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Wavefronts for a non-local reaction–diffusion population model with general distributive maturity

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We consider the Al-Omari and Gourley non-local reaction–diffusion population model with distributed maturity. Existence of monotone wavefronts for some particular probability distribution of maturity that permits the linear chain trick was previously obtained; here, we consider the most general form of such a distribution using some comparison arguments for abstract functional differential equations with infinite delay and a fixed point approach combined with the upper and lower solutions technique.

Keywords: non-local distribution; reaction–diffusion equation; infinite distributed maturity; monotone travelling front.

1. Introduction

Much has been done for the existence of monotone travelling waves of population models in the form of delay reaction–diffusion equations with non-local effects that describe the interaction of spatial diffusion and time delay (see, e.g. Britton, 1990; Gourley & Britton, 1996; Ma, 2001; Schaaf, 1987; Smith & Zhao, 2000; So *et al.*, 2001; Thieme, 1979; Thieme & Zhao, 2003; Weinberger, 2002; Weng *et al.*, 2003; Wu & Zou, 2001, and the survey paper by Gourley & Wu, 2006).

In particular, Al-Omari & Gourley (2005a) derived the following model for a single-species population with stage structure and distributed maturation:

$$\frac{\partial u(t,x)}{\partial t} = d_m \frac{\partial^2 u(t,x)}{\partial x^2} - \beta u^2(t,x) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, x, y) f(s) \mathrm{e}^{-\gamma s} u(t-s,y) \mathrm{d}y \, \mathrm{d}s, \quad (1.1)$$

for $t > 0$, $x \in \mathbb{R}$, where

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{4t}}, \quad (1.2)$$

$u(t,x)$ is the density of the matured individuals at time t and location x, $d_m > 0$ and $d_i \geqslant 0$ are the constant diffusion coefficients for matured individuals and juveniles, respectively, $\gamma > 0$ is the death

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rate of juveniles and f is the probability density function so that

$$f(s) \geqslant 0, \quad \forall s \in [0, \infty), \qquad \int_0^\infty f(s)\mathrm{d}s = 1. \tag{1.3}$$

Model (1.1) includes, with a particular probability function f, the following non-local reaction–diffusion equation with discrete delay $\tau > 0$:

$$\frac{\partial u(t,x)}{\partial t} = d_m \frac{\partial^2 u(t,x)}{\partial x^2} - \beta u^2(t,x) + \alpha \int_{-\infty}^{\infty} G(d_i\tau, x, y)\mathrm{e}^{-\gamma\tau} u(t-\tau, y)\mathrm{d}y. \tag{1.4}$$

Also, when $d_i \to 0$, $G(d_i\tau, x, y) \to G(0, x, y) = \delta(x-y)$ and (1.4) reduces to the local reaction–diffusion equation with delay:

$$\frac{\partial u(t,x)}{\partial t} = d_m \frac{\partial^2 u(t,x)}{\partial x^2} - \beta u^2(t,x) + \alpha\, \mathrm{e}^{-\gamma\tau} u(t-\tau, x). \tag{1.5}$$

Using a monotone iterative scheme, Al-Omari & Gourley (2002) concluded that there exists $c^* > 0$ (depending on τ) such that (1.5) has a monotone travelling wavefront with speed c for every $c > c^*$. The persistence of such a wavefront when $d_i > 0$ is small was obtained using a perturbation argument based on the Fredholm alternative theory. In Al-Omari & Gourley (2005b), model (1.1) with general distributed maturity was studied and an asymptotic expression of wavefronts with large wave speeds was derived using the idea of Canosa (1973). When $f(s) = \frac{s}{\tau^2}\mathrm{e}^{-\frac{s}{\tau}}$, the existence of wavefronts of (1.1) with small $\tau > 0$ was established. In addition, for the special form $f(s) = \frac{1}{\tau} g_0\left(\frac{s}{\tau}\right)$ with $0 \leqslant g_0(s) \leqslant A\,\mathrm{e}^{-Bs}$ and $\int_0^\infty g_0(s)\mathrm{d}s = 1$ (for some constants A and B), an asymptotic expression for the minimum wave speed $c_{\min}$ of (1.1) with small $\tau > 0$ was derived.

The purpose of this paper is to establish the existence of monotone wavefronts for (1.1) with the most general probability function f. Our main techniques include the general setting and comparison argument developed in Ruan & Wu (1994) for abstract functional differential equations with infinite delay, the fixed point argument developed by Ma (2001) that establishes the existence of wavefronts once an appropriate pair of weak upper and weak lower solutions are constructed and the usual procedure of constructing a pair of weak upper and weak lower solutions from the linearization of the system at the trivial solution.

In what follows, we focus on the wavefront $u(t,x) = U(x+ct)$ with wave speed $c > 0$ that connects 0 and $u^+ = \frac{\alpha}{\beta}\int_0^\infty \mathrm{e}^{-\gamma s} f(s)\mathrm{d}s$ at $\pm\infty$, where 0 and u^+ are the non-negative equilibria, and the solutions of $\alpha u \int_0^\infty \mathrm{e}^{-\gamma s} f(s)\mathrm{d}s = \beta u^2$. We note that

$$\begin{aligned} &\alpha u \int_0^\infty \mathrm{e}^{-\gamma s} f(s)\mathrm{d}s < \beta u^2 \quad \text{as } u > u^+, \\ &\alpha u \int_0^\infty \mathrm{e}^{-\gamma s} f(s)\mathrm{d}s > \beta u^2 \quad \text{as } u \in (0, u^+). \end{aligned} \tag{1.6}$$

2. Well posedness and comparisons

In this section, we discuss the existence, uniqueness and positivity of solutions for the initial problem of (1.1), following the abstract framework of Ruan & Wu (1994).

Let $X = \mathrm{BUC}(\mathbb{R}, \mathbb{R})$ be the space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|_X$, $X_+ = \{\varphi \in X\colon \varphi(x) \geqslant 0, x \in \mathbb{R}\}$. Then, X_+ is a closed cone of X and X is a Banach lattice under the partial ordering $\leqslant_X$ induced by X_+.

Suppose $g\colon (-\infty, 0] \to [1, \infty)$ is a function satisfying the following conditions:

(g1) g is continuous, non-increasing and $g(0) = 1$;
(g2) $\frac{g(s+\theta)}{g(s)} \to 1$ uniformly for $s \in (-\infty, 0]$ as $\theta \to 0^+$;
(g3) $g(s) \to \infty$ as $s \to -\infty$.

Define

$$\mathrm{UC}_g = \left\{\phi;\, \phi\colon (-\infty, 0] \to X \text{ is continuous}, \frac{\phi}{g} \text{ is uniformly continuous on } (-\infty, 0], \sup_{s\leqslant 0} \frac{|\phi(s)|_X}{g(s)} < \infty\right\},$$

$$\mathrm{UC}_g^+ = \{\phi \in \mathrm{UC}_g;\, \phi(\theta) \geqslant_X 0 \text{ for } \theta \leqslant 0\}.$$

Similarly, UC_g^+ induces a partial ordering $\leqslant_{\mathrm{UC}_g}$ on UC_g. Let UC_g be equipped with the norm

$$|\phi|_g = |\phi|_{\mathrm{UC}_g} := \sup_{s\leqslant 0} \frac{|\phi(s)|_X}{g(s)}, \quad \text{for } \phi \in \mathrm{UC}_g.$$

According to Ruan & Wu (1994, p. 494), $(\mathrm{UC}_g, |\cdot|_g)$ is a Banach space satisfying the required fading memory space conditions (A1–A5) in Ruan & Wu (1994). As usual, we identify an element $\phi \in \mathrm{UC}_g$ as a function from $\mathbb{R} \times (-\infty, 0] \to \mathbb{R}$ by $\phi(x, s) = \phi(s)(x)$. For any continuous function $u\colon(-\infty, b) \to X$, where $b > 0$, we define u_t by $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$.

For any $L \geqslant u^+$, define some subsets of X and UC_g by

$$[0, L]_X := \{\varphi \in X;\, 0 \leqslant \varphi(x) \leqslant L, x \in \mathbb{R}\},$$

$$[0, L]_{\mathrm{UC}_g} := \{\phi \in \mathrm{UC}_g\colon 0 \leqslant_X \phi(\theta) \leqslant_X L \text{ for } \theta \leqslant 0\},$$

$$D := [0, \infty) \times [0, L]_X, \quad D(t) := [0, L]_X, \quad \text{for } t \in [0, \infty),$$

$$\mathcal{D} := [0, \infty) \times [0, L]_{\mathrm{UC}_g}, \quad \mathcal{D}(t) := [0, L]_{\mathrm{UC}_g}, \quad \text{for } t \in [0, \infty).$$

Then, conditions (D1) and (D2) in Ruan & Wu (1994) are satisfied. As $D(t) = [0, L]_X$ is convex, condition (D3) in Ruan & Wu (1994) also holds (see Ruan & Wu, 1994, p. 510).

Note that the solution of the initial-value problem for the parabolic partial differential equation

$$\begin{aligned} \frac{\partial w(t, x)}{\partial t} &= d_m \frac{\partial^2 w(t, x)}{\partial x^2} - 2L\beta w(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ w(0, \cdot) &= \varphi(\cdot) \in X \end{aligned} \tag{2.1}$$

is given by

$$w(t, x) = [T(t)\varphi](x) := \frac{\mathrm{e}^{-2L\beta t}}{\sqrt{4\pi d_m t}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-y)^2}{4d_m t}} \varphi(y)\mathrm{d}y.$$

$T(t)\colon X \to X$ is an analytic semigroup (see Lunardi, 1995) with $T(t)X_+ \subset X_+$ for all $t \geqslant 0$.

Rewrite (1.1) as

$$\frac{\partial u(t,x)}{\partial t} - d_m \frac{\partial^2 u(t,x)}{\partial x^2} + 2\beta L u(t,x)$$
$$= \beta u(t,x)(2L - u(t,x)) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, x, y) f(s) \mathrm{e}^{-\gamma s} u(t-s, y) \mathrm{d}y \, \mathrm{d}s. \tag{2.2}$$

Define a functional $F\colon \mathrm{UC}_g \to X$ as follows:

$$F(\phi)(x) = \beta\phi(0,x)(2L - \phi(0,x)) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, x, y) f(s) \mathrm{e}^{-\gamma s} \phi(-s, y) \mathrm{d}y \, \mathrm{d}s.$$

Assume that $\phi \in \mathcal{D}(0) = [0, L]_{\mathrm{UC}_g}$ is a given continuous initial function with $\phi(\theta, x) \in [0, L]$, $\forall\, (\theta, x) \in (-\infty, 0] \times \mathbb{R}$. Then, the abstract equivalent integral form of (1.1) is

$$\begin{aligned} u(t) &= T(t)\phi(0,\cdot) + \int_0^t T(t-\tau) F(u_\tau) \mathrm{d}\tau, && t > 0,\\ u(t) &= \phi(t,\cdot), && t \in (-\infty, 0]. \end{aligned} \tag{2.3}$$

DEFINITION 2.1 A continuous function $v\colon (-\infty, b) \to X$ is called a supersolution (subsolution) of (1.1) on $[0, b)$ if

$$v(t) \geqslant (\leqslant) T(t)\phi(0,\cdot) + \int_0^t T(t-\tau) F(v_\tau) \mathrm{d}\tau, \quad 0 \leqslant t < b. \tag{2.4}$$

If v is both a supersolution and a subsolution on $[0, b)$, then it is said to be a (mild) solution of (1.1).

REMARK 2.1 Assume that there is a bounded and continuous $v\colon \mathbb{R} \times (-\infty, b) \to \mathbb{R}$, with $b > 0$ and such that v is C^2in $x \in \mathbb{R}$, C^1 in $t \in (0, b)$, and

$$\frac{\partial v(t,x)}{\partial t} \geqslant (\leqslant) d_m \frac{\partial^2 v(t,x)}{\partial x^2} - \beta v^2(t,x) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, x, y) f(s) \mathrm{e}^{-\gamma s} v(t-s, y) \mathrm{d}y \, \mathrm{d}s,$$

for $(t, x) \in (0, b) \times \mathbb{R}$. Then, by the fact that $T(t)X_+ \subset X_+$, it follows that (2.4) holds, and hence $v(t, x)$ is a supersolution (subsolution) of (1.1) on $[0, b)$.

THEOREM 2.1 Assume that (1.3) holds and $L \geqslant u^+$ is given. Then, the following conclusions hold.

(i) For any ϕ with $\phi \in [0, L]_{\mathrm{UC}_g}$, (1.1) has a unique solution $u(t, x) = u(t, x; \phi)$ defined on $[0, \infty)$ such that $u(t, \cdot) \in [0, L]_X$ and $u_t \in [0, L]_{\mathrm{UC}_g}$ for $t \geqslant 0$.

(ii) For any pair of supersolution $w^+(t, x)$ and subsolution $w^-(t, x)$ of (1.1) on $\mathbb{R} \times \mathbb{R}$ with $0 \leqslant w^+(t, x), w^-(t, x) \leqslant L$ for $(t, x) \in \mathbb{R} \times \mathbb{R}$ and $0 \leqslant w^-(s, x) \leqslant w^+(s, x) \leqslant L$ for $(s, x) \in (-\infty, 0] \times \mathbb{R}$, there holds $0 \leqslant w^-(t, x) \leqslant w^+(t, x) \leqslant L$ for $(t, x) \in (0, \infty) \times \mathbb{R}$.

Proof. Let $T(t, \tau) = T(t - \tau)$ and $S(t, \tau) = T(t - \tau)$. Then, one can verify that conditions (T1–T4), (S1) and (S2) in Ruan & Wu (1994) are satisfied. For any $l > 0$, define

$$L_{1,l} := (2\beta L + \beta l) + \alpha \int_0^\infty f(s) g(-s) \mathrm{e}^{-\gamma s} \, \mathrm{d}s.$$

Then, for any $\phi_1, \phi_2 \in [0, L]_{\mathrm{UC}_g}$, we have

$$|F(\phi_1) - F(\phi_2)|_X = \sup_{x \in \mathbb{R}} |F(\phi_1)(x) - F(\phi_2)(x)| \leqslant L_{1,l} |\phi_1 - \phi_2|_{\mathrm{UC}_g}$$

as long as $|\phi_1|_{\mathrm{UC}_g} \leqslant l$ and $|\phi_2|_{\mathrm{UC}_g} \leqslant l$. Therefore, condition (F) in Ruan & Wu (1994) is satisfied.

Let $\hat{L}$ and $\hat{0}$ stand for the constant functions with value L and 0 on $\mathbb{R} \times \mathbb{R}$, respectively. In view of Definition 2.1 and Remark 2.1, one can easily verify that $u = \hat{L}$ and $u = \hat{0}$ is a pair of supersolution and subsolution of (1.1). Let $v^+ = \hat{L}$ and $v^- = \hat{0}$. Then, (C3–C5) in Ruan & Wu (1994) are satisfied.

Furthermore, if $\phi_1, \phi_2 \in [0, L]_{\mathrm{UC}_g}$ and $\phi_1 \geqslant_{\mathrm{UC}_g} \phi_2$, then

$$\begin{aligned} F(\phi_1)(x) - F(\phi_2)(x) &= 2\beta L[\phi_1(x,0) - \phi_2(x,0)] - \beta[\phi_1^2(x,0) - \phi_2^2(x,0)] \\ &\quad + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, x, y) f(s) \mathrm{e}^{-\gamma s} [\phi_1(y,-s) - \phi_2(y,-s)] \mathrm{d}y \, \mathrm{d}s \\ &= \beta[2L - (\phi_1(x,0) + \phi_2(x,0))](\phi_1(x,0) - \phi_2(x,0)) \\ &\quad + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, x, y) f(s) \mathrm{e}^{-\gamma s} [\phi_1(y,-s) - \phi_2(y,-s)] \mathrm{d}y \, \mathrm{d}s \geqslant 0. \end{aligned}$$

Thus, $F(\phi)$ is non-decreasing on $\phi \in [0, L]_{\mathrm{UC}_g}$.

For any $\phi_1, \phi_2 \in [0, L]_{\mathrm{UC}_g}$ with $\phi_1 \geqslant_{\mathrm{UC}_g} \phi_2$, we have

$$[\phi_1(0) - \phi_2(0)] + h[F(\phi_1) - F(\phi_2)] \geqslant_X 0, \quad \text{for } h \geqslant 0,$$

which leads to

$$\lim_{h \to 0^+} \operatorname{dist}\{[\phi_1(0) - \phi_2(0)] + h[F(\phi_1) - F(\phi_2)], X_+\} = 0.$$

For each $b > 0$, the existence and uniqueness of a solution $u(t, x; \phi)$ on $[0, b)$ follows from Theorem 5.2 in Ruan & Wu (1994) with $S(t,s) = T(t,s) = T(t-s)$ for $t \geqslant s \geqslant 0$ and $v^+ = \hat{L}, v^- = \hat{0}$. As $0 \leqslant u(t, \cdot; \phi) \leqslant L$ on $[0, b)$, we conclude that the maximal interval of existence is $[0, \infty)$.

We now prove the conclusion (ii). Since $w^+, w^- \in [0, L]_{\mathrm{UC}_g}$ and $w^- \leqslant_{\mathrm{UC}_g} w^+$, it follows from Theorem 5.2 in Ruan & Wu (1994) that

$$0 \leqslant u(t, x; w^-) \leqslant u(t, x; w^+) \leqslant L, \quad \text{for } x \in \mathbb{R}, \ \ t \geqslant 0. \tag{2.5}$$

Again by applying Theorem 5.2 in Ruan & Wu (1994) with $[v_-(t,x) = w^-(t,x)$ and $v^+(t,x) = \hat{L}]$ and $[v_-(t,x) = \hat{0}$ and $v^+(t,x) = w^+(t,x)]$, respectively, we get

$$w^-(t,x) \leqslant u(t, x; w^-) \leqslant L, \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}, \tag{2.6}$$

and

$$0 \leqslant u(x, t; w^+) \leqslant w^+(t,x), \quad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}, \tag{2.7}$$

from which it follows that $w^-(t,x) \leqslant w^+(t,x)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$. □

3. Monotone wavefronts

Consider a solution of (1.1) with the form $u(t,x) = U(x+ct)$, $c > 0$, connecting the two equilibria $u \equiv 0$ and $u \equiv u^+$. Let $z = x + ct$, then we have from (1.1) that $U(z)$ satisfies

$$cU'(z) = d_m U''(z) - \beta U^2(z) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, z, \xi) f(s) \mathrm{e}^{-\gamma s} U(\xi - cs) \mathrm{d}\xi \, \mathrm{d}s \tag{3.1}$$

subject to the boundary condition

$$U(-\infty) = 0, \quad U(\infty) = u^+. \tag{3.2}$$

Here, we are interested in non-decreasing solutions of (3.1) and (3.2).

For any given $L \geqslant u^+$, let

$$C_{[0,L]}(\mathbb{R}, \mathbb{R}) := \{\varphi; \varphi \in C(\mathbb{R}, \mathbb{R}), 0 \leqslant \varphi(z) \leqslant L \text{ for } z \in \mathbb{R}\},$$

$$[QU](z) := 2\beta L U(z) - \beta U^2(z) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, z, \xi) f(s) \mathrm{e}^{-\gamma s} U(\xi - cs) \mathrm{d}\xi \, \mathrm{d}s.$$

Then, (3.1) can be written as

$$-d_m U''(z) + cU'(z) + 2\beta L U(z) = [QU](z). \tag{3.3}$$

The eigenvalues of $-d_m U''(z) + cU'(z) + 2\beta L U(z) = 0$ are

$$\lambda_1 = \frac{c - \sqrt{c^2 + 8 d_m \beta L}}{2 d_m} < 0 \quad \text{and} \quad \lambda_2 = \frac{c + \sqrt{c^2 + 8 d_m \beta L}}{2 d_m} > 0.$$

Define an operator A: $C_{[0,L]}(\mathbb{R}, \mathbb{R}) \to C^2(\mathbb{R}, \mathbb{R})$ by

$$[A\varphi](z) := \frac{1}{d_m(\lambda_2 - \lambda_1)} \left\{ \int_{-\infty}^z \mathrm{e}^{\lambda_1 (z-s)} [Q\varphi](s) \mathrm{d}s + \int_z^\infty \mathrm{e}^{\lambda_2 (z-s)} [Q\varphi](s) \mathrm{d}s \right\}.$$

LEMMA 3.1 We have the following:

(i) $[Q\varphi_1](z) \leqslant [Q\varphi_2](z)$ for $z \in \mathbb{R}$, if $\varphi_1, \varphi_2 \in C_{[0,L]}(\mathbb{R}, \mathbb{R})$ with $\varphi_1(z) \leqslant \varphi_2(z)$ for $z \in \mathbb{R}$;
(ii) $0 \leqslant [Q\varphi](z) \leqslant 2\beta L^2$ for $\varphi \in C_{[0,L]}(\mathbb{R}, \mathbb{R})$;
(iii) $[Q\varphi](z)$ is non-decreasing in $z \in \mathbb{R}$, if $\varphi \in C_{[0,L]}(\mathbb{R}, \mathbb{R})$ is non-decreasing in $z \in \mathbb{R}$.

Proof. The proof of (i) is similar to the proof of the monotonicity of F in Theorem 2.1. If $\varphi \in C_{[0,L]}(\mathbb{R}, \mathbb{R})$, then in view of conclusion (i), we have

$$0 \leqslant [Q\varphi_1](z) \leqslant 2\beta L^2 - \beta L^2 + \alpha L \int_0^\infty f(s) \mathrm{e}^{-\gamma s} \, \mathrm{d}s \leqslant 2\beta L^2,$$

which implies conclusion (ii).

Assume that $\varphi \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ is non-decreasing in $z \in \mathbb{R}$. Note that

$$\alpha \int_0^\infty \int_{\mathbb{R}} G(d_i s, z, \xi) f(s) \mathrm{e}^{-\gamma s} \varphi(\xi - cs) \mathrm{d}\xi \, \mathrm{d}s = \alpha \int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i s}} \mathrm{e}^{-\frac{(z-\xi)^2}{4 d_i s}} f(s) \mathrm{e}^{-\gamma s} \varphi(\xi - cs) \mathrm{d}\xi \, \mathrm{d}s$$

$$= \alpha \int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i s}} \mathrm{e}^{-\frac{t^2}{4 d_i s}} f(s) \mathrm{e}^{-\gamma s} \varphi(z - t - cs) \mathrm{d}t \, \mathrm{d}s.$$

For given $z_1, z_2 \in \mathbb{R}$, $z_1 \leqslant z_2$, we have

$$\begin{aligned} [Q\varphi](z_1) - [Q\varphi](z_2) &= [2\beta L - \beta(\varphi(z_1) + \varphi(z_2))][\varphi(z_1) - \varphi(z_2)] \\ &\quad + \alpha \int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i s}} \mathrm{e}^{-\frac{t^2}{4 d_i s}} f(s) \mathrm{e}^{-\gamma s} [\varphi(z_1 - t - cs) \\ &\quad - \varphi(z_2 - t - cs)] \mathrm{d}t \, \mathrm{d}s \leqslant 0. \end{aligned}$$

Therefore, conclusion (iii) follows. □

It is easy to show that $A\colon C_{[0,L]}(\mathbb{R},\mathbb{R}) \to C^2(\mathbb{R},\mathbb{R})$ is a well-defined map, and for any $\varphi \in C_{[0,L]}(\mathbb{R},\mathbb{R})$, one has

$$-d_m[A\varphi]'' + c[A\varphi]' + 2\beta L[A\varphi] = Q\varphi. \tag{3.4}$$

Thus, a fixed point of A is a solution of (3.3). The following lemma is a straightforward consequence of Lemma 3.1.

LEMMA 3.2 We have the following:

(i) $[A\varphi_1](z) \leqslant [A\varphi_2](z)$ for $z \in \mathbb{R}$, if $\varphi_1, \varphi_2 \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ with $\varphi_1(z) \leqslant \varphi_2(z)$ for $z \in \mathbb{R}$;
(ii) $0 \leqslant [A\varphi](z) \leqslant L$ for $\varphi \in C_{[0,L]}(\mathbb{R},\mathbb{R})$;
(iii) $[A\varphi](z)$ is non-decreasing in $z \in \mathbb{R}$, if $\varphi \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ is non-decreasing in $z \in \mathbb{R}$.

Let $0 < \mu < \min\{-\lambda_1, \lambda_2\}$ and

$$B_\mu(\mathbb{R},\mathbb{R}) = \left\{ \varphi \in C(\mathbb{R},\mathbb{R});\ \sup_{z\in\mathbb{R}} |\varphi(z)| \mathrm{e}^{-\mu|z|} < \infty \right\},$$

with the norm $|\varphi|_\mu = \sup_{z\in\mathbb{R}} |\varphi(z)| \mathrm{e}^{-\mu|z|}$. Then, it is easy to verify that $(B_\mu(\mathbb{R},\mathbb{R}), |\cdot|_\mu)$ is a Banach space.

LEMMA 3.3 Assume that $L \geqslant u^+$ be given. Then, $A\colon B_\mu(\mathbb{R},\mathbb{R}) \to B_\mu(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R})$.

Proof. We first claim that $Q: C_{[0,L]}(\mathbb{R},\mathbb{R}) \to B_\mu(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R})$. For any $\varphi_1, \varphi_2 \in C_{[0,L]}(\mathbb{R},\mathbb{R})$, we have

$$\begin{aligned}
&|[Q\varphi_1](z) - [Q\varphi_2](z)|\mathrm{e}^{-\mu|z|} \\
&\quad \leqslant |2\beta L[\varphi_1(z) - \varphi_2(z)] - \beta[\varphi_1^2(z) - \varphi_2^2(z)]|\mathrm{e}^{-\mu|z|} \\
&\qquad + \alpha\,\mathrm{e}^{-\mu|z|}\int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{(z-\xi)^2}{4d_i s}} f(s)\mathrm{e}^{-\gamma s}|\varphi_1(\xi - cs) - \varphi_2(\xi - cs)|\mathrm{d}\xi\,\mathrm{d}s \\
&\quad = [2\beta L - \beta(\varphi_1(z) + \varphi_2(z))][\varphi_1(z) - \varphi_2(z)]\mathrm{e}^{-\mu|z|} \\
&\qquad + \alpha\,\mathrm{e}^{-\mu|z|}\left\{\int_0^\tau \int_{-\infty}^{-M} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}} f(s)\mathrm{e}^{-\gamma s}|\varphi_1(z - y - cs) - \varphi_2(z - y - cs)|\mathrm{d}y\,\mathrm{d}s\right. \\
&\qquad\qquad + \int_0^\tau \int_{-M}^{M} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}} f(s)\mathrm{e}^{-\gamma s}|\varphi_1(z - y - cs) - \varphi_2(z - y - cs)|\mathrm{d}y\,\mathrm{d}s \\
&\qquad\qquad + \int_0^\tau \int_{M}^{\infty} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}} f(s)\mathrm{e}^{-\gamma s}|\varphi_1(z - y - cs) - \varphi_2(z - y - cs)|\mathrm{d}y\,\mathrm{d}s \\
&\qquad\qquad \left. + \int_\tau^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}} f(s)\mathrm{e}^{-\gamma s}|\varphi_1(z - y - cs) - \varphi_2(z - y - cs)|\mathrm{d}y\,\mathrm{d}s\right\}.
\end{aligned} \tag{3.5}$$

For any $\epsilon > 0$, let $\tau > 0$ and $M > 0$ be large enough such that

$$2\alpha L \int_\tau^\infty f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}s < \frac{\epsilon}{4},$$

$$2\alpha L\left\{\int_{-\infty}^{-M} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}}\,\mathrm{d}y + \int_M^\infty \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}}\,\mathrm{d}y\right\} < \frac{\epsilon}{4}. \tag{3.6}$$

For such chosen τ and M, if $y \in [-M, M]$, $s \in [0, \tau]$, one has $-y - cs \in [-M - c\tau, M]$. Choose $\delta > 0$ such that

$$\delta < \min\left\{\frac{\epsilon}{4\alpha}\mathrm{e}^{-\mu(M+c\tau)}, \frac{\epsilon}{8\beta L}\right\}. \tag{3.7}$$

If $|\varphi_1 - \varphi_2|_\mu < \delta$, we have

$$|\varphi_1(z-y-cs)-\varphi_2(z-y-cs)| \leqslant \delta\,\mathrm{e}^{\mu|z|}\,\mathrm{e}^{\mu(M+c\tau)} < \frac{\epsilon}{4\alpha}\mathrm{e}^{\mu|z|}, \quad \text{for } y \in [-M, M],\quad s \in [0, \tau]. \tag{3.8}$$

Therefore, we have from (3.5–3.8) that

$$
\begin{aligned}
&|[Q\varphi_1](z) - [Q\varphi_2](z)|\mathrm{e}^{-\mu|z|} \\
&\leqslant 2\beta L|\varphi_1 - \varphi_2|_\mu + 2L\alpha \int_0^\tau f(s)\mathrm{e}^{-\gamma s}\left\{\int_{-\infty}^{-M} \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}}\,\mathrm{d}y + \int_M^\infty \frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}}\,\mathrm{d}y\right\}\mathrm{d}s \\
&\quad + \alpha\delta\,\mathrm{e}^{\mu(M+c\tau)}\int_0^\tau f(s)\mathrm{e}^{-\gamma s}\left[\int_{-M}^{M}\frac{1}{\sqrt{4\pi d_i s}}\mathrm{e}^{-\frac{y^2}{4d_i s}}\,\mathrm{d}y\right]\mathrm{d}s + 2\alpha L\int_\tau^\infty f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}s \\
&\leqslant 2\beta L\delta + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon,
\end{aligned}
$$

if $|\varphi_1 - \varphi_2|_\mu < \delta$. Hence, Q is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R})$.

Now, we want to show that A is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R})$. For any $z \geqslant 0$, we have

$$
\begin{aligned}
|[A\varphi_1](z) - [A\varphi_2](z)| &\leqslant \frac{1}{d_m(\lambda_2-\lambda_1)}\left\{\int_{-\infty}^{z}\mathrm{e}^{\lambda_1(z-s)}|(Q\varphi_1)(s) - (Q\varphi_2)(s)|\mathrm{d}s\right. \\
&\qquad\left. + \int_z^\infty \mathrm{e}^{\lambda_2(z-s)}|(Q\varphi_1)(s) - (Q\varphi_2)(s)|\mathrm{d}s\right\} \\
&\leqslant \frac{1}{d_m(\lambda_2-\lambda_1)}\left\{\int_{-\infty}^{z}\mathrm{e}^{\lambda_1(z-s)+\mu|s|}\,\mathrm{d}s + \int_z^\infty \mathrm{e}^{\lambda_2(z-s)+\mu|s|}\,\mathrm{d}s\right\}|Q\varphi_1 - Q\varphi_2|_\mu \\
&= \frac{1}{d_m(\lambda_2-\lambda_1)}\left\{\int_{-\infty}^{0}\mathrm{e}^{\lambda_1(z-s)-\mu s}\,\mathrm{d}s + \int_0^z \mathrm{e}^{\lambda_1(z-s)+\mu s}\,\mathrm{d}s\right. \\
&\qquad\left. + \int_z^\infty \mathrm{e}^{\lambda_2(z-s)+\mu s}\,\mathrm{d}s\right\}|Q\varphi_1 - Q\varphi_2|_\mu \\
&= \frac{1}{d_m(\lambda_2-\lambda_1)}\left[\frac{\lambda_2-\lambda_1}{(\mu-\lambda_1)(\lambda_2-\mu)}\mathrm{e}^{\mu z} + \frac{2\mu}{\lambda_1^2-\mu^2}\mathrm{e}^{\lambda_1 z}\right]|Q\varphi_1 - Q\varphi_2|_\mu.
\end{aligned}
$$

Therefore, for $z \geqslant 0$, it follows that

$$
\begin{aligned}
|[A\varphi_1](z) - [A\varphi_2](z)|\mathrm{e}^{-\mu|z|} &\leqslant \frac{1}{d_m(\lambda_2-\lambda_1)}\left[\frac{\lambda_2-\lambda_1}{(\mu-\lambda_1)(\lambda_2-\mu)} + \frac{2\mu}{\lambda_1^2-\mu^2}\mathrm{e}^{(\lambda_1-\mu)z}\right]|Q\varphi_1 - Q\varphi_2|_\mu \\
&\leqslant \frac{1}{d_m(\lambda_2-\lambda_1)}\left[\frac{\lambda_2-\lambda_1}{(\mu-\lambda_1)(\lambda_2-\mu)} + \frac{2\mu}{\lambda_1^2-\mu^2}\right]|Q\varphi_1 - Q\varphi_2|_\mu. \qquad (3.9)
\end{aligned}
$$

If $z < 0$, we have

$$|[A\varphi_1](z) - [A\varphi_2](z)| \leqslant \frac{1}{d_m(\lambda_2 - \lambda_1)} \left\{ \int_{-\infty}^{z} e^{\lambda_1(z-s)-\mu s}\, ds + \int_{z}^{0} e^{\lambda_2(z-s)-\mu s}\, ds \right.$$
$$\left. + \int_0^\infty e^{\lambda_2(z-s)+\mu s}\, ds \right\} |Q\varphi_1 - Q\varphi_2|_\mu$$
$$= \frac{1}{d_m(\lambda_2 - \lambda_1)} \left[\frac{\lambda_2 - \lambda_1}{-(\mu + \lambda_1)(\lambda_2 + \mu)} e^{-\mu z} + \frac{2\mu}{\lambda_2^2 - \mu^2} e^{\lambda_2 z} \right] |Q\varphi_1 - Q\varphi_2|_\mu.$$

Therefore, for $z < 0$, it follows that

$$|[A\varphi_1](z) - [A\varphi_2](z)| e^{-\mu|z|} \leqslant \frac{1}{d_m(\lambda_2 - \lambda_1)} \left[\frac{\lambda_2 - \lambda_1}{-(\mu + \lambda_1)(\lambda_2 + \mu)} + \frac{2\mu}{\lambda_2^2 - \mu^2} e^{(\lambda_2+\mu)z} \right] |Q\varphi_1 - Q\varphi_2|_\mu$$
$$\leqslant \frac{1}{d_m(\lambda_2 - \lambda_1)} \left[\frac{\lambda_2 - \lambda_1}{-(\mu + \lambda_1)(\lambda_2 + \mu)} + \frac{2\mu}{\lambda_2^2 - \mu^2} \right] |Q\varphi_1 - Q\varphi_2|_\mu. \tag{3.10}$$

Putting together (3.9) and (3.10), we conclude that A is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R})$. □

DEFINITION 3.1 A function $\xi \in C(\mathbb{R}, \mathbb{R})$ is called a weak upper solution of (3.1) if it is twice differentiable on $\mathbb{R}$ except for a set with finitely many points, $S = \{z_1, z_2, \ldots, z_m\}$, and satisfies

$$d_m \xi''(z) - c\xi'(z) - \beta\xi^2(z) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, z, y) f(s) e^{-\gamma s} \xi(y - cs) dy\, ds \leqslant 0, \quad \text{for } z \in \mathbb{R} \setminus S. \tag{3.11}$$

A weak lower solution of (3.1) is defined in a similar way with a reversing inequality in (3.11).

DEFINITION 3.2 A function $\rho \in C^2(\mathbb{R}, \mathbb{R})$ is called an upper solution of (3.1) if it satisfies

$$d_m \rho''(z) - c\rho'(z) - \beta\rho^2(z) + \alpha \int_0^\infty \int_{-\infty}^\infty G(d_i s, z, y) f(s) e^{-\gamma s} \rho(y - cs) dy\, ds \leqslant 0, \quad \text{for } z \in \mathbb{R}. \tag{3.12}$$

A lower solution of (3.1) is defined in a similar way with a reversing inequality in (3.12).

LEMMA 3.4 If $\xi \in C(\mathbb{R}, \mathbb{R})$ is a weak upper solution (weak lower solution) of (3.1) and $\xi'(z^+) \leqslant (\geqslant) \xi'(z^-)$ for $z \in \mathbb{R}$, then $[A\xi](z) \leqslant (\geqslant) \xi(z)$ for $z \in \mathbb{R}$, and $\rho = A\xi$ is an upper (a lower) solution of (3.1).

Proof. We only verify the conclusion for the upper solution. Assume that $z_0 = -\infty < z_1 < z_2 < \cdots < z_m < z_{m+1} = \infty$. By Lemma 3.2 and the definition of $A\xi$, we know that $A\xi \in C_{[0,L]}(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}, \mathbb{R})$. Since

$$-d_m \xi''(z) + c\xi'(z) + 2\beta L\xi(z) \geqslant [Q\xi](z), \quad \text{for } z \in \mathbb{R} \setminus S,$$

for any $z \in (z_k, z_{k+1})$, $k = 0, 1, \ldots, m$, we have

$$
\begin{aligned}
[A\xi](z) &\leqslant \frac{1}{d_m(\lambda_2-\lambda_1)}\left\{\int_{-\infty}^{z} e^{\lambda_1(z-s)}[-d_m\xi''(s)+c\xi'(s)+2\beta L\xi(z)]ds \right. \\
&\qquad \left. + \int_{z}^{\infty} e^{\lambda_2(z-s)}[-d_m\xi''(s)+c\xi'(s)+2\beta L\xi(z)]ds\right\} \\
&= \xi(z) + \frac{1}{\lambda_2-\lambda_1}\left\{\sum_{j=1}^{k} e^{\lambda_1(z-z_j)}[\xi'(z_j^+)-\xi'(z_j^-)] + \sum_{j=k+1}^{m} e^{\lambda_2(z-z_j)}[\xi'(z_j^+)-\xi'(z_j^-)]\right\} \\
&\leqslant \xi(z).
\end{aligned} \tag{3.13}
$$

On the other hand, we have from Lemma 3.2 and (3.4) that

$$
\begin{aligned}
& d_m[A\xi]''(z) - c[A\xi]'(z) - \beta[A\xi]^2(z) + \alpha\int_0^\infty\int_{-\infty}^{\infty} G(d_i s, z, y) f(s) e^{-\gamma s}[A\xi](y-cs)dy\, ds \\
&= d_m[A\xi]''(z) - c[A\xi]'(z) - 2\beta L[A\xi](z) + [Q(A\xi)](z) \\
&\leqslant d_m[A\xi]''(z) - c[A\xi]'(z) - 2\beta L[A\xi](z) + [Q\xi](z) = 0.
\end{aligned}
$$

Therefore, $A\xi$ is an upper solution of (3.1). □

THEOREM 3.1 Let $L = u^+$. If (3.1) has an upper solution $\bar{\rho} \in C_{[0,L]}(\mathbb{R},\mathbb{R}) \cap C^2(\mathbb{R},\mathbb{R})$ and a lower solution $\underline{\rho} \in C_{[0,L]}(\mathbb{R},\mathbb{R}) \cap C^2(\mathbb{R},\mathbb{R})$ such that $\bar{\rho} \neq \hat{u}^+$, $\underline{\rho} \neq \hat{0}$ and $\sup_{s\leqslant z}\underline{\rho}(s) \leqslant \bar{\rho}(z)$ for $z \in \mathbb{R}$. Then, there exists at least one monotone solution of (3.1) satisfying (3.2). That is, (1.1) has a monotone wavefront connecting $\hat{0}$ and $\hat{u}^+$.

Proof. Let $M = \frac{8\beta L^2}{d_m(\lambda_2-\lambda_1)}$. Define a set

$$
\Gamma = \left\{ \varphi \in C_{[0,L]}(\mathbb{R},\mathbb{R});\quad \begin{array}{l} (i)\ \varphi \text{ is non-decreasing on } \mathbb{R}; \\ (ii)\ \underline{\rho}(z) \leqslant \varphi(z) \leqslant \bar{\rho}(z) \text{ for } z \in \mathbb{R}; \\ (iii)\ |\varphi(u)-\varphi(v)| \leqslant M|u-v| \text{ for } u, v \in \mathbb{R}. \end{array} \right\}
$$

It is easy to see that Γ is a convex closed subset in $B_\mu(\mathbb{R},\mathbb{R})$. Further, we can verify that Γ is a compact set in $B_\mu(\mathbb{R},\mathbb{R})$. In fact, assume that $\{\varphi_n(\cdot)\} \subset \Gamma$ is a sequence. For any given $\epsilon > 0$, choose $M_1 > 0$ large enough such that

$$
\sup_{|z|\geqslant M_1} |\varphi_n(z) - \varphi_m(z)| e^{-\mu|z|} \leqslant 2L\, e^{-\mu M_1} < \frac{\epsilon}{2}. \tag{3.14}
$$

Since $\{\varphi_n(\cdot)\}$ is uniformly bounded and equicontinuous on $[-M_1, M_1]$, by Arzera–Ascoli theorem, $\{\varphi_n(z)\}$ has a subsequence which is convergent on $[-M_1, M_1]$ with respect to the supremum norm. Without loss of generality, we denote this subsequence by $\{\varphi_n(\cdot)\}$. This leads to the conclusion that $\{\varphi_n(\cdot)\}$ is a Cauchy sequence on $[-M_1, M_1]$ with respect to the supremum norm. Therefore, there exists $K > 0$ such that

$$
\sup_{|z|\leqslant M_1} |\varphi_n(z) - \varphi_m(z)| e^{-\mu|z|} \leqslant \sup_{|z|\leqslant M_1} |\varphi_n(z) - \varphi_m(z)| < \frac{\epsilon}{2}, \quad \text{for } n, m > K.
$$

This, together with (3.14), leads to the conclusion that $\{\varphi_n(z)\}$ is a Cauchy sequence in $B_\mu(\mathbb{R},\mathbb{R})$. As $B_\mu(\mathbb{R},\mathbb{R})$ is a Banach space, thus $\{\varphi_n(\cdot)\}$ is convergent in $B_\mu(\mathbb{R},\mathbb{R})$.

Similarly to (3.13), we can show that

$$[A\bar{\rho}](z) \leqslant \bar{\rho}(z), \quad [A\underline{\rho}](z) \geqslant \underline{\rho}(z), \quad \text{for } z \in \mathbb{R}.$$

Let $\phi(z) = \sup_{s \leqslant z} \underline{\rho}(s)$. Then, $\phi(z)$ is non-decreasing on $\mathbb{R}$ and

$$\underline{\rho}(z) \leqslant \phi(z) \leqslant \bar{\rho}(z), \quad \text{for } z \in \mathbb{R}.$$

By Lemma 3.2 and the above inequalities, we have

$$\underline{\rho}(z) \leqslant [A\underline{\rho}](z) \leqslant [A\phi(z)] \leqslant [A\bar{\rho}](z) \leqslant \bar{\rho}(z), \quad \text{for } z \in \mathbb{R}. \tag{3.15}$$

For any $u, v \in \mathbb{R}$, assuming that $u \geqslant v$, note that $|[Q\phi](z)| \leqslant 2\beta L^2$ for $z \in \mathbb{R}$ (see Lemma 3.1), $\lambda_1 < 0$ and then

$$\begin{aligned}
&\left| \int_{-\infty}^{u} e^{\lambda_1(u-s)}[Q\phi](s)ds - \int_{-\infty}^{v} e^{\lambda_1(v-s)}[Q\phi](s)ds \right| \\
&\leqslant \left| \int_{v}^{u} e^{\lambda_1(u-s)}[Q\phi](s)ds \right| + \left| \int_{-\infty}^{v} [e^{\lambda_1(u-s)} - e^{\lambda_1(v-s)}][Q\phi](s)ds \right| \\
&\leqslant 2\beta L^2(u-v) + \int_{-\infty}^{v} [e^{\lambda_1(v-s)} - e^{\lambda_1(u-s)}][Q\phi](s)ds \\
&\leqslant 2\beta L^2(u-v) + 2\beta L^2 \int_{-\infty}^{v} [e^{\lambda_1(v-s)} - e^{\lambda_1(u-s)}]ds \\
&= 2\beta L^2(u-v) + 2\beta L^2(e^{\lambda_1 v} - e^{\lambda_1 u}) \int_{-\infty}^{v} e^{-\lambda_1 s}\, ds \\
&= 2\beta L^2(u-v) + 2\lambda_1\beta L^2 e^{\lambda_1\zeta}(v-u) \cdot \left(-\frac{1}{\lambda_1}\right) e^{-\lambda_1 v} \\
&\leqslant 2\beta L^2(u-v) + 2\beta L^2(u-v) = 4\beta L^2(u-v),
\end{aligned}$$

where $\zeta \in (v, u)$ and $e^{\lambda_1\zeta} \leqslant e^{\lambda_1 v}$. Similarly, we have

$$\left| \int_{u}^{\infty} e^{\lambda_1(u-s)}[Q\phi](s)ds - \int_{v}^{\infty} e^{\lambda_1(v-s)}[Q\phi](s)ds \right| \leqslant 4\beta L^2(u-v).$$

Therefore,

$$\begin{aligned}
|[A\phi](u) - [A\phi](v)| &= \frac{1}{d_m(\lambda_2-\lambda_1)} \left| \int_{-\infty}^{u} e^{\lambda_1(u-s)}[Q\phi](s)\, ds - \int_{-\infty}^{v} e^{\lambda_1(v-s)}[Q\phi](s)ds \right. \\
&\qquad \left. + \int_{u}^{\infty} e^{\lambda_1(u-s)}[Q\phi](s)\, ds - \int_{v}^{\infty} e^{\lambda_1(v-s)}[Q\phi](s)ds \right| \\
&\leqslant M|u-v|, \quad \text{for } z \in \mathbb{R}.
\end{aligned}$$

Now, we have shown $A\phi \in \Gamma$, and thus Γ is not empty. On the other hand, we have from (3.15) and Lemma 3.2 that $A\Gamma \subset \Gamma$. Therefore, using Schauder fixed point theorem, we conclude that A has a fixed point $\varphi \in \Gamma$ which is a solution of (3.1).

In what follows, we verify that φ satisfies (3.2). Since φ is non-decreasing and $A\varphi = \varphi$, we have

$$0 \leqslant \varphi_{-\infty} := \lim_{z\to-\infty} \varphi(z) = \lim_{z\to-\infty} [A\varphi](z) \leqslant \inf_{z\in\mathbb{R}} \bar{\rho}(z),$$

$$\sup_{z\in\mathbb{R}} \underline{\rho}(z) \leqslant \varphi_{\infty} := \lim_{z\to\infty} \varphi(z) = \lim_{z\to\infty} [A\varphi](z) \leqslant L.$$

Let $\hat{\varphi}_{-\infty}$ and $\hat{\varphi}_{\infty}$ be the constant functions on $z \in \mathbb{R}$ with the value $\varphi_{-\infty}$ and φ_{∞}, respectively. We can show

$$\hat{\varphi}_{-\infty} = A(\hat{\varphi}_{-\infty}) \quad \text{and} \quad \hat{\varphi}_{\infty} = A(\hat{\varphi}_{\infty}). \tag{3.16}$$

We need to show

$$\lim_{z\to-\infty} [A\varphi](z) = A(\hat{\varphi}_{-\infty}) \quad \text{and} \quad \lim_{z\to\infty} [A\varphi](z) = A(\hat{\varphi}_{\infty}). \tag{3.17}$$

We only verify the first equality in (3.17). By using L'Hôspital's rule, we obtain

$$\begin{aligned}
\lim_{z\to-\infty} [A\varphi](z) &= \frac{1}{d_m(\lambda_2-\lambda_1)} \lim_{z\to-\infty} \left\{ \frac{\int_{-\infty}^{z} \mathrm{e}^{-\lambda_1 s}[Q\varphi](s)\mathrm{d}s}{\mathrm{e}^{-\lambda_1 z}} + \frac{\int_{z}^{\infty} \mathrm{e}^{-\lambda_2 s}[Q\varphi](s)\mathrm{d}s}{\mathrm{e}^{-\lambda_2 z}} \right\} \\
&= \frac{1}{d_m(\lambda_2-\lambda_1)} \left\{ \lim_{z\to-\infty} \frac{[Q\varphi](z)}{-\lambda_1} + \lim_{z\to-\infty} \frac{-[Q\varphi](z)}{-\lambda_2} \right\} \\
&= \frac{1}{d_m(\lambda_2-\lambda_1)} \cdot \frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2} \lim_{z\to-\infty} [Q\varphi](z) \\
&= \frac{-1}{d_m\lambda_1\lambda_2} \lim_{z\to-\infty} [Q\varphi](z).
\end{aligned} \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}
A(\hat{\varphi}_{-\infty}) &= \frac{Q(\hat{\varphi}_{-\infty})}{d_m(\lambda_2-\lambda_1)} \left\{ \int_{-\infty}^{z} \mathrm{e}^{\lambda_1(z-s)}\,\mathrm{d}s + \int_{z}^{\infty} \mathrm{e}^{\lambda_2(z-s)}\,\mathrm{d}s \right\} \\
&= \frac{Q(\hat{\varphi}_{-\infty})}{d_m(\lambda_2-\lambda_1)} \cdot \frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2} = \frac{-Q(\hat{\varphi}_{-\infty})}{d_m\lambda_1\lambda_2}.
\end{aligned} \tag{3.19}$$

Since φ is non-decreasing, by Lebesgue's dominated convergence theorem, we derive

$$\begin{aligned}
&\lim_{z\to-\infty} \alpha \int_0^{\infty} \int_{-\infty}^{\infty} G(d_i s, z, \xi) f(s)\mathrm{e}^{-\gamma s}\varphi(\xi - cs)\mathrm{d}\xi\,\mathrm{d}s \\
&= \lim_{z\to-\infty} \alpha \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_i s}} \mathrm{e}^{-\frac{t^2}{4d_i s}} f(s)\mathrm{e}^{-\gamma s}\varphi(z - t - cs)\mathrm{d}t\,\mathrm{d}s \\
&= \alpha\varphi_{-\infty} \int_0^{\infty} f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}s,
\end{aligned}$$

which, together with the definition of $Q\varphi$, implies that

$$\lim_{z\to-\infty}[Q\varphi](z) = 2\beta L\varphi_{-\infty} - \beta\varphi_{-\infty}^2 + \alpha\varphi_{-\infty}\int_0^\infty f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}s = Q(\hat{\varphi}_{-\infty}). \tag{3.20}$$

Now, we have from (3.18–3.20) that (3.17) and thus (3.16) hold. That is, $\hat{\varphi}_{-\infty}$ and $\hat{\varphi}_{\infty}$ are fixed points of A. Since A has only two constant fixed points $\hat{0}$ and $\hat{u}^+$, we know $\hat{\varphi}_{-\infty} = \hat{0}$ and $\hat{\varphi}_{\infty} = \hat{u}^+$. This completes the proof. □

THEOREM 3.2 Let $L = u^+$. If (3.1) has a weak upper solution $\bar{\xi} \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ and a weak lower solution $\underline{\xi} \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ such that

(i) $\bar{\xi} \neq \hat{u}^+, \underline{\xi} \neq \hat{0}$ and $\sup_{s\leqslant z}\underline{\xi}(s) \leqslant \bar{\xi}(z)$ for $z \in \mathbb{R}$,
(ii) $\bar{\xi}'(z^+) \leqslant \bar{\xi}'(z^-)$ and $\underline{\xi}'(z^+) \geqslant \underline{\xi}'(z^-)$ for $z \in \mathbb{R}$,

then there exists at least one monotone solution of (3.1) satisfying (3.2). That is, (1.1) has a monotone wavefront connecting $\hat{0}$ and $\hat{u}^+$.

Proof. Let $\bar{\rho}(z) = [A\bar{\xi}](z)$ and $\underline{\rho}(z) = [A\underline{\xi}](z)$. Then, by (ii) and Lemma 3.4, we know that $\bar{\rho}(z) \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ and $\underline{\rho}(z) \in C_{[0,L]}(\mathbb{R},\mathbb{R})$ are an upper solution and a lower solution of (3.1), respectively. Furthermore, they satisfy

$$\underline{\xi}(z) \leqslant \underline{\rho}(z), \quad \bar{\rho}(z) \leqslant \bar{\xi}(z), \quad \text{for } z \in \mathbb{R}.$$

Define $\tilde{\xi}(z) = \sup_{s\leqslant z}\underline{\xi}(s)$. Then, $\tilde{\xi}(z)$ is non-decreasing on $\mathbb{R}$ and $\underline{\xi}(z) \leqslant \tilde{\xi}(z) \leqslant \bar{\xi}(z)$ for $z \in \mathbb{R}$. It follows from Lemma 3.2 and (i) that $[A\tilde{\xi}](z)$ is non-decreasing on $\mathbb{R}$ and

$$\sup_{s\leqslant z}\underline{\rho}(s) \leqslant \sup_{s\leqslant z}[A\tilde{\xi}](s) = [A\tilde{\xi}](z) \leqslant [A\bar{\xi}](z) = \bar{\rho}(z), \quad \text{for } z \in \mathbb{R}.$$

In view of Theorem 3.1, the proof is complete. □

The linearized equation of (3.1) at $U = \hat{0}$ is

$$cU'(z) = d_m U''(z) + \alpha\int_0^\infty\int_{-\infty}^\infty G(d_i s, z, \xi) f(s)\mathrm{e}^{-\gamma s}U(\xi - cs)\mathrm{d}\xi\,\mathrm{d}s. \tag{3.21}$$

The corresponding characteristic equation is

$$c\lambda - d_m\lambda^2 = \alpha\int_0^\infty f(s)\mathrm{e}^{-s(\gamma + c\lambda - d_i\lambda^2)}\,\mathrm{d}s. \tag{3.22}$$

Let $p(\lambda, c) := d_m\lambda^2 - c\lambda + \alpha\int_0^\infty f(s)\mathrm{e}^{-s(\gamma+c\lambda-d_i\lambda^2)}\,\mathrm{d}s$. Then, we have

$$\frac{\partial p}{\partial\lambda} = 2d_m\lambda - c + \alpha\int_0^\infty f(s)[-s(c - 2d_i\lambda)]\mathrm{e}^{-s(\gamma+c\lambda-d_i\lambda^2)}\,\mathrm{d}s,$$

$$\frac{\partial p^2}{\partial\lambda^2} = 2d_m + \alpha\int_0^\infty f(s)s^2(c - 2d_i\lambda)^2\,\mathrm{e}^{-s(\gamma+c\lambda-d_i\lambda^2)}\,\mathrm{d}s + 2d_i\alpha\int_0^\infty sf(s)\mathrm{e}^{-s(\gamma+c\lambda-d_i\lambda^2)}\,\mathrm{d}s > 0,$$

$$p(0, c) = \alpha\int_0^\infty f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}s > 0,$$

$$p(\lambda, 0) = d_m\lambda^2 + \alpha \int_0^{\infty} f(s)\mathrm{e}^{-s(\gamma - d_i\lambda^2)}\,\mathrm{d}s > 0,$$

$$p(\lambda, \infty) = -\infty, \quad \text{for any given } \lambda > 0, \quad p(\infty, c) = \infty, \quad \text{for any given } c > 0,$$

$$\frac{\partial p}{\partial c} = -\lambda - \alpha\lambda \int_0^{\infty} s f(s)\mathrm{e}^{-s(\gamma + c\lambda - d_i\lambda^2)}\,\mathrm{d}s < 0, \quad \text{for } \lambda > 0. \tag{3.23}$$

Thus, we conclude the following lemma.

LEMMA 3.5 There exists a pair of (λ^*, c^*) such that

(i) $p(\lambda^*, c^*) = 0$, $\frac{\partial p}{\partial \lambda}(\lambda^*, c^*) = 0$;
(ii) $p(\lambda, c) > 0$ for $0 < c < c^*$ and $\lambda > 0$;
(iii) $p(\lambda, c) = 0$ has two zero $0 < \lambda_a < \lambda_b < \infty$ for $c > c^*$. Furthermore, there exists $\epsilon_0 > 0$ such for any $\epsilon \in (0, \epsilon_0)$ with $0 < \lambda_a < \lambda_a + \epsilon < \lambda_b$, we have

$$p(\lambda_a + \epsilon, c) < 0. \tag{3.24}$$

Assume that $c > c^*$. Define functions $\bar{\xi}(z)$ and $\underline{\xi}(z)$ by

$$\bar{\xi}(z) = \min\{u^+, u^+\,\mathrm{e}^{\lambda_a z}\} \quad \text{and} \quad \underline{\xi}(z) = \max\{0, \sigma_1\,\mathrm{e}^{\lambda_a z} - \sigma_2\,\mathrm{e}^{(\lambda_a+\epsilon)z}\}, \quad \forall z \in \mathbb{R}, \tag{3.25}$$

where λ_a and ϵ are given as in Lemma 3.5 and $0 < \sigma_1 \leqslant \sigma_2$ are chosen so that $\sup_{s\leqslant z}\underline{\xi}(s) \leqslant \bar{\xi}(z)$, $z \in \mathbb{R}$.

LEMMA 3.6 Assume that $c > c^*$. Then, $\bar{\xi}(z)$ and $\underline{\xi}(z)$ given by (3.25) are a pair of weak upper solution and weak lower solution of (3.1).

Proof. If $z > 0$, we have

$$\begin{aligned}
&d_m\bar{\xi}''(z) - c\bar{\xi}'(z) - \beta[\bar{\xi}(z)]^2 + \alpha \int_0^{\infty}\int_{-\infty}^{\infty} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\bar{\xi}(y - cs)\mathrm{d}y\,\mathrm{d}s \\
&= -\beta[u^+]^2 + \alpha u^+ \int_0^{\infty}\int_{cs}^{\infty} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \alpha u^+ \int_0^{\infty}\int_{-\infty}^{cs} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\,\mathrm{e}^{\lambda_a(y-cs)}\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant -\beta[u^+]^2 + \alpha u^+ \int_0^{\infty}\int_{-\infty}^{\infty} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\,\mathrm{d}y\,\mathrm{d}s = 0.
\end{aligned}$$

If $z < 0$, since $\bar{\xi}(z) \leqslant u^+\,\mathrm{e}^{\lambda_a z}$ for any $z \in \mathbb{R}$, we have

$$\begin{aligned}
&d_m\bar{\xi}''(z) - c\bar{\xi}'(z) - \beta[\bar{\xi}(z)]^2 + \alpha \int_0^{\infty}\int_{-\infty}^{\infty} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\bar{\xi}(y - cs)\mathrm{d}y\,\mathrm{d}s \\
&\leqslant d_m(\lambda_a)^2 u^+\,\mathrm{e}^{\lambda_a z} - c\lambda_a u^+\,\mathrm{e}^{\lambda_a z} - \beta[u^+\,\mathrm{e}^{\lambda_a z}]^2 + \alpha \int_0^{\infty}\int_{-\infty}^{\infty} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}u^+\,\mathrm{e}^{\lambda_a(y-cs)}\mathrm{d}y\,\mathrm{d}s \\
&\leqslant u^+\,\mathrm{e}^{\lambda_a z}\left\{d_m\lambda_a^2 - c\lambda_a + \alpha \int_0^{\infty} f(s)\mathrm{e}^{-s(\gamma + c\lambda_a - d_i\lambda_a^2)}\,\mathrm{d}s\right\} = 0.
\end{aligned}$$

Therefore, $\bar{\xi}(z)$ is a weak upper solution of (3.1).

Now, we want to verify that $\underline{\xi}(z)$ is a weak lower solution of (3.1). Let

$$\lambda_\epsilon := \lambda_a + \epsilon, \quad g(z) := \sigma_1\, \mathrm{e}^{\lambda_a z} - \sigma_2\, \mathrm{e}^{\lambda_\epsilon z}, \quad z_0 = \frac{1}{\epsilon}\ln\left(\frac{\sigma_1}{\sigma_2}\right) \leqslant 0.$$

Then, we have

$$g(z_0) = 0, \quad g(z) > 0 \;\text{ as } z < z_0, \quad g(z) < 0 \;\text{ as } z > z_0.$$

If $z > z_0$, we have

$$d_m\underline{\xi}''(z) - c\underline{\xi}'(z) - \beta[\underline{\xi}(z)]^2 + \alpha\int_0^\infty\int_{-\infty}^\infty G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\underline{\xi}(y - cs)\mathrm{d}y\,\mathrm{d}s$$

$$= \alpha\int_0^\infty\int_{-\infty}^{cs+z_0} G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\{\sigma_1\,\mathrm{e}^{\lambda_a(y-cs)} - \sigma_2\,\mathrm{e}^{\lambda_\epsilon(y-cs)}\}\,\mathrm{d}y\,\mathrm{d}s \geqslant 0.$$

Since

$$\underline{\xi}(z) \geqslant \sigma_1\,\mathrm{e}^{\lambda_a z} - \sigma_2\,\mathrm{e}^{\lambda_\epsilon z}, \quad [\underline{\xi}(z)]^2 \leqslant \sigma_1^2\,\mathrm{e}^{\lambda_\epsilon z}, \quad \forall z \in \mathbb{R}, \tag{3.26}$$

if $z < z_0$, we have

$$\begin{aligned}
&d_m\underline{\xi}''(z) - c\underline{\xi}'(z) - \beta[\underline{\xi}(z)]^2 + \alpha\int_0^\infty\int_{-\infty}^\infty G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\underline{\xi}(y - cs)\mathrm{d}y\,\mathrm{d}s \\
&\quad\geqslant d_m\lambda_a^2\sigma_1\,\mathrm{e}^{\lambda_a z} - d_m(\lambda_\epsilon)^2\sigma_2\,\mathrm{e}^{\lambda_\epsilon z} - c\lambda_a\sigma_1\,\mathrm{e}^{\lambda_a z} \\
&\qquad + c\lambda_\epsilon\sigma_2\,\mathrm{e}^{\lambda_\epsilon z} - \beta\sigma_1^2\,\mathrm{e}^{\lambda_\epsilon z} + \alpha\int_0^\infty\int_{-\infty}^\infty G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}[\sigma_1\,\mathrm{e}^{\lambda_a(y-cs)} - \sigma_2\,\mathrm{e}^{\lambda_\epsilon(y-cs)}]\mathrm{d}y\,\mathrm{d}s \\
&\quad= -d_m(\lambda_\epsilon)^2\sigma_2\,\mathrm{e}^{\lambda_\epsilon z} + c\lambda_\epsilon\sigma_2\,\mathrm{e}^{\lambda_\epsilon z} - \beta\sigma_1^2\,\mathrm{e}^{\lambda_\epsilon z} - \alpha\int_0^\infty\int_{-\infty}^\infty G(d_i s, z, y) f(s)\mathrm{e}^{-\gamma s}\sigma_2\,\mathrm{e}^{\lambda_\epsilon(y-cs)}\,\mathrm{d}y\,\mathrm{d}s \\
&\quad= \sigma_2\,\mathrm{e}^{\lambda_\epsilon z}\left\{-d_m(\lambda_\epsilon)^2 + c\lambda_\epsilon - \alpha\int_0^\infty f(s)\mathrm{e}^{-s[\gamma + c\lambda_\epsilon - d_i(\lambda_\epsilon)^2]}\,\mathrm{d}s\right\} - \beta\sigma_1^2\,\mathrm{e}^{\lambda_\epsilon z} \\
&\quad= -\sigma_2\,\mathrm{e}^{\lambda_\epsilon z}p(\lambda_\epsilon, c) - \beta\sigma_1^2\,\mathrm{e}^{\lambda_\epsilon z}.
\end{aligned}$$

We have from Lemma 3.5 that $p(\lambda_\epsilon, c) < 0$, and therefore, we can choose $0 < \sigma_1 < \sigma_2$ such that $-\sigma_2 p(\lambda_\epsilon, c) - \beta\sigma_1^2 \geqslant 0$. Thus, $\underline{\xi}(z)$ is a weak lower solution of (3.1). □

Now, we can state the main result of this paper.

THEOREM 3.3 For every $c \geqslant c^*$, (1.1) has a monotone wavefront connecting 0 and u^+.

Proof. If $c > c^*$, the existence of monotone wavefront connecting 0 and u^+ for (1.1) follows from Lemmas 3.4 and 3.5 and Theorem 3.2. In the case of $c = c^*$, we use a limit argument similar to Zhao & Wang (2004). Let $\{c_k\} \subset (c^*, c^* + 1]$ with $\lim_{k\to\infty} c_k = c^*$. Since $c_k > c^*$, (3.1) with $c = c_k$ admits a non-decreasing solution $U_k(z)$ such that $U_k(-\infty) = 0$ and $U_k(\infty) = u^+$. Without loss of generality, we may assume that $U_k(0) = \frac{u^+}{2}$. Note that U_k satisfies

$$U_k(z) = \frac{1}{d_m(\lambda_2^k - \lambda_1^k)} \left\{ \int_{-\infty}^{z} e^{\lambda_1^k (z-s)} [QU_k](s) ds + \int_z^{\infty} e^{\lambda_2^k (z-s)} [QU_k](s) ds \right\}, \tag{3.27}$$

where

$$\lambda_1^k = \frac{c_k - \sqrt{c_k^2 + 8 d_m \beta L}}{2 d_m} < 0 \quad \text{and} \quad \lambda_2^k = \frac{c_k + \sqrt{c_k^2 + 8 d_m \beta L}}{2 d_m} > 0.$$

Since $\{U_k\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$ (see Γ in Theorem 3.1), using Arzela–Ascoli theorem and the standard diagonalization procedure, we can obtain a subsequence of functions $\{U_{k_m}\}$ which converges to U^*, as $m \to \infty$, uniformly for z in any bounded subset of $\mathbb{R}$. Clearly, $U^*(z)$ is non-decreasing and $U^*(0) = \frac{u^+}{2}$. By the dominated convergence theorem and (3.27), it follows that

$$U^*(z) = \frac{1}{d_m(\lambda_2^* - \lambda_1^*)} \left\{ \int_{-\infty}^{z} e^{\lambda_1^* (z-s)} [QU^*](s) ds + \int_z^{\infty} e^{\lambda_2^* (z-s)} [QU^*](s) ds \right\}, \tag{3.28}$$

where

$$\lambda_1^* = \frac{c^* - \sqrt{(c^*)^2 + 8 d_m \beta L}}{2 d_m} < 0 \quad \text{and} \quad \lambda_2^* = \frac{c^* + \sqrt{(c^*)^2 + 8 d_m \beta L}}{2 d_m} > 0.$$

Since $\lim_{z \to \pm\infty} U^*(z)$ exist, L'Hôspital rule implies $U^*(-\infty) = 0$ and $U^*(\infty) = u^+$. Thus, $U^*(x + ct)$ is a monotone wavefront of (1.1) connecting 0 to u^+. □

4. Conclusions and remarks

We considered the existence of wavefronts $u(t, x) = U(x + ct)$ for the population model (1.1) with a general probability density function f satisfying (1.3). We derived the existence of $c^* > 0$ such that for every $c \geqslant c^*$, (1.1) has monotone wavefronts connecting two equilibria 0 and u^+ with the main result stated in Theorem 3.3 and the minimal wave speed given in Lemma 3.5.

Recently, Wang *et al.* (2006) considered similar issues for more systems of reaction–diffusion equations with non-local delayed non-linearities, their restrictions on the lower–upper solution pair being slightly different from ours here.

We have, from Lemma 3.5, a system of algebraic equations $p(\lambda^*, c^*) = 0$, $\frac{\partial p}{\partial \lambda}(\lambda^*, c^*) = 0$ that determine the minimal wave speed c^*. As

$$p(\lambda, c) = d_m \lambda^2 - c\lambda + \alpha \int_0^{\infty} f(s) e^{-s(\gamma + c\lambda - d_i \lambda^2)} ds,$$

we see that c^* depends on key model parameters d_m, d_i, α, γ and probability density function f in a complicated way.

As a final remark, we found that a recent article (Fang *et al.*, 2007) approached similar issues from an interesting angle: regarding system (1.1) with distributed maturity as a limiting situation of the finite-delay system and then using available results for the non-local reaction–diffusion equations with finite delay. This study also addressed the connection of the minimal wave speed to the propagation rate.

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