gamma_1}(\mathbb{R}_-, A^2)$ and $f \in L_{2,\gamma_0}(\mathbb{R}_+, H_1)$ with some γ_0, $\gamma_1 \ge 0$, there exists $\gamma^* > \max(\gamma_0, \gamma_1)$ such that for all $\gamma > \gamma^*$, the problem (1)-(2) *has a unique solution u satisfying the estimate*

$$(6) \qquad \|u\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \le d\left(\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|^2_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}\right)^{1/2}$$

with a certain $d > 0$ independent of f and φ.

Remark 1. *Conditions* (4) *are necessary to ensure solvability, as shown in examples of [7]. See also the example 2 at the end of the second part of the article.*

In the case when $B_j(t)$ and $D_j(t)$ are independent of t (i.e. $B_j(t) \equiv B_j$; $D_j(t) \equiv D_j$) and where $g_j(t) \equiv t - h_j$, for positive constants h_j: $0 < h_1 < h_2 < \ldots$, we consider the following operator-valued function

$$\mathcal{L}(\lambda) = \lambda^2 I + A^2 + \sum_{j=1}^{\infty} (B_j A + \lambda D_j)\, e^{-\lambda h_j} + \widehat{K}(\lambda) A + \lambda \widehat{Q}(\lambda), \tag{7}$$

where $\widehat{K}(\lambda)$ and $\widehat{Q}(\lambda)$ are the Laplace transforms of $K(t)$ and $Q(t)$, and I is indentity operator in the space H.

Suppose there exists a constant $\nu_0 \geq 0$ such that

$$\sum_{j=1}^{\infty} e^{-\nu_0 h_j} \left(\|B_j\| + \|D_j\|\right) < +\infty. \tag{8}$$

Proposition 1. *Suppose the condition* (8) *is satisfied and* $K(t)$ *and* $Q(t)$ *satisfy the estimate* (3). *Then there exists positive* $\varkappa^*$ *such that the operator-valued function* $\mathcal{L}(\lambda)$ *satisfies the inequalities*

$$\begin{aligned} &\left\| \mathcal{L}(\lambda) \left(\lambda^2 I + A^2\right)^{-1} - I \right\| < 1, \\ &\|\lambda \mathcal{L}^{-1}(\lambda)\| + \|A \mathcal{L}^{-1}(\lambda)\| \leq \tfrac{const}{\Re\lambda}, \quad \Re\lambda > \varkappa^*. \end{aligned} \tag{9}$$

Proposition 2. *Suppose the conditions of Proposition 1 are satisfied and the operator* A *has a compact inverse. Suppose also that there exist* $N \in \mathbb{N}$ *and* $h \in \mathbb{R}_+$ *such that* $B_j \equiv D_j \equiv 0$, $j \equiv N+1, N+2, \ldots$ *and* $K(t) \equiv Q(t) = 0$ *for* $t > h$. *Then the operator-valued function* $\mathcal{L}^{-1}(\lambda)$ *is finite meromorphic and spectrum of* $\mathcal{L}(\lambda)$ *consists of isolated eigenvalues of finite algebraic multiplicity that are the finite dimensional poles of* $\mathcal{L}^{-1}(\lambda)$.

We give the proofs of Propositions 1 and 2 at the third part of the article.

Remark 2. *An example shows that it is impossible to substitute the first derivative in the third term of equation* (1) *by the second derivative, and replacing operator* A *by the operator* A^2 *is given at the end of the the third part of the article.*

In the case $B_j \equiv D_j \equiv 0$, $j = 1, 2, \ldots$ operator-valued function $\mathcal{L}(\lambda)$ was investigated in [21], see also [24].

We also consider a similar to (1), (2) initial-valued problem in an autonomous case of the following type:

$$\begin{aligned}(10)\qquad &\tfrac{d^2u}{dt^2}(t) + A^2u(t) + \textstyle\sum_{j=1}^{N}\left[B_jAu(t-h_j) + D_ju^{(1)}(t-h_j)\right] \\ &+ \int_{-\infty}^{t} K(t-s)A^2u(s)ds + \int_{-\infty}^{t} Q(t-s)u^{(1)}(s)ds = f(t), \quad t>0,\end{aligned}$$

subject to

$$(11)\qquad u(t) = \varphi(t), \quad t\in\mathbb{R}_-.$$

Theorem 2. *Suppose operators B_j, D_j, $\widetilde{B}_j = AB_jA^{-1}$, $\widetilde{D}_j = AD_jA^{-1}$ are bounded in the space H, scalar functions K and Q are given so that $K\in W_1^1(\mathbb{R}_+)$, $Q\in L_1(\mathbb{R}_+)$ and functions f and φ are given so that $f\in L_{2,\gamma_0}(\mathbb{R}_+,H_1)$ and $\varphi\in W_{2,\gamma_1}^3(\mathbb{R}_-,A^3)$ for certain γ_0, $\gamma_1\in\mathbb{R}$. Then there exists $\gamma^*>\max(\gamma_0,\gamma_1)$ such that for all $\gamma>\gamma^*$ the problem (10)-(11) has a unique solution $u\in W_{2,\gamma}^2(\mathbb{R}_+,A^2)$ satisfying the estimate*

$$\|u\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)} \le d_1\left(\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|^2_{W_{2,\gamma}^3(\mathbb{R}_-,A^3)}\right)^{1/2}$$

with a constant d_1 independent of f and φ.

In the particular case where $Q(t)\equiv 0$, $B_j\equiv D_j\equiv 0$ for $j=1,2,\ldots,N$, the operator A has a compact inverse and the kernel K has the the following representation

$$K(t) = -\sum_{j=1}^{m} c_je^{-\gamma_jt}$$

with real constants c_j and γ_j so that $c_j>0$ and $0<\gamma_1<\gamma_2<\cdots<\gamma_m$, we can give a rather detailed picture of the resolvent set and the spectrum of the operator-valued function $\mathcal{L}(\lambda)$. More precisely we have that the spectrum of the operator-valued function $\mathcal{L}$ is:

$$\sigma(\mathcal{L}) \equiv \overline{\left(\bigcup_{k=1}^{m}\bigcup_{n=1}^{\infty}\{\lambda_{n_k}\}\right)} \bigcup \left(\bigcup_{n=1}^{\infty}\lambda_n^{\pm}\right),$$

where $\{\lambda_{n_k}\}_{n=1}^{\infty}$, $1\le k\le m$ are real eigenvalues $\lambda_{n_k}\in\mathbb{R}$ with cluster points $\sigma_k = \lim_{n\to\infty}\lambda_{n_k}$, where σ_k ($\sigma_m<\sigma_{m-1}<\cdots<\sigma_1$) are the solutions of the equations

$$\sum_{k=1}^{m}\frac{c_k}{x+\gamma_k} = 1.$$

In addition, the complex eigenvalues $\lambda_n^{\pm} = \mu_n \pm i\nu_n$ with $\mu_n, \nu_n \in \mathbb{R}$ satisfy $\lim_{n\to\infty} \mu_n = -\frac{1}{2}\sum_{k=1}^{m} c_k$ and $\nu_n = a_n + \underline{o}\left(\frac{1}{a_n^2}\right)$ with a_n being the eigenvalues of the operator A: $Ae_n = a_n e_n$; and $\{e_n\}_{n=1}^{\infty}$ the orthonormal basis of the corresponding eigenvectors. The eigenvalues $\{a_n\}_{n=1}^{\infty}$ are numerated in increasing order $(0 < a_1 \leq a_2 \leq a_3 \leq \dots)$ counting its multiplicities.

The result describing the structure of the spectrum of operator-valued function $\mathcal{L}(\lambda)$ may be obtained in the following way. Using the fact that $\{e_n\}_{n=1}^{\infty}$ is the orthonormal basis of the space H we can divide the integrodifferential equation (10) (where $Q(t) \equiv 0$, $B_j \equiv D_j \equiv 0$, $j = 1, 2, \dots, N$) into infinite number of scalar integrodifferential equations (projections on one-dimensional spaces formed by vectors e_n):

$$\frac{d^2 u_n(t)}{dt^2} + a_n^2 u_n(t) - \int_{-\infty}^{t} \sum_{k=1}^{m} c_k e^{-\gamma_k(t-s)} a_n^2 u_n(s)ds = f_n(t),$$

where

$$u_n(t) = (u(t), e_n), \quad f_n(t) = (f(t), e_n), \quad n = 1, 2, \dots.$$

Then using Laplace transform we have the following problem for determination eigenvalues $\{\lambda_n^{\pm}\} \bigcup \{\bigcup_{k=1}^{m} \{\lambda_{nk}\}\}$

$$l_n(\lambda) = \lambda^2 + a_n^2 - a_n^2\Big(\sum_{k=1}^{m} \frac{c_k}{\lambda + \gamma_k}\Big) = 0, \quad n = 1, 2, \dots.$$

On the base of Vieta theorem we find the spectrum of these scalar problems and the spectrum $\sigma(\mathcal{L})$ of operator-valued function $\mathcal{L}(\lambda)$ like closure of union of spectrum of these scalar problems.

It is important to underline that the spectrum of operator-valued function $\mathcal{L}(\lambda)$ is lying in left half-plane, if the following inequality

$$\sum_{j=1}^{m} \frac{c_j}{\gamma_j} < 1 \tag{12}$$

is valid. If the inequality

$$\sum_{j=1}^{m} \frac{c_j}{\gamma_j} > 1 \tag{13}$$

holds then there is the cluster-point σ_1 of eigenvalues of operator-valued function $\mathcal{L}$ which is lying in right half-plane. Thus if the inequality (13) holds the problem (10)–(11) (where $Q(t) \equiv 0$, $B_j \equiv D_j \equiv 0$, $j = 1, 2, \ldots, N$) is unstable.

We note that in the case where $B_j \equiv 0$ and $D_j \equiv 0$ for $j = 1, 2, \ldots, N$; equation (10) is the abstract form of the heat equation with memory introduced by Gurtin and Pipkin in [11], and there have been extensive studies of this equation (see [11] - [13] and references therein).

3. The proofs of certain results and examples.

Example 1. Now, consider the operator pencil

$$\mathcal{L}_1(\lambda) = \lambda^2 + A^2 + D\lambda^s e^{-\lambda}, \quad s = 1, 2, \ldots$$

When $s = 1$, it corresponds to the special case of the pencil $\mathcal{L}(\lambda)$ with $n = h = 1$, $D_1 = D$, and $B_1 = 0$. When $s = 2, \ldots$, the estimate in Proposition 1 is not valid; moreover, one can show that $\mathcal{L}_1^{-1}(\lambda)$ may not exist at the points λ with arbitrarily large real parts.

Indeed, let $D = I$ and $s = 2$. Consider a sequence of real numbers $y_m = \frac{3\pi}{2} + 2\pi m$, $m = 0, 1, \ldots$. For a given $y = y_m$, the equation

$$2xye^x + x^2 - y^2 = 0 \tag{14}$$

defines a bijection x to y: $\mathbb{R}_+ \to \mathbb{R}_+$. Using y_m, we define a sequence $\{x_m\}_{m=0}^{\infty}$ such that the pairs (x_m, y_m) satisfy (14). Define a positive operator A by the rule

$$Af = \sum_{m=1}^{\infty} a_m^{\frac{1}{2}}(f, e_m)e_m,$$

where $\{e_m\}_{m=1}^{\infty}$ is an orthonormal basis in the separable space H and $a_m = 4x_m y_m \cosh x_m$. Since A has a discrete spectrum, the problem of the existence of $\mathcal{L}_1^{-1}(\lambda)$ splits into a countable set scalar problems:

$$\lambda^2(1 + e^{-\lambda}) + a_m = 0, \quad m = 1, 2, \ldots. \tag{15}$$

Obviously, $\lambda_m = x_m + iy_m$ is a solution to the mth scalar problem in (15), so that $\mathcal{L}_1^{-1}(\lambda)$ does not exist at the points λ_m. Moreover, $\Re\lambda_m \to +\infty$ as $m \to \infty$. Thus, the solvability result cannot be extended to equations with deviating argument that involve terms with the second derivative with respect to t.

A similar situation occurs for the operator pencil

$$\mathcal{L}_2(\lambda) = \lambda^2 I + A^2 + BA^{\Theta}e^{-\lambda}, \quad \Theta \in \mathbb{R}_+.$$

Following a procedure analogous to the describe above, we can construct an operator A for which the operator-valued function $\mathcal{L}_2^{-1}(\lambda)$ is not holomorphic in the half-plane $\Re\lambda > \gamma$ for any γ and $\Theta > 1$.

Example 2. The following example shows the importance of conditions (4). Consider the initial value problem

$$\frac{d^2u(t)}{dt^2} + u(t) - 4(t^2+t+1)e^{2t}u(t-1) = 0, \quad t \in \mathbb{R}_+,$$

$$u(t) = e^{t^2+t}, \quad t \in (-1, 0],$$

which is a special case of the problem (1)–(2) (here, $H = \mathbb{C}$, $n = 1$, $h_1 = 1$, $D_1(t) \equiv 0$, and $A \equiv I$). This problem has a unique solution $u(t) = e^{t^2+t}$, which does not belong to $W^2_{2,\gamma}((-1,+\infty), A^2)$ for any $\gamma > 0$. In this case, $\sup_{t\in[0,+\infty)} \|B_1(t)\| = +\infty$; i. e., conditions (4) is violated.

We give now the proofs of Propositions 1 and 2, and Theorem 2.

Proof of the proposition 1. The spectrum of A is contained in $\mathbb{R}_+$. Applying the well-known estimate of the resolvent in term of distance from the spectrum of the normal operators $\pm iA$ we obtain

$$\|\lambda(\lambda^2 I + A^2)^{-1}\| \le |\lambda| \|(\lambda I + iA)^{-1}\| \|(\lambda I - iA)^{-1}\| \le (\Re\lambda)^{-1}, \quad \Re\lambda > 0. \tag{16}$$

Hence we have

$$\|A(\lambda^2 I + A^2)^{-1}\| \le \|(\lambda I + iA)^{-1}\| + |\lambda| \|(\lambda^2 I + A^2)^{-1}\| \le 2(\Re\lambda)^{-1}, \quad \Re\lambda > 0. \tag{17}$$

On the base of estimates (16), (17) we obtain the estimate

$$\|\sum_{j=1}^{\infty} e^{-\lambda h_j}[B_j A + \lambda D_j](\lambda^2 I + A^2)^{-1}\| \le$$

$$\le const(\sum_{j=1}^{\infty} e^{-\nu_0 h_j}(\|B_j\| + \|D_j\|)(\Re\lambda)^{-1}, \quad \Re\lambda > \nu_0 > 0. \tag{18}$$

Using the well-known properties of the Laplace transform

$$\|\widehat{K}(\lambda)\| \to 0, \quad \|\widehat{Q}(\lambda)\| \to 0 \quad (\Re\lambda \to +\infty) \tag{19}$$

and estimates (16), (17) , we deduce

$$\|[\widehat{K}(\lambda)A + \lambda\widehat{Q}(\lambda)](\lambda^2 I + A^2)^{-1}\| \to 0, \quad \Re\lambda \to +\infty. \tag{20}$$

Choosing suitable $\varkappa_+$ and using (18), (20), we obtain the first estimate (9) for $\Re\lambda > \varkappa_+$. The second inequality (9) is the corollary of the first one.

From the representation

$$\mathcal{L}^{-1}(\lambda) = (\lambda^2 I + A^2)^{-1}(I + (\mathcal{L}(\lambda) - (\lambda^2 I + A^2))(\lambda^2 I + A)^{-1})^{-1},$$

first inequality (9), and (16), (17) we obtain the second inequality (9). □

Proof of the proposition 2. The proof of this result follows from M. V. Keldyš lemma (see [22]) in view of the fact that $(\mathcal{L}(\lambda)A^{-2} - I)$ takes values in the ring of compact operators, $\mathcal{L}(\lambda)A^{-2}$ - is entire operator-valued function and $\mathcal{L}(\lambda)A^{-2}$ is invertible (by Proposition 1) for λ with a sufficiently large real part. □

The proof of the Theorem 2.

We will give the proof of the theorem 2 under the following additional assumptions: $\varphi(t) = 0$, $t \leq 0$, and operators $B_j \equiv D_j \equiv 0$, $j = 1, 2, \dots N$.

Let us introduce the function $v(t) = e^{-\gamma t}u(t)$. Hence the equation (10) for the function u is equivalent to the following equation for the function v:

$$\begin{aligned} &v^{(2)}(t) + 2\gamma v^{(1)}(t) + (A^2 + \gamma^2 I)v(t) + \int_0^t e^{-\gamma(t-s)}K(t-s)A^2 v(s)\,ds + \\ (21)\qquad &+ \int_0^t e^{-\gamma(t-s)}Q(t-s)v^{(1)}(s)\,ds + \gamma\int_0^t e^{-\gamma(t-s)}Q(t-s)v(s)\,ds, = \\ &= e^{-\gamma t}f(t),\ t > 0; \end{aligned}$$

$$v(+0) = 0,\ v^{(1)}(+0) = 0.$$

We look for a solution of the equation (21) in the form

$$v(t) = \mathcal{L}z(t) = \int_0^t e^{-\gamma(t-s)}A^{-1}sin(A(t-s))z(s)\,ds, \quad t > 0;$$

with new unknown function z. Then we have

$$v^{(1)}(t) = -\gamma\int_0^t e^{-\gamma(t-s)}A^{-1}sin(A(t-s))z(s)\,ds +$$

$$+ \int_0^t e^{-\gamma(t-s)}cos(A(t-s))z(s)\,ds,$$

$$
\begin{aligned}
v^{(2)}(t) = \qquad & \gamma^2 \int_0^t e^{-\gamma(t-s)} A^{-1} sin(A(t-s)) z(s) ds - \\
& -2\gamma \int_0^t e^{-\gamma(t-s)} cos(A(t-s)) z(s)\, ds + \\
& +z(t) - \int_0^t e^{-\gamma(t-s)} A sin(A(t-s)) z(s)\, ds, \quad t > 0.
\end{aligned}
$$

Substituting $v = \mathcal{L}z$ into the equation (21) we obtain that the equation (21) is equivalent to the following integral equation for the function z:

$$
\begin{aligned}
& z(t) + \int_0^t e^{-\gamma(t-s)} K(t-s) A^2 \left(\int_0^s e^{-\gamma(s-\theta)} A^{-1} sin(A(s-\theta)) z(\theta)\, d\theta\right) ds + \\
& + \int_0^t e^{-\gamma(t-s)} Q(t-s) \left(\int_0^s e^{-\gamma(s-\theta)} cos(A(s-\theta)) z(\theta)\, d\theta\right) ds = \\
& = e^{-\gamma t} f(t), \ t > 0.
\end{aligned} \tag{22}
$$

We will consider the equation (22) in the space $L_2(R_+, H_1)$. Let denote $\omega(t) = Az(t)$. Then function ω belongs to the space $L_2(R_+, H)$ and satisfies the following equation

$$
\begin{aligned}
& \omega(t) + \int_0^t e^{-\gamma(t-s)} K(t-s) A \left(\int_0^s e^{-\gamma(s-\theta)} A sin(A(s-\theta)) \omega(\theta) d\theta\right) ds + \\
& + \int_0^t e^{-\gamma(t-s)} Q(t-s) \left(\int_0^s e^{-\gamma(s-\theta)} cos(A(s-\theta)) \omega(\theta)\, d\theta\right) ds = \\
& = e^{-\gamma t} F(t), \ t > 0,
\end{aligned} \tag{23}
$$

where $F(t) = Af(t)$.

Under assumptions on the kernel K it's Laplace transform:

$$
\hat{K}(\lambda) = \int_0^{+\infty} e^{-\lambda t} K(t)\, dt
$$

satisfies the estimate

$$
|\hat{K}((\mu+\gamma) + i\nu)| \le \frac{\text{const}}{((\mu+\gamma)^2 + \nu^2)^{\frac{1}{2}}}, \quad \mu \ge 0. \tag{24}
$$

In turn the Laplace transform $\hat{Q}$ of the function Q satisfies the inequality

$$
|\hat{Q}((\mu+\gamma) + i\nu)| \le \text{const}, \qquad \mu \ge 0. \tag{25}
$$

Applying the Laplace transform to the equation (25) we obtain

$$
\begin{aligned}
&((I+\hat{K}(\lambda+\gamma)A^2((\lambda+\gamma)^2I+A^2)^{-1}+ \\
(26)\qquad &+(\lambda+\gamma)\hat{Q}(\lambda+\gamma)((\lambda+\gamma)^2I+A^2)^{-1})\hat{\omega}(\lambda)= \\
&=\hat{F}(\lambda+\gamma),\ \mathrm{Re}\lambda>0,\ \lambda=\mu+i\nu,
\end{aligned}
$$

where $\hat{\omega}(\lambda)$ is the Laplace transform of the function ω. Let's assume $\tau=\mu+\gamma$ and evaluate the norm of operator $\hat{K}(\tau+i\nu)A^2((\tau+i\nu)^2I+A^2)^{-1})$. We evaluate the function

$$a^2(|(\tau+i\nu)^2+a^2|)^{-1}(\tau^2+\nu^2)^{-\frac{1}{2}}$$

where $a\in[\alpha_0,+\infty)$, $\alpha_0=\inf\limits_{\|u\|_H=1}(Au,u)>0$ and $\nu\in\mathbb{R}$. We choose certain number $d\in(0,1)$ and estimate the function $f(a,\nu,\tau)=(|(\tau+i\nu)^2+a^2|)^2(\tau^2+\nu^2)$ from below:

$$
\begin{aligned}
f(a,\nu,\tau)&=((\tau^2+a^2-\nu^2)^2+4\tau^2\nu^2)(\tau^2+\nu^2)\geq \\
&\geq\min[\min_{\nu^2\in[0,da^2]}f(a,\nu,\tau),\min_{\nu^2\in[da^2,+\infty]}f(a,\nu,\tau)]\geq \\
&\geq\min[(\tau^2+(1-d)a^2)^2\tau^2,(\tau^2+da^2)^2 4da^2\tau^2].
\end{aligned}
$$

Hence using the inequality (24) and the theorem about spectral representation of the selfadjoint operator A we obtain:

$$
\begin{aligned}
&\|\hat{K}(\tau+i\nu)A^2((\tau+i\nu)^2I+A^2)^{-1}\|\leq \\
&\leq\mathrm{const}\max\left[\frac{a^2}{(\tau^2+(1-d)a^2)\tau},\frac{a^2}{2a\tau(d(\tau^2+da^2))^{\frac{1}{2}}}\right]\leq \\
&\leq\mathrm{const}\max\left[\frac{1}{(\frac{\tau^2}{a^2}+(1-d))\tau},\frac{1}{2\tau(d(\frac{\tau^2}{a^2}+d))^{\frac{1}{2}}}\right]\leq \\
&\leq\mathrm{const}\max\left[\frac{1}{(1-d)\tau},\frac{1}{2\tau d}\right].
\end{aligned}
$$

Choosing $d=\frac{1}{3}$ we get the following estimate

$$
(27)\qquad \|\hat{K}(\tau+i\nu)A^2((\tau+i\nu)^2I+A^2)^{-1}\|\leq\mathrm{const}(\frac{3}{2\tau})\leq\frac{\mathrm{const}_1}{\mu+\gamma}.
$$

Now we will obtain the estimate of the following expression:

$$\|\hat{Q}(\tau+i\nu)(\tau+i\nu)((\tau+i\nu)^2I+A^2)^{-1}\|.$$

In order to do this we evaluate the scalar function

$$g(\tau,\nu,a)=\left|\frac{(\tau+i\nu)}{(\tau+i\nu)^2+a^2}\right|^2,$$

where $a \in [\alpha_0, +\infty)$, $\tau = \mu + \gamma$, $\mu \geq 0$, $\nu \in \mathbb{R}$. It is rather clear that the following chain of the inequalities is valid:

$$g(\tau,\nu,a) = \tfrac{\tau^2+\nu^2}{(a^2-\nu^2)^2+\tau^4+2a^2\tau^2+2\tau^2\nu^2} \leq \tfrac{(\tau^2+\nu^2)}{\tau^4+2a^2\nu^2+2\nu^2\tau^2} \leq$$
$$\leq \tfrac{1}{\tau^2}\left[\tfrac{\tau^2+\nu^2}{\tau^2+2\nu^2}\right] \leq \tfrac{1}{\tau^2} = \tfrac{1}{(\mu+\gamma)^2}. \tag{28}$$

On the base of the estimates (25), (28) and theorem about the spectral representation of the selfadjoint operator A we have

$$\|\hat{Q}(\tau+i\nu)(\tau+i\nu)((\tau+i\nu)^2 I + A^2)^{-1}\| \leq \frac{\text{const}}{\mu+\gamma}. \tag{29}$$

Then choosing big enough γ_0 from the estimates (27), (29) and representation (26) we obtain that function $\hat{\omega}$ belongs to Hardy space $H_2(\Re\lambda > 0)$ in the right half-plane $(\Re\lambda > 0)$. Hence on the base of Hardy theorem we have the the unique solvability of the integral equation (23) in the space $L_2(\mathbb{R}_+, H)$ for arbitrary $\gamma \geq \gamma_0$.

Then the integral equation(22) has a unique solution $z \in L_2(\mathbb{R}_+, H_1)$ for arbitrary $\gamma \geq \gamma_0$. In turn, from the representation $v = \mathcal{L}z$ and Lemma 1 from [7] we have the unique solubility of the Cauchy problem (10), (11) in the space $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ for arbitrary $\gamma \geq \gamma_0$.

In order to make the description more complete and for the convenience of the readers we give here the formulation of Lemma 1 from [7].

Denote by $|||\cdot|||$ the norms of operators acting in the space $L_2(\mathbb{R}_+, H)$.

Lemma 1. *([7]) The operators $\mathcal{L}$ and $\mathcal{M}$ defined as*

$$(\mathcal{L}z)(t) = \begin{cases} \int_0^t e^{-\gamma(t-s)} A^{-1} sin(A(t-s))z(s)\,ds, & t \geq 0 \\ 0 & t < 0; \end{cases}$$

$$(\mathcal{M}z)(t) = \begin{cases} \int_0^t e^{-\gamma(t-s)} cos(A(t-s))z(s)\,ds, & t \geq 0 \\ 0 & t < 0; \end{cases}$$

satisfy the estimates

$$|||A\mathcal{L}||| \leq \frac{1}{2\gamma}, \quad |||\mathcal{M}||| \leq \frac{1}{\gamma},$$

for $\gamma > 0$.

Thus on the base of Lemma 1 ([7]) it is not difficult to obtain that function $v = \mathcal{L}z$ belong to the space $W_2^2(\mathbb{R}_+, A^2)$ and the following estimate

$$\|v\|_{W_2^2(\mathbb{R}_+,A^2)} \leq \mathrm{const}\|z\|_{L_2(\mathbb{R}_+,H_1)}, \qquad \gamma \geq \gamma_0$$

is valid (see [7] for more details) with const independent on function $z \in L_2(\mathbb{R}_+, H_1)$.

Now we are going to give another versions of the statement for the initial-value problem of the Gurtin-Pipkin itegrodifferential equation. Suppose that operators $B_j \equiv D_j \equiv 0$, for $j = 1, 2, \ldots, N$, kernels $K \in W_1^1(\mathbb{R}_+)$, $Q \in L_1(\mathbb{R}_+)$.

Then the initial-value problem (10), (11) may be reduced to the following form

$$u^{(2)} + A^2u(t) + \int_0^t K(t-s)A^2u(s)\,ds + \tag{30}$$
$$+ \int_0^t Q(t-s)u^{(1)}(s)\,ds = f_1(t),\ t > 0$$

$$u(+0) = \varphi(-0) = \varphi_0,\ u^{(1)}(+0) = \varphi^{(1)}(-0) = \varphi_1 \tag{31}$$

with new right part

$$f_1(t) = f(t) + h(t),$$

where

$$h(t) = -\int_{-\infty}^{0} K(t-s)A^2\varphi(s)\,ds - \int_{-\infty}^{0} Q(t-s)\varphi^{(1)}(s)\,ds. \tag{32}$$

On the base of Hausdorf-Young inequality we obtain that vector-function $h \in L_{2,\gamma_1}(\mathbb{R}_+, H_1)$ and the following inequalities:

$$\|h\|_{L_{2,\gamma}(\mathbb{R}_+,H_1)} \leq$$

$$\leq \|Aq_1\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|Aq_2\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \leq$$

$$\leq K_1\,\|\varphi\|_{W^3_{2,\gamma}(\mathbb{R}_-,A^3)}, \qquad \gamma \geq \gamma_1;\ K_1 = \mathrm{const} > 0;$$

where

$$e^{-\gamma t}q_1(t) = \int_0^{-\infty} e^{-\gamma(t-s)}K(t-s)\left[e^{-\gamma s}A^2\varphi(s)\right]\,ds,$$

$$e^{-\gamma t}q_2(t) = \int_0^{-\infty} e^{-\gamma(t-s)}Q(t-s)\left[e^{-\gamma s}\varphi^{(1)}(s)\right]\,ds,$$

are valid.

Owing to the trace theorem (see [16], chapter I, for more details) we have $\varphi(-0) \in H_{5/2}$, $\varphi^{(1)}(-0) \in H_{3/2}$, and the following estimates

$$\|\varphi(-0)\|_{5/2} \leq K_2 \|\varphi\|_{W^3_{2,\gamma}(\mathbb{R}_-,A^3)}, \quad K_2 = \text{const} > 0,$$

$$\left\|\varphi^{(1)}(-0)\right\|_{3/2} \leq K_3 \|\varphi\|_{W^3_{2,\gamma}(\mathbb{R}_-,A^3)}, \quad K_3 = \text{const} > 0,$$

are satisfied. Introducing the new function $y(t) = e^{-\gamma t}u(t)$ we obtain the following problem for function y:

$$\begin{aligned} y^{(2)}(t) + 2\gamma y^{(1)}(t) + (A^2+\gamma^2 I)y(t) + \int_0^t e^{-\gamma(t-s)}K(t-s)A^2y(s)\,ds+ \\ +\int_0^t e^{-\gamma(t-s)}Q(t-s)y^{(1)}(s)\,ds+ \\ +\gamma\int_0^t e^{-\gamma(t-s)}Q(t-s)y(s)\,ds = e^{-\gamma t}f_1(t),\ t>0. \end{aligned} \tag{33}$$

$$y(+0) = \varphi_0, \quad y^{(1)}(+0) = \varphi_1 - \gamma\varphi_0. \tag{34}$$

In turn using the substitution

$$y(t) = e^{-\gamma t}\cos(At)\varphi_0 + e^{-\gamma t}A^{-1}\sin(At)\varphi_1 + v(t)$$

we reduce that function v satisfies the problem (21). Thus the assumption $\varphi(t) = 0$, $t \leq 0$ do not restrict the generality of our considerations in the proof of the theorem 2.

Now let us suppose that kernel $Q \equiv 0$. Then after integrating the equation (30) from 0 till to t we have

$$\begin{aligned} \tfrac{du}{dt}(t) + \int_0^t A^2u(\theta)\,d\theta + \int_0^t \left(\int_0^s K(s-\theta)A^2u(\theta)\,d\theta\right)ds = \\ = \int_0^t f_1(\theta)\,d\theta + u^{(1)}(+0). \end{aligned} \tag{35}$$

Changing the order of integrating in the third term of (35) we obtain the following equation

$$\frac{du}{dt}(t) + \int_0^t A^2u(\theta)\,d\theta + \int_0^t Q(t-\theta)A^2u(\theta)\,d\theta = \\ = f_2(\theta), \tag{36}$$

where

$$Q(t) = \int_0^t K(\zeta)\,d\zeta,\ f_2(t) = \int_0^t f_1(\theta)\,d\theta + \varphi_1$$

Hence we obtain that function u satisfies the following integrodifferential equation of the first order:

$$\frac{du}{dt}(t) + \int_0^t G(t-\theta)A^2u(\theta)\,d\theta = f_2(t), \\ u(+0) = \varphi_0, \tag{37}$$

where the kernel

$$G(t) = 1 + Q(t).$$

It is relevant to underline that the equation (37) is an abstract form of Gurtin-Pipkin equation proposed by Pandolfi in [11], [13].

Remark 3. . *In spite the fact that the problem* (1), (2) *and problem* (10), (11) *looks like the similar problems the perturbations of abstract hyperbolic equation in the equation* (1) *and in the equation* (10) *are different. So in the equation* (1) *the conditions on the kernel* $K(t)$ *are more general but it is possible to include only first power of operator* A *in the perturbation term. On the order hand in the equation* (10) *the conditions on the kernel* K *are more restrictive* ($K \in W_1^1(\mathbb{R}_+)$) *but it is possible to include the second power of operator* A *in the perturbation term.*

4. Comments and remarks. An essential feature of our results such as Theorem 1 is to allow variable delays as well as variable coefficients $B_j(t)A$ and $D_j(t)$, while in [3], [17]-[20] the case of constant delays under more stringent constants on coefficients is investigated.

We also consider unbounded operator coefficients $B_j(t)A$ on the delay terms, in comparison with the existing results in hyperbolic case [14]-[15].

The method for providing Theorem 1 is different from those adopted in [3], [14]-[15], [17]-[20], [26] and is analogous to the method used in [7]-[10], [23], [25]. Our results generalize those in [7]. More precisely, the argument in our proof of Theorem 1 is similar to that in the proof of Theorem 1 from [7]. Like in [7] we use a reduction of the problem (1)–(2) to an equivalent functional integral equation in the space $L_2(\mathbb{R}_+, H_1)$. The argument, based on the fact that this functional integral equation (analogous to equation (29) from [7]) has a bounded operator coefficients, enables us to estimate the norms of above mentioned operators in the space $L_2(\mathbb{R}_+, H_1)$. In the process of estimating the norms of integral operators, we essentially need to use Lemma 1 and Lemma 2 from [7].

It is relevant to underline here that our method of proving the correct solvability of the initial-value problem (10)–(11) is seriously different in the comparison with approach used by L. Pandolfi in [11]. The proof of the theorem 2 is based on the reduction of the problem (10), (11) to equivalent in the sense of solubility functional integral equation (22) of convolution type in the space $L_2(\mathbb{R}_+, H_1)$. Then in order to prove it's unique solubility we estimate the Laplace transforms of the kernels of the integral terms of this equations. In particular case $B_j \equiv D_j \equiv 0$, $Q(t) \equiv 0$, $j = 1, 2, \dots, N$, this procedure was provided in the work of V. V. Vlasov and D. A. Medvedev [23] (Part I). In [11] L. Pandolfi studied the Gurtin-Pipkin equation having the form (37). Using sine and cosine operator-valued functions he reduced the problem (37) to the integral equation on function u which is equivalent in the sence of solubility to the problem (37). In his investigators L. Pandolfi studied the solvability of the inregral equation functional spaces on finite interval $(0, T)$ of the time variable t. In comparison we study the problem (10), (11) in weighted Sobolev space $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ on the semiaxis $\mathbb{R}_+$.

In our consideration we essentially use the Hilbert structure of the spaces $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$, $L_{2,\gamma}(\mathbb{R}_+, H)$ and Hardy theorem, while L. Pandolfi works in more general Banach spaces. Another results concerned the investigation of the equations of Gurtin-Pipkin type are described in [13].

Our study here continues the studies of parabolic functional differential equations in [2], [8]-[10], [23], [25] and the most important feature of equation (1) considered here is that the unperturbed equation is an abstract hyperbolic equation in a Hilbert space, where the unperturbed equation in relevant works [17]-[20] is an abstract parabolic equation with the operator A as a generator of a bounded holomorphic semigroup.

It is relevant to underline that problems similar to (10), (11), arising in the theory of viscoelasticity, studied in [21], [24].

More details of the proofs, the generalization of the results and comparison with existing results will be provided elsewhere.

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