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Threshold dynamics of a delayed reaction diffusion equation subject to the Dirichlet condition

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We establish the threshold dynamics of a delayed reaction diffusion equation subject to the homogeneous Dirichlet boundary condition when the delayed reaction term is non-monotone. We illustrate the main results by two examples, including the delayed Nicholson's blowflies diffusion equation.

Keywords: reaction diffusive equation; global stability; Dirichlet boundary value; non-monotone dynamical system

2000 Mathematics Subject Classifications: Primary: 35B40; Secondary: 35B35; 92D25

1. Introduction

Our focus here is on the threshold dynamics of reaction diffusion equations with nonlinear and delayed non-monotone reaction terms under the Dirichlet boundary condition. This type of system, including Nicholson's blowflies equation and the logistic-type delay diffusion equation, arises naturally from modelling population dynamics and spatial ecology.

Much has been done in the case where the space is a one-dimensional unbounded domain in association with the consideration of traveling wavefronts, see [7,8,9] and references therein. There are also substantial studies about asymptotic properties of solutions to the logistic-type delayed diffusion equation under the Dirichlet/Neumann boundary condition, see, for example, [1,2,4,12,14,20,29]. Unfortunately, results about the global dynamics for Nicholson's blowflies equation under the Dirichlet boundary condition seem to be quite scarce [22,28] due to the difficulty in describing the non-negative non-trivial steady-state solution analytically.

By applying the theory of monotone dynamical systems and the Krein–Rutman theorem, So and Zhao [24] obtained a threshold result on the global dynamics of scalar reaction diffusion equations with time delays, and they also applied their general results to the diffusive Nicholson's blowflies equation where the nonlinearity satisfies a certain monotonicity property in the domain under consideration. In [22], So and Yang studied the asymptotic behaviour of the diffusive

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Nicholson's blowflies equation subject to the Neumann boundary condition and with delayed non-monotone feedbacks. In [28], Wu and Zhao introduced a new ordering in the phase space with respect to which some reaction diffusion equations with non-linear and delayed reaction terms (which are not necessarily monotone) can generate monotone semiflows so that the global dynamics (in particular, threshold dynamics) of some diffusive population models with delayed and non-local effects can be described by using the order-preserving property and some general results of monotone dynamical systems.

None of these results can be applied to examine the global dynamical behaviours of solutions to the following delayed reaction diffusion equation subject to the Dirichlet boundary condition:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= d\Delta u(t,x) - \mu u(t,x) + \mu f(u(t-\tau,x)) \quad \text{in } D \equiv (0,\infty)\times\Omega, \\ u|_{\partial\Omega} &= 0 \quad \text{on } \Gamma \equiv (0,\infty)\times\partial\Omega, \\ u(\theta,x) &= \varphi(\theta,x) \quad \text{for } (\theta,x)\in[-\tau,0]\times\overline{\Omega}. \end{aligned} \tag{1}$$

Here, Ω is a bounded domain with smooth boundary $\partial\Omega$, Δ is the Laplacian operator, d, μ, and τ are positive parameters, $f:\mathbb{R}_+\equiv[0,\infty)\to\mathbb{R}_+$ is a local Lipschitz function with $f(0)=0$, and the initial function φ is assumed to be a continuous and non-negative function with respect to the two variables. Special cases of Equation (1) have been studied. See [5,6,10,17,18,23,25,30]. Recently, by combining a dynamical system argument with the maximum principle as well as some subtle inequalities, Yi and Zou [31] established the global attractivity of the positive steady state of the diffusive Nicholson's equation with the homogeneous Neumann boundary condition, and their result can be applied to the case where the reaction is non-monotone.

The purpose of this paper is to modify some of the arguments in Yi and Zou [31] to describe the threshold dynamics of Equation (1) that will include results in [22] as special cases. As shall be shown, the modification seems to be significant, and some rather different approaches must be taken while dealing with the Dirichlet problem. In Section 2, we establish the threshold dynamics of Equation (1), and in Section 3, we provide illustrative examples.

2. Threshold dynamics

Let $\mathbb{R}$ denote the set of all real numbers. Let $C_0=\{\varphi\in C(\overline{\Omega},\mathbb{R}):\varphi|_{\partial\Omega}=0\}$ and $X=\{\phi\in C([-\tau,0]\times\overline{\Omega},\mathbb{R}):\phi|_{[-\tau,0]\times\partial\Omega}=0\}$ be equipped with the usual supremum norm $\|\cdot\|$. Note that we use φ for an element in $C(\overline{\Omega})$, and ϕ for an element in $C([-\tau,0]\times\overline{\Omega})$.

Also let $C_+=\{\varphi\in C_0:\varphi|_{\overline{\Omega}}\geq 0\}$ and $X_+=\{\phi\in X:\phi|_{[-\tau,0]\times\overline{\Omega}}\geq 0\}$. For $a\in\mathbb{R}$, $\hat{a}\in C_0$ is defined as $\hat{a}(x)=a$ for $x\in\overline{\Omega}$. Similarly, $\hat{\hat{a}}\in X$ is defined as $\hat{\hat{a}}(\theta,x)=a$ for $(\theta,x)\in[-\tau,0]\times\overline{\Omega}$. For the simplicity of notation, we shall write $a\equiv\hat{a}$ and $a\equiv\hat{\hat{a}}$ if no confusion arises.

For an interval $I\subseteq\mathbb{R}$, let $I+[-\tau,0]=\{t+\theta:t\in I \text{ and } \theta\in[-\tau,0]\}$. For $u:(I+[-\tau,0])\times\overline{\Omega}\to\mathbb{R}$ and $t\in I$, we write $u_t(\cdot,\cdot)$ for the element of X defined by $u_t(\theta,x)=u(t+\theta,x)$ for $-\tau\leq\theta\leq 0$ and $x\in\overline{\Omega}$.

Let $T(\cdot)$ be the semigroup on C_0 generated by the operator $d\Delta$ under the Dirichlet boundary condition. It is well known that $T(\cdot)$ is an analytic, compact, and positive semigroup on C_0 [19,21,27].

Define $F:X\to C_0$ by

$$(F(\phi))(x)=\begin{cases}-\mu\phi(0,x)+\mu f(\phi(-\tau,x)), & x\in\overline{\Omega} \text{ and } \phi\in X_+,\\ 0, & \text{otherwise.}\end{cases}$$

We consider the following integral equation with a given initial condition,

$$\begin{aligned} u(t,\cdot) &= T(t)\phi(0,\cdot) + \int_0^t T(t-s)F(u_s)\mathrm{d}s, \quad t \geq 0, \\ u_0 &= \phi \in X. \end{aligned} \tag{2}$$

By the standard theory [15,27], for each $\phi \in X$, Equation (2) admits a unique solution $u^{\phi}(t,\cdot)$ (with values in C_0) on its maximal interval $[0, \sigma_{\phi})$. As usual, $u^{\phi}(t,x)$ is called a mild solution of Equation (1). For details, see [15,16,27].

Definition 2.1 *For given $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1$, a continuous function $u : [t_1 - \tau, t_2) \times \overline{\Omega} \to \mathbb{R}$ is called a classical solution of Equation* (1) *for $t \in [t_1, t_2)$ if all involved derivatives exist and for $i, j \in \{1, \ldots, m\}$, $\partial u/\partial t$ and $\partial^2 u/\partial x_i \partial x_j$ are continuous for $(t,x) \in (t_1, t_2) \times \Omega$, and $\partial u/\partial x_i$ is continuous for $(t,x) \in (t_1,t_2) \times \overline{\Omega}$, and u satisfies Equation* (1) *for $(t,x) \in (t_1,t_2) \times \overline{\Omega}$.*

Let λ_1 be the first eigenvalue of the operator $-\Delta$ with the homogeneous Dirichlet boundary condition. To investigate the threshold dynamics of Equation (1) or its abstract form (2), we introduce the following assumptions:

(H1) There exists $M > 0$ such that $f(x) \leq M$ for $x \in \mathbb{R}_+$.
(H2) There exists $u^* > 0$ such that $f(u^*) = u^*$.
(H3) If $a \geq 0$, $b \geq 0$, and $u^* \geq k > 0$, then $a - 1 \geq |b-1|$ implies $-af(k) + f(bk) \leq 0$. Moreover, $-af(k) + f(bk) = 0$ if and only if $a = b = 1$.
(H4) If $a \geq 0$, $b \geq 0$, and $u^* \geq k > 0$, then $1 - a \geq |b-1|$ implies $-af(k) + f(bk) \geq 0$. Moreover, $-af(k) + f(bk) = 0$ if and only if either $a = b = 1$ or $a = b = 0$.
(H5) There exists a continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(\xi) = \xi g(\xi)$ for $\xi \in \mathbb{R}_+$.

We refer to Lemma 2.5 below for the motivation of assumptions (H3) and (H4).

By (H1) and Theorem 2.2.3 in [27], we have $\sigma_{\phi} = \infty$ for $\phi \in X$. Since $\lim_{h\to 0^+} \mathrm{dist}(\psi(0) + hF(\psi), C_+) = \lim_{h\to 0^+} \inf\{\|\psi(0) + hF(\psi) - \tilde{\psi}\| : \tilde{\psi} \in C_+\} = 0$ for $\psi \in X_+$, it follows from Proposition 3 and Remark 2.4 in [15] that $(u^{\phi})_t \in X_+$ for $t \in \mathbb{R}_+$ and $\phi \in X_+$. In the sequel, we always assume that (H1) holds.

Define the map $U : \mathbb{R}_+ \times X_+ \to X_+$ by $U(t,\phi) = (u^{\phi})_t$ for $(t,\phi) \in X$. The following result is a direct consequence of the abstract results in [3,26,27].

Lemma 2.2 *The map U is a semiflow defined on X_+ and satisfies the following properties*:

(i) *For a given $t > \tau$, $U(t,\cdot) : X_+ \to X_+$ is completely continuous. More precisely, if $B \subset X_+$ is bounded, then $U(t,\cdot)B$ is precompact for $t > \tau$.*
(ii) *For a given $\phi \in X_+$, $U(t,\phi)(0,\cdot)$ is a classical solution of Equation* (2) *for $t > \tau$.*

The proof of the following lemma can be found in [19,21].

Lemma 2.3 *Let $T > 0$ and $W \subseteq \overline{\Omega}$ be an open domain with a smooth boundary ∂W. Let $u(t,x)$ be a continuous function on $[0,T] \times \overline{\Omega}$ with derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$, and $\partial u/\partial t$ existing and being continuous on $(0,T] \times \Omega$. Let $Lu(t,x) = d\Delta u(t,x) - \partial u/\partial t(t,x)$. Then the following statements are true*:

(i) *If $Lu(t,x) > 0$ for $(t,x) \in (0,T) \times W$, then u cannot attain a local maximum in $(0,T) \times W$;*
(ii) *If $Lu(t,x) \geq 0$ for $(t,x) \in (0,T) \times W$ and if u attains its maximum in $[0,T] \times \overline{W}$ at a point $(t^*,x^*) \in (0,T) \times W$, then $u(t,x) = u(t^*,x^*)$ for $(t,x) \in [0,T] \times \overline{W}$;*

(iii) *Suppose that the first derivatives of u with respect to the x_i exist and are continuous on* $(0,T]\times\overline{W}$. *Let* $c\geq 0$ *and* $Lu(t,x)-cu(t,x)\geq 0$ *for* $(t,x)\in(0,T)\times W$. *If there exist* $(t^*,x^*)\in(0,T)\times\partial W$, $\varepsilon^*\in(0,T)$ *and an open ball* $S^*\subseteq W$ *such that* $S^*\cap\partial W=\{x^*\}$ *and* $u(t^*,x^*)>u(t,x)$ *for* $(t,x)\in[t^*-\varepsilon^*,t^*+\varepsilon^*]\times S^*$, *then* $\partial u/\partial n|_{(t^*,x^*)}>0$.

LEMMA 2.4 *Suppose* $\phi\in X_+\backslash\{0\}$. *Then*

(i) $U(t,\phi)(0,x)>0$ *for* $(t,x)\in[3\tau,\infty)\times\Omega$;
(ii) $\partial U(t,\phi)(0,x)/\partial n<0$ *for* $(t,x)\in[3\tau,\infty)\times\partial\Omega$;
(iii) *there exists* $K=K_{\phi,M}>0$ *such that* $\|U(t,\phi)\|\leq K$ *for* $t\in[0,\infty)$.

Proof Let $u(t,x)=u^{\phi}(t,x)$ for $(t,x)\in[-\tau,\infty)\times\overline{\Omega}$. According to Lemma 2.1(ii), $u(t,x)$ is a classical solution of Equation (1) for $t>\tau$.

To prove (i), we first claim that $U(\tau,\phi)\in X_+\setminus\{0\}$. If not, then $U(\tau,\phi)=0$ and hence $u(t,x)=0$ for $(t,x)\in[0,\tau]\times\overline{\Omega}$. This, combined with Equation (2), gives $\int_0^t T(t-s)F(u_s)\mathrm{d}s=0$ for $t\in[0,\tau]$. Thus, $\int_0^1 T(t-s)[u(s-\tau,\cdot)e^{-u(s-\tau,\cdot)}]\mathrm{d}s=0$ for $t\in[0,\tau]$. Since $T(\cdot)$ is a strongly positive semigroup, that is, $T(t)(\psi)(x)>0$ for $(t,x,\psi)\in(0,\infty)\times\Omega\times C_+\setminus\{0\}$ [22, Corollary 7.2.3], we can deduce that $u(s-\tau,\cdot)e^{-u(s-\tau,\cdot)}=0$ for $s\in[0,\tau]$ and thus $\phi=U(0,\phi)=u_0=0$, a contradiction. Similarly, we can prove $U(2\tau,\phi)\in X_+\setminus\{0\}$. Hence, there exists $(\tilde{t},\tilde{x})\in(\tau,2\tau)\times\overline{\Omega}$ such that $u(\tilde{t},\tilde{x})>0$. Suppose now that statement (i) is not true. Then there is $(t^*,x^*)\in[3\tau,\infty)\times\Omega$ such that $u(t^*,x^*)=0$. This implies that u attains its minimum in $[\tau,\infty)\times\overline{\Omega}$ at $(t^*,x^*)\in(\tau,\infty)\times\Omega$. On the other hand, it follows from Equation (1) that $d\Delta u(t,x)-(\partial u(t,x)/\partial t)-\mu u(t,x)\leq 0$ for $(t,x)\in(\tau,\infty)\times\Omega$. Let $v(t,x)=-u(t,x)e^{\mu t}$ for $(t,x)\in[-\tau,\infty)\times\overline{\Omega}$. Then $Lv(t,x)\geq 0$ for $(t,x)\in(\tau,\infty)\times\Omega$ (where L is defined in Lemma 2.3) and v attains its maximum in $[\tau,\infty)\times\overline{\Omega}$ at $(t^*,x^*)\in(\tau,\infty)\times\Omega$. Then, by Lemma 2.3 (ii), we have $v(t,x)=v(t^*,x^*)=0$ for $(t,x)\in[\tau,t^*]\times\overline{\Omega}$ and hence $u(t,x)=0$ for $(t,x)\in[\tau,t^*]\times\overline{\Omega}$, which shows $U(2\tau,\phi)=0$, a contradiction. This proves (i).

We next prove statement (ii). From Equation (1), we have $d\Delta u(t,x)-(\partial u(t,x)/\partial t)-\mu u(t,x)\leq 0$ for all $(t,x)\in[3\tau,\infty)\times\Omega$. By (i) proved above and Lemma 2.3 (iii), we obtain that $\partial U(t,\phi)(0,x)/\partial n=\partial u(t,x)/\partial n<0$ for $(t,x)\in[3\tau,\infty)\times\partial\Omega$, that is, statement (ii) holds.

Finally, we prove statement (iii). By the definition of U, there exists $K_0=K_0(\phi)>0$ such that $\|U(t,\phi)\|\leq K_0$ for $t\in[0,3\tau]$. Let $K=K_{\phi,M}=3K_0+3M$, where M is the number in assumption (H1). We claim $\|U(t,\phi)\|\leq K$ for $t\in[0,\infty)$. Otherwise, there exists $(t^*,x^*)\in(3\tau,\infty)\times\Omega$ such that $u(t^*,x^*)>K$. Let $t^{**}=\inf\{t\in[3\tau,\infty):u(t,x)=K$ for some $x\in\Omega\}$. Then $t^{**}>3\tau$, and there exists $x^{**}\in\Omega$ such that $u(t^{**},x^{**})=K$ and $u(t,x)<K$ for $(t,x)\in(0,t^{**})\times\overline{\Omega}$. Hence, $\partial u(t,x)/\partial t|_{(t,x)=(t^*,x^*)}\geq 0$ and $d\Delta u(t,x)|_{(t,x)=(t^*,x^*)}\leq 0$. But, it follows from Equation (1) that $\partial u(t,x)/\partial t\leq d\Delta u(t,x)-\mu u(t,x)+\mu M$, in particular, $\partial u(t,x)/\partial t|_{(t,x)=(t^*,x^*)}\leq\mu(-K+M)<0$, a contradiction. This completes the proof of statement (iii). ■

For $\phi\in X_+$, we define $O(\phi)=\{U(t,\phi):t\geq 0\}$ and $\omega(\phi)=\bigcap_{t\geq 0}\overline{O(U(t,\phi))}$. By Lemma 2.2 (i) and Lemma 2.4 (iii), we know that $\overline{O(\phi)}$ is compact and hence $\omega(\phi)$ is non-empty, compact, connected and invariant. According to the invariant property of $\omega(\phi)$, for $\psi\in\omega(\phi)$, there is a global solution $u:\mathbb{R}\times\overline{\Omega}\to\mathbb{R}$ with $u_0=\psi$ and $u_t\in\omega(\phi)$ for $t\in R$ [12].

LEMMA 2.5 *Assume that* (H2), (H3) *and* (H5) *hold. Then the following statements hold*:

(i) $g(ak)<g(k)$ *for* $k\in(0,u^*]$ *and* $a>1$.
(ii) $g(\xi)$ *is strictly decreasing in* $\xi\in[0,u^*]$.
(iii) $g(0)>g(\xi)$ *for* $\xi>0$.
(iv) $g(\xi)\leq 1$ *for* $\xi\in[u^*,\infty)$.

Proof It follows from (H3) with $a = b$ and $a > 1$ that $af(k) > f(ak)$ for $k \in (0, u^*]$. Thus (H5) implies $g(ak) < g(k)$ for $k \in (0, u^*]$ and $a > 1$, that is, statement (i) holds.

For $\xi, \eta \in (0, u^*]$ with $\xi > \eta$, by taking $a = \xi/\eta > 1$, we obtain from statement (i) that $g(\eta) > g(a\eta) = g(\xi)$, that is, $g(\xi)$ is strictly decreasing in $\xi \in [0, u^*]$. This proves statement (ii).

We now prove statement (iii) by way of contradiction. Suppose statement (iii) is not true. Then there exists $\xi^* > 0$ such that $g(\xi^*) \geq g(0)$. Let $\xi^{**} = \sup\{\xi \in [0, \xi^*] : g(\xi) = \max\{g(\eta) : \eta \in [0, \xi^*]\}\}$. Then $\xi^{**} > 0$ and $g(\xi^{**}) = \max\{g(\eta) : \eta \in [0, \xi^*]\}$. It follows from statement (i) that $\xi^{**} > u^*$. But, statement (i) implies $g(\xi^{**}) = g(\xi^{**}/u^* u^*) < g(u^*)$, a contradiction. Thus, statement (iii) holds.

Finally, we prove statement (iv). Indeed, we can easily see from statement (i) that $g(\xi) = g(\xi/u^* u^*) \leq g(u^*) = 1$ for $\xi \geq u^*$. This completes the proof. ■

Applying Theorem and Remark in [13] and Lemma 2.5, we have the following result:

Proposition 2.6 *Assume that* (H2), (H3) *and* (H5) *hold. If* $\mu g(0) > d\lambda_1 + \mu$, *then Equation* (1) *has a unique positive steady-state solution* (*to be denoted by* $u_+(x)$ *in what follows*).

To proceed further, we now introduce the following notations:

$$M^\circ(\phi)(x) = \frac{\phi(0, x)}{u_+(x)} - 1 \quad \text{for } x \in \Omega \text{ and } \phi \in X_+;$$

$$M^\partial(\phi)(x) = \frac{\partial\phi(0, x)/\partial n}{\partial u_+(x)/\partial n} - 1 \quad \text{for } x \in \partial\Omega \text{ and } \phi \in X_+;$$

$$M(\phi)(x) = \begin{cases} M^\circ(\phi)(x), & x \in \Omega \text{ and } \phi \in X_+, \\ M^\partial(\phi)(x), & x \in \partial\Omega \text{ and } \phi \in X_+. \end{cases}$$

Obviously, $\Pi(\psi) = \max\{\Pi_+(\psi), \Pi_-(\psi)\}$.

Lemma 2.7 *Let* $\psi \in X_+$. *Then* $M(\cdot)(\cdot)$ *is a continuous function with respect to* $(x, \phi) \in \overline{\Omega} \times \overline{O(u_{2\tau}(\psi))}$.

Proof According to the definition of $M(\phi)(x)$, we know that, for a given $x \in \overline{\Omega}$, $M^\circ(\phi)(x)$ is a continuous function with respect to $\phi \in \overline{O(u_{2\tau}(\psi))}$.

It suffices to prove that for a given $\phi \in \overline{O(u_{2\tau}(\psi))}$, $M(\phi)(x)$ is a continuous function with respect to $x \in \overline{\Omega}$. In the following, let $\{x_n\}_{n=1}^\infty$ be a sequence of Ω such that $\lim_{n\to\infty} x_n = x_0$ for some $x_0 \in \partial\Omega$. Note that for sufficient large n, there exist $x_n^0 \in \partial\Omega$ and $s_n > 0$ such that $x_n = x_n^0 + s_n n_{x_n^0}$ and $x_n^0 + \eta s_n n_{x_n^0} \in \overline{\Omega}$ for $\eta \in [0, \tau]$. Thus,

$$\begin{aligned}
\lim_{n\to\infty} M^\circ(\phi)(x_n) &= \lim_{n\to\infty} \frac{\phi(0, x_n)}{u_+(x_n)} - 1 \\
&= \lim_{n\to\infty} \frac{\int_0^1 \partial\phi(0, x_n^0 + \eta s_n n_{x_n})/\partial\eta \mathrm{d}\eta}{\int_0^1 \partial u_+(x_n^0 + \eta s_n n_{x_n^0})/\partial\eta \mathrm{d}\eta} - 1 \\
&= \lim_{n\to\infty} \frac{\int_0^1 \nabla_x\phi(0, x_n^0 + \eta s_n n_{x_n^0}) n_{x_n^0} \mathrm{d}\eta}{\int_0^1 \nabla_x u_+(x_n^0 + \eta s_n n_{x_n^0}) n_{x_n^0} \mathrm{d}\eta} - 1 \\
&= M^\partial(\phi)(x_0).
\end{aligned}$$

On the other hand, it follows from the definition of $M(\phi)(x)$ that $M(\phi)(\cdot)|_\Omega$ and $M(\phi)(\cdot)|_{\partial\Omega}$ are continuous functions. Thus, $M(\phi)(\cdot)$ is a continuous function in $\overline{\Omega}$. This completes the proof. ■

Proposition 2.8 *Assume that* (H1)–(H5) *hold. If* $\mu g(0) > d\lambda_1 + \mu$, *then the positive steady-state solution* $u_+(x)$ *of Equation* (1) *attracts all solutions of Equation* (1) *with the initial value* $\psi \in X_+ \setminus \{0\}$.

Proof By way of contradiction, suppose that there exists $\psi \in X_+ \setminus \{0\}$ such that $\Pi(\psi) > 0$. Without loss of generality, we may assume that $\Pi_-(\psi) \geq \Pi_+(\psi)$. Then, $1 \geq \Pi_-(\psi) = \Pi(\psi) > 0$.

We first prove $u_+(x) \leq u^*$ for $x \in \Omega$. Otherwise, there is a $u^{**} > u^*$ such that $u^{**} = \sup\{u_+(x) : x \in \Omega\}$. It follows from Equation (1) that $f(u^{**}) \geq u^{**}$. Let $a = b = u^{**}/u^*$ and $k = u^*$. Then, by assumption (H3), $f(u^{**}) = f(ak) < af(k) = u^{**}$, a contradiction.

We complete the proof by discussing four possible cases.

Case 1 There exist $x^* \in \Omega$ and $\phi^* \in \omega(\psi) \setminus \{0\}$ such that $-M^\circ(\phi^*)(x^*) = \Pi_-(\psi)$. It follows from Lemma 2.4 (i) that $-M^\circ(\phi^*)(x^*) < 1$. Let $v(t,x) = u^{\phi^*}(t,x)/u_+(x)$ for $x \in \Omega$ and $t \in \mathbb{R}$, where $u^{\phi^*}(t,x)$ denotes a full orbit in $\omega(\psi)$ with $(u^{\phi^*})_0 = \phi^*$. Obviously, $v(t,x)$ attains the minimum value $v(0,x^*)$ in $\mathbb{R} \times \Omega$ (since $v(t,x) - v(0,x^*) = (1 - v(0,x^*)) - (1 - v(t,x)) \geq \Pi_-(\psi) - \Pi(\psi) = 0$). Hence, $\Delta v(0,x^*) \geq 0$, $dv(0,x^*)/\partial t = 0$ and $\partial v(0,x^*)/\partial x_j = 0$ for $j \in \{1,2,\ldots,n\}$. On the other hand, for $t \in \mathbb{R}$, we can deduce from Equation (1) that

$$\begin{aligned}
\frac{\partial v(t,x^*)}{\partial t} &= \frac{1}{u_+(x^*)} \cdot \frac{\partial u^{\phi^*}(t,x^*)}{\partial t} \\
&= \frac{1}{u_+(x^*)}[d\Delta u^{\phi^*}(t,x^*) - \mu u^{\phi^*}(t,x^*) + \mu f(u^{\phi^*}(t-\tau,x^*))] \\
&= \frac{1}{u_+(x^*)}[du_+(x^*)\Delta v(t,x^*) + dv(t,x^*)\Delta u_+(x^*) \\
&\quad - \mu u^{\phi^*}(t,x^*) + \mu f(u^{\phi^*}(t-\tau,x^*))] \\
&= d\Delta v(t,x^*) + \frac{\mu}{u_+(x^*)}[v(t,x^*)(u_+(x^*) - f(u_+(x^*))) \\
&\quad - u^{\phi^*}(t,x^*) + f(u^{\phi^*}(t-\tau,x^*))] \\
&= d\Delta v(t,x^*) + \frac{\mu}{u_+(x^*)}[-v(t,x^*)f(u_+(x^*)) + f(u^{\phi^*}(t-\tau,x^*))] \\
&= d\Delta v(t,x^*) + \frac{\mu}{u_+(x^*)}\left[-v(t,x^*)f(u_+(x^*)) + f\left(u_+(x^*)\frac{u^{\phi^*}(t-\tau,x^*)}{u_+(x^*)}\right)\right] \\
&= d\Delta v(t,x^*) + \frac{\mu}{u_+(x^*)}[-v(t,x^*)f(u_+(x^*)) + f(u_+(x^*)v(t-\tau,x^*))].
\end{aligned}$$

By assumption (H4), $|v(-\tau,x^*) - 1| \leq 1 - v(0,x^*)$ implies $\partial v(0,x^*)/\partial t - d\Delta v(0,x^*) > 0$, a contradiction.

Case 2 There exist $x^* \in \partial\Omega$ and $\phi^* \in \omega(\psi) \setminus \{0\}$ such that $-M^\partial(\phi^*)(x^*) = \Pi_-(\psi)$ and $-M^\circ(\phi)(x) < \Pi_-(\psi)$ for $(x,\phi) \in \Omega \times \omega(\psi)$. In this case, it follows from Lemma 2.4 (ii) that $a \equiv 1 + M^\partial(\phi^*)(x^*) \in (0,1)$. Let $v(t,x) = u^{\phi^*}(t,x) - au_+(x)$ for $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, where $u^{\phi^*}(t,x)$ denotes a full orbit in $\omega(\psi)$ with $u_0(\phi^*) = \phi^*$. Obviously, $v(0,x^*) = 0$, $\partial v/\partial n|_{x=x^*} = 0$ and $v(t,x) > 0$ for $t \in \mathbb{R}$ and $x \in \Omega$. On the other hand, for $(x,t) \in \Omega \times \mathbb{R}$, it follows from

Equation (1) that

$$\begin{aligned}
\frac{\partial v(t,x)}{\partial t} &= \frac{\partial u^{\phi^*}(t,x)}{\partial t} \\
&= d\Delta u^{\phi^*}(t,x) - \mu u^{\phi^*}(t,x) + \mu f(u^{\phi^*}(t-\tau,x)) \\
&= d\Delta v(t,x) + ad\Delta u_+(x) - \mu u^{\phi^*}(t,x) + \mu f(u^{\phi^*}(t-\tau,x)) \\
&= d\Delta v(t,x) + a[\mu u_+(x) - \mu f(u_+(x))] - \mu u^{\phi^*}(t,x) + \mu f(u^{\phi^*}(t-\tau,x)) \\
&= d\Delta v(t,x) - \mu v(t,x) + \mu[-af(u_+(x)) + f(u^{\phi^*}(t-\tau,x))] \\
&= d\Delta v(t,x) - \mu v(t,x) + \mu\left[-af(u_+(x)) + f\left(u_+(x)\frac{u^{\phi^*}(t-\tau,x)}{u_+(x)}\right)\right].
\end{aligned}$$

Since $1-a = \Pi(\psi) \geq |u^{\phi^*}(t-\tau,x)/u_+(x) - 1|$, assumption (H4) implies $\partial v(t,x)/\partial t - d\Delta v(t,x) + \mu v(t,x) > 0$ for $(x,t) \in \Omega \times \mathbb{R}$. Applying Lemma 2.3 (iii), we can deduce $\partial v/\partial n|_{(0,x^*)} < 0$, a contradiction.

Case 3 $\omega(\psi) \setminus \{0\} \neq \emptyset$ and $-M(\phi)(x) < \Pi_-(\psi)$ for $x \in \overline{\Omega}$ and $\phi \in \omega(\psi) \setminus \{0\}$. In this case, we have $0 \in \omega(\psi)$ and $\Pi(\psi) = |M(0,x)| = 1$. If $\Pi_+(\psi) = 1$, then similar arguments as those in Case 1 or Case 2 will produce a contradiction. Now we assume $M_+ \equiv \Pi_+(\psi) < 1$. Then $-1 \leq M_+ < 1$. Let $v(t,x) = u^{\psi}(t,x)$ for $(t,x) \in [-\tau,\infty) \times \overline{\Omega}$. To complete the proof, we need the following claims.

Claim 1 There is an $s^* > 5\tau$ such that $1 > M_+^* \equiv \sup\{M(v_t)(x) : (t,x) \in [s^*,+\infty) \times \overline{\Omega}\}$. Indeed, by the definition of $\omega(\psi)$ and Lemma 2.7, there exists $s^* > 5\tau$ such that $\inf\{\|M(v_t)(\cdot) - M(\phi)(\cdot)\| : \phi \in \omega(\psi)\} < (1-M_+)/3$ for $t \in [s^*,+\infty)$. It follows from the compactness of $\omega(\psi)$ that for every $t \in [s^*,+\infty)$ there exists $\varphi^t \in \omega(\psi)$ such that $\|M(v_t)(\cdot) - M(\phi^t)(\cdot)\| = \inf\{\|M(v_t)(\cdot) - M(\phi)(\cdot)\| : \phi \in \omega(\psi)\}$. Therefore, according to the choice of ϕ^t, we obtain

$$\begin{aligned}
M_+^* &= \sup\{M(v_t)(x) : (t,x) \in [s^*,+\infty) \times \overline{\Omega}\} \\
&\leq \sup\{M(v_t)(x) - M(\varphi^t)(x) : (t,x) \in [s^*,+\infty) \times \overline{\Omega}\} \\
&\quad + \sup\{M(\varphi^t)(x) : (t,x) \in [s^*,+\infty) \times \overline{\Omega}\} \\
&\leq \sup\{\|M(v_t)(\cdot) - M(\varphi^t)(\cdot)\| : t \in [s^*,+\infty)\} \\
&\quad + \sup\{M(\psi^t)(x) : (t,x) \in [s^*,+\infty) \times \overline{\Omega}\} \\
&\leq \frac{1-M_+}{3} + M_+ \\
&< 1.
\end{aligned}$$

This completes the proof of Claim 1.

Since $\omega(\psi) \setminus \{0\} \neq \emptyset$ and $\omega(\psi) \setminus \{0\} \subseteq \mathrm{Int}(X_+)$, there exist $\varepsilon_0 \in (0, 1/8\min\{1, 1-M_+, 1-M_+^*\})$ and $\phi^* \in \omega(\psi)$ such that $\phi^* - \varepsilon_0 u^+ \in X_+$. In view of $\{0,\phi^*\} \subseteq \omega(\psi)$ and the definition of $\omega(\psi)$, there exist $s_1 > s^* + \tau$, $s_2 > s^* + \tau$ and $s_3 > s^* + \tau$ such that $s_1 < s_2 - \tau < s_2 < s_3 - \tau < s_3$, $\|M(v_{s_2})(\cdot)\| > 1 - 1/8\varepsilon_0$, $\sup\{\|M(v_{s_1+\theta})(\cdot) - M(u_\theta(\phi^*))(\cdot)\| : \theta \in [-\tau,0]\} < 1/8\varepsilon_0$ and $\|M(v_{s_3})(\cdot) - M(\phi^*)(\cdot)\| < 1/8\varepsilon_0$. Let $M_1 = \sup\{|M(v_t)(x)| : (t,x) \in [s_1-\tau, s_3] \times \overline{\Omega}\}$. Then it follows from $M_+^* < 1$, the definition of M_1 and Lemma 2.4 (i, ii) and Lemma 2.7 that $M_1 < 1$. By the choice of s_i and the definition of M_1, we have $M_1 \geq \|M(v_{s_2})(\cdot)\| > 1 - 1/8\varepsilon_0$. In view of the choice of ε_0, we have $M_1 > M_+$ and $M_1 > M_+^*$, and hence $1 > M_1 > \max\{M_+, M_+^*\}$.

Claim 2 $M_1 > |M(v_t)(x)|$ for $(t, x) \in ([s_1 - \tau, s_1] \bigcup \{s_3\}) \times \overline{\Omega}$. Indeed, it follows from the choices of s_1 and ϕ^* that, for $\theta \in [-\tau, 0]$,

$$\begin{aligned}
\|M(v_{s_1+\theta})(\cdot)\| &\le \|M((u^{\phi^*})_\theta)(\cdot)\| + \|M(v_{s_1+\theta})(\cdot) - M((u^{\phi^*})_\theta)(\cdot)\| \\
&= \max\{\sup\{M((u^{\phi^*})_\theta)(x) : x \in \overline{\Omega}\}, \sup\{-M((u^{\phi^*})_\theta)(x) : x \in \overline{\Omega}\}\} \\
&\quad + \|M(v_{s_1+\theta})(\cdot) - M((u^{\phi^*})_\theta)(\cdot)\| \\
&\le \max\{M_+, 1 - \varepsilon_0\} + \|M(v_{s_1+\theta})(\cdot) - M((u^{\phi^*})_\theta)(\cdot)\| \\
&< \max\{M_+, 1 - \varepsilon_0\} + \frac{1}{8}\varepsilon_0 \\
&= \max\left\{M_+ + \frac{1}{8}\varepsilon_0, 1 - \frac{7}{8}\varepsilon_0\right\}.
\end{aligned}$$

According to the choice of ε_0, we have $M_+ + 1/8\varepsilon_0 < 1 - 1/8\varepsilon_0$, and hence

$$\sup\{\|M(v_{s_1+\theta})(\cdot)\| : \theta \in [-\tau, 0]\} \le 1 - \frac{1}{8}\varepsilon_0 < M_1.$$

A similar argument shows that $\|M(v_{s_3})(\cdot)\| < M_1$. This proves Claim 2.

Claim 3 $|M(v_t)(x)| < M_1$ for $(t, x) \in [s_1 - \tau, s_3] \times \Omega$. Otherwise, it follows from Claim 2 and $M_1 > M_+^*$ that there exists $(t_2, x_2) \in (s_1, s_3) \times \Omega$ such that $M_1 = -M(v_{t_2})(x_2) \in (0, 1)$. Using a similar discussion as that in Case 1, we can deduce a contradiction. Consequently, Claim 3 holds.

It follows from Claims 2 and 3 and $M_1 > M_+^*$ that there exists $(t^{**}, x^{**}) \in (s_1, s_3) \times \partial\Omega$ such that $M_1 = -M(v_{t^{**}})(x^{**}) > 0$. By a similar discussion as that in Case 2, we also can deduce a contradiction.

Case 4 $\omega(\psi) = \{0\}$. It follows from $\mu g(0) > d\lambda_1 + \mu$ and assumption (H5) that there is a $K^* > 0$ such that $f(x) \ge (d\lambda_1 + \mu)/\mu x$ for $x \in [0, K^*]$. From the definition of $\omega(\psi)$, there exists $T > 5\tau$ such that $0 < u^\psi(t, x) < \min\{u^*/3, K^*/3\}$ for $x \in \Omega$ and $t > T$. Let $H(\phi) = -\mu\phi(0) + (d\lambda_1 + \mu)\phi(-\tau)$ for $\phi \in X_+$. Then H is quasimonotone in the sense of [15,27]. It follows from Equation (2) that, for $t > 5\tau$,

$$u^\psi(t, \cdot) \ge T(t)\phi(0, \cdot) + \int_0^t T(t - s)H((u^\psi)_s)\mathrm{d}s.$$

Note that there exists $\phi^* \in C_+$ such that $-\Delta\phi^* = \lambda_1\phi^*$ and $\phi^*(x) > 0$ for $x \in \Omega$. Let $\epsilon_0 > 0$ such that $u(T + \theta, \cdot) - \epsilon_0\phi^* \in C_+$ for $\theta \in [-\tau, 0]$. Denote $v(t, x) = \epsilon_0\phi^*(x)$ for all $(t, x) \in [-\tau, \infty) \times \overline{\Omega}$. Then $v(t) = T(t)\phi(0, \cdot) + \int_0^t T(t - s)H(v_s)\mathrm{d}s$. Thus by Corollary 8.1.11 in [27], we have $u(t + T, x) \ge v(t, x) = \epsilon_0\phi^*(x)$ for $(t, x) \in [-\tau, \infty) \times \overline{\Omega}$, which implies $\omega(\phi) \ne \{0\}$, a contradiction.

Summarizing Cases 1–4, we see that $\Pi(\psi) > 0$ is impossible and hence $\Pi(\psi) = 0$. This completes the proof. ■

Proposition 2.9 *Assume that* (H1)–(H5) *hold. If* $\mu g(0) \le d\lambda_1 + \mu$, *then the trivial steady-state solution* 0 *of Equation* (1) *attracts all solutions of Equation* (1) *with the initial value* $\psi \in X_+$.

Proof By way of contradiction, we assume that there exists $u_0 \in X_+$ such that $\omega(u_0) \ne \{0\}$. Then, by the compactness of $\omega(u_0)$, there exist $M^* > 0$ and $\phi^* \in \omega(u_0)$ such that $M^* = \sup\{\|\phi(0, \cdot)\|_{L^2} : \phi \in \omega(u_0)\}$ and $M^* = \|\phi^*(0, \cdot)\|_{L^2}$. By the invariance of $\omega(u_0)$, there exists a global classical solution $w(t,x) : \mathbb{R} \times \overline{\Omega} \to [0,\infty)$ of Equation (1) such that $w_0 = \phi^*$. We multiply Equation (1) by $w(t,x)$ and integrate it over Ω. Using integration by parts, the Poincaré inequality, the Hölder inequality and Lemma 2.5(iii), we can obtain

$$\frac{\mathrm{d}\|u(t,\cdot)\|_{L^2}}{\mathrm{d}t} < -(d\lambda_1 + \mu)\|u(t,\cdot)\|_{L^2} + \mu g(0)\|u(t-\tau,\cdot)\|_{L^2}.$$

It follows from the choice of w that $\mathrm{d}\|u(t,\cdot)\|_{L^2}/\mathrm{d}t|_{t=0} = 0$ and $\|u(0,\cdot)\|_{L^2} \geq \|u(-\tau,\cdot)\|_{L^2}$, a contradiction. ■

According to Proposition 2.6, Proposition 2.8 and Proposition 2.9, we have the following main results:

THEOREM 2.10 *Assume that assumptions* $(H1)$–$(H5)$ *hold.*

(i) *If* $\mu g(0) \leq d\lambda_1 + \mu$, *then the trivial steady-state solution* 0 *of Equation* (1) *attracts all solutions of Equation* (1) *with the initial value in* X_+.
(ii) *If* $\mu g(0) > d\lambda_1 + \mu$, *then the non-trivial steady-state solution* u_+ *of Equation* (1) *attracts all solutions of Equation* (1) *with the initial value in* $X_+ \setminus \{0\}$.

Remark 2.11 Theorem 2.10 still holds if we replace (H1) with

(H1*). There exists a sequence $\{u_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} u_n = \infty$ and $f([0,u_n]) \subseteq [0,u_n]$.

Remark 2.12 Similar results as those in Theorem 2.10 can be established if we replace $d\Delta$ with a uniformly elliptic operator.

3. Applications

In this section, we illustrate Theorem 2.10 with two examples.

Example 3.1 Consider Nicholson's blowflies equation with diffusion,

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + pu(t-\tau,x)e^{-\rho u(t-\tau,x)}, \quad t>0, \quad x \in \Omega, \\ &u(0,\cdot)|_{\partial\Omega} = 0, \end{aligned} \tag{3}$$

where $d, \rho, \tau, \beta, \delta \in (0,\infty)$.

In Example 3.1, we have $\mu = \delta$, $u^* = (1/\rho)\ln p/\delta$ and $f(x) = p/\delta x e^{-\rho x}$ for $x \in [0,\infty)$.

LEMMA 3.2 *If* $1 < p/\delta \leq e^2$, *then* $f(x) = p/\delta x e^{-\rho x}$ *satisfies assumptions* $(H1)$–$(H5)$.

Proof We can easily see that assumptions (H1), (H2) and (H5) hold. It suffices to prove assumption (H3) since assumption (H4) can be proved similarly. In the following, we assume that $a \geq 0$, $b \geq 0$, $0 < k \leq (1/\rho)\ln(p/\delta)$ and $a - 1 \geq |b-1|$. If $\rho k \leq 1$, then a simple computation proves assumption (H3). Now suppose $\rho k > 1$. Let $\beta^* = e^{\rho k}$, $a^* = a\rho k$ and $b^* = b\rho k$. Then $\beta^* \in (1, e^2]$, $a^* - \ln\beta^* \geq |b^* - \ln\beta^*|$ and $-af(k) + f(bk) = p/\rho\delta e^{-\rho k}(-a^* + \beta^* b^* e^{-b^*})$. Hence, by Lemma 2.4 in [32], we have $-a^* + \beta^* b^* e^{-b^*} \leq 0$. Moreover, $-a^* + \beta^* b^* e^{-b^*} = 0$ if and only if $a^* = b^* = \ln\beta$. This completes the proof. ■

Theorem 2.10, combined with Lemma 3.2, gives the following theorem.

THEOREM 3.3 *Assume* $p/\delta \in (1, e^2]$.

(i) *If* $p \leq d\lambda_1 + \delta$, *then the trivial steady-state solution* 0 *of Equation* (3) *attracts all solutions of Equation* (3) *with the initial value in* X_+.
(ii) *If* $p > d\lambda_1 + \delta$, *then there exists a unique nontrivial steady-state solution of Equation* (3) *which attracts all solutions of Equation* (3) *with the initial value in* $X_+\backslash\{0\}$.

Example 3.4 Consider the following scalar equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + \frac{pu(t-\tau,x)}{1+u(t-\tau,x)},\\
&u(0,\cdot)|_{\partial\Omega} = 0,
\end{aligned} \tag{4}$$

where $d, \tau, \delta, p > 0$ and $p/\delta > 1$.

In Example 3.4, $\mu = \delta$, $u^* = p/\delta - 1$ and $f(x) = (p/\delta)(x/1+x)$ for $x \in [0,\infty)$.

LEMMA 3.5 *If* $1 < p/\delta$, *then* $f(x) = (p/\delta)(x/1+x)$ *satisfies assumptions* (H1)–(H5).

Proof It is easy to see that assumptions (H1), (H2) and (H5) hold. It suffices to prove assumption (H4) since assumption (H3) can be proved similarly. Assume that $a \geq 0, b \geq 0, 0 < k \leq p/\delta - 1$ and $1 - a \geq |b-1|$. Then $b \geq a$ and $a \leq 1$. By a simple computation, we obtain that $-af(k) + f(bk) = (pk/\delta(1+k)(1+bk))(b + bk - a - abk)$, and thus $-af(k) + f(bk) \geq 0$. Moreover, $-af(k) + f(bk) = 0$ if and only if either $a = b = 1$ or $a = b = 0$. This completes the proof. ■

Theorem 3.6 follows immediately from Theorem 2.10 and Lemma 3.5.

THEOREM 3.6 *Suppose* $p/\delta > 1$.

(i) *If* $p \leq d\lambda_1 + \delta$, *then the trivial steady-state solution* 0 *of Equation* (4) *attracts all solutions of Equation* (4) *with the initial value in* X_+.
(ii) *If* $p > d\lambda_1 + \delta$, *then there exists a unique non-trivial steady-state solution of Equation* (4) *which attracts all solutions of Equation* (4) *with the initial value in* $X_+\backslash\{0\}$.

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