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PARTIAL DIFFERENTIAL EQUATIONS

Spectral Analysis and Solvability of Abstract Hyperbolic Equations with Aftereffect

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Abstract—We analyze functional-differential equations with unbounded operator coefficients in a Hilbert space whose leading part is an abstract hyperbolic equation perturbed by terms with a retarded argument and by terms with Volterra integral operators.

We consider spectral problems for the operator functions that are the symbols of above-mentioned equations in the autonomous case.

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Numerous papers dealing with the analysis of the solvability and asymptotic behavior of solutions of functional-differential and integro-differential equations in Banach spaces and, in particular, in the Hilbert space have been published recently (see [1–16] and the bibliography therein).

In the present paper, we consider functional-differential and integro-differential equations with unbounded operator coefficients in the Hilbert space. The leading part of these equations is an abstract hyperbolic equation perturbed by terms with retarded arguments and by terms with Volterra integral operators.

Unlike other studies known to the authors in this direction, the present paper deals with the analysis of variable delays and variable operator coefficients, while, in most of the papers known to the authors, the case of constant delays (see [1–5]) and bounded operator coefficients multiplying the delays (see [8, 9] and the bibliography therein) was considered.

We prove the well-posed solvability of initial-boundary value problems for the above-mentioned equations in weighted Sobolev spaces on the positive half-line.

In the final part of the present paper, by way of application, we consider some results for the Gurtin–Pipkin integro-differential equations describing the propagation of heat at a finite velocity in media with memory.

1. DEFINITIONS, NOTATION, AND STATEMENT OF THE MAIN RESULTS

Let H be a separable Hilbert space, and let A be a positive self-adjoint operator with bounded inverse in H. The domain $\mathrm{Dom}(A^\beta)$ of the operator A^β, $\beta > 0$, is a Hilbert space H_β with respect to the norm $\|\cdot\|_\beta = \|A^\beta \cdot\|$ on $\mathrm{Dom}(A^\beta)$, which is equivalent to the graph norm of the operator A^β.

By $L_{2,\gamma}((a,b),X)$, $-\infty \le a < b \le +\infty$, we denote the space of measurable vector functions ranging in a Hilbert space X; it is equipped with the norm

$$\|f\|_{L_{2,\gamma}((a,b),X)} \equiv \left(\int_a^b \exp(-2\gamma t)\|f(t)\|_X^2 \, dt \right)^{1/2}, \qquad \gamma \ge 0.$$

By $W_{2,\gamma}^m((a,b),A^m)$, $m \in \mathbb{N}$, we denote the space of vector functions ranging in H such that $A^{pm}u^{(1-p)m} \in L_{2,\gamma}((a,b),H)$ $(p = 0,1)$; this space is equipped with the norm

$$\|u\|_{W_{2,\gamma}^m((a,b),A^m)} \equiv (\|u^{(m)}\|_{L_{2,\gamma}((a,b),H)}^2 + \|A^m u\|_{L_{2,\gamma}((a,b),H)}^2)^{1/2}.$$

Here and throughout the following, $u^{(m)}(t) = \dfrac{d^m}{dt^m}u(t)$, $m \in \mathbb{N}$. For a more detailed description of the space $W^m_{2,\gamma}((a,b), A^m)$, see [17, Chap. 1]. Throughout the following, we omit the corresponding index if $\gamma = 0$.

Consider the initial-boundary value problem for the functional-differential equation of the form

$$u^{(2)}(t) + A^2u(t) + \sum_{j=1}^{\infty}[B_j(t)Au(g_j(t)) + D_j(t)u^{(1)}(g_j(t))] + \int_{-\infty}^{t} K(t-s)Au(s)\,ds + \int_{-\infty}^{t} Q(t-s)u^{(1)}(s)\,ds = f(t), \qquad t > 0, \tag{1}$$

with the initial condition

$$u(t) = \varphi(t), \qquad t \in \mathbb{R}_- = (-\infty, 0]. \tag{2}$$

Here $B_j(t)$ and $D_j(t)$, $j = 1, 2, \ldots$, are operator functions ranging in the ring of bounded operators in H, and $K(t)$ and $Q(t)$ are operator functions ranging in the ring of bounded operators in H and Bochner integrable on the half-line $\mathbb{R}_+$ with weight $e^{-\varkappa t}$,

$$\int_0^{+\infty} \exp(-\varkappa t)\|K(t)\|\,dt < +\infty, \qquad \int_0^{+\infty} \exp(-\varkappa t)\|Q(t)\|\,dt < +\infty; \tag{3}$$

further, the $g_j(t)$ are scalar real-valued continuously differentiable functions on the half-line $\mathbb{R}_+$ such that $g_j(t) \le t$; $g_j^{(1)}(t) > 0$, $t \in \mathbb{R}_+$ $(j = 1, 2, \ldots)$.

By g_j^{-1} we denote the inverse functions of g_j. In the following, we assume that $f \in L_{2,\gamma_0}(\mathbb{R}_+, H_1)$ and $\varphi \in W^2_{2,\gamma_1}(\mathbb{R}_-, A^2)$ for some γ_0 and $\gamma_1 \in \mathbb{R}$.

Definition 1. A function u is called a *strong solution* of problem (1), (2) if it belongs to the space $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ for some $\gamma \ge 0$ and satisfies Eq. (1) almost everywhere on the half-line $\mathbb{R}_+$ and the initial condition (2).

The following theorem on the well-posed solvability of problem (1), (2) is one of the main results of the present paper.

Theorem 1. *Suppose that operator functions* $B_j(t)$, $D_j(t)$, $\tilde{B}_j(t) = AB_j(t)A^{-1}$, *and* $\tilde{D}_j(t) = AD_j(t)A^{-1}$ *range in the ring of bounded operators in* H, *are strongly continuous, and satisfy the inequalities*

$$\sum_{j\ge1} \sup_{t\in\mathbb{R}_+} \exp(-\delta(t - g_j(t))) \left[\sup_{t\in\mathbb{R}_+} \frac{\|B_j(t)\|}{(g_j^{(1)}(t))^{1/2}} + \sup_{t\in\mathbb{R}_+} \frac{\|D_j(t)\|}{(g_j^{(1)}(t))^{1/2}} \right] < +\infty, \tag{4}$$

$$\sum_{j\ge1} \sup_{t\in\mathbb{R}_+} \exp(-\delta(t - g_j(t))) \left[\sup_{t\in\mathbb{R}_+} \frac{\|\tilde{B}_j(t)\|}{(g_j^{(1)}(t))^{1/2}} + \sup_{t\in\mathbb{R}_+} \frac{\|\tilde{D}_j(t)\|}{(g_j^{(1)}(t))^{1/2}} \right] < +\infty \tag{5}$$

for some $\delta \ge 0$, *where the* $g_j(t)$ *are functions such that* $t - g_j(t) \ge \alpha_0 = \text{const} > 0$, $t \ge 0$, $j = 1, 2, \ldots$ *In addition, suppose that the operator functions* $K(t)$, $Q(t)$, $\tilde{K}(t) = AK(t)A^{-1}$, *and* $\tilde{Q}(t) = AQ(t)A^{-1}$ *range in the ring of bounded operators in the space* H *and satisfy inequalities* (3) *and the inequalities*

$$\int_0^{+\infty} \exp(-\varkappa_1 t)\|\tilde{K}(t)\|\,dt < +\infty, \qquad \int_0^{+\infty} \exp(-\varkappa_1 t)\|\tilde{Q}(t)\|\,dt < +\infty \tag{6}$$

for some $\varkappa_1 \geq 0$. *Then, for arbitrary functions* $\varphi \in W^2_{2,\gamma_1}(\mathbb{R}_-, A^2)$ *and* $f \in L_{2,\gamma_0}(\mathbb{R}_+, H_1)$ *with some* $\gamma_0, \gamma_1 \geq 0$, *there exists a constant* $\gamma^* \geq \max(\gamma_0, \gamma_1)$ *such that, for all* $\gamma > \gamma^*$, *problem* (1), (2) *has a unique solution* $u \in W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ *satisfying the inequality*

$$\|u\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \leq d(\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|^2_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)})^{1/2} \tag{7}$$

with a constant d *independent of* f *and* φ.

Remark. Conditions (4) and (5) are essential for the well-posed solvability of problem (1), (2). The corresponding examples can be found in [13].

If the coefficients $B_j(t)$ and $D_j(t)$ are independent of t [i.e., $B_j(t) \equiv B_j$ and $D_j(t) \equiv D_j$], and the delays $g_j(t)$ have the form $g_j(t) = t - h_j$ with positive constants $0 < h_1 < h_2 < \cdots$, we consider the operator function

$$\mathcal{L}(\lambda) = \lambda^2 I + A^2 + \sum_{j=1}^{\infty}(B_j A + \lambda D_j)\exp(-\lambda h_j) + \hat{K}(\lambda)A + \lambda\hat{Q}(\lambda), \tag{8}$$

where $\hat{K}(\lambda)$ and $\hat{Q}(\lambda)$ stand for the Laplace transforms of the operator functions $K(t)$ and $Q(t)$ and I is the identity operator in H. Note that the operator function $\mathcal{L}(\lambda)$ is the symbol (an analog of the characteristic quasipolynomial) of Eq. (1) in the autonomous case.

Suppose that there exists a constant ν_0 such that

$$\sum_{j=1}^{\infty}\exp(-\nu_0 h_j)(\|B_j\| + \|D_j\|) < +\infty. \tag{9}$$

Assertion 1. *Let condition* (9) *be satisfied, and let the operator functions* $K(t)$ *and* $Q(t)$ *satisfy condition* (3). *Then there exists a* $\varkappa^* > 0$ *such that the operator function* $\mathcal{L}(\lambda)$ *satisfies the inequality*

$$\|\lambda\mathcal{L}^{-1}(\lambda)\| + \|A\mathcal{L}^{-1}(\lambda)\| < \frac{\text{const}}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > \varkappa^*. \tag{10}$$

Along with problem (1), (2), consider the functional-differential equation

$$u^{(2)}(t) + A^2u(t) + \sum_{j=1}^{N}[B_jAu(t-h_j) + D_ju^{(1)}(t-h_j)] + \int_{-\infty}^{t} K(t-s)A^2u(s)\,ds + \int_{-\infty}^{t} Q(t-s)u^{(1)}(s)\,ds = f(t), \qquad t > 0, \tag{11}$$

with the initial condition

$$u(t) = \varphi(t), \qquad t \in \mathbb{R}_- = (-\infty, 0]. \tag{12}$$

Theorem 2. *Let the operators* B_j, D_j, $\tilde{B}_j = AB_jA^{-1}$, *and* $\tilde{D}_j = AD_jA^{-1}$, $j = 1, 2, \ldots, N$, *be bounded in the space* H, *let* $K(t)$ *and* $Q(t)$ *be complex-valued functions in the spaces* $W^1_1(\mathbb{R}_+)$ *and* $L_1(\mathbb{R}_+)$, *respectively, and let* f *and* φ *be vector functions such that* $f \in L_{2,\gamma_0}(\mathbb{R}_+, H_1)$ *and* $\varphi \in W^3_{2,\gamma_1}(\mathbb{R}_-, A^3)$ *for some* γ_0, $\gamma_1 \in \mathbb{R}$. *Then there exists a* $\gamma^* > \max(\gamma_0, \gamma_1)$ *such that, for all* $\gamma > \gamma^*$, *problem* (11), (12) *has a unique solution* $u \in W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ *satisfying the estimate*

$$\|u\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \leq d_1(\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|^2_{W^3_{2,\gamma}(\mathbb{R}_-,A^3)})^{1/2} \tag{13}$$

with a constant d_1 *independent of* f *and* φ.

The proof of Theorem 2 can be found in [15].

Note that although problems (1), (2) and (11), (12) are similar, they have differences in statements. In particular, the integral terms are different: only the first power of the operator A is admitted in Eq. (1) under much more general conditions on the operator function $K(t)$, while the second power of the operator A is admitted in Eq. (11) under much more restricted conditions for the kernel $K(t)$.

Equation (11) is closely related to applications, since, for the case in which $B_j \equiv D_j \equiv 0$, $j = 1, 2, \ldots, N$, it is an abstract form of the Gurtin–Pipkin integro-differential equation describing the propagation of heat at a finite velocity in media with memory.

This integro-differential equation was derived by Gurtin and Pipkin in [19].

Presently, such equations are studied by numerous authors (see [20] and the bibliography therein).

Under additional constraints imposed on the coefficients of the equation, one can give a very detailed description of the spectrum of the corresponding operator function $\mathcal{L}(\lambda)$. We make the following additional assumptions: $B_j \equiv D_j \equiv 0$, $j = 1, 2, \ldots, N$, $Q(t) \equiv 0$, and the kernel $K(t)$ can be represented in the form

$$K(t) = -\sum_{j=1}^{\infty} c_j \exp(-\gamma_j t),$$

where $c_j > 0$, $\gamma_j > 0$, $j = 1, 2, \ldots$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n < \cdots$, $\gamma_j \to +\infty$ $(j \to +\infty)$, and

$$\sum_{j=1}^{\infty} c_j < +\infty, \qquad \sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} < 1.$$

Under these assumptions, the spectrum of the operator function $\mathcal{L}(\lambda)$ is the set

$$\sigma(\mathcal{L}) \equiv \overline{\left(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{\lambda_{n_k}\}\right)} \bigcup \left(\bigcup_{n=1}^{\infty} \lambda_n^{\pm}\right),$$

where the λ_{n_k} are real eigenvalues with condensation point $\sigma_k = \lim_{n\to+\infty} \lambda_{n_k}$ and the σ_k $(\sigma_1 > \sigma_2 > \cdots > \sigma_m > \cdots)$ are the solutions of the equation

$$\sum_{k=1}^{\infty} \frac{c_k}{x + \gamma_k} = 1.$$

In turn, the complex eigenvalues $\lambda_n^{\pm} = \mu_n \pm i\nu_n$ with μ_n, $\nu_n \in \mathbb{R}$, satisfy the relations

$$\lim_{n\to\infty} \mu_n = -\frac{1}{2}\sum_{k=1}^{\infty} c_k, \qquad \nu_n = a_n + \bar{o}\left(\frac{1}{a_n^2}\right),$$

where the a_n are the eigenvalues of the operator A: $Ae_n = a_n e_n$, and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of the corresponding eigenvectors. The eigenvalues $\{a_n\}_{n=1}^{\infty}$ are arranged in increasing order $(0 < a_1 \le a_2 \le a_3 \le \cdots)$ counting multiplicities. This result on the structure of the operator function $\mathcal{L}(\lambda)$ was proved by G.R. Kabirova.

It is of interest to note that if

$$\sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} < 1,$$

then the spectrum of the operator function $\mathcal{L}(\lambda)$ lies in the open left half-plane. But if the opposite inequality

$$\sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} > 1$$

holds, then the accumulation point σ_1 of the eigenvalues λ_{n_1} lies in the right half-plane, and in this case, problem (11), (12) is unstable.

It is also reasonable to note that the complex branch of the roots $\{\lambda_n^\pm\}_{n=1}^\infty$ is responsible for the wave character of heat propagation at a finite velocity in media with memory.

2. PROOF OF THE MAIN RESULTS

We prerequisite the proof of the main results with a number of auxiliary results.

Throughout the following, we denote norms of operators acting in the spaces $L_2(\mathbb{R}_+, H)$ and $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ by $|||\cdot|||$ and $|||\cdot|||_W$, respectively.

Lemma 1. *The operators L and M defined by the relations*

$$(Lz)(t) = \int_0^t \exp(-\gamma(t-s))A^{-1}\sin(A(t-s))z(s)\,ds, \qquad t \ge 0,$$

$$(Mz)(t) = \int_0^t \exp(-\gamma(t-s))\cos(A(t-s))z(s)\,ds, \qquad t \ge 0,$$

satisfy the estimates

$$|||AL||| \le \frac{1}{2\gamma}, \qquad |||M||| \le \frac{1}{\gamma}, \qquad \gamma > 0. \tag{14}$$

Note that the proof of Lemma 1 can be found in [13]. We introduce the operators acting by the rule

$$(\mathcal{S}_{g_j}u)(t) = \begin{cases} u(g_j(t)) & \text{for } t \text{ such that } g_j(t) \ge 0 \\ 0 & \text{for } t \text{ such that } g_j(t) < 0, \end{cases}$$

$$(\mathcal{T}^{g_j}u)(t) = \begin{cases} 0 & \text{for } t \text{ such that } g_j(t) \ge 0 \\ u(g_j(t)) & \text{for } t \text{ such that } g_j(t) < 0. \end{cases}$$

Obviously,

$$u(g_j(t)) = (\mathcal{S}_{g_j}u)(t) + (\mathcal{T}^{g_j}u)(t), \qquad t \ge 0.$$

By using the above-introduced operators, we rewrite Eq. (1) as

$$u^{(2)}(t) + A^2u(t) + \sum_{j=1}^{\infty}[B_j(t)S_{g_j}(Au)(t) + D_j(t)S_{g_j}(u^{(1)})(t)] + \int_0^t K(t-s)Au(s)\,ds + \int_0^t Q(t-s)u^{(1)}(s)\,ds = f_1(t), \qquad t > 0, \tag{15}$$

where the right-hand side $f_1(t)$ has the form

$$f_1(t) = f(t) - \sum_{j=1}^{\infty}[B_j(t)T^{g_j}(A\varphi)(t) + D_j(t)T^{g_j}(\varphi^{(1)})(t)] - \int_{-\infty}^0 K(t-s)A\varphi(s)\,ds - \int_{-\infty}^0 Q(t-s)\varphi^{(1)}(s)\,ds = f_1(t), \qquad t > 0, \tag{16}$$

since $u(t) = \varphi(t)$ for $t < 0$.

Our next step is to analyze the well-posed solvability of Eq. (15) with the initial conditions

$$u(+0) = \varphi(-0) = \varphi_0, \qquad u^{(1)}(+0) = \varphi^{(1)}(-0) = \varphi_1. \tag{17}$$

It is important that Eq. (15) is a linear equation with Volterra operators.

We pass from problem (15), (17) to a problem with homogeneous (zero) initial conditions. To this end, we introduce the new unknown function $\omega(t)$ as follows:

$$u(t) = \cos(At)\varphi_0 + A^{-1}\sin(At)\varphi_1 + \omega(t).$$

Since Eq. (15) is linear, for the function $\omega(t)$, we obtain the problem

$$\omega^{(2)}(t) + A^2\omega(t) + \sum_{j=1}^{\infty}[B_j(t)S_{g_j}(A\omega)(t) + D_j(t)S_{g_j}(\omega^{(1)})(t)] + \int_0^t K(t-s)A\omega(s)\,ds + \int_0^t Q(t-s)\omega^{(1)}(s)\,ds = f_2(t), \qquad t > 0, \tag{18}$$

where

$$f_2(t) = f_1(t) - \sum_{j=1}^{\infty}[B_j(t)S_{g_j}(Ap)(t) + D_j(t)S_{g_j}(p^{(1)})(t)] - \int_0^t K(t-s)Ap(s)\,ds + \int_0^t Q(t-s)p^{(1)}(s)\,ds, \tag{19}$$

and

$$p(t) = A^{-1}\sin(At)\varphi_1 + \cos(At)\varphi_0,$$

with the initial conditions

$$\omega(+0) = 0, \qquad \omega^{(1)}(+0) = 0. \tag{20}$$

We seek the solution of problem (18)–(20) in the form $\omega(t) = \exp(\gamma t)v(t)$ with a new unknown function $v(t)$. Then for the function $v(t)$, we obtain the problem

$$v^{(2)}(t) + 2\gamma\frac{dv}{dt}(t) + (A^2 + \gamma^2 I)v(t) + \int_0^t \exp(-\gamma(t-s))K(t-s)Av(s)\,ds + \int_0^t \exp(-\gamma(t-s)) \times Q(t-s)(v^{(1)}(s) + \gamma v(s))\,ds + \sum_{j=1}^{\infty}\exp(-\gamma(t-g_j(t))) \times [B_j(t)S_{g_j}(Av)(t) + D_j(t)S_{g_j}(v^{(1)} + \gamma v)(t)] = \exp(-\gamma t)f_2(t), \qquad t > 0, \tag{21}$$

$$v(+0) = 0, \qquad v^{(1)}(+0) = 0, \tag{22}$$

whose solution will be sought in the form $v = Lz$ with a new unknown function z.

By substituting this function into Eq. (21), we obtain the integro-differential equation

$$z(t) + \int_0^t \exp(-\gamma(t-s))K(t-s)A(Lz)(s)\,ds + \int_0^t \exp(-\gamma(t-s))Q(t-s)(Mz)(s)\,ds + \sum_{j=1}^{\infty}\exp(-\gamma(t-g_j(t)))[B_j(t)S_{g_j}(ALz)(t) + D_j(t)S_{g_j}(Mz)(t)] = \exp(-\gamma t)f_2(t), \qquad t \in \mathbb{R}_+, \tag{23}$$

which is equivalent to problem (21), (22) in the sense of solvability.

Our next aim is to prove the well-posed solvability of Eq. (23) in the space $L_2(\mathbb{R}_+, H_1)$. To this end, we show that norms of the integral operators occurring on the left-hand side in Eq. (23) can also be made small by an appropriate choice of a sufficiently large $\gamma > 0$.

To prove this result, we need the following assertion.

Assertion 2. *Let $g(t)$ be a continuously differentiable real-valued function such that $g(t) \le t$ and $g^{(1)}(t) > 0$, $t \in \mathbb{R}_+$. Let $B(t)$ be an operator function that ranges in the ring of bounded operators in H, is strongly continuous, and satisfies the condition*

$$\sup_{t\in[g^{-1}(0),+\infty]} \left(\|B(t)\|^2 \frac{1}{g^{(1)}(t)} \right) = b_0 < +\infty. \tag{24}$$

Then the operator $(Sv)(t) = B(t)(S_g v)(t)$, where S_g is the internal superposition operator

$$(\mathcal{S}_g v)(t) = \begin{cases} v(g(t)) & \text{for} \quad t \text{ such that } g(t) \ge 0 \\ 0 & \text{for} \quad t \text{ such that } g(t) < 0, \end{cases}$$

is bounded in the space $L_2(\mathbb{R}_+, H)$, and its norm can be estimated as

$$|||S||| \le \sqrt{b_0}. \tag{25}$$

Note that close assertions can be found in [18, Chap. I, pp. 20–28]. From Eq. (23), we pass to the following equation for the new unknown function $y(t) = Az(t)$:

$$\begin{aligned} y(t) &+ \int_0^t \exp(-\gamma(t-s))AK(t-s)A^{-1}(Ly)(s)\,ds \\ &+ \int_0^t \exp(-\gamma(t-s))AQ(t-s)A^{-1}(My)(s)\,ds \\ &+ \sum_{j=1}^{\infty} \exp(-\gamma(t-g_j(t)))[\tilde{B}_j(t)S_{g_j}(ALy)(t) + \tilde{D}_j(t)S_{g_j}(My)(t)] \\ &= \exp(-\gamma t)Af_2(t), \qquad t \in \mathbb{R}_+, \end{aligned} \tag{26}$$

which will be considered in the space $L_2(\mathbb{R}_+, H)$.

By using Assertion 2, Lemma 1, and assumptions (4) and (5), we obtain the estimate

$$\begin{aligned} &\left\| \sum_{j=1}^{\infty} \exp(-\gamma(t-g_j(t)))[\tilde{B}_j(t)S_{g_j}(ALy)(t) + \tilde{D}_j(t)S_{g_j}(My)(t)] \right\|_{L_2(\mathbb{R}_+,H)} \\ &\qquad \le \sum_{j\ge 1} \sup_{t\in\mathbb{R}_+} \exp(-\gamma(t-g_j(t))) \left[\sup_{t\in\mathbb{R}_+} \frac{\|\tilde{B}_j(t)\|}{(g_j^{(1)}(t))^{1/2}} + \sup_{t\in\mathbb{R}_+} \frac{\|\tilde{D}_j(t)\|}{(g_j^{(1)}(t))^{1/2}} \right] \\ &\qquad \times \frac{1}{\gamma} \|y\|_{L_2(\mathbb{R}_+,H)}, \qquad \gamma > \delta > 0. \end{aligned} \tag{27}$$

In turn, for the second and third terms on the left-hand side in Eq. (26), we obtain the estimates

$$\left\| \int_0^t \exp(-\gamma(t-s))\tilde{K}(t-s)(Ly)(s)\,ds \right\|_{L_2(\mathbb{R}_+,H)} \le \sup_{\nu\in\mathbb{R}} \|\tilde{K}(\gamma+i\nu)\| \frac{1}{2\gamma} \|y\|_{L_2(\mathbb{R}_+,H)}, \quad \gamma > \varkappa_1, \tag{28}$$

$$\left\| \int_0^t \exp(-\gamma(t-s))\tilde{Q}(t-s)(My)(s)\,ds \right\|_{L_2(\mathbb{R}_+,H)} \le \sup_{\nu\in\mathbb{R}} \|\tilde{Q}(\gamma+i\nu)\| \frac{1}{\gamma} \|y\|_{L_2(\mathbb{R}_+,H)}, \quad \gamma > \varkappa_1. \tag{29}$$

By taking into account the well-known properties

$$\sup_{\nu\in\mathbb{R}} \|\tilde{K}(\gamma+i\nu)\| \to 0, \qquad \sup_{\nu\in\mathbb{R}} \|\tilde{Q}(\gamma+i\nu)\| \to 0, \quad \gamma\to+\infty, \tag{30}$$

of the Laplace transform of the functions $\tilde{K}(t)$ and $\tilde{Q}(t)$ and inequality (14), we obtain the unique solvability of Eq. (26) in the space $L_2(\mathbb{R}_+, H)$ for a sufficiently large $\gamma \ge 0$. Note that, by virtue of the conditions imposed on the right-hand side $f(t)$ of Eq. (1), the representations (16) and (19), and the conditions imposed on the initial function φ, we conclude that the vector function $\exp(-\gamma t)Af(t)$ belongs to the space $L_2(\mathbb{R}_+, H)$ for $\gamma \ge \gamma^*$, where $\gamma^* > \max(\gamma_0, \gamma_1)$.

Moreover, by a straightforward verification with the use of relations (4)–(6), one can show that

$$\|Af_2\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le \text{const} \times (\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}) \tag{31}$$

with a constant independent of the functions f and φ.

Then, by using inequality (31) and by taking into account the well-posed solvability of Eq. (26), we obtain the estimate

$$\|y\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le c_1(\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}) \tag{32}$$

with a constant c_1 independent of the functions f and φ.

By using the representations $v = Lz$ and $y = Az$ and Lemma 1, we show that, under the above-stipulated assumptions, $v \in W^2_2(\mathbb{R}_+, A^2)$ and

$$\|v\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \le c_2(\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|\varphi\|_{W^2_{2,\gamma}(\mathbb{R}_-,A^2)}) \tag{33}$$

with a constant c_2 independent of the functions f and φ. This estimate readily implies the desired estimate (7). The proof of Theorem 1 is complete.

Proof of Assertion 1. The spectrum of the operator A lies on the semiaxis $\mathbb{R}_+$. By using the well-known estimate for the resolvent via the distance to the spectra of the normal operators $\pm iA$, we obtain the inequalities

$$\|\lambda(\lambda^2 I + A^2)^{-1}\| \le |\lambda| \|(\lambda I + iA)^{-1}\| \, \|(\lambda I - iA)^{-1}\| \le (\operatorname{Re}\lambda)^{-1}, \qquad \operatorname{Re}\lambda > 0. \tag{34}$$

Then from the previous inequality, we have

$$\|A(\lambda^2 I + A^2)^{-1}\| \le \|(\lambda I + iA)^{-1}\| + |\lambda| \, \|(\lambda^2 I + A^2)^{-1}\| \le 2(\operatorname{Re}\lambda)^{-1}, \qquad \operatorname{Re}\lambda > 0. \tag{35}$$

By using (34) and (35), we obtain the inequality

$$\left\| \sum_{j=1}^{\infty} \exp(-\lambda h_j)[B_j A + \lambda D_j](\lambda^2 I + A^2) \right\| \le \text{const} \times \left(\sum_{j=1}^{\infty} \exp(\nu_0 h_j)(\|B_j\| + \|D_j\|)(\operatorname{Re}\lambda)^{-1} \right), \qquad \operatorname{Re}\lambda > \nu_0 > 0. \tag{36}$$

From the well-known properties

$$\|\hat{K}(\lambda)\| \to 0, \qquad \|\hat{Q}(\lambda)\| \to 0 \qquad (\operatorname{Re}\lambda \to +\infty)$$

of the Laplace transform and the estimates (34) and (35), we obtain

$$\|[\hat{K}(\lambda)A + \lambda\hat{Q}(\lambda)](\lambda^2 I + A^2)^{-1}\| \to 0, \qquad \operatorname{Re}\lambda \to +\infty. \tag{37}$$

If we take a sufficiently large $\varkappa_+ > 0$ and use relations (36) and (37), then we obtain the estimate

$$\|\mathcal{L}(\lambda)(\lambda^2 I + A^2)^{-1} - I\| < 1, \qquad \operatorname{Re}\lambda > \varkappa^*. \tag{38}$$

In turn, on the basis of the estimate (38), the representations

$$\mathcal{L}^{-1}(\lambda) = (\lambda^2 I + A^2)^{-1}(I + (\mathcal{L}(\lambda) - (\lambda^2 I + A^2))(\lambda^2 I + A^2)^{-1})^{-1},$$

and the estimates (34) and (35), we obtain the desired inequality (10). The proof of the assertion is complete.

3. REMARKS AND COMMENTS

Our results are characterized by the fact that we consider the case of variable coefficients $B_j(t)A$ and $D_j(t)$ and variable delays $g_j(t)$ (Theorem 1), while the case of constant delays and constant operator coefficients multiplying the retarded terms was considered in [1–9].

The method of proof of Theorem 1 essentially differs from the proof of solvability in [1–9, 21]. But it is mainly similar to the proof of well-posed solvability in [13–16]. Theorem 1 generalizes the corresponding result in [13]. Consider this aspect in more detail. Following [13], we reduce problem (1), (2) to the equivalent (in the sense of the solvability) functional-integral equation (23) in the space $L_2(\mathbb{R}_+, H)$. This functional-integral equation (by analogy with Eq. (29) in [13]) is characterized by bounded operator coefficients, which essentially simplifies its analysis and, in particular, the derivation of estimates for the norms of operators in the space $L_2(\mathbb{R}_+, H)$.

It is reasonable to note that the method of proof of Theorem 2 on the well-posed solvability of problem (11), (12) also essentially differs from the approach used in [20]. The proof of Theorem 2 can be found in [15, 16] and is based on the reduction of problem (11), (12) to an equivalent (in the sense of solvability) functional-integral equation of convolution type in the space $L_2(\mathbb{R}_+, H)$.

In turn, to prove the unique solvability of this equation, we estimate the Laplace transforms of kernels of integral operators in Hardy spaces. The corresponding case in which

$$B_j \equiv D_j \equiv 0, \qquad j = 1, 2, \ldots, N,$$

was presented in [15, 16]. In [20], the Gurtin–Pipkin equation represented in a different form and obtained by the integration of Eq. (11) with respect to the time variable t was considered (for details, see [15]). By using the operator functions $\sin(At)$ and $\cos(At)$, the author reduced the original problem to an integral equation for the unknown function u, which is equivalent to the original problem in the sense of solvability. In addition, unlike the results of the present paper, Pandolfi considered the solvability in function spaces defined on a finite interval $(0, T)$ of the time variable t, while in the present paper, we analyze the solvability in Sobolev weighted spaces $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ on the half-line $\mathbb{R}_+$.

In the proof of Theorem 2, we essentially use the Hilbert structure of the spaces $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ and $L_{2,\gamma}(\mathbb{R}_+, H)$ and the Hardy theorem, while in [20], considerations were performed in Banach spaces of functions on a finite interval $(0, T)$ of the time variable.

As was mentioned above, the results of the present paper generalize the results in [13] and provide a natural development of the results in the papers [10–12] dealing with the analysis of functional-differential equations with unbounded operator coefficients whose leading part is given by an abstract parabolic equation.

In conclusion, note that problems arising in the theory of oscillation propagation in viscoelastic media with memory naturally lead to the investigation of integro-differential equations of the form (11) with

$$B_j \equiv D_j \equiv 0, \qquad j = 1, 2, \ldots, N$$

(see [22] and the bibliography therein).

Problems arising in the theory of oscillation propagation in strongly inhomogeneous media (the Darcy law) also lead to equations of the form (11). The description of such problems can be found in [23] (see also the related bibliography).

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