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ENTIRE SOLUTIONS IN DELAYED LATTICE DIFFERENTIAL EQUATIONS WITH MONOSTABLE NONLINEARITY*

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Abstract. We construct new types of entire solutions for a class of monostable delayed lattice differential equations with global interaction by mixing a heteroclinic orbit of the spatially averaged ordinary differential equations with traveling wave fronts with different speeds. We also establish the uniqueness of entire solutions and the continuous dependence of such an entire solution on parameters, such as wave speeds, for the spatially discrete Fisher-KPP equation.

Key words. entire solution, traveling wave front, heteroclinic orbit, delayed lattice differential equation, monostable nonlinearity

AMS subject classifications. 35B40, 35R10, 37L60, 58D25

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1. Introduction.

We consider the following delayed lattice differential equations:

$$u_n'(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] - du_n(t) + \sum_{i\in\mathbb{Z}} J(i)\, b\left(u_{n-i}(t-\tau)\right), \tag{1.1}$$

where $D > 0$ is a given constant, $\tau \ge 0$, $I(i) = I(-i) \ge 0$, $J(i) = J(-i) \ge 0$, $\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i) = 1$, $\sum_{i\in\mathbb{Z}} J(i) = 1$, $\sum_{i\in\mathbb{Z}\setminus\{0\}} e^{\lambda|i|}I(i) < \infty$, and $\sum_{i\in\mathbb{Z}} e^{\lambda|i|}J(i) < \infty$ for every $\lambda \ge 0$. The birth function $b \in C^2(\mathbb{R})$, and we assume that there exists a constant $K > 0$ such that

$$b(0) = dK - b(K) = 0$$

and that

(H1) for $u \in (0, K)$, there hold $b(u) > du$, $b'(u) \ge 0$, and $b(u) \le b'(0)u$;

(H2) $b'(K) < d < b'(0)$.

A specific function $b(u) = pue^{-\alpha u}$ with $p > 0$ and $\alpha > 0$, which has been widely used in the mathematical biology literature, satisfies the above conditions for a wide range of parameters p and α.

A special case when $I(i) = 0$ for $|i| \ne 1$ and $I(1) = \frac{1}{2}$ is

$$u_n' = \frac{D}{2}\left[u_{n+1} + u_{n-1} - 2u_n\right] - du_n + \sum_{i\in\mathbb{Z}} J(i)\, b\left(u_{n-i}(t-\tau)\right). \tag{1.2}$$

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This system was derived by Weng, Huang, and Wu [40] for the dynamics of growth of a single species population with two age classes distributed over a patchy environment consisting of all integer nodes of a one-dimensional (1-D) lattice. Another special case when $\tau = 0$, $J(0) = 1$, and $J(i) = 0$ for $|i| > 0$ is

$$u_n'(t) = D \sum_{i \in \mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] + f(u_n(t)), \tag{1.3}$$

which was derived by Bates and Chmaj [1] as an l_2-gradient flow for a Helmholtz-free energy functional with general long range interactions. Both lattice systems include, as a special example, the following spatially discrete Fisher-KPP equation:

$$u_n'(t) = \frac{D}{2}\left[u_{n+1} + u_{n-1} - 2u_n\right] + f(u_n(t)). \tag{1.4}$$

It is shown in [25] that (1.1) admits a nondecreasing traveling wave front $\phi_c(n+ct)$ satisfying $\phi_c(-\infty) = 0$ and $\phi_c(+\infty) = K$ for every $c \geq c^* > 0$. Furthermore, $\lim_{\xi\to-\infty} \phi_c(\xi)e^{-\lambda_1(c)\xi} = 1$ and $\lim_{\xi\to-\infty} \phi_c'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$ for $c > c^*$, where c and $\lambda_1(c)$ satisfy

$$\Delta(\lambda, c) = D \sum_{i \in \mathbb{Z}\setminus\{0\}} I(i)\left(e^{-\lambda i} - 1\right) - c\lambda - d + b'(0)\, e^{-\lambda c\tau} \sum_{i\in\mathbb{Z}} J(i)\, e^{-\lambda i} = 0 \tag{1.5}$$

and c^* is determined by $\Delta(\lambda, c) = 0$ and $\frac{\partial}{\partial\lambda}\Delta(\lambda, c) = 0$. More precisely, there exist $c^* > 0$ and $\lambda^* > 0$ such that

(D1) if $0 < c < c^*$ and $\lambda > 0$, then $\Delta(\lambda, c) > 0$;

(D2) if $c = c^*$, then the equation $\Delta(\lambda, c^*) = 0$ has a double real root $\lambda_1(c^*) = \lambda_2(c^*)$ with $0 < \lambda_1(c^*) = \lambda_2(c^*) = \lambda^*$ such that $\Delta(\lambda, c^*) > 0$ for $\lambda \neq \lambda^*$;

(D3) if $c > c^*$, then the equation $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda^* < \lambda_2(c)$ such that $\lambda_1'(c) < 0$, $\lambda_2'(c) > 0$, $\frac{d}{dc}\{c\lambda_1(c)\} < 0$, and

$$\Delta(\lambda, c) = \begin{cases} > 0 & \text{for } \lambda < \lambda_1(c), \\ < 0 & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)), \\ > 0 & \text{for } \lambda > \lambda_2(c). \end{cases}$$

We note that in [25] (see also [1], where bistable waves were considered) there is a further assumption on the kernel I; that is, *the support of I contains either $i = 1$ or two relatively prime integers*, to ensure $\phi_c'(\xi) > 0$ for every $c \geq c^*$. It is interesting to note that, for $c > c^*$, we can confirm $\phi_c'(\xi) > 0$ for any $\xi \in \mathbb{R}$ without this assumption. In fact, for a fixed $c > c^*$, their proof of the existence of nondecreasing traveling wave fronts ϕ_c with $\phi_c(-\infty) = 0$, $\phi_c(+\infty) = K$, and $\lim_{\xi\to-\infty} \phi_c(\xi)e^{-\lambda_1(c)\xi} = 1$ is independent of this assumption; so is the proof of $\lim_{\xi\to-\infty} \phi_c'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$. We now note that $\phi_c'(\xi) \geq 0$ for $\xi \in \mathbb{R}$. Assume that there exists $\xi_0 \in \mathbb{R}$ such that $\phi_c'(\xi_0) = 0$. Then there must be $\phi_c''(\xi_0) = 0$. It is obvious that ϕ_c satisfies

$$\begin{aligned} c\phi_c''(\xi) = D &\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[\phi_c'(\xi - i) - \phi_c'(\xi)\right] - d\phi_c'(\xi) \\ &+ \sum_{i\in\mathbb{Z}} J(i) b'\left(\phi_c\left(\xi - i - c\tau\right)\right) \phi_c'\left(\xi - i - c\tau\right), \end{aligned}$$

which implies that $\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\phi_c'(\xi_0 - i) = 0$. By $\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i) = 1$, there exists a $i_0 \in \mathbb{N}$ such that $I(i_0) > 0$ and $\phi_c'(\xi_0 - i_0) = 0$. Let $\xi_1 = \xi_0 - i_0$. A similar argument yields $\phi_c'(\xi_0 - 2i_0) = \phi_c'(\xi_1 - i_0) = 0$. Continuing this procedure, we have

$\phi_c'(\xi_0 - mi_0) = 0$ for all $m \in \mathbb{N}$, a contradiction to the fact $\lim_{\xi\to-\infty} \phi_c'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$. Therefore, we have $\phi_c'(\xi) > 0$ for $\xi \in \mathbb{R}$. Of course, to prove $\phi_c'(\xi) > 0$ for the case $c = c^*$ the above assumption on the support of I seems necessary. Since in what follows in this paper we use only the traveling wave fronts ϕ_c with $c > c^*$, we shall not require this assumption.

In the remainder of this paper, we always normalize the traveling wave front $\phi_c(n + ct)$ so that $\phi_c(0) = \frac{K}{2}$. Then, for each $c \in (c^*, +\infty)$, we set

$$\alpha_c = \lim_{z\to-\infty} \phi_c(z)e^{-\lambda_1(c)z}. \tag{1.6}$$

Furthermore, we define $A_c > 0$ for each $c \in (c^*, +\infty)$ by

$$A_c = \inf\left\{A > 0 : A \geq \phi_c(z)e^{-\lambda_1(c)z} \quad \text{for any} \quad z \in \mathbb{R}\right\}. \tag{1.7}$$

It is easy to see that $A_c \geq \alpha_c$.

Our focus is on the so-called entire solutions; here an entire solution of (1.1) is a solution defined for all $(n, t) \in \mathbb{Z} \times \mathbb{R}$. In what follows, we say that a sequence of functions $\Phi_p(t) = \{\Phi_{n,p}(t)\}_{n\in\mathbb{Z}}$ converges to a function $\Phi_{p_0}(t) = \{\Phi_{n,p_0}(t)\}_{n\in\mathbb{Z}}$ in $\mathcal{T}$ if, for every compact set $S \subset \mathbb{Z} \times \mathbb{R}$, the functions $\Phi_{n,p}(t)$ and $\frac{d}{dt}\Phi_{n,p}(t)$ converge uniformly in $(n, t) \in S$ to $\Phi_{n,p_0}(t)$ and $\frac{d}{dt}\Phi_{n,p_0}(t)$ as $p \to p_0$.

One of our main results can be stated as follows.

THEOREM 1.1. *Let $\Gamma(t)$ be a heteroclinic orbit of the following functional differential equation:*

$$\frac{d}{dt}u(t) = -du(t) + b(u(t - \tau)),$$

which is increasing and satisfies $\Gamma(-\infty) = 0$, $\Gamma(+\infty) = K$, $\lim_{t\to-\infty} e^{-\lambda_* t}\Gamma(t) = K$, and $\Gamma(t) \leq Ke^{\lambda_* t}$ for all $t \in \mathbb{R}$, where $\lambda_* > 0$ is the unique real root of the equation $\lambda + d - b'(0)e^{-\lambda\tau} = 0$. Then for every $c_1, \ldots, c_m, c_1', \ldots, c_l' > c^*$, $\theta_0, \theta_1, \ldots, \theta_m, \theta_1', \ldots, \theta_l' \in \mathbb{R}$, and $\chi \in \{0, 1\}$, there exists an entire solution $\Phi(t) = \{\Phi_n(t)\}_{n\in\mathbb{Z}}$ of (1.1) such that

$$\begin{aligned} &\max\left\{\max_{1\leq i\leq m} \phi_{c_i}(n + c_i t + \theta_i), \max_{1\leq j\leq l} \phi_{c_j'}\left(-n + c_j' t + \theta_j'\right), \chi\Gamma(t + \theta_0)\right\} \\ &\quad \leq \Phi_n(t) \leq \min\left\{\vartheta_m^+(n, t), \vartheta_l^-(n, t), \vartheta^0(n, t)\right\} \end{aligned} \tag{1.8}$$

on $(n, t) \in \mathbb{Z} \times \mathbb{R}$, where

$$\begin{aligned} \vartheta_m^+(n, t) = \min_{1\leq i\leq m} \Bigg\{ &\phi_{c_i}(n + c_i t + \theta_i) + \chi K e^{\lambda_*(t+\theta_0)} \\ &+ \sum_{1\leq j\leq m, j\neq i} A_{c_j} e^{\lambda_1(c_j)(n+c_j t+\theta_j)} + \sum_{1\leq j\leq l} A_{c_j'} e^{\lambda_1(c_j')(-n+c_j' t+\theta_j')} \Bigg\}, \end{aligned}$$

$$\begin{aligned} \vartheta_l^-(n, t) = \min_{1\leq i\leq l} \Bigg\{ &\phi_{c_i'}(-n + c_i' t + \theta_i') + \chi K e^{\lambda_*(t+\theta_0)} \\ &+ \sum_{1\leq j\leq m} A_{c_j} e^{\lambda_1(c_j)(n+c_j t+\theta_j)} + \sum_{1\leq j\leq l, j\neq i} A_{c_j'} e^{\lambda_1(c_j')(-n+c_j' t+\theta_j')} \Bigg\}, \end{aligned}$$

$$\vartheta^0(n, t) = \chi\Gamma(t + \theta_0) + \sum_{1\leq j\leq m} A_{c_j} e^{\lambda_1(c_j)(n+c_j t+\theta_j)} + \sum_{1\leq j\leq l} A_{c_j'} e^{\lambda_1(c_j')(-n+c_j' t+\theta_j')},$$

and $m, l \in \mathbb{N} \cup \{0\}$ with $\chi + m + l \geq 2$. Moreover, the following statements hold:

(i) *For any* $n \in \mathbb{Z}$, $\Phi_n'(t) > 0$ *for* $t \in \mathbb{R}$.

(ii) $\lim_{t\to\infty} \sup_{n\in\mathbb{Z}} |\Phi_n(t) - K| = 0$ *and* $\lim_{t\to-\infty} \sup_{|n|\le N_0} |\Phi_n(t)| = 0$ *for every given* $N_0 \in \mathbb{N}$.

(iii) *If* $m \ge 1$*, then* $\lim_{n\to\infty} \|\Phi_n(\cdot) - K\|_{L^\infty[a,+\infty)} = 0$ *for every* $a \in \mathbb{R}$*; if* $l \ge 1$*, then* $\lim_{n\to-\infty} \|\Phi_n(\cdot) - K\|_{L^\infty[a,+\infty)} = 0$ *for every* $a \in \mathbb{R}$.

(iv) *If* $\chi = 1$ *and* $m = 0\,(l = 0$*, respectively), then* $\Phi_n(t)$ *converges uniformly on* $t \in [a,b]$ *to* $\Gamma(t+\theta_0)$ *as* $n \to +\infty\,(n \to -\infty$*, respectively) for any* $a, b \in \mathbb{R}$ *with* $a < b$.

(v) *If* $\chi = 1$*, then* $\Phi_n(t) \sim Ke^{\lambda_*(t+\theta_0)}$ *as* $t \to -\infty$ *for every* $n \in \mathbb{Z}$.

(vi) *If* $\chi = 0$*, then, for every* $n \in \mathbb{Z}$*, there exist* $B_2(n) > B_1(n) > 0$ *such that*

$$B_1(n)e^{c_{\max}\lambda_1(c_{\max})t} < \Phi_n(t) < B_2(n)e^{c_{\max}\lambda_1(c_{\max})t} \quad \text{for every } t \ll -1,$$

where $c_{\max} = \max\{\max_{1\le i\le m} c_i, \max_{1\le j\le l} c_j'\}$.

(vii) *If we denote* $\Phi(t)$ *by* $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$ *when* $\chi = 1$ *and denote* $\Phi(t)$ *by* $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l'}(t)$ *when* $\chi = 0$*, then*

$$\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$$

converges to $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l'}(t)$ *as* $\theta_0 \to -\infty$ *in* $\mathcal{T}$ *and uniformly on* $(n,t) \in \mathbb{Z} \times (-\infty, a]$ *for every* $a \in \mathbb{R}$*;* $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$ *converges to* K *as* $\theta_0 \to +\infty$ *in* $\mathcal{T}$ *and uniformly on* $(n,t) \in \mathbb{Z} \times [a, +\infty)$ *for every* $a \in \mathbb{R}$.

(viii) $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$ *converges to*

$$\Phi_{c_1,\dots,c_{i-1},c_{i+1},\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_{i-1},\theta_{i+1},\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$$

as $\theta_i \to -\infty$ *in* $\mathcal{T}$ *and uniformly on* $(n,t) \in \{n : n \le N_0, n \in \mathbb{Z}\} \times (-\infty, a]$ *for every* $N_0 \in \mathbb{Z}$ *and* $a \in \mathbb{R}$. $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$ *converges to*

$$\Phi_{c_1,\dots,c_m;c_1',\dots,c_{j-1}',c_{j+1}',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_{j-1}',\theta_{j+1}',\dots,\theta_l';\theta_0}(t)$$

as $\theta_j' \to -\infty$ *in* $\mathcal{T}$ *and uniformly on* $(n,t) \in \{n : n \ge N_0, n \in \mathbb{Z}\} \times (-\infty, a]$ *for every* $N_0 \in \mathbb{Z}$ *and* $a \in \mathbb{R}$*. Similar results hold for* $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l'}(t)$.

(ix) $\Phi(t)$ *converges to* K *as* $\theta_i \to +\infty$ *in* $\mathcal{T}$ *and uniformly on* $(n,t) \in \{n : n \ge N_0, n \in \mathbb{Z}\} \times [a, +\infty)$ *for every* $N_0 \in \mathbb{Z}$ *and* $a \in \mathbb{R}$*;* $\Phi(t)$ *converges to* K *as* $\theta_j' \to +\infty$ *in* $\mathcal{T}$ *and uniformly on* $(n,t) \in \{n : n \le N_0, n \in \mathbb{Z}\} \times [a, +\infty)$ *for every* $N_0 \in \mathbb{Z}$ *and* $a \in \mathbb{R}$.

From (iv) and (v) of Theorem 1.1 and the fact $\lambda_* < c_{\max}\lambda_1(c_{\max})$, it follows that $\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$ are completely different from

$$\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l'}(t).$$

Theorem 1.1 applies to the spatially discrete Fisher-KPP equation (1.4), where $f \in C^2$ satisfies $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, $f(u) > 0$, and $f(u) \le f'(0)u$ for $u \in (0,1)$. In this case, $K = 1$, $d = \max_{u\in[0,1]} |f'(u)|$, and $b(u) = du + f(u)$. The existence, uniqueness, and stability of traveling wave fronts of (1.4) were studied in Chen, Fu, and Guo [3], Chen and Guo [4, 5], and Zinner [44]; the entire solutions of (1.4) were studied by Guo and Morita [18] and Guo [19]. However, there seem to be no results on the uniqueness of entire solutions of (1.4) and the continuous dependence on parameters $c_1, \dots, c_m, c_1', \dots, c_l', \theta_0, \theta_1, \dots, \theta_m, \theta_1', \dots, c_l'$ given by Theorem 1.1. The following theorem is devoted to this topic and is a spatially discrete version of results of Hamel and Nadirashvili [20], where the reaction-diffusion Fisher-KPP equation was considered.

THEOREM 1.2. *For any $c, c' > c^*$, $\theta_0, \theta, \theta' \in \mathbb{R}$, and $\varrho, \varrho', \chi \in \{0,1\}$ with $\varrho + \varrho' + \chi \geq 2$, there exists a unique entire solution $\Phi(t) = \{\Phi_n(t)\}_{n\in\mathbb{Z}}$ of* (1.4) *such that* (i)–(ix) *of Theorem* 1.1 *hold and*

$$\begin{aligned}(1.9)\qquad & \max\left\{\varrho\phi_c\left(n+ct+\theta\right), \varrho'\phi_{c'}\left(-n+c't+\theta'\right), \chi\Gamma\left(t+\theta_0\right)\right\}\\ & \leq \Phi_n(t)\\ & \leq \min\Big\{\varrho\phi_c\left(n+ct+\theta\right)+\chi e^{f'(0)(t+\theta_0)}+\varrho' A_{c'}e^{\lambda_1(c')(-n+c't+\theta')},\\ &\qquad \varrho'\phi_{c'}\left(-n+c't+\theta'\right)+\chi e^{f'(0)(t+\theta_0)}+\varrho A_c e^{\lambda_1(c)(n+ct+\theta)},\\ &\qquad \chi\Gamma\left(t+\theta_0\right)+\varrho A_c e^{\lambda_1(c)(n+ct+\theta)}+\varrho' A_{c'}e^{\lambda_1(c')(-n+c't+\theta')}\Big\}\end{aligned}$$

on $(n,t) \in \mathbb{Z}\times\mathbb{R}$. In particular, when $\varrho = \varrho' = \chi = 1$, the entire solutions $\Phi = \Phi_{c,c',\theta,\theta',\theta_0}$ depend continuously on $(c, c', \theta, \theta', \theta_0) \in (c^*, +\infty)^2 \times \mathbb{R}^3$ in $\mathcal{T}$; when $\varrho = \varrho' = 1$ and $\chi = 0$, the entire solutions $\Phi = \Phi_{c,c',\theta,\theta'}$ depend continuously on $(c, c', \theta, \theta') \in (c^*, +\infty)^2 \times \mathbb{R}^2$ in $\mathcal{T}$; when $\varrho = \chi = 1$ and $\varrho' = 0$, the entire solutions $\Phi = \Phi_{c,\theta,\theta_0}$ depend continuously on $(c, \theta, \theta_0) \in (c^*, +\infty) \times \mathbb{R}^2$ in $\mathcal{T}$; when $\varrho' = \chi = 1$ and $\varrho = 0$, the entire solutions $\Phi = \Phi_{c',\theta',\theta_0}$ depend continuously on $(c', \theta', \theta_0) \in (c^*, +\infty) \times \mathbb{R}^2$ in $\mathcal{T}$.

We note that when $\varrho' = \chi = 0$ and $\varrho = 1$, $\Phi_n(t) = \Phi_{n;c,\theta}(t) = \phi(n + ct + \theta)$ for $(n,t) \in \mathbb{Z}\times\mathbb{R}$; when $\varrho = \chi = 0$ and $\varrho' = 1$, $\Phi_n(t) = \Phi_{n;c',\theta'}(t) = \phi(-n + c't + \theta')$ for $(n,t) \in \mathbb{Z}\times\mathbb{R}$; and when $\varrho = \varrho' = 0$ and $\chi = 1$, $\Phi_n(t) = \Phi_{n;\theta_0}(t) = \Gamma(t + \theta_0)$ for $(n,t) \in \mathbb{Z}\times\mathbb{R}$. Therefore, similarly to the discussions in Hamel and Nadirashvili [20], it follows from Theorems 1.1 and 1.2 that the functions $\Phi_{c,c',\theta,\theta',\theta_0}(t)$ ($\Phi_{c,c',\theta,\theta'}(t)$, $\Phi_{c,\theta,\theta_0}(t)$, $\Phi_{c',\theta',\theta_0}(t)$, respectively) established by Theorem 1.2 are the 5-D (4-D, 3-D, and 3-D, respectively) manifold of entire solutions of (1.4). In addition, (1.4) possesses two 2-D manifolds of entire solutions of traveling wave type, namely, $\Phi^+_{c,\theta}(t) = \{\phi_c(n+ct+\theta)\}_{n\in\mathbb{Z}}$ and $\Phi^-_{c',\theta'}(t) = \{\phi_{c'}(-n + c't + \theta')\}_{n\in\mathbb{Z}}$, and a 1-D manifold of spatially homogeneous entire solutions, namely, $\Gamma(t + \theta_0)$. Let $\mathcal{M}_5$ ($\mathcal{M}_4$, $\mathcal{M}^+_3$, $\mathcal{M}^-_3$, $\mathcal{M}^+_2$, $\mathcal{M}^-_2$, and $\mathcal{M}_1$, respectively) be the above 5-D (4-D, 3-D, 3-D, 2-D, 2-D, and 1-D, respectively) manifold of entire solutions. Then, from Theorems 1.1 and 1.2, it follows that $\mathcal{M}_4$ is on the boundary of $\mathcal{M}_5$ (via taking the limit $\theta_0 \to -\infty$) and $\mathcal{M}^+_3$ (or $\mathcal{M}^-_3$) is on the boundary of $\mathcal{M}_5$ (via taking the limit $\theta \to -\infty$) (or $\theta' \to -\infty$). $\mathcal{M}^+_2$ (or $\mathcal{M}^-_2$) is on the boundary of $\mathcal{M}_4$ (via taking the limit $\theta' \to -\infty$) (or $\theta \to -\infty$) and is also on the boundary of $\mathcal{M}^+_3$ (or $\mathcal{M}^-_3$) (via taking the limit $\theta_0 \to -\infty$). $\mathcal{M}_1$ is on the boundary of $\mathcal{M}^+_3$ (or $\mathcal{M}^-_3$) (via taking the limit $\theta \to -\infty$) (or $\theta' \to -\infty$). In particular, $\mathcal{M}^+_2$ (or $\mathcal{M}^-_2$) is on the boundary of $\mathcal{M}_5$ (via taking the limits $\theta' \to -\infty$ and $\theta_0 \to -\infty$) (or $\theta \to -\infty$ and $\theta_0 \to -\infty$), and $\mathcal{M}_1$ is on the boundary of $\mathcal{M}_5$ (via taking the limits $\theta \to -\infty$ and $\theta' \to -\infty$). We can also easily show that the functions $\Phi_{c,c',\theta,\theta',\theta_0}$ converge to $\Phi^+_{c,\theta}$ as $\theta' \to -\infty$ and $\theta_0 \to -\infty$ in $\mathcal{T}$ and to $\Phi^-_{c',\theta'}$ as $\theta \to -\infty$ and $\theta_0 \to -\infty$ in $\mathcal{T}$, and that $\Phi_{c,c',\theta,\theta',\theta_0}$ converge to Φ_{θ_0} as $\theta \to -\infty$ and $\theta' \to -\infty$ in $\mathcal{T}$.

Contrasting to [18, 20], we require only $f(u) \leq f'(0)u$ for any $u \in (0,1)$ other than $f'(u) \leq f'(0)$. We also note some differences on the uniqueness of entire solutions up to a spatial-temporal translation between a reaction-diffusion equation and its spatially discrete analogue (see a similar remark for the bistable nonlinearity reported by Wang, Li, and Ruan [39]). Namely, consider the reaction-diffusion KPP equation

$$(1.10)\qquad \frac{d}{dt}u(x,t) = D\Delta u(x,t) + f(u),$$

for which the existence of entire solutions was established by Hamel and Nadirashvili ([20, Theorems 1.1, 1.3, and 1.4 and Corollary 1.5]). For comparison, we consider only the entire solutions established by [20, Theorem 1.3], corresponding to the case $\chi = 0$ and $\varrho = \varrho' = 1$ in our Theorem 1.2. The entire solution $v_{c,c',h,h'}(x,t)$ of (1.10) established by [20, Theorem 1.3] and satisfying (1.4) of [20] is unique for each given $(c, c', h, h') \in (c^*, +\infty)^2 \times \mathbb{R}^2$. Consequently, it is easy to see that for any $(\bar{h}, \bar{h}') \neq (h, h')$,

$$v_{c,c',\bar{h},\bar{h}'}(x,t) = v_{c,c',\bar{h},\bar{h}'}(x + x_0, t + t_0) \quad \text{for} \quad (x,t) \in \mathbb{R}^2,$$

where

$$x_0 = \frac{c\left(\bar{h}' - h'\right) - c'\left(\bar{h} - h\right)}{c + c'}, \quad t_0 = \frac{\left(\bar{h}' - h'\right) + \left(\bar{h} - h\right)}{c + c'}.$$

But for (1.4) if $(\bar{\theta}, \bar{\theta}') \neq (\theta, \theta')$, then $\Phi_{n;c,c',\bar{\theta},\bar{\theta}'}(t) = \Phi_{n+n_0;c,c',\theta,\theta'}(t + t_0)$ for any $(n,t) \in \mathbb{Z} \times \mathbb{R}$ if and only if

$$\frac{c\left(\bar{h}' - h'\right) - c'\left(\bar{h} - h\right)}{c + c'} \in \mathbb{Z},$$

and, hence, $n_0 = \frac{c(\bar{h}' - h') - c'(\bar{h} - h)}{c + c'}$, $t_0 = \frac{(\bar{h}' - h') + (\bar{h} - h)}{c + c'}$. When $(\bar{c}, \bar{c}') \neq (c, c')$, as proved by [24, Theorem 1.1], there exists no $(x_0, t_0) \in \mathbb{R}^2$ such that $v_{\bar{c},\bar{c}',h,h'}(\cdot,\cdot) = v_{c,c',h,h'}(\cdot + x_0, \cdot + t_0)$ on $\mathbb{R}^2$ for (1.10). Similarly, for (1.4), there exists no $(n_0, t_0) \in \mathbb{Z} \times \mathbb{R}$ such that $\Phi_{n;\bar{c},\bar{c}',h,h'}(t) = v_{n+n_0;c,c',h,h'}(t)$ for any $(n,t) \in \mathbb{Z} \times \mathbb{R}$.

There have been extensive studies about the dynamics of lattice delay systems (1.1), as reported in a recent survey by Gourley and Wu [16]. In particular, the asymptotic speed of propagation and the existence of monotone traveling waves were studied in [25, 40]. The existence, uniqueness, and stability of traveling wave solutions of (1.1) and (1.2) with monostable and bistable nonlinearities have henceforth been studied; see Ma and Zou [26] for the bistable case and Ma and coworkers [25, 27] for the monostable case. Also, Gourley and Wu [17] proved for (1.2) that if the birth rate is so small that a patch alone cannot sustain a positive equilibrium, then the whole population in the patchy environment will become extinct; and if the birth rate is large enough that each patch can sustain a positive equilibrium and if the maturation time is moderate, then the model exhibits nonlinear oscillations characterized by the occurrence of multiple periodic traveling waves. A stage-structured model for a single species on a finite 1-D spatial lattice was also studied in [22]. Related results on traveling waves of lattice differential equations (without delay) can be found in Cahn, Chow, and Van Vleck [2], Chen, Fu, and Guo [3], Chen and Guo [4, 5], Chow [10], Mallet-Paret [28], Wu and Zou [42], and references therein. We note that some progress has been made as well for 2-D lattice delay differential equations; see, for example, Cheng, Li, and Wang [8, 9], Shi, Li, and Cheng, [32], and Weng et al. [41]. In addition, Wang, Li, and Ruan [35, 36, 37] studied traveling wave solutions of reaction-diffusion equations with spatial-temporal delay.

The aforementioned studies also suggest that these wave solutions $\phi_c(n + ct)$ are defined for all $t \in \mathbb{R}$. They often determine the long time behavior of the solutions of Cauchy-type problems and constitute an important part of global attractors, which consist of *entire solutions*. However, the global attractors can be quite complicated, and recent studies for reaction-diffusion equations with continuous spatial variables have showed the existence of many new types of entire solutions arising from the simple traveling wave fronts, and these entire solutions combined provide essential information about the global attractors; see Chen and Guo [6], Chen, Guo, and Nimomiya [7], Fukao, Morita, and Nimomiya [14], Guo and Morita [18], Hamel and Nadirashvili [20, 21], and Yagisita [43]. For the Fisher-KPP nonlinearity and bistable nonlinearity, these entire solutions behave as two (opposite) wave fronts of positive speed(s) approaching each other from both sides of the x-axis and then annihilate in a finite time. Similar results hold true for nonlocal reaction-diffusion equations with delayed monostable and bistable nonlinearities ([24, 38]). Morita and Ninomiya [30] and Guo [19] have constructed other types of entire solutions for reaction-diffusion equations and discrete diffusive equations with bistable nonlinearity, respectively, which are different from those obtained in [6, 7, 14, 18, 20, 21, 24, 38, 43]. In particular, Li, Liu, and Wang [23] established the existence of entire solutions for reaction-advection-diffusion equations in cylinders, where the ignition temperature nonlinearity has been studied. As reported in [30], entire solutions play also very important roles in some other areas, for example, transient dynamics, distinct history of two solutions, etc.

The remainder of this paper is organized as follows: In section 2, we show how systems (1.1) arise from some areas, such as population biology. In section 3, we establish some existence and comparison results, which are needed in what follows. In section 4, we show the existence of heteroclinic orbit $\Gamma(t)$ connecting two equilibria 0 and K. Section 5 is devoted to Theorem 1.1, and then Theorem 1.2 is proved in section 6.

2. Important particular cases. In this section, we derive from a structured population model a particular case of systems along with an explicit formula to calculate $J(i)$.

Consider a single species population with age structures distributed over a patchy environment consisting of all integer nodes of a 1-D lattice. Let $w_n(t)$ be the density of juvenile individuals in the nth patch and at time t, $v_n(t,a)$ be the density of individuals with age a in the nth patch and at time t, and $\tau > 0$ the length of a juvenile period. Then

$$w_n(t) = \int_0^\tau v_n(t,a)\,da.$$

Let $u_n(t)$ be the density of mature individuals in the nth patch and at time t. Assume that the spatial dispersal of juvenile individuals and mature individuals is isotropic and can be long range (see Murray [31]). Assume that the diffusion rate of juvenile individuals with age a is $\overline{D}(a) \geq 0$ and the diffusion rate of mature individuals is a constant $D > 0$. Let $K(n-i)$ and $I(n-i)$ be the probability distributions of juvenile individuals and mature individuals traveling from the ith patch to the nth patch, respectively. Then we have

$$K(i) \geq 0,\ I(i) \geq 0,\ K(i) = K(-i),\ I(i) = I(-i),\ \sum_{i\in\mathbb{Z}\setminus\{0\}} K(i) = 1,\ \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i) = 1.$$

Since only the mature population can reproduce, we have

$$\text{(2.1)}\quad \begin{cases} \frac{\partial}{\partial t}v_n(t,a) + \frac{\partial}{\partial a}v_n(t,a) = \overline{D}(a)\sum_{i\in\mathbb{Z}\setminus\{0\}} K(n-i)\left[v_i(t,a) - v_n(t,a)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad -\mu(a)\,v_n(t,a), \quad 0 < a < \tau, \\ v_n(t,0) = \widehat{b}(u_n(t)), \\ \frac{d}{dt}u_n(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(n-i)\left[u_i(t) - u_n(t)\right] - \widehat{d}(u_n(t)) + v_n(t,\tau), \end{cases}$$

where $\mu(a)$ denotes the death rate of the juvenile individuals with age $a \in (0,\tau)$, $\widehat{b}:\mathbb{R}_+ \to \mathbb{R}_+$ is the birth function, and $\widehat{d}:\mathbb{R}_+ \to \mathbb{R}_+$ is the death function of mature individuals.

For fixed $s \geq -\tau$, let $V_n^s(t) = v_n(t, t-s)$ for $s \leq t \leq s+\tau$. Then $V_n^s(s) = v_n(s,0) = \widehat{b}(u_n(s))$. From (2.1),

$$\begin{aligned}\frac{d}{dt}V_n^s(t) &= \left.\frac{\partial}{\partial t}v_n(t,a)\right|_{a=t-s} + \left.\frac{\partial}{\partial a}v_n(t,a)\right|_{a=t-s} \\ &= \overline{D}(t-s)\sum_{i\in\mathbb{Z}\setminus\{0\}} K(i)\left[V_{n-i}^s(t) - V_n^s(t)\right] - \mu(t-s)V_n^s(t).\end{aligned} \tag{2.2}$$

Note that the grid function $V_n^s(t)$ can be viewed as the discrete spectral of a periodic function $v^s(t,\omega)$ by discrete Fourier transform [15, 34]:

$$v^s(t,\omega) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{-i(n\omega)}V_n^s(t), \tag{2.3}$$

$$V_n^s(t) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{i(n\omega)}v^s(t,\omega)\,d\omega, \tag{2.4}$$

where i is the imaginary unit. Applying (2.2) and (2.3) yields

$$\begin{aligned}\frac{\partial}{\partial t}v^s(t,\omega) &= \left[\overline{D}(t-s)\sum_{k\in\mathbb{Z}\setminus\{0\}} K(k)\left(e^{-ik\omega}-1\right) - \mu(t-s)\right]v^s(t,\omega) \\ &= \left[-2\overline{D}(t-s)\sum_{k\in\mathbb{Z}\setminus\{0\}} K(k)\sin^2\left(\frac{k\omega}{2}\right) - \mu(t-s)\right]v^s(t,\omega).\end{aligned}$$

Solving the equation, we get

$$\begin{aligned}v^s(t,\omega) = \exp&\left\{-2\sum_{k\in\mathbb{Z}\setminus\{0\}} K(k)\sin^2\left(\frac{k\omega}{2}\right)\int_s^t \overline{D}(z-s)\,dz - \int_s^t \mu(z-s)\,dz\right\} \\ &\times v^s(s,\omega).\end{aligned}$$

By the inverse discrete Fourier transform (2.4), we obtain

$$\begin{aligned}V_n^s(t) &= \frac{1}{\sqrt{2\pi}}e^{-\int_s^t \mu(z-s)dz} \\ &\quad\times \int_{-\pi}^{\pi} e^{i(n\omega)}\exp\left\{-2\alpha_s\sum_{k\in\mathbb{Z}\setminus\{0\}} K(k)\sin^2\left(\frac{k\omega}{2}\right)\right\}v^s(s,\omega)\,d\omega,\end{aligned}$$

where $\alpha_{t-s} = \int_s^t \overline{D}(z-s)dz = \int_0^{t-s}\overline{D}(z)dz$. Noting that $V_n^s(s) = v_n(s,0) = \widehat{b}(u_n(s))$, by (2.3) we have

$$v_s(s,\omega) = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} e^{-i(j\omega)}V_j^s(t) = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} e^{-i(j\omega)}\widehat{b}(u_j(s)).$$

Hence,

$$V_n^s(t) = \frac{1}{2\pi} e^{-\int_s^t \mu(z-s)dz} \sum_{j\in\mathbb{Z}} \widehat{b}(u_j(s)) \times \int_{-\pi}^{\pi} e^{i((n-j)\omega)} \exp\left\{-2\alpha_s \sum_{k\in\mathbb{Z}\setminus\{0\}} K(k)\sin^2\left(\frac{k\omega}{2}\right)\right\} d\omega. \tag{2.5}$$

Let $t = s+\tau$, $\widehat{\mu} = e^{-\int_s^t \mu(z-s)dz} = e^{-\int_0^\tau \mu(z)dz}$, and $\alpha = \int_0^\tau \overline{D}(z)dz$. Then (2.5) yields

$$v_n(t,\tau) = \frac{\widehat{\mu}}{2\pi}\sum_{j\in\mathbb{Z}} \beta_\alpha(n-j)\widehat{b}(u_j(s)) = \frac{\widehat{\mu}}{2\pi}\sum_{j\in\mathbb{Z}} \beta_\alpha(n-j)\widehat{b}(u_j(t-\tau)),$$

where

$$\beta_\alpha(j) = \int_{-\pi}^{\pi} e^{i(j\omega)} \exp\left\{-2\alpha \sum_{k\in\mathbb{Z}\setminus\{0\}} K(k)\sin^2\left(\frac{k\omega}{2}\right)\right\} d\omega.$$

Thus, the last equality of (2.1) becomes

$$\frac{d}{dt}u_n(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] - \widehat{d}(u_n(t)) + \frac{\widehat{\mu}}{2\pi}\sum_{j\in\mathbb{Z}} \beta_\alpha(n-j)\widehat{b}(u_j(t-\tau)), \quad t>0. \tag{2.6}$$

Let $\widehat{d}(u) = du$, $b(u) = \widehat{\mu}\widehat{b}(u)$, and $J(i) = \frac{1}{2\pi}\beta_\alpha(i)$; then (2.6) reduces to (1.1).

In particular, the case when $I(i) = K(i) = 0$ for $|i| \neq 1$ and $I(1) = K(1) = \frac{1}{2}$ was studied by Weng, Huang, and Wu [40]. In this case, (2.6) reduces to (1.2). When $\overline{D}(a) \equiv 0$, we have $\alpha = 0$, and it follows that (2.6) reduces to

$$\frac{d}{dt}u_n(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] - \widehat{d}(u_n(t)) + \widehat{\mu}\widehat{b}(u_n(t-\tau)), \quad t>0.$$

When $\tau = 0$, $\alpha = 0$, and $\widehat{\mu} = 1$, we have

$$\frac{d}{dt}u_n(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] - \widehat{d}(u_n(t)) + \widehat{b}(u_j(t)), \quad t>0,$$

which coincides with (1.3).

When the diffusion rate $\overline{D}(a)$ and death rate $\mu(a)$ of the juvenile individuals are independent of age a, namely, $D_0 \equiv \overline{D}(a)$ and $\gamma \equiv \mu(a)$ for $a \in [0,\tau]$, we have

$$\frac{d}{dt}w_n(t) = D_0\sum_{i\in\mathbb{Z}\setminus\{0\}} K(i)\left[w_{n-i}(t) - w_n(t)\right] - \gamma u_n(t) + \widehat{b}(u_n(t)) - \frac{e^{-\gamma\tau}}{2\pi}\sum_{j\in\mathbb{Z}} \beta_\alpha(n-j)\widehat{b}(u_j(t-\tau)), \quad t>0. \tag{2.7}$$

We note that it is easy to prove that $\sum_{j\in\mathbb{Z}}\frac{1}{2\pi}\beta_\alpha(j) = 1$. It seems difficult to prove $\beta_\alpha(j) \geq 0$ for general kernel $\sum_{i\in\mathbb{Z}\setminus\{0\}} K(i) = 1$ though it was proved by Weng, Huang, and Wu [40] for the case when $K(i) = 0$ for $|i| \neq 1$ and $K(\pm 1) = \frac{1}{2}$. Nevertheless, in the remainder of this paper, we consider (1.1) for general kernel functions $I(i)$ and $J(i)$ satisfying the assumptions in section 1.

3. Preliminaries. Consider the initial value problem

$$\begin{cases} u_n'(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] - du_n(t) + \sum_{i\in\mathbb{Z}} J(i)\, b\left(u_{n-i}(t-\tau)\right), \\ u_n(s) = \varphi_n(s), \end{cases} \tag{3.1}$$

where $n \in \mathbb{Z}$, $t > 0$, and $s \in [-\tau, 0]$.

DEFINITION 3.1. *A sequence of continuous differentiable functions* $\{v_n(t)\}_{n\in\mathbb{Z}}$, $t \in [-\tau, l)$, $l > 0$, *is called a supersolution (subsolution) of* (3.1) *on* $[0, l)$ *if*

$$v_n'(t) \geq (\leq)\, D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[v_{n-i}(t) - v_n(t)\right] - dv_n(t) + \sum_{i\in\mathbb{Z}} J(i)\, b\left(v_{n-i}(t-\tau)\right) \tag{3.2}$$

for all $t \in [0, l)$.

LEMMA 3.2. *For any* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ *with* $\varphi_n \in C([-\tau, 0], [0, K])$, (3.1) *admits a unique solution* $u(t;\varphi) = \{u_n(t;\varphi)\}_{n\in\mathbb{Z}}$ *on* $[0, +\infty)$ *satisfying* $u_n(s) = \varphi_n(s)$ *and* $0 \leq u_n(t) \leq K$ *for* $s \in [-\tau, 0]$, $t \in [-\tau, +\infty)$, *and* $n \in \mathbb{Z}$. *For any pair of supersolution* $w_n^+(t)$ *and subsolution* $w_n^-(t)$ *of* (3.1) *on* $[0, +\infty)$ *with* $0 \leq w_n^-(t) \leq K$, $0 \leq w_n^+(t) \leq K$ *for* $t \in [-\tau, +\infty)$, $n \in \mathbb{Z}$, *and* $w_n^+(s) \geq w_n^-(s)$ *for* $s \in [-\tau, 0]$, $n \in \mathbb{Z}$, *there holds* $w_n^+(t) \geq w_n^-(t)$ *for* $t \geq 0$, $n \in \mathbb{Z}$.

Note that (3.1) is equivalent to

$$\begin{cases} u_n(t) = \varphi_n(0)\, e^{-(D+d)t} + \int_0^t e^{(D+d)(s-t)} H_n[u](s)\, ds, \quad t > 0, \\ u_n(t) = \varphi_n(t), \quad t \in [-\tau, 0], \end{cases}$$

where $H_n[u](t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)u_{n-i}(t) + \sum_{i\in\mathbb{Z}} J(i)b(u_{n-i}(t-\tau))$. Lemma 3.2 can be proved using an argument used in [26, Lemma 4.1].

Consider also the following linear initial value problem:

$$\begin{cases} u_n'(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] - du_n(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\, u_{n-i}(t-\tau), \\ u_n(s) = \varphi_n(s) \in C\left([-\tau, 0], \mathbb{R}\right), \end{cases} \tag{3.3}$$

where $n \in \mathbb{Z}$, $t > 0$, and $s \in [-\tau, 0]$.

Before stating the following theorem, we first define a Banach space l^∞ by

$$l^\infty = \left\{\xi = \{\xi_i\}_{i\in\mathbb{Z}}, \xi_i \in \mathbb{R} : \sup_{i\in\mathbb{Z}} |\xi_i| < \infty\right\}$$

with the norm $\|\xi\|_{l^\infty} = \sup_{i\in\mathbb{Z}} |\xi_i|$.

THEOREM 3.3. *For any* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ *with* $\varphi \in C([-\tau, 0], l^\infty)$, (3.3) *admits a unique solution* $u(t) := u(t;\varphi) = \{u_n(t;\varphi)\}_{n\in\mathbb{Z}}$ *on* $[0, +\infty)$. *Furthermore, if* $\varphi^1, \varphi^2 \in C([-\tau, 0], l^\infty)$ *satisfy* $\varphi_n^1(s) \leq \varphi_n^2(s)$ *for any* $n \in \mathbb{Z}$ *and* $s \in [-\tau, 0]$, *then* $u_n(t;\varphi^1) \leq u_n(t;\varphi^2)$ *holds for any* $n \in \mathbb{Z}$ *and* $t > 0$.

Proof. Let $X = l^\infty$. Set $X^+ = \{\xi \in l^\infty : \xi_i \geq 0 \text{ for each } i \in \mathbb{Z}\}$. Then it is easy to see that X^+ is a closed cone of X. Let $T(t) = e^{-(D+d)t}$; it is obvious that $\{T(t)\}$ is a strongly continuous semigroup on X. In particular, it is strongly positive. Now let $\mathcal{C} = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm. Set $\mathcal{C}^+ = \{\Phi \in \mathcal{C} : \Phi(s) \in X^+, s \in [-\tau, 0]\}$. Then $\mathcal{C}^+$ is a positive cone of $\mathcal{C}$. For any continuous function $w : [-\tau, +\infty) \to X$, define $w_t \in \mathcal{C}$,

$t \in [0, +\infty)$, by $w_t(s) = w(t+s)$, $s \in [-\tau, 0]$. Then the map $t \mapsto w_t$ is a continuous function from $[0, +\infty)$ to $\mathcal{C}$.

Define $f : \mathcal{C} \to X$ by

$$f(w) = \{f_i(w)\}_{i \in \mathbb{Z}}$$

for $w = \{w_i\}_{i \in \mathbb{Z}} \in \mathcal{C}$, where

$$f_i(w) = D \sum_{j \in \mathbb{Z} \setminus \{0\}} I(j)\, w_{i-j}(0) + b'(0) \sum_{k \in \mathbb{Z}} J(k)\, w_{i-k}(-\tau).$$

It is not difficult to verify that $f : \mathcal{C} \to X$ is globally Lipschitz continuous. Furthermore, since for any $v, w \in \mathcal{C}$ with $v \geq w$ in $\mathcal{C}$,

$$\begin{aligned} f_i(v) - f_i(w) = {} & D \sum_{j \in \mathbb{Z} \setminus \{0\}} I(j)\, v_{i-j}(0) - D \sum_{j \in \mathbb{Z} \setminus \{0\}} I(j)\, w_{i-j}(0) \\ & + b'(0) \sum_{k \in \mathbb{Z}} J(k)\, v_{i-k}(-\tau) - b'(0) \sum_{k \in \mathbb{Z}} J(k)\, w_{i-k}(-\tau) \\ \geq {} & 0, \end{aligned}$$

it follows that $f(v) \geq f(w)$ in X for any $v, w \in \mathcal{C}$ with $v \geq w$, which implies that $f : \mathcal{C} \to X$ is quasi-monotone in the sense that

$$\lim_{h \to 0} \frac{1}{h} \operatorname{dist}\left((v(0) - w(0)) + h\left[f(v) - f(w) \right], X^+ \right) = 0$$

for any $v, w \in \mathcal{C}$ with $v \geq w$.

Note that (3.3) is equivalent to

$$\text{(3.4)} \qquad \begin{cases} u(t) = T(t)\, u(0) + \int_0^t T(t-s)\, f(u_s)\, ds, & t > 0, \\ u(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases}$$

Take $M_0 = \max_{t \in [-\tau, 0]} \|\varphi(t)\|_{l^\infty}$. Furthermore, define a vector-valued function $v^+(\cdot) = \{v_n^+(\cdot)\}_{n \in \mathbb{Z}} : [-\tau, +\infty) \to X$ by

$$\text{(3.5)} \qquad \begin{cases} v_n^+(t) = M_0, & t \in [-\tau, 0], \\ v_n^+(t) = M_0 e^{(b'(0) - d)t}, & t > 0 \text{ for any } n \in \mathbb{Z}. \end{cases}$$

It is easy to verify that v^+ satisfies

$$\text{(3.6)} \qquad v^+(t) \geq T(t)\, v^+(s) + \int_s^t T(t-r)\, f\left(v_r^+\right) dr \text{ for any } t > s \geq 0.$$

Define $v^-(\cdot) = \{v_n^-(\cdot)\}_{i \in \mathbb{Z}} : [-\tau, +\infty) \to X$ by $v^-(\cdot) = -v^+(\cdot)$. Then v^- satisfies

$$\text{(3.7)} \qquad v^-(t) \leq T(t)\, v^-(s) + \int_s^t T(t-r)\, f\left(v_r^-\right) dr \text{ for any } t > s \geq 0.$$

Now we use the conclusions of [29]. By setting $S(t,s) = T(t,s) = T(t-s)$ for any $t \geq s \geq 0$ and $B(t, \Phi) = f(\Phi)$, the existence and uniqueness of the solution $u(t; \varphi)$ follows from [29, Corollary 5].

For any $\varphi^1, \varphi^2 \in \mathcal{C}$ with $\varphi^1 \leq \varphi^2$ in $\mathcal{C}$, again applying [29, Corollary 5], we have

$$v^-(t) \leq u\left(t; \varphi^1\right) \leq u\left(t; \varphi^2\right) \leq v^+(t) \text{ in } X \text{ for any } t \geq 0,$$

via letting

$$M_0 = \max\left\{\max_{s\in[-\tau,0]}\left\|\varphi^1(s)\right\|_{l^\infty}, \max_{s\in[-\tau,0]}\left\|\varphi^2(s)\right\|_{l^\infty}\right\}$$

in (3.5). This implies the solution semiflow is order preserving. The proof is complete. ∎

Remark 3.1. Assume that the continuous functions $w^\pm = \{w_n^\pm\}_{n\in\mathbb{Z}} : [-\tau, +\infty) \to l^\infty$ satisfy (3.6) and (3.7), respectively, and $w_n^+(s) \geq w_n^-(s)$ for any $(n, s) \in \mathbb{Z}\times[-\tau, 0]$; then we have $w_n^+(t) \geq w_n^-(t)$ for any $(n, t) \in \mathbb{Z} \times [0, +\infty)$.

THEOREM 3.4. *Assume that*

$$w_n^-(t) \in C([-\tau, \infty), (-\infty, K]) \quad \textit{and} \quad w_n^+(t) \in C([-\tau, \infty), [0, \infty))$$

satisfy $w_n^-(t) \leq w_n^+(t)$ for any $t \in [-\tau, 0]$ and $n \in \mathbb{Z}$ and

$$\frac{d}{dt}w_n^+(t) \geq D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[w_{n-i}^+(t) - w_n^+(t)\right] - dw_n^+(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\, w_{n-i}^+(t-\tau), \tag{3.8}$$

$$\frac{d}{dt}w_n^-(t) \leq D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[w_{n-i}^-(t) - w_n^-(t)\right] - dw_n^-(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\, w_{n-i}^-(t-\tau) \tag{3.9}$$

for any $t > 0$ and $n \in \mathbb{Z}$. Then there holds $w_n^+(t) \geq w_n^-(t)$ for any $t > 0$ and $n \in \mathbb{Z}$.

Proof. Put $w_n(t) := w_n^-(t) - w_n^+(t)$, $n \in \mathbb{Z}$, $t \in [-\tau, +\infty)$. Then $w_n(t)$ is continuous and bounded from above by K, and $\overline{w}(t) := \sup_{n\in\mathbb{Z}} w_n(t)$ is continuous on $[-\tau, \infty)$. We use a contradiction argument to prove the assertion. Suppose that the assertion is not true. Let $M_0 > 0$ be such that $M_0 + d - b'(0)e^{-M_0\tau} > 0$. Then there exists $t_0 > 0$ such that $\overline{w}(t_0) > 0$ and

$$\overline{w}(t_0)\, e^{-M_0 t_0} = \sup_{t\geq-\tau}\left\{\overline{w}(t)\, e^{-M_0 t}\right\} > \overline{w}(s)\, e^{-M_0 s} \text{ for all } s \in [-\tau, t_0). \tag{3.10}$$

Let $\{n_j\}_{j\in\mathbb{N}}$ be a sequence so that $w_{n_j}(t_0) > 0$ for all $j \geq 1$ and $\lim_{j\to\infty} w_{n_j}(t_0) = \overline{w}(t_0)$. Let $\{t_j\}_{j\in\mathbb{N}} \subset (0, t_0]$ so that

$$w_{n_j}(t_j)\, e^{-M_0 t_j} = \max_{t\in[0,t_0]}\left\{w_{n_j}(t)\, e^{-M_0 t}\right\}. \tag{3.11}$$

Since

$$w_{n_j}(t_0)\, e^{-M_0 t_0} \leq w_{n_j}(t_j)\, e^{-M_0 t_j} \leq \overline{w}(t_j)\, e^{-M_0 t_j} \leq \overline{w}(t_0)\, e^{-M_0 t_0},$$

we have $\lim_{j\to+\infty} \overline{w}(t_j)e^{-M_0 t_j} = \overline{w}(t_0)e^{-M_0 t_0}$. Then there must be $\lim_{j\to+\infty} t_j = t_0$ due to (3.10). In view of $w_{n_j}(t_0)e^{-M_0(t_0-t_j)} \leq w_{n_j}(t_j) \leq \overline{w}(t_0)e^{-M_0(t_0-t_j)}$, we obtain $\lim_{j\to+\infty} w_{n_j}(t_j) = \overline{w}(t_0)$.

Following (3.11), for each $j \geq 1$, we have

$$0 \leq \frac{d}{dt}\left\{w_{n_j}(t)\, e^{-M_0 t}\right\}_{t=t_j} = \left[w'_{n_j}(t_j) - M_0 w_{n_j}(t_j)\right] e^{-M_0 t_j},$$

and, hence, $w'_{n_j}(t_j) \geq M_0 w_{n_j}(t_j)$. Then it follows from (3.8) and (3.9) that

$$\begin{aligned} 0 &\geq w'_{n_j}(t_j) - D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[w_{n_j-i}(t_j) - w_{n_j}(t_j)\right] + d w_{n_j}(t_j) \\ &\quad - b'(0) \sum_{i\in\mathbb{Z}} J(i)\left[w^-_{n_j-i}(t_j-\tau) - w^+_{n_j-i}(t_j-\tau)\right] \\ &\geq (M_0 + D + d)\, w_{n_j}(t_j) - D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\, w_{n_j-i}(t_j) - b'(0)\, \overline{w}(t_j-\tau) \\ &\geq (M_0 + D + d)\, w_{n_j}(t_j) - D\overline{w}(t_j) - b'(0)\, \overline{w}(t_j-\tau). \end{aligned}$$

Taking $j \to +\infty$, we have

$$\begin{aligned} 0 &\geq (M_0 + D + d)\, \overline{w}(t_0) - D\overline{w}(t_0) - b'(0)\, e^{M_0(t_0-\tau)}\left[\overline{w}(t_0-\tau)\, e^{-M_0(t_0-\tau)}\right] \\ &\geq (M_0 + d)\, \overline{w}(t_0) - b'(0)\, e^{M_0(t_0-\tau)}\overline{w}(t_0)\, e^{-M_0 t_0} \\ &= \left[M_0 + d - b'(0)\, e^{-M_0\tau}\right] \overline{w}(t_0). \end{aligned}$$

In view of $M_0 + d - b'(0)e^{-M_0\tau} > 0$, we obtain that $\overline{w}(t_0) \leq 0$, which contradicts to $\overline{w}(t_0) > 0$. Consequently, we conclude that $w^+_n(t) \geq w^-_n(t)$ for all $n \in \mathbb{Z}$ and $t \in (0, +\infty)$. This completes the proof. ∎

4. Existence of heteroclinic orbits. In this section, we show the existence of a heteroclinic orbit connecting the equilibria $u \equiv 0$ and $u \equiv K$ for the following functional differential equation:

$$\frac{d}{dt}u(t) = -du(t) + b(u(t-\tau)). \tag{4.1}$$

There are now various methods developed to establish the existence of such a heteroclinic orbit, for example, Faria, Huang, and Wu [11], Faria and Trofimchuk [12, 13], Li, Wang, and Wu [24], and Smith [33]. However, except for Faria and Trofimchuk [13], these results do not give the exponential decay rate of the heteroclinic orbit connecting the equilibria $u \equiv 0$ and $u \equiv K$ at minus infinity. At the same time, the results in [13] are not directly applicable (see the condition (A1) of [13]) and do not ensure the monotonicity of the heteroclinic orbit.

Define

$$\Lambda(\lambda) = \lambda + d - b'(0)\, e^{-\lambda\tau};$$

then it is easy to prove that the equation $\Lambda(\lambda) = 0$ has one and only one real root $\lambda_* > 0$ such that $\Lambda(\lambda) < 0$ for any $\lambda < \lambda_*$ and $\Lambda(\lambda) > 0$ for any $\lambda > \lambda_*$.

Define an operator $S : C(\mathbb{R}, [0, K]) \to C(\mathbb{R}, [0, K])$ by

$$S(u)(t) = e^{-dt} \int_{-\infty}^{t} e^{ds} b(u(s-\tau))\, ds \text{ for any } u \in C(\mathbb{R}, [0, K]).$$

PROPOSITION 4.1.
(i) *If* $u \in C(\mathbb{R}, [0, K])$*, then* $S(u) \in C^1(\mathbb{R}, [0, K])$*.*
(ii) *For any* $u, v \in C(\mathbb{R}, [0, K])$ *with* $u \leq v$*,* $S(u) \leq S(v)$*.*
(iii) *For any* $u \in C(\mathbb{R}, [0, K])$*, if* $u(\cdot)$ *is increasing in* $\mathbb{R}$*, then so is* $S(u)(\cdot)$*.*
Let $b''_{\max} = \max_{u\in[0,K]} |b''(u)|$. Define

$$\overline{u}(t) = K \min\left\{e^{\lambda_* t}, 1\right\} \text{ and } \underline{u}(t) = \max\left\{K e^{\lambda_* t}\left(1 - M e^{\epsilon t}\right), 0\right\},$$

where $\epsilon \in (0, \lambda_*)$ and $M > 1$ with

$$1 - \frac{(d + \lambda_*)\, e^{-\epsilon\tau}}{d + \lambda_* + \epsilon} - \frac{(d + \lambda_*)\, K e^{-\lambda_*\tau} b''_{\max}}{M b'(0)\,(d + 2\lambda_*)} > 0.$$

LEMMA 4.2. *For any* $t \in \mathbb{R}$*,* $S(\overline{u})(t) \leq \overline{u}(t)$ *and* $\underline{u}(t) \leq S(\underline{u})(t)$*.*

Proof. First, we prove $S(\overline{u})(t) \leq \overline{u}(t)$. When $t \geq 0$, $\overline{u}(t) = K$. Therefore,

$$\begin{aligned} S(\overline{u})(t) &= e^{-dt}\int_{-\infty}^{t} e^{ds} b(\overline{u}(s - \tau))\, ds \\ &\leq e^{-dt}\int_{-\infty}^{t} e^{ds} b(K)\, ds = dKe^{-dt}\int_{-\infty}^{t} e^{ds} ds = K = \overline{u}(t). \end{aligned}$$

When $t < 0$, $\overline{u}(t) = K e^{\lambda_* t}$. Noting that $d + \lambda_* = b'(0)e^{-\lambda_*\tau}$, we have

$$\begin{aligned} S(\overline{u})(t) &= e^{-dt}\int_{-\infty}^{t} e^{ds} b(\overline{u}(s - \tau))\, ds \leq e^{-dt}\int_{-\infty}^{t} e^{ds} b'(0)\overline{u}(s - \tau)\, ds \\ &\leq b'(0) K e^{-dt}\int_{-\infty}^{t} e^{ds} e^{\lambda_*(s-\tau)} ds = \frac{b'(0)\, e^{-\lambda_*\tau} K}{d + \lambda_*} e^{\lambda_* t} = \overline{u}(t). \end{aligned}$$

Now we prove $\underline{u}(t) \leq S(\underline{u})(t)$. Let $t_0 = \frac{1}{\epsilon}\ln\frac{1}{M} < 0$ such that $1 - M e^{\epsilon t_0} = 0$. When $t \geq t_0$, $\underline{u}(t) = 0$, and, hence, $\underline{u}(t) \leq S(\underline{u})(t)$. When $t < t_0$, $\underline{u}(t) = K e^{\lambda_* t}(1 - M e^{\epsilon t}) \leq K e^{\lambda_* t}$. In this case, we have

$$\begin{aligned} S(\underline{u})(t) &= e^{-dt}\int_{-\infty}^{t} e^{ds} b(\underline{u}(s - \tau))\, ds \\ &\geq e^{-dt}\int_{-\infty}^{t} e^{ds}\left[b'(0)\,\underline{u}(s - \tau) - b''_{\max}\underline{u}^2(s - \tau)\right] ds \\ &\geq e^{-dt}\int_{-\infty}^{t} e^{ds}\left[b'(0) K e^{\lambda_*(s-\tau)}\left(1 - M e^{\epsilon(s-\tau)}\right) - b''_{\max} K^2 e^{2\lambda_*(s-\tau)}\right] ds \\ &= \frac{b'(0) K e^{-\lambda_*\tau}}{d + \lambda_*} e^{\lambda_* t} - \frac{M b'(0) K e^{-(\lambda_*+\epsilon)\tau}}{d + \lambda_* + \epsilon} e^{(\lambda_*+\epsilon)t} - \frac{b''_{\max} K^2 e^{-2\lambda_*\tau}}{d + 2\lambda_*} e^{2\lambda_* t} \\ &\geq \underline{u}(t) + \frac{M b'(0) K e^{-\lambda_*\tau}}{d + \lambda_*}\left[1 - \frac{(d + \lambda_*)\, e^{-\epsilon\tau}}{d + \lambda_* + \epsilon} - \frac{(d + \lambda_*)\, K e^{-\lambda_*\tau} b''_{\max}}{M b'(0)\,(d + 2\lambda_*)}\right] e^{(\lambda_*+\epsilon)t} \\ &\geq \underline{u}(t). \end{aligned}$$

The proof is complete. ∎

THEOREM 4.3. *There exists a heteroclinic solution* $\Gamma(t)$ *of* (4.1)*, which is increasing on* $\mathbb{R}$ *and satisfies* $\lim_{t\to-\infty} e^{-\lambda_* t}\Gamma(t) = K$, $\Gamma(+\infty) = K$, $\Gamma(t) \le Ke^{\lambda_* t}$*, and* $\Gamma'(t) > 0$ *for every* $t \in \mathbb{R}$.

Proof. By an argument similar to that of [42, Theorem 3.1], we can get a nondecreasing solution $\Gamma(t)$ which meets the theorem except $\Gamma'(t) > 0$ for any $t \in \mathbb{R}$. Since $\Gamma'(t)$ satisfies

$$\Gamma''(t) = -d\Gamma'(t) + b'\left(\Gamma(t-\tau)\right)\Gamma'(t-\tau) \quad \forall t \in \mathbb{R},$$

we have

$$\Gamma'(t) = e^{-d(t-s)}\Gamma'(s) + \int_s^t e^{-d(t-r)} b'\left(\Gamma(t-\tau)\right)\Gamma'(r-\tau)dr \quad \text{for any} \quad s < t.$$

Note that $\Gamma'(t) \ge 0$ for any $t \in \mathbb{R}$. Then it is easy to see that if $\Gamma'(t_0) > 0$ for some $t_0 \in \mathbb{R}$, then $\Gamma'(t) > 0$ for all $t > t_0$. In view of $\lim_{t\to-\infty} e^{-\lambda_* t}\Gamma(t) = K$, we know that there exists a sequence $\{t_i\}$ with $t_i \to -\infty$ as $i \to +\infty$ such that $\Gamma(t_i) > 0$ for any $i \in \mathbb{N}$. Hence, we conclude $\Gamma'(t) > 0$ for any $t \in \mathbb{R}$. This completes the proof. ∎

5. Proof of Theorem 1.1.

In this section, we prove Theorem 1.1.

LEMMA 5.1. *Suppose that* $u(t;\varphi) = \{u_n(t;\varphi)\}_{n\in\mathbb{Z}}$ *is a solution of* (1.1) *with initial value* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ *with* $\varphi_n \in C([-\tau,0],[0,K])$*; then there exists a positive constant* $M_* > 0$ *such that for any* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ *with* $\varphi_n \in C([-\tau,0],[0,K])$ *and* $t > \tau$, $|u_n'(t;\varphi)| \le M_*$ *and* $|u_n''(t;\varphi)| \le M_*$.

Proof. Denote $u_n(t;\varphi)$ by $u_n(t)$. Let $M' = 2DK + 2dK$. It is easy to see that $|u_n'(t;\varphi)| \le M'$ for any $t > 0$. For $t > \tau$, there is

$$\begin{aligned} u_n''(t) = D &\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}'(t) - u_n'(t)\right] \\ &- du_n'(t) + \sum_{i\in\mathbb{Z}} J(i)\, b'\left(u_{n-i}(t-\tau)\right) u_{n-i}'(t-\tau). \end{aligned}$$

Set $M'' = 2DM' + dM' + M'b'(0)$. Then $|u_n''(t;\varphi)| \le M''$. Note that M' and M'' are independent of φ and $t > \tau$. Take $M_* = \max\{M', M''\}$. This completes the proof. ∎

LEMMA 5.2. *Let* $u^k(t;\varphi^k) = \{u_n^k(t;\varphi^k)\}_{n\in\mathbb{Z}}$ *be a solution of the following initial value problem:*

$$\begin{cases} \frac{d}{dt}u_n^k(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[u_{n-i}(t) - u_n(t)\right] \\ \qquad\qquad -du_n(t) + \sum_{i\in\mathbb{Z}} J(i)\, b\left(u_{n-i}(t-\tau)\right), \ t > 0, \\ u_n^k(t) = \varphi_n^k(t), \ t \in [-\tau, 0], \end{cases}$$

where

$$\begin{aligned} \varphi_n^k(t) = \max\Bigg\{ &\max_{1\le i\le m} \phi_{c_i}\left(n + c_i(t-k) + \theta_i\right), \\ &\max_{1\le j\le l} \phi_{c_j'}\left(-n + c_j'(t-k) + \theta_j'\right), \chi\Gamma\left((t-k) + \theta_0\right) \Bigg\}. \end{aligned}$$

Then $v^k\left(t;\varphi^k\right)=\left\{u_n^k\left(t+k;\varphi^k\right)\right\}_{n\in\mathbb{Z}}$ *satisfies*

$$\begin{aligned}\limsup_{t>-k,k\to+\infty} v_n^k\left(t\right) \le \phi_{c_i}\left(n+c_it+\theta_i\right)+\chi Ke^{\lambda_*(t+\theta_0)}+\sum_{1\le j\le m,j\ne i}A_{c_j}e^{\lambda_1(c_j)(n+c_jt+\theta_j)}\\ +\sum_{1\le j\le l}A_{c_j'}e^{\lambda_1\left(c_j'\right)\left(-n+c_j't+\theta_j'\right)} \qquad \textit{for } 1\le i\le m,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\limsup_{t>-k,k\to+\infty} v_n^k\left(t\right) \le \phi_{c_i'}\left(-n+c_i't+\theta_i'\right)+\chi Ke^{\lambda_*(t+\theta_0)}+\sum_{1\le j\le m}A_{c_j}e^{\lambda_1(c_j)(n+c_jt+\theta_j)}\\ +\sum_{1\le j\le l,j\ne i}A_{c_j'}e^{\lambda_1\left(c_j'\right)\left(-n+c_j't+\theta_j'\right)} \qquad \textit{for } 1\le i\le l,\end{aligned} \tag{5.2}$$

$$\begin{aligned}\limsup_{t>-k,k\to+\infty} v_n^k\left(t\right) \le \chi\Gamma\left(t+\theta_0\right)+\sum_{1\le j\le m}A_{c_j}e^{\lambda_1(c_j)(n+c_jt+\theta_j)}\\ +\sum_{1\le j\le l}A_{c_j'}e^{\lambda_1\left(c_j'\right)\left(-n+c_j't+\theta_j'\right)}.\end{aligned} \tag{5.3}$$

Proof. We prove only (5.1), because the proofs of (5.2) and (5.3) are similar to that of (5.1). Assume $m\ge 1$. Consider $i=1$. Let

$$w_n^k\left(t\right)=u_n^k\left(t\right)-\phi_{c_1}\left(n+c_1(t-k)+\theta_1\right).$$

Then $w_n^k(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}w_n^k\left(t\right) &= D\sum_{i\in\mathbb{Z}\backslash\{0\}}I\left(i\right)\left[w_{n-i}\left(t\right)-w_n\left(t\right)\right]-dv_n\left(t\right)+\sum_{i\in\mathbb{Z}}J\left(i\right)b\left(u_{n-i}\left(t-\tau\right)\right)\\ &\quad -\sum_{i\in\mathbb{Z}}J\left(i\right)b\left(\phi_{c_1}\left(n-i+c_1\left(t-\tau\right)+\theta_1\right)\right)\\ &\le D\sum_{i\in\mathbb{Z}\backslash\{0\}}I\left(i\right)\left[w_{n-i}\left(t\right)-w_n\left(t\right)\right]-dw_n\left(t\right)+b'\left(0\right)\sum_{i\in\mathbb{Z}}J\left(i\right)w_{n-i}\left(t-\tau\right).\end{aligned}$$

Since

$$\chi Ke^{\lambda_*((t-k)+\theta_0)}+\sum_{2\le j\le m}A_{c_j}e^{\lambda_1(c_j)(n+c_j(t-k)+\theta_j)}+\sum_{1\le j\le l}A_{c_j'}e^{\lambda_1\left(c_j'\right)\left(-n+c_j'(t-k)+\theta_j'\right)}$$

is a solution of (3.3) with

$$\begin{aligned}\varphi_n\left(s\right)=\chi Ke^{\lambda_*((s-k)+\theta_0)}+\sum_{2\le j\le m}A_{c_j}e^{\lambda_1(c_j)(n+c_j(s-k)+\theta_j)}\\ +\sum_{1\le j\le l}A_{c_j'}e^{\lambda_1\left(c_j'\right)\left(-n+c_j'(s-k)+\theta_j'\right)}\end{aligned}$$

for any $s \in [-\tau, 0]$ and $n \in \mathbb{Z}$, then by Theorem 3.4 and

$$\begin{aligned}
&\chi K e^{\lambda_*((s-k)+\theta_0)} + \sum_{2\le j\le m} A_{c_j} e^{\lambda_1(c_j)(n+c_j(s-k)+\theta_j)} \\
&\quad + \sum_{1\le j\le l} A_{c'_j} e^{\lambda_1(c'_j)\left(-n+c'_j(s-k)+\theta'_j\right)} \\
&\ge \max\left\{\max_{1\le i\le m} \phi_{c_i}\left(n+c_i\left(t-k\right)+\theta_i\right),\right. \\
&\qquad \left.\max_{1\le j\le l} \phi_{c'_j}\left(-n+c'_j\left(t-k\right)+\theta'_j\right), \chi\Gamma\left(\left(t-k\right)+\theta_0\right)\right\} \\
&\quad -\phi_{c_1}\left(n+c_1(s-k)+\theta_1\right)
\end{aligned}$$

for any $s \in [-\tau, 0]$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}
w_n^k(t) \le \chi K e^{\lambda_*((t-k)+\theta_0)} &+ \sum_{2\le j\le m} A_{c_j} e^{\lambda_1(c_j)(n+c_j(t-k)+\theta_j)} \\
&+ \sum_{1\le j\le l} A_{c'_j} e^{\lambda_1(c'_j)\left(-n+c'_j(t-k)+\theta'_j\right)} \quad \text{for any } t>0.
\end{aligned}$$

That is,

$$\begin{aligned}
u_n^k(t) \le \phi_{c_1}\left(n+c_1(t-k)+\theta_1\right) + \chi K e^{\lambda_*((t-k)+\theta_0)} + \sum_{2\le j\le m} A_{c_j} e^{\lambda_1(c_j)(n+c_j(t-k)+\theta_j)} \\
+ \sum_{1\le j\le l} A_{c'_j} e^{\lambda_1(c'_j)\left(-n+c'_j(t-k)+\theta'_j\right)} \quad \text{for any } t>0.
\end{aligned}$$

By the arbitrariness of $k \in \mathbb{N}$, we have that (5.1) holds.

When $m = 0$, the inequality (5.1) reduces to the following:

$$\limsup_{t>-k, k\to+\infty} v_n^k(t) \le \chi K e^{\lambda_*(t+\theta_0)} + \sum_{1\le j\le l} A_{c'_j} e^{\lambda_1(c'_j)\left(-n+c'_j t+\theta'_j\right)},$$

which holds obviously. This completes the proof. ∎

Proof of Theorem 1.1. Define $v^k(t) = \{v_n^k(t)\}_{n\in\mathbb{Z}}$ with $v_n^k(t) := u_n(t+k;\psi^k)$ for any $(n,t) \in \mathbb{Z} \times [-\tau-k, +\infty)$, where

$$\begin{aligned}
\psi^k &= \left\{\psi_n^k(s)\right\}_{k\in\mathbb{Z}}, \\
\psi_n^k(s) &= \max\left\{\max_{1\le i\le m} \phi_{c_i}\left(n+c_i\left(s-k\right)+\theta_i\right),\right. \\
&\qquad \left.\max_{1\le j\le l} \phi_{c'_j}\left(-n+c'_j\left(s-k\right)+\theta'_j\right), \chi\Gamma\left(\left(s-k\right)+\theta_0\right)\right\} < K
\end{aligned}$$

for any $(n,s) \in \mathbb{Z} \times [-\tau, 0]$. Note that

$$\begin{aligned}
&\max\left\{\max_{1\le i\le m} \phi_{c_i}\left(n+c_i t+\theta_i\right), \max_{1\le j\le l} \phi_{c'_j}\left(-n+c'_j t+\theta'_j\right), \chi\Gamma\left(t+\theta_0\right)\right\} \\
&\le v_n^k(t) \le v_n^{k+1}(t) \le \min\left\{K, \vartheta_m^+(n,t), \vartheta_l^-(n,t), \vartheta^0(n,t)\right\}
\end{aligned} \tag{5.4}$$

for any $(n,t) \in \mathbb{Z} \times [-\tau-k, +\infty)$. From Lemma 5.1 and by a diagonal extraction process, there exists a subsequence $\{v^{k_i}(t) = \{v_n^{k_i}(t)\}_{n\in\mathbb{Z}} : i \in \mathbb{N}\}$ such that $v^{k_i}(t)$

converges to a function $\Phi(t)=\{\Phi_n(t)\}_{n\in\mathbb{Z}}$ in $\mathcal{T}$; that is, for any compact set $S\subset\mathbb{Z}\times\mathbb{R}$, $v_n^{k_i}(t)$ and $\frac{d}{dt}v_n^{k_i}(t)$ converge uniformly in $(n,t)\in S$ to $\Phi_n(t)$ and $\frac{d}{dt}\Phi_n(t)$, respectively. In view of $v_n^k(t)\le v_n^{k+1}(t)$ for any $t>-k$, we have $\lim_{k\to+\infty}v_n^k(t)=\Phi_n(t)$ for any $(n,t)\in\mathbb{Z}\times\mathbb{R}$. The limit function is unique, whence all of the functions $v^k(t)$ converge to the function $\Phi(t)$ in $\mathcal{T}$ as $k\to+\infty$. Since $u^{k_i}(t)=\{u_n^{k_i}(t)\}_{n\in\mathbb{Z}}$ satisfies (1.1), the limit function $\Phi(t)=\{\Phi_n(t)\}_{n\in\mathbb{Z}}$ is an entire solution of (1.1). In particular, it follows from (5.4) that (1.8) holds on $(n,t)\in\mathbb{Z}\times\mathbb{R}$.

Now we show (i); that is, $\frac{d}{dt}\Phi_n(t)>0$ on $\mathbb{R}$ for every $n\in\mathbb{Z}$. Since

$$\begin{aligned}\psi_n^k(s)&=\max\Big\{\max_{1\le i\le m}\phi_{c_i}\left(n+c_i(s-k)+\theta_i\right),\\&\qquad\max_{1\le j\le l}\phi_{c_j'}\left(-n+c_j'(s-k)+\theta_j'\right),\chi\Gamma\left((s-k)+\theta_0\right)\Big\}\\&\le\max\Big\{\max_{1\le i\le m}\phi_{c_i}\left(n+c_i(s+\varepsilon-k)+\theta_i\right),\\&\qquad\max_{1\le j\le l}\phi_{c_j'}\left(-n+c_j'(s+\varepsilon-k)+\theta_j'\right),\chi\Gamma\left((s+\varepsilon-k)+\theta_0\right)\Big\}\\&=\psi_n^k(s+\varepsilon)\end{aligned}$$

for any $\varepsilon>0$, $s\in[-\tau,0]$, and $n\in\mathbb{Z}$, we have $u_n^k(t;\psi^k(\cdot))\le u_n^k(t;\psi^k(\cdot+\varepsilon))$ for any $(n,t)\in\mathbb{Z}\times[-\tau,+\infty)$. On the other hand, $\psi_n^k(s+\varepsilon)\le u_n^k(s+\varepsilon;\psi^k(\cdot))$ for any $\varepsilon>0$, $s\in[-\tau,0]$, and $n\in\mathbb{Z}$, and, hence,

$$u_n^k\left(t;\psi^k(\cdot)\right)\le u_n^k\left(t;u_n^k\left(\cdot+\varepsilon;\psi^k(\cdot)\right)\right)=u_n^k\left(t+\varepsilon;\psi^k(\cdot)\right)$$

for any $(n,t)\in\mathbb{Z}\times[-\tau,+\infty)$. Thus, it follows from the arbitrariness of $\epsilon>0$; that $u_n^k(t)$ is increasing on t; that is, $v^k(t)$ is increasing on t. Therefore, $\Phi_n'(t)\ge 0$ on $\mathbb{R}$ for every $n\in\mathbb{Z}$. Since $\Phi_n'(t)$ satisfies

$$\begin{aligned}\Phi_n''(t)&=D\sum_{i\in\mathbb{Z}\setminus\{0\}}I(i)\left[\Phi_{n-i}'(t)-\Phi_n'(t)\right]\\&\quad-d\Phi_n'(t)+\sum_{i\in\mathbb{Z}}J(i)\,b'\left(\Phi_{n-i}(t-\tau)\right)\Phi_{n-i}'(t-\tau),\end{aligned}\tag{5.5}$$

we have that $\Phi_n'(t)$ satisfies

$$\Phi_n'(t)=\Phi_n'(s)\,e^{-(D+d)(t-s)}+\int_s^t e^{-(D+d)(t-r)}R_n(\Phi)(r)\,dr\qquad\text{for any } s<t,$$

where $R_n(\Phi)(t)=D\sum_{i\in\mathbb{Z}\setminus\{0\}}I(i)\Phi_{n-i}'(t)+\sum_{i\in\mathbb{Z}}J(i)b'(\Phi_{n-i}(t-\tau))\Phi_{n-i}'(t-\tau)\ge 0$. Obviously, for each $n\in\mathbb{Z}$, if there exists $t_0\in\mathbb{R}$ such that $\Phi_n'(t_0)>0$, then $\Phi_n'(t)>0$ for any $t>t_0$. Therefore, there must be $\Phi_n'(t)>0$ for any $(n,t)\in\mathbb{Z}\times\mathbb{R}$. We argue by a contradiction. In fact, assume that, for some $n_1\in\mathbb{Z}$, there is t_1 such that $\Phi_{n_1}'(t_1)=0$ and, hence, then $\Phi_{n_1}'(t)=0$ for any $t\le t_1$, which implies that $\lim_{t\to-\infty}\Phi_{n_1}(t)=\Phi_{n_1}(t_1)>0$. But following from (1.8), we have $\lim_{t\to-\infty}\Phi_{n_1}(t)=0$, which yields a contradiction.

Now we prove (vii). For the sake of convenience, we denote

$$\Phi_{c_1,\dots,c_m;c_1',\dots,c_l';\theta_1,\dots,\theta_m;\theta_1',\dots,\theta_l';\theta_0}(t)$$

by $\Phi(t;\theta_0)$ and $\Phi_{c_1,\dots,c_m;c'_1,\dots,c'_l;\theta_1,\dots,\theta_m;\theta'_1,\dots,\theta'_l}(t)$ by $\Phi(t;-\infty)$. For $\chi\in\{0,1\}$, let

$$\psi^k(t)_\chi = \left\{\psi_n^k(s)_\chi\right\}_{k\in\mathbb{Z}},$$
$$\psi_n^k(s)_\chi = \max\left\{\max_{1\le i\le m}\phi_{c_i}\left(n+c_i(s-k)+\theta_i\right),\right.$$
$$\left.\max_{1\le j\le l}\phi_{c'_j}\left(-n+c'_j(s-k)+\theta'_j\right), \chi\Gamma\left((s-k)+\theta_0\right)\right\},$$

and $v^k(t)_\chi = \{v_n^k(t)_\chi\}_{n\in\mathbb{Z}}$ with $v_n^k(t)_\chi := u_n(t+k;\psi^k(\cdot)_\chi)$ for any $(n,t)\in\mathbb{Z}\times[-\tau-k,+\infty)$. Set $\bar v^k(t) = v^k(t)_1 - v^k(t)_0 = \{v_n^k(t)_1 - v_n^k(t)_0\}_{n\in\mathbb{Z}}$. Then $\bar v^k(t)$ satisfies $0\le \bar v^k(t)\le K$ for any $t\in[-\tau-k,+\infty)$ and

$$\frac{d}{dt}\bar v_n^k(t) \le D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[\bar v_{n-i}^k(t) - \bar v_n^k(t)\right] - d\bar v_n^k(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\,\bar v_{n-i}^k(t-\tau).$$

Noting that $\bar v_n^k(s)\le Ke^{\lambda_*(s+\theta_0)}$ for any $s\in[-\tau-k,-k]$ and $w_n^k(t) = Ke^{\lambda_*(t+\theta_0)}$ satisfies

$$\frac{d}{dt}w_n^k(t) = D\sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[w_{n-i}^k(t) - w_n^k(t)\right] - dw_n^k(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\,w_{n-i}^k(t-\tau)$$

for any $t\in[-\tau-k,+\infty)$, it follows from Theorem 3.4 that $0\le\bar v_n^k(t)\le Ke^{\lambda_*(t+\theta_0)}$ for any $(n,t)\in\mathbb{Z}\times[-\tau-k,+\infty)$ and $k\in\mathbb{N}$. Note that $\lim_{k\to+\infty} v_n^k(t)_0 = \Phi_n(t;-\infty)$ and $\lim_{k\to+\infty} v_n^k(t)_1 = \Phi_n(t;\theta_0)$ for any $(n,t)\in\mathbb{Z}\times\mathbb{R}$. Therefore, there must be $0<\Phi_n(t;\theta_0)-\Phi_n(t;-\infty)\le Ke^{\lambda_*(t+\theta_0)}$ for all $(n,t)\in\mathbb{Z}\times\mathbb{R}$, which implies that $\Phi(t;\theta_0)$ converges uniformly on $(n,t)\in\mathbb{Z}\times(-\infty,a]$ to $\Phi(t;-\infty)$ as $\theta_0\to-\infty$ for any $a\in\mathbb{R}$. For any sequence $\theta_0^k\to-\infty$ $(k\to+\infty)$, the functions $\Phi(t;\theta_0^k)$ converge to a solution of (1.1) in $\mathcal{T}$, which turns out to be $\Phi(t;-\infty)$. The limit does not depend on the sequence θ_0^k, whence all of the functions $\Phi(t;\theta_0)$ converge to $\Phi(t;-\infty)$ in $\mathcal{T}$ as $\theta_0\to-\infty$. The assertion as $\theta_0\to+\infty$ is obvious.

We next prove (viii). Assume $\chi=1$. Similarly to that in (vii), we denote

$$\Phi_{c_1,\dots,c_m;c'_1,\dots,c'_l;\theta_1,\dots,\theta_m;\theta'_1,\dots,\theta'_l;\theta_0}(t)$$

by $\Phi(t)_{\theta_i}$ and $\Phi_{c_1,\dots,c_{i-1},c_{i+1},\dots,c_m;c'_1,\dots,c'_l;\theta_1,\dots,\theta_{i-1},\theta_{i+1},\dots,\theta_m;\theta'_1,\dots,\theta'_l;\theta_0}(t)$ by $\Phi(t)_\infty$. Set

$$\psi^k(t)_{\theta_i} = \left\{\psi_n^k(s)_{\theta_i}\right\}_{k\in\mathbb{Z}},$$
$$\psi_n^k(s)_{\theta_i} = \max\left\{\max_{1\le j\le m}\phi_{c_j}\left(n+c_j(s-k)+\theta_j\right),\right.$$
$$\left.\max_{1\le j\le l}\phi_{c'_j}\left(-n+c'_j(s-k)+\theta'_j\right), \Gamma\left((s-k)+\theta_0\right)\right\},$$

and $v^k(t)_{\theta_i} = \{v_n^k(t)_{\theta_i}\}_{n\in\mathbb{Z}}$ with $v_n^k(t)_{\theta_i} := u_n(t+k;\psi^k(\cdot)_{\theta_i})$ for any $(n,t)\in\mathbb{Z}\times[-\tau-k,+\infty)$. Take

$$\psi^k(t)_\infty = \left\{\psi_n^k(s)_\infty\right\}_{k\in\mathbb{Z}},$$
$$\psi_n^k(s)_\infty = \max\left\{\max_{j\in\{1,\dots,i-1,i+1,\dots,m\}}\phi_{c_j}\left(n+c_j(s-k)+\theta_j\right),\right.$$
$$\left.\max_{1\le j\le l}\phi_{c'_j}\left(-n+c'_j(s-k)+\theta'_j\right), \Gamma\left((s-k)+\theta_0\right)\right\},$$

and $v^k(t)_\infty = \{v_n^k(t)_\infty\}_{n\in\mathbb{Z}}$ with $v_n^k(t)_\infty := u_n(t+k;\psi^k(\cdot)_\infty)$ for any $(n,t) \in \mathbb{Z} \times [-\tau-k, +\infty)$. Set $\hat{v}^k(t) = v^k(t)_{\theta_i} - v^k(t)_\infty = \{v_n^k(t)_{\theta_i} - v_n^k(t)_\infty\}_{n\in\mathbb{Z}}$. Then $\hat{v}^k(t)$ satisfies $0 \le \hat{v}_n^k(t) \le K$ for any $(n,t) \in \mathbb{Z} \times [-\tau-k, +\infty)$ and

$$\frac{d}{dt}\hat{v}_n^k(t) \le D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[\hat{v}_{n-i}^k(t) - \hat{v}_n^k(t)\right] - d\hat{v}_n^k(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\,\hat{v}_{n-i}^k(t-\tau).$$

Noting that $\hat{v}_n^k(s) \le \phi_{c_i}(n + c_i s + \theta_i) \le A_{c_i} e^{\lambda_1(c_i)(n+c_i s+\theta_i)}$ for any $s \in [-\tau-k, -k]$ and that $\bar{w}_n^k(t) = A_{c_i} e^{\lambda_1(c_i)(n+c_i t+\theta_i)}$ satisfies

$$\frac{d}{dt}\bar{w}_n^k(t) = D \sum_{i\in\mathbb{Z}\setminus\{0\}} I(i)\left[\bar{w}_{n-i}^k(t) - \bar{w}_n^k(t)\right] - d\bar{w}_n^k(t) + b'(0)\sum_{i\in\mathbb{Z}} J(i)\,\bar{w}_{n-i}^k(t-\tau)$$

for any $t \in [-\tau-k, +\infty)$, it follows from Theorem 3.4 that $0 \le \hat{v}_n^k(t) \le A_{c_i} e^{\lambda_1(c_i)(n+c_i t+\theta_i)}$ for any $(n,t) \in \mathbb{Z} \times [-\tau-k, +\infty)$ and $k \in \mathbb{N}$. Since $\lim_{k\to+\infty} v^k(t)_{\theta_i} = \Phi(t)_{\theta_i}$ and $\lim_{k\to+\infty} v^k(t)_\infty = \Phi(t)_\infty$, we have $0 < \Phi_n(t)_{\theta_i} - \Phi_n(t)_\infty \le A_{c_i} e^{\lambda_1(c_i)(n+c_i t+\theta_i)}$ for all $(n,t) \in \mathbb{Z} \times \mathbb{R}$, which implies that $\Phi(t)_{\theta_i}$ converges uniformly on $(n,t) \in \{n : n \le N_0, n \in \mathbb{Z}\} \times (-\infty, a]$ to $\Phi(t)_\infty$ as $\theta_i \to -\infty$ for any $N_0 \in \mathbb{Z}$ and $a \in \mathbb{R}$. For any sequence $\theta_i^k \to -\infty$ $(k \to +\infty)$, the functions $\Phi(t)_{\theta_i^k}$ converge to a solution of (1.1), which must be $\Phi(t)_\infty$. Since the limit is independent of the sequence θ_i^k, all of the functions $\Phi(t)_{\theta_i}$ converge to $\Phi(t)_\infty$ in $\mathcal{T}$ as $\theta_i \to -\infty$. The assertion as $\theta_j' \to -\infty$ and the case $\chi = 0$ can be proved similarly.

Using the inequality (1.8), we can prove (ii)–(vi) and (ix) of Theorem 1.1. ∎

6. Proof of Theorem 1.2.

In this section, we prove Theorem 1.2. We prove only the continuous dependence of the entire solution on the parameters c, c', θ, θ', and θ_0 and the uniqueness of the entire solutions satisfies (1.9). Other conclusions follow immediately from Theorem 1.1.

Consider (1.4) or

$$u_n'(t) = \frac{D}{2}\left[u_{n+1} + u_{n-1} - 2u_n\right] + f\left(u_n(t)\right), \tag{6.1}$$

where f satisfies the conditions given after (1.4). Let $\phi_c(n+ct)$ be a traveling wave front of (6.1) with wave speed $c > c^*$. As done in section 1, we normalize $\phi_c(n+ct)$ so that $\phi(0) = \frac{1}{2}$. Then the functions $\phi_c(z)$ are continuous with respect to $c \in (c^*, +\infty)$ in the norms $C^1_{loc}(\mathbb{R})$ (see [20, p. 1267] for the definition of these norms). Indeed, if $c_l \to c \in (c^*, +\infty)$, then by the unique boundedness of $|\phi_{c_l}'(z)|$ and $|\phi_{c_l}''(z)|$ in $z \in \mathbb{R}$ on $l \in \mathbb{N}$ and by a diagonal extraction process, there exists a subsequence c_{l_i} such that $\phi_{c_{l_i}} \to \phi$ in $C^1_{loc}(\mathbb{R})$, where ϕ is a solution of

$$c\phi'(z) = \frac{D}{2}\left[\phi(z+1) + \phi(z-1) - 2\phi(z)\right] + f(\phi(z)) \text{ in } z \in \mathbb{R}.$$

Obviously, ϕ is nondecreasing in $\mathbb{R}$ and is not a constant and $\phi(0) = \frac{1}{2}$. By the assumptions of f, we have $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Thus, ϕ is a traveling wave front of (6.1) with wave speed c. Following Chen and Guo [5], we have $\phi \equiv \phi_c$. Finally, the whole sequence $\phi_{c_l} \to \phi_{c_0}$ in $C^1_{loc}(\mathbb{R})$ as $l \to +\infty$. In view of [4, 5], we know that α_c and A_c defined by (1.6) and (1.7), respectively, are still valid for (6.1). In addition, there exactly is $\lambda_* = f'(0)$ for (6.1).

LEMMA 6.1. *For* (6.1), $\alpha_c = \lim_{z\to-\infty} \phi_c(z)\, e^{-\lambda_1(c)z}$ *is continuous in* $c \in (c^*, +\infty)$.

Proof. Fix $c_0 \in (c^*, +\infty)$ and let $c_l \to c_0$ as $l \to +\infty$ with $c_l > c^*$ for each $l \in \mathbb{N}$. Then by Chen and Guo [4] (see also Ma, Weng, and Zou [25]), we know that, for each $c \in (c^*, +\infty)$, there exists a unique traveling wave front $\widetilde{\phi}_c$ such that $\widetilde{\phi}_c'(\cdot) > 0$, $\widetilde{\phi}_c(-\infty) = 0$, $\widetilde{\phi}_c(+\infty) = 1$, and

$$\lim_{z \to -\infty} \widetilde{\phi}_c(z) e^{-\lambda_1(c)z} = 1.$$

Then we have that $\widetilde{\phi}_{c_l} \to \widetilde{\phi}_{c_0}$ in $C^1_{loc}(\mathbb{R})$ as $l \to +\infty$. In fact, since $c_l \to c_0$ as $l \to +\infty$, then there exist a subsequence c_{l_i} and a function $\widetilde{\phi}$ such that $\widetilde{\phi}_{c_{l_i}} \to \widetilde{\phi}$ in $C^1_{loc}(\mathbb{R})$, where $\widetilde{\phi}$ is nondecreasing and satisfies

$$c\widetilde{\phi}'(z) = \frac{D}{2}\left[\widetilde{\phi}(z+1) + \widetilde{\phi}(z-1) - 2\widetilde{\phi}(z)\right] + f\left(\widetilde{\phi}(z)\right) \text{ in } z \in \mathbb{R}.$$

On the other hand, by Chen and Guo [4], there exist two constants $q > 1$ and $\beta > 1$, independent of c_l, such that

$$e^{\lambda_1(c_l)z} - qe^{\beta\lambda_1(c_l)z} \le \phi_{c_l}(z) \le e^{\lambda_1(c_l)z} + qe^{\beta\lambda_1(c_l)z} \text{ for any } z \in \mathbb{R}$$

and therefore, as $l \to \infty$,

$$e^{\lambda_1(c_0)z} - qe^{\beta\lambda_1(c_0)z} \le \widetilde{\phi}(z) \le e^{\lambda_1(c_0)z} + qe^{\beta\lambda_1(c_0)z} \text{ for any } z \in \mathbb{R},$$

which implies that $\widetilde{\phi}(z)$ is not a constant and satisfies $\widetilde{\phi}(z)e^{-\lambda_1(c_0)z} = 1$. Then it follows from Chen and Guo [5] (see also Ma, Weng, and Zou [25]) that $\widetilde{\phi} \equiv \widetilde{\phi}_{c_0}$. Consequently, the whole sequence $\widetilde{\phi}_{c_l} \to \widetilde{\phi}_{c_0}$ in $C^1_{loc}(\mathbb{R})$ as $l \to +\infty$.

Now let $\widetilde{\phi}_{c_0}(\varsigma_0) = \frac{1}{2}$ and $\widetilde{\phi}_{c_l}(\varsigma_l) = \frac{1}{2}$. Then we have that $\varsigma_l \to \varsigma_0$ as $l \to +\infty$. Assume that this assertion is not true. Take $\varsigma_l \to \overline{\varsigma} \neq \varsigma_0$ as $l \to +\infty$ (up to extraction of some subsequence). If $|\overline{\varsigma}| < \infty$, then, by $\widetilde{\phi}_{c_l} \to \widetilde{\phi}_{c_0}$ in $C^1_{loc}(\mathbb{R})$ as $l \to +\infty$, we have $\widetilde{\phi}_{c_0}(\overline{\varsigma}) = \frac{1}{2} = \widetilde{\phi}_{c_0}(\varsigma_0)$, which is impossible since $\frac{d}{dz}\widetilde{\phi}_{c_0}(z) > 0$ for any $z \in \mathbb{R}$ and $\overline{\varsigma} \neq \varsigma_0$. If $\overline{\varsigma} = +\infty$, then $\widetilde{\phi}_{c_l}(\varsigma_0 + 1) < \widetilde{\phi}_{c_l}(\varsigma_l) = \frac{1}{2}$ for sufficiently large l implies that $\widetilde{\phi}_{c_0}(\varsigma_0 + 1) \le \frac{1}{2} = \widetilde{\phi}_{c_0}(\varsigma_0)$, which is also impossible. Similarly, $\overline{\varsigma} = -\infty$ is impossible. Thus, we conclude that $\varsigma_l \to \varsigma_0$ as $l \to +\infty$.

Again applying Chen and Guo [5], we have that $\phi_{c_0}(\cdot) = \widetilde{\phi}_{c_0}(\varsigma_0 + \cdot)$ and $\phi_{c_l}(\cdot) = \widetilde{\phi}_{c_l}(\varsigma_l + \cdot)$. Since

$$\lim_{z \to -\infty} \phi_{c_0}(z) e^{-\lambda_1(c_0)(z+\varsigma_0)} = \lim_{z \to -\infty} \widetilde{\phi}_{c_0}(z + \varsigma_0) e^{-\lambda_1(c_0)(z+\varsigma_0)} = 1,$$

we have $\lim_{z\to-\infty} \phi_{c_0}(z)e^{-\lambda_1(c_0)z} = e^{\lambda_1(c_0)\varsigma_0} = \alpha_{c_0}$. Similarly, we have

$$\lim_{z \to -\infty} \phi_{c_l}(z) e^{-\lambda_1(c_l)z} = e^{\lambda_1(c_l)\varsigma_l} = \alpha_{c_l}.$$

Finally, there holds $\alpha_{c_l} \to \alpha_{c_0}$ as $l \to +\infty$. This completes the proof. ∎

Recall that

$$A_c = \inf\left\{A > 0 : Ae^{\lambda_1(c)z} \ge \phi_c(z) \text{ in } z \in \mathbb{R}\right\}.$$

LEMMA 6.2. *For* (6.1), A_c *is continuous on* $c \in (c^*, +\infty)$.

Proof. Fix $c_0 \in (c^*, +\infty)$ and let $c_l \to c_0$ as $l \to +\infty$ with $c_l > c^*$ for each $l \in \mathbb{N}$. We prove the theorem by way of contradiction. Assume $A_{c_l} \to A_0 \in \mathbb{R} \cup \{\infty\}$

as $l \to \infty$ (up to extraction of some subsequence) and $A_0 \neq A_{c_0}$. Since $A_{c_l} \geq e^{-\lambda_1(c_l)z}\phi_{c_l}(z)$ for any $z \in \mathbb{R}$, $A_0 \geq e^{-\lambda_1(c_0)z}\phi_{c_0}(z)$ and, hence, $A_0 > A_{c_0}$. Fix $b = \min\{\frac{A_0+A_{c_0}}{2}, A_{c_0}+1\}$. Then there exists $L \in \mathbb{N}$ such that for any $l > L$, $A_{c_l} > b$. On the other hand, since $\alpha_{c_l} \to \alpha_{c_0} \leq A_{c_0}$ and $\lambda_1(c_l) \to \lambda_1(c_0)$, there exists a constant $Z_0 > 0$, independent of c_l, such that $\phi_{c_l}(z)e^{-\lambda_1(c_l)z} \leq b$ for any $|z| > Z_0$. For $z \in [-Z_0, Z_0]$, by $\phi_{c_0}(z)e^{-\lambda_1(c_0)z} \leq A_{c_0}$, $\phi_{c_l}(z) \to \phi_{c_0}(z)$ in $C^1_{loc}(\mathbb{R})$, and the equicontinuity of $e^{-\lambda_1(c_l)z}$ on l, there exists $L' > L$ such that $\phi_{c_l}(z)e^{-\lambda_1(c_l)z} \leq b$ for any $l > L'$ and $z \in [-Z_0, Z_0]$. Therefore, $\phi_{c_l}(z)e^{-\lambda_1(c_l)z} \leq b$ for any $l > L'$ and $z \in \mathbb{R}$, which contradicts $A_{c_l} > b$ for any $l > L$. ∎

Before proving Theorem 1.2, we first consider the following linear Cauchy problem:

$$\begin{cases} \frac{d}{dt}u_n(t) = \frac{D}{2}\left[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)\right] + f'(0)\,u_n(t), \quad t > 0, \\ u_n(0) = u_n^0, \end{cases} \tag{6.2}$$

where $u^0 = \{u_n^0\}_{n\in\mathbb{Z}} \in l^\infty$. By Theorem 3.3, we know that (6.2) admits a unique solution $u(t) = u(t;u^0) = \{u_n(t)\}_{n\in\mathbb{Z}}$ on $t \in [0,+\infty)$. By using the discrete Fourier transformation, we can exactly solve the solution $u(t) = \{u_n(t)\}_{n\in\mathbb{Z}}$ of (6.2) as follows:

$$u_n(t) = \frac{1}{\pi}e^{f'(0)t}\sum_{i=-\infty}^{+\infty} u_i^0 \int_0^\pi \cos\left((i-n)\omega\right)e^{Dt(\cos\omega - 1)}d\omega. \tag{6.3}$$

This formulation is very crucial for the proof of Theorem 1.2.

Proof of Theorem 1.2. We prove only the case $\varrho = \varrho' = \chi = 1$. Consider a sequence

$$(c_k, c_k', \theta_k, \theta_k', \theta_{0,k}) \to (c, c', \theta, \theta', \theta_0) \in (c^*, +\infty)^2 \times \mathbb{R}^3 \quad \text{as} \quad k \to +\infty.$$

For given $(c_k, c_k', \theta_k, \theta_k', \theta_{0,k})$ and $(c, c', \theta, \theta', \theta_0)$, it follows from Theorem 1.1 that there exist entire solutions $\Phi_{c_k;c_k';\theta_k;\theta_k';\theta_{0,k}}(t)$ and $\Phi_{c;c';\theta;\theta';\theta_0}(t)$ of (1.4) satisfying (1.9). For the sake of convenience, set $\Phi^k(t) = \{\Phi_n^k(t)\}_{n\in\mathbb{Z}} = \Phi_{c_k;c_k';\theta_k;\theta_k';\theta_{0,k}}(t)$ and $\Phi(t) = \{\Phi_n(t)\}_{n\in\mathbb{Z}} = \Phi_{c;c';\theta;\theta';\theta_0}(t)$.

Using Lemma 5.1, there exists a function $\widetilde{\Phi}(t) = \{\widetilde{\Phi}_n(t)\}_{n\in\mathbb{Z}}$ such that $\Phi_n^k(t) \to \widetilde{\Phi}_n(t)$ as $k \to \infty$ (up to extraction of some subsequence) in $\mathcal{T}$. In particular, the function $\widetilde{\Phi}(t) = \{\widetilde{\Phi}_n(t)\}_{n\in\mathbb{Z}}$ is also an entire solution of (6.1) (or (1.4)). By passage to the limit $k \to +\infty$ in (1.9), the function $\widetilde{\Phi}(t) = \{\widetilde{\Phi}_n(t)\}_{n\in\mathbb{Z}}$ fulfills the estimates

$$\begin{aligned} &\max\left\{\phi_c(n+ct+\theta), \Gamma(t+\theta_0), \phi_{c'}(-n+c't+\theta')\right\} \\ &\leq \widetilde{\Phi}_n(t) \leq \min\Big\{1, \phi_c(n+ct+\theta) + e^{\lambda_*(t+\theta_0)} + A_{c'}e^{\lambda_1(c')(-n+c't+\theta')}, \\ &\qquad \Gamma(t+\theta_0) + A_c e^{\lambda_1(c)(n+ct+\theta)} + A_{c'}e^{\lambda_1(c')(-n+c't+\theta')}, \\ &\qquad \phi_{c'}(-n+c't+\theta') + e^{\lambda_*(t+\theta_0)} + A_c e^{\lambda_1(c)(n+ct+\theta)}\Big\} \end{aligned} \tag{6.4}$$

for any $(n,t) \in \mathbb{Z}\times\mathbb{R}$.

Let us now prove that $\widetilde{\Phi}_n(t) = \Phi_n(t)$ for any $(n,t) \in \mathbb{Z}\times\mathbb{R}$. Recall that the functions $v^k(t) = \{v_n^k(t)\}_{n\in\mathbb{N}}$, which are solutions of the Cauchy problems

$$\frac{d}{dt}v_n^k(t) = \frac{D}{2}\left[v_{n+1}^k(t) + v_{n-1}^k(t) - 2v_n^k(t)\right] + f\left(v_n^k(t)\right), \; t > -k, \; n \in \mathbb{N},$$

with the initial conditions

$$v_n^k(-k) = v_{n,0}^k = \max\left\{\phi_c(n-ck+\theta), \Gamma(-k+\theta_0), \phi_{c'}(-n-c'k+\theta')\right\},$$

converge to the function $\Phi(t) = \{\Phi_n(t)\}_{n\in\mathbb{N}}$ in $\mathcal{T}$; see the proof of Theorem 1.1. Let us now compare the functions $\widetilde{\Phi}(t)$ to the functions $v^k(t)$ for $t > -k$. Following (6.4), we get that

$$\text{(6.5)}\qquad \begin{aligned} &\left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| \\ &\le \begin{cases} e^{f'(0)(-k+\theta_0)} + A_{c'}e^{\lambda_1(c')(-n-c'k+\theta')} & \text{if } n \ge n_k^+ + 2, \\ A_c e^{\lambda_1(c)(n-ck+\theta)} + A_{c'}e^{\lambda_1(c')(-n-c'k+\theta')} & \text{if } n_k^- + 1 \le n \le n_k^+ - 1, \\ e^{f'(0)(-k+\theta_0)} + A_c e^{\lambda_1(c)(n-ck+\theta)} & \text{if } n \le n_k^- - 2 \end{cases} \end{aligned}$$

for sufficiently large k, where n_k^- and n_k^+ are two integers defined as follows:

$$n_k^- = \operatorname{int}\left[-c'k + \frac{f'(0)}{\lambda_1(c')}k + \theta' - \frac{1}{\lambda_1(c')}\ln\frac{1}{\alpha_{c'}} - \frac{f'(0)}{\lambda_1(c')}\theta_0\right],$$
$$n_k^+ = \operatorname{int}\left[ck - \frac{f'(0)}{\lambda_1(c)}k - \theta + \frac{1}{\lambda_1(c)}\ln\frac{1}{\alpha_c} + \frac{f'(0)}{\lambda_1(c)}\theta_0\right].$$

n_k^- is obtained by comparing $\phi_{c'}(-n-c'k+\theta')$ and $\Gamma(-k+\theta_0)$ and using the asymptotic behaviors of $\phi_{c'}$ and Γ. Similarly, we obtain n_k^+.

For any $x \in \mathbb{R}$, define

$$\operatorname{int}[x] = \max\{m : m \in \mathbb{Z}, m \le x\}.$$

Fix $t_0 > -k$. For any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $k > -t_0$, define

$$a_n^k = \frac{1}{2\pi}e^{-D(t_0+k)}\int_0^\pi \cos(nw)\, e^{D(t_0+k)\cos w}dw.$$

By Weng, Huang, and Wu [40], we know that $a_n^k = a_{-n}^k > 0$ for any $n \in \mathbb{Z}$ and $\sum_{n=-\infty}^{+\infty} a_n^k = 1$. Furthermore, for $n > 1$ there is

$$\begin{aligned} a_n^k &= \frac{1}{2\pi}\int_0^\pi \cos(nw)\, e^{D(t_0+k)[\cos w - 1]}dw = \frac{1}{2n\pi}\int_0^\pi e^{D(t_0+k)[\cos w-1]}d\sin(nw) \\ &= \frac{D(t_0+k)}{2n\pi}\int_0^\pi \sin(nw)\sin w e^{D(t_0+k)[\cos w-1]}dw \\ &= \frac{D(t_0+k)}{4n\pi}\int_0^\pi \left[\cos((n-1)w) - \cos((n+1)w)\right]e^{D(t_0+k)[\cos w-1]}dw \\ &= \frac{D(t_0+k)}{2n}\left(a_{n-1}^k - a_{n+1}^k\right), \end{aligned}$$

which implies that $a_{n-1}^k > a_{n+1}^k$ for any $n \in \mathbb{N}$. By the symmetry, $a_{-n-1}^k < a_{-n+1}^k$ for any $n \in \mathbb{N}$.

We claim that, for any given $\eta \in (0, +\infty)$, there hold

$$\text{(6.6)}\qquad a_{\operatorname{int}[\eta k]}^k \to 0 \quad\text{and}\quad a_{\operatorname{int}[\eta k]-1}^k \to 0 \quad\text{as}\quad k \to +\infty.$$

Consider $a^k_{\text{int}[\eta k]} \to 0$ as $k \to +\infty$. If the assertion is false, then we can assume that $a^k_{\text{int}[\eta k]} \to \delta > 0$ as $k \to +\infty$ (up to extraction of some subsequence). Taking k sufficiently large such that $\frac{\text{int}[\eta k]}{2} > \frac{4}{\delta} + 1$ and $a^k_{\text{int}[\eta k]} > \frac{\delta}{2}$, by $a^k_{n-1} > a^k_{n+1}$ for $n \in \mathbb{N}$, we have $\sum_{n=0}^{\text{int}[\eta k]} a^k_n > 1$, which contradicts the fact $\sum_{n=-\infty}^{+\infty} a^k_n < 1$. Therefore, $\lim_{k\to+\infty} a^k_{\text{int}[\eta k]} = 0$. Similarly, we have $\lim_{k\to+\infty} a^k_{\text{int}[\eta k]-1} = 0$. For any $m > N$,

$$\begin{aligned}
\sum_{n=N\geq 1}^{m} a^k_n &= \frac{D\,(t_0+k)}{2N}\left(a^k_{N-1} - a^k_{N+1}\right) + \frac{D\,(t_0+k)}{2(N+1)}\left(a^k_N - a^k_{N+2}\right) \\
&\quad + \frac{D\,(t_0+k)}{2(N+2)}\left(a^k_{N+1} - a^k_{N+3}\right) \\
&\quad + \cdots + \frac{D\,(t_0+k)}{2(m-1)}\left(a^k_{m-2} - a^k_m\right) + \frac{D\,(t_0+k)}{2m}\left(a^k_{m-1} - a^k_{m+1}\right) \\
&= \frac{D\,(t_0+k)}{2N}a^k_{N-1} + \frac{D\,(t_0+k)}{2(N+1)}a^k_N - \frac{D\,(t_0+k)}{N\,(N+2)}a^k_{N+1} \\
&\quad - \frac{D\,(t_0+k)}{(N+1)\,(N+3)}a^k_{N+2} - \cdots - \frac{D\,(t_0+k)}{(m-3)\,(m-1)}a^k_{m-2} \\
&\quad - \frac{D\,(t_0+k)}{(m-2)\,m}a^k_{m-1} - \frac{D\,(t_0+k)}{2(m-1)}a^k_m - \frac{D\,(t_0+k)}{2m}a^k_{m+1} \\
&\leq \frac{D\,(t_0+k)}{2N}a^k_{N-1} + \frac{D\,(t_0+k)}{2(N+1)}a^k_N.
\end{aligned}$$

Therefore,

$$\sum_{n=N\geq 1}^{\infty} a^k_n \leq \frac{D\,(t_0+k)}{2N}a^k_{N-1} + \frac{D\,(t_0+k)}{2(N+1)}a^k_N.$$

Consequently, we obtain that for any given $\eta > 0$,

$$\sum_{n=\text{int}[\eta k]}^{\infty} a^k_n \to 0 \quad \text{and} \quad \sum_{n=\text{int}[\eta k]+1}^{\infty} a^k_n \to 0 \quad \text{as} \quad k \to +\infty \tag{6.7}$$

due to $\frac{D(t_0+k)}{[\eta k]} \to \frac{D}{\eta}$ as $k \to +\infty$.

Now fix $(n_0, t_0) \in \mathbb{Z} \times \mathbb{R}$; we estimate $\widetilde{\Phi}_{n_0}(t_0) - v^k_{n_0}(t_0)$ as $k \to +\infty$. We compare $\widetilde{\Phi}_n(t) - v^k_n(t)$ with a solution of the linear equation

$$\frac{d}{dt}u_n\,(t) = \frac{D}{2}\left[u_{n+1}\,(t) + u_{n-1}\,(t) - 2u_n\,(t)\right] + f'\,(0)\,u_n\,(t)\,, \quad t > -k,$$

with the initial condition $u_n(-k) = |\widetilde{\Phi}_n(-k) - v^k_n(-k)|$. Using (6.3), we deduce that

$$
\begin{aligned}
&\left|\widetilde{\Phi}_{n_0}(t_0) - v_{n_0}^k(t_0)\right| \\
&\le \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=-\infty}^{+\infty} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\le \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^+ +2}^{+\infty} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\quad + \frac{1}{\pi} e^{f'(0)(t_0+k)} \int_0^\pi \cos\left(\left(n_k^+ + 1 - n_0\right) w\right) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\quad + \frac{1}{\pi} e^{f'(0)(t_0+k)} \int_0^\pi \cos\left(\left(n_k^+ - n_0\right) w\right) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\quad + \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^- +1}^{n_k^+ -1} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\quad + \frac{1}{\pi} e^{f'(0)(t_0+k)} \int_0^\pi \cos\left(\left(n_k^- - n_0\right) w\right) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\quad + \frac{1}{\pi} e^{f'(0)(t_0+k)} \int_0^\pi \cos\left(\left(n_k^- - 1 - n_0\right) w\right) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\quad + \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=-\infty}^{n_k^- -2} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw.
\end{aligned}
$$

Call I, II, III, IV, V, VI, and VII the seven terms on the right-hand side of this last inequality. We have

$$
\begin{aligned}
\mathrm{I} &= \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^+ +2}^{+\infty} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\le \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^+ +2}^{+\infty} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} e^{f'(0)(-k+\theta_0)} dw \\
&\quad + \frac{A_{c'}}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^+ +2}^{+\infty} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} e^{\lambda_1(c')(-n-c'k+\theta')} dw \\
&= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
\mathrm{I}_1 &= \frac{1}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^+ +2}^{+\infty} \int_0^\pi \cos((n-n_0)w) e^{D(t_0+k)[\cos w-1]} e^{f'(0)(-k+\theta_0)} dw \\
&= 2e^{f'(0)(t_0+\theta_0)} \sum_{n=n_k^+ +2}^{+\infty} a_{n-n_0}^k \to 0 \text{ as } k \to +\infty
\end{aligned}
$$

due to the fact $\frac{n_k^+ + 2 - n_0}{k} > \frac{c\lambda_1(c) - f'(0)}{2\lambda_1(c)} > 0$ for sufficiently large k and (6.7). In addition, we have

$$\begin{aligned}
\mathrm{I}_2 &= \frac{A_{c'}}{\pi} e^{f'(0)(t_0+k)} \sum_{n=n_k^+ +2}^{+\infty} \int_0^\pi \cos\left((n-n_0)\,w\right) e^{D(t_0+k)[\cos w - 1]} e^{\lambda_1(c')\left(-n-c'k+\theta'\right)} dw \\
&= 2A_{c'} e^{\left(f'(0)-\lambda_1(c')c'\right)k} e^{f'(0)\left(t_0+\theta'\right)} \sum_{n=n_k^+ +2}^{+\infty} a_{n-n_0}^k e^{-\lambda_1(c')n} \\
&\le 2A_{c'} e^{f'(0)\left(t_0+\theta'\right)} \sum_{n=n_k^+ +2}^{+\infty} a_{n-n_0}^k \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Consider II. In this case, let

$$0 \le \delta = \left(ck - \frac{f'(0)}{\lambda_1(c)}k - \theta + \frac{1}{\lambda_1(c)} \ln \frac{1}{\alpha_c} + \frac{f'(0)}{\lambda_1(c)}\theta_0\right) - n_k^+ < 1.$$

Then we have

$$\begin{aligned}
\mathrm{II} &= 2e^{f'(0)(t_0+k)} a_{n_k^+ +1-n_0}^k \left|\widetilde{\Phi}_{n_k^+ +1}(-k) - v_{n_k^+ +1}^k(-k)\right| \\
&\le 2e^{f'(0)(t_0+k)} a_{n_k^+ +1-n_0}^k \left[e^{f'(0)(-k+\theta_0)} + A_{c'} e^{\lambda_1(c')\left(-n_k^+ -1-c'k+\theta'\right)}\right] \\
&\quad + 2e^{f'(0)(t_0+k)} a_{n_k^+ +1-n_0}^k \left[A_{c'} e^{\lambda_1(c')\left(-n_k^+ -1-c'k+\theta'\right)} + A_c e^{\lambda_1(c)\left(n_k^+ +1-ck+\theta\right)}\right] \\
&= 2e^{f'(0)(t_0+k)} a_{n_k^+ +1-n_0}^k \left[e^{f'(0)(-k+\theta_0)} + 2A_{c'} e^{\lambda_1(c')\left(-n_k^+ -1-c'k+\theta'\right)}\right] \\
&\quad + 2e^{f'(0)(t_0+k)} a_{n_k^+ +1-n_0}^k A_c e^{\lambda_1(c)\left\{\left(ck - \frac{f'(0)}{\lambda_1(c)}k - \theta + \frac{1}{\lambda_1(c)} \ln \frac{1}{\alpha_c} + \frac{f'(0)}{\lambda_1(c)}\theta_0\right) - \delta + 1 - ck + \theta\right\}} \\
&= 2e^{f'(0)(t_0+k)} a_{n_k^+ +1-n_0}^k \left[e^{f'(0)(-k+\theta_0)} + 2A_{c'} e^{\lambda_1(c')\left(-n_k^+ -1-c'k+\theta'\right)}\right] \\
&\quad + \frac{2A_c}{\alpha_c} e^{\lambda_1(c)(1-\delta)+f'(0)(t_0+\theta_0)} a_{n_k^+ +1-n_0}^k \\
&\to 0 \qquad \text{as} \qquad k \to +\infty.
\end{aligned}$$

Similarly, we have that III$\to 0$ as $k \to +\infty$. For IV, we have

$$\begin{aligned}
\mathrm{IV} &= \frac{1}{\pi} e^{f'(0)(t_0+k)} \\
&\quad \times \sum_{n=n_k^- +1}^{n_k^+ -1} \int_0^\pi \cos\left((n-n_0)\,w\right) e^{D(t_0+k)[\cos w - 1]} \left|\widetilde{\Phi}_n(-k) - v_n^k(-k)\right| dw \\
&\le 2e^{f'(0)(t_0+k)} \sum_{n=n_k^- +1}^{n_k^+ -1} a_{n-n_0}^k \left[A_c e^{\lambda_1(c)(n-ck+\theta)} + A_{c'} e^{\lambda_1(c')\left(-n-c'k+\theta'\right)}\right] \\
&= 2e^{f'(0)(t_0+k)} \sum_{n=0}^{n_k^+ -1} a_{n-n_0}^k \left[A_c e^{\lambda_1(c)(n-ck+\theta)} + A_{c'} e^{\lambda_1(c')\left(-n-c'k+\theta'\right)}\right] \\
&\quad + 2e^{f'(0)(t_0+k)} \sum_{n=n_k^- +1}^{-1} a_{n-n_0}^k \left[A_c e^{\lambda_1(c)(n-ck+\theta)} + A_{c'} e^{\lambda_1(c')\left(-n-c'k+\theta'\right)}\right] \\
&= \mathrm{IV}_1 + \mathrm{IV}_2.
\end{aligned}$$

Consider IV_1. Then

$$
\begin{aligned}
\mathrm{IV}_1 &= 2e^{f'(0)(t_0+k)} \sum_{n=0}^{n_k^+-1} a_{n-n_0}^k \left[A_c e^{\lambda_1(c)(n-ck+\theta)} + A_{c'} e^{\lambda_1(c')(-n-c'k+\theta')}\right] \\
&\le 2A_{c'} e^{-(\lambda_1(c')c'-f'(0))k} e^{f'(0)t_0+\lambda_1(c')\theta'} \sum_{n=0}^{n_k^+} a_{n-n_0}^k \\
&\quad + 2A_c e^{f'(0)(t_0+k)} \sum_{n=\mathrm{int}[n_k^+/2]}^{n_k^+-1} a_{n-n_0}^k e^{\lambda_1(c)(n-ck+\theta)} \\
&\quad + 2A_c e^{f'(0)(t_0+k)} \sum_{n=0}^{\mathrm{int}[n_k^+/2]} a_{n-n_0}^k e^{\lambda_1(c)(n-ck+\theta)} \\
&\le 2A_{c'} e^{-(\lambda_1(c')c'-f'(0))k} e^{f'(0)t_0+\lambda_1(c')\theta'} \\
&\quad + 2A_c e^{f'(0)(t_0+k)} e^{\lambda_1(c)((n_k^+-1)-ck+\theta)} \sum_{n=\mathrm{int}[n_k^+/2]}^{\infty} a_{n-n_0}^k \\
&\quad + 2A_c e^{\lambda_1(c)(\mathrm{int}[n_k^+/2]-ck+\theta)} e^{f'(0)(t_0+k)} \\
&= \mathrm{IV}_1^1 + \mathrm{IV}_1^2 + \mathrm{IV}_1^3.
\end{aligned}
$$

It is easy to see that $\mathrm{IV}_1^1 \le 2A_{c'} e^{-(\lambda_1(c')c'-f'(0))k} e^{f'(0)t_0+\lambda_1(c')\theta'} \to 0$ as $k \to +\infty$, because $\lambda_1(c')c' - f'(0) > 0$. By virtue of

$$
\begin{aligned}
&e^{f'(0)(t_0+k)} e^{\lambda_1(c)((n_k^+-1)-ck+\theta)} \\
&\quad \le A_c e^{f'(0)(t_0+k)} e^{\lambda_1(c)\left\{\left(ck-\frac{f'(0)}{\lambda_1(c)}k-\theta+\frac{1}{\lambda_1(c)}\ln\frac{1}{\alpha_c}+\frac{f'(0)}{\lambda_1(c)}\theta_0\right)-1-ck+\theta\right\}} \\
&\quad = \frac{1}{\alpha_c} e^{f'(0)(t_0+\theta_0)-\lambda_1(c)}
\end{aligned}
$$

and $\frac{n_k^+}{2} - n_0 > \frac{c\lambda_1(c)-f'(0)}{4\lambda_1(c)}k$ for sufficiently large k, we have

$$
\mathrm{IV}_1^2 = 2A_c e^{f'(0)(t_0+k)} e^{\lambda_1(c)((n_k^+-1)-ck+\theta)} \sum_{n=\mathrm{int}[n_k^+/2]}^{\infty} a_{n-n_0}^k \to 0 \text{ as } k \to +\infty.
$$

Moreover, we have

$$
\begin{aligned}
\mathrm{IV}_1^3 &= 2A_c e^{\lambda_1(c)(\mathrm{int}[n_k^+/2]-ck+\theta)} e^{f'(0)(t_0+k)} \\
&\le 2A_c e^{\lambda_1(c)\left(\frac{1}{2}\left[ck-\frac{f'(0)}{\lambda_1(c)}k-\theta+\frac{1}{\lambda_1(c)}\ln\frac{1}{\alpha_c}+\frac{f'(0)}{\lambda_1(c)}\theta_0\right]-ck+\theta\right)} e^{f'(0)(t_0+k)} \\
&= 2A_c e^{f'(0)t_0+\frac{1}{2}\ln\frac{1}{\alpha_c}+\frac{1}{2}f'(0)\theta_0+\frac{1}{2}\lambda_1(c)\theta} e^{-\frac{1}{2}[\lambda_1(c)c-f'(0)]k} \\
&\to 0 \text{ as } k \to +\infty.
\end{aligned}
$$

We can prove that IV_2, V, VI, and VII converge to zero as $k \to +\infty$ by arguments similar to those for IV_1, III, II, and I, respectively.

Eventually, $|\widetilde{\Phi}_{n_0}(t_0) - v_{n_0}^k(t_0)| \to 0$ as $k \to +\infty$. Since $v_{n_0}^k(t_0) \to \Phi_{n_0}(t_0)$ and $(n_0, t_0) \in \mathbb{Z} \times \mathbb{R}$ is arbitrary, we obtain that $\Phi_n(t) = \widetilde{\Phi}_n(t)$ for any $(n,t) \in \mathbb{Z} \times \mathbb{R}$. The limit function being unique, the whole sequence Φ^k converges to Φ as $k \to +\infty$.

Using the same estimates as above, we can prove that the entire solution of (1.4) satisfying (1.9) is unique. ∎

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