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RESEARCH ARTICLE

Backward/Hopf bifurcations in SIS models with delayed nonlinear incidence rates*

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Abstract The classical SIS model with a constant transmission rate exhibits simple dynamic behaviors fully determined by the basic reproduction number. Behavioral changes and intervention measures influenced by the level of infection, likely with a time lag, require the transmission rate to be a nonlinear function of the total infectives. This nonlinear transmission, as shown in this paper via a combination of qualitative and numerical analysis, can generate interesting dynamical behaviors at the population level including backward and Hopf bifurcations. We conclude that sustained infections and periodic outbreaks can be consequences of delayed changes in behaviors or human intervention.

Keywords Delayed epidemic model, nonlinear incidence, periodic outbreak, backward bifurcation, behavior change
MSC 34K45, 34K25, 92D30, 34K60

1 Introduction

The following classical SIS model without vital dynamics (death and birth)

$$\begin{cases} S'(t) = -\beta \dfrac{S(t)I(t)}{N(t)} + \gamma I(t), \\ I'(t) = \beta \dfrac{S(t)I(t)}{N(t)} - \gamma I(t), \end{cases} \tag{1.1}$$

with $N(t) = S(t) + I(t)$, and a constant transmission rate β as well as

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a constant recovery rate γ exhibits simple dynamics. See, for example, Ref. [2]. It has been observed in a number of studies [3,7,9,10,12] that the transmission rate should be a nonlinear function of the total infectives $I(t)$, since the number of effective contacts between infectives and susceptibles may saturate at a high infection level due to crowding effects (of infectives) or due to some protection measures taken by the susceptibles. Such a nonlinear function can lead to complicated dynamics, including Bogdanov-Takens singularity, multiple equilibria and limit cycles.

In this paper, we consider the situation where the nonlinear transmission rate $\beta = \beta(T(t-\tau))$ is a nonlinear function of I with a time delay, since the behavior changes of susceptible individuals occurs with a time delay. Such a time lag can be a result of a long incubation period (for example, some sexually transmitted diseases) or ineffectiveness of the surveillance system and the public education program. A detailed qualitative analysis (Sections 2 and 3), based on the normal form and center manifold theory [6], shows that backward bifurcation and Hopf bifurcation of stable periodic oscillations can take place. Numerical simulations in Section 4 confirm that such oscillations can be easily observed in models with realistic epidemiological parameters.

2 Model formulation

We consider the following SIS model with delayed nonlinear incidence rate:

$$\begin{cases} S'(t) = -\widetilde{\beta}(I(t-\tau))\dfrac{S(t)I(t)}{N(t)} + \gamma I(t), \\ I'(t) = \widetilde{\beta}(I(t-\tau))\dfrac{S(t)I(t)}{N(t)} - \gamma I(t), \end{cases} \tag{2.1}$$

where $N(t) = S(t) + I(t)$ is the total population at time t, $S(t)$ and $I(t)$ are respectively the sizes of the susceptible and infective populations, γ is the recovery rate. We assume that infected individuals recover without immunity. It is assumed that the transmission rate $\widetilde{\beta}$ depends on the size of infective population, with a time delay τ. We assume that $\widetilde{\beta}$: $\mathbb{R}_+ \to \mathbb{R}_+ := [0, \infty)$ is a continuous function with $\widetilde{\beta}(0) \neq 0$.

It follows immediately that $N'(t) = 0$ for $t \geqslant 0$, and hence

$$N(t) = S(t) + I(t) = S(0) + I(0) \quad \text{for } t \geqslant 0.$$

Therefore, the dynamic behavior of solutions of (2.1) is fully determined by the following scalar delay differential equation:

$$I'(t) = \widetilde{\beta}(I(t-\tau))\frac{[N(0) - I(t)]I(t)}{N(0)} - \gamma I(t).$$

Normalizing by

$$x(t) = \frac{I(t)}{N(0)}, \quad \beta(y) = \widetilde{\beta}(yN(0)) \quad \text{for } y \in \mathbb{R}, \tag{2.2}$$

we obtain

$$x'(t) = \beta(x(t-\tau))[1-x(t)]x(t) - \gamma x(t). \tag{2.3}$$

To determine a unique solution of (2.3) for all future time $t \geqslant 0$, we need to prescribe the initial value of x on the interval $[-\tau, 0]$. We assume

$$x(t) = \varphi(t) \in [0,1], \quad t \in [-\tau, 0], \tag{2.4}$$

where $\varphi \in C([-\tau, 0]; \mathbb{R})$, the Banach space of continuous functions on $[-\tau, 0]$ equipped with the super-norm. It is straightforward to verify that $x(t) \in [0,1]$ for all $t \geqslant 0$. Furthermore, if β is a non-increasing function, then

$$x'(t) \leqslant \beta(0)[1-x(t)]x(t) - \gamma x(t).$$

Consequently, under the condition that

$$\mathscr{R}_0 = \frac{\beta(0)}{\gamma} < 1, \tag{2.5}$$

$x(t) \to 0$ as $t \to \infty$, so the *disease-free* equilibrium ($x \equiv 0$) is globally asymptotically stable.

Complicated dynamics occurs, however, if $\mathscr{R}_0 > 1$ due to the interaction of the nonlinearity of β and the time delay $\tau > 0$.

Recall that $\mathscr{R}_\mathbf{0}$ is normally called the basic reproduction number.

2.1 $\mathscr{R}_0$ and stability of equilibria

We note that equilibria of (2.3) are given by the following algebraic equation:

$$\beta(x)x(1-x) - \gamma x = 0.$$

Let

$$\beta^* = \max\{\beta(x)(1-x);\ x \in [0,1]\}. \tag{2.6}$$

Then (2.3) has only one equilibrium 0 if $\gamma > \beta^*$ and this corresponds to the disease free equilibrium $x = 0$. If $\gamma < \beta^*$, then (2.3) has at least one equilibrium $x = x^* \in (0,1)$. This *endemic* equilibrium $x = x^*$ satisfies

$$\beta(x^*)(1-x^*) = \gamma.$$

The linearization of (2.3) at a given equilibrium $\widetilde{x}$ is

$$y'(t) = \beta'(\widetilde{x})\overline{x}(1-\widetilde{x})y(t-\tau) + [\beta(\widetilde{x})(1-2\widetilde{x}) - \gamma]y(t). \tag{2.7}$$

In particular, the linearization of (2.3) at the disease free equilibrium $x = 0$ is

$$y'(t) = [\beta(0) - \gamma]y(t).$$

Therefore, the equilibrium $x = 0$ is asymptotically stable if $\beta(0) < \gamma$ and unstable if $\beta(0) > \gamma$. In other words, $x = 0$ is asymptotically stable/unstable if the basic reproduction number $\mathscr{R}_0$ is less/larger than 1.

Note also that with

$$\beta_{\max} = \max\{\beta(x); 0 \leqslant x \leqslant 1\}, \tag{2.8}$$

we have

$$x'(t) \leqslant \beta_{\max}[1 - x(t)]x(t) - \gamma x(t).$$

Consequently, $x = 0$ is globally asymptotically stable if

$$\beta_{\max} < \gamma. \tag{2.9}$$

$\beta_{\max}$ and $\beta(0)$ can have different ordering relationships, and this seems to be the mechanism behind backward bifurcations. A simple example where $\beta(0) = \beta_{\max}$ is when β is a non-increasing function such as

$$\beta_M(x) = \mathrm{e}^{-\alpha x}\beta_0 \quad \text{with } \alpha > 0,\ \beta_0 > 0. \tag{2.10}$$

An example where $\beta(0) < \beta_{\max}$ is

$$\beta_N(x) = \mathrm{e}^{-\alpha x}(\beta_0 + \beta_1 x) \quad \text{with } \alpha > 0,\ \beta_0 > 0,\ \beta_1 > 0. \tag{2.11}$$

For $\beta_N(x)$, $\beta(0) < \beta_{\max}$ if $\beta_1 > \alpha\beta_0$. In particular, if

$$0 < \frac{\beta_1 - \alpha\beta_0}{\alpha\beta_1} < 1, \tag{2.12}$$

then

$$\beta_{\max} = \frac{\beta_1}{\alpha}\mathrm{e}^{-(1-\alpha\frac{\beta_0}{\beta_1})}. \tag{2.13}$$

See Fig. 1 for the graphs of $\beta_M(x)$ and $\beta_N(x)$.

We now summarize the above discussion as follows.

Theorem 2.1 *Let $\mathscr{R}_0$, β^* and $\beta_{\max}$ be defined as in* (2.5), (2.6) *and* (2.8). *Then*

(i) *the zero solution of* (2.3) *is asymptotically stable if $\beta(0) < \gamma$ and unstable if $\beta(0) > \gamma$;*

(ii) *the zero solution of* (2.3) *is globally attractive if $\beta_{\max} < \gamma$;*

(iii) *system* (2.3) *has at least one endemic equilibrium $x = x^* \in (0, 1)$ if and only if $\gamma < \beta^*$.*

As $\beta(x^*)(1 - x^*) = \gamma$, at an endemic equilibrium x^*, the linearization of (2.3) can be written as

$$y'(t) = -B_1 y(t) + B_2 y(t - \tau), \tag{2.14}$$

with

$$B_1 = \beta(x^*)x^* > 0, \quad B_2 = \beta'(x^*)(1 - x^*)x^*. \tag{2.15}$$

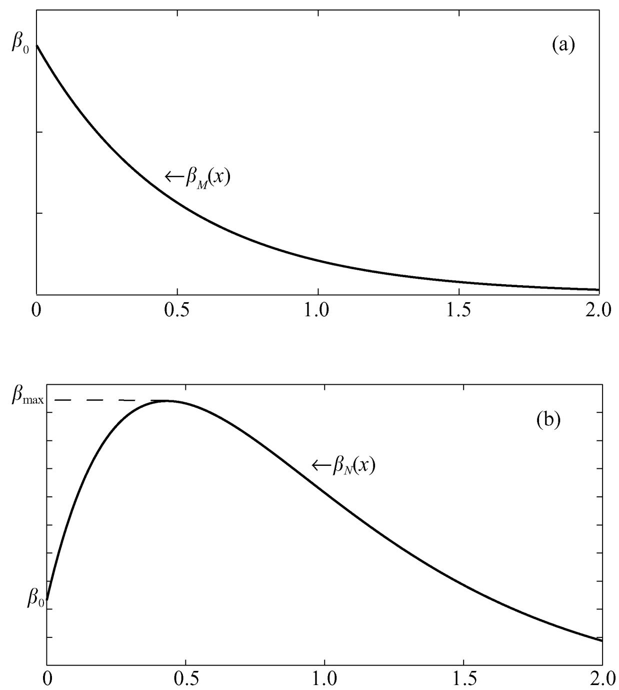


Fig. 1 (a) Graph of $\beta_M(x) = \beta_0 \mathrm{e}^{-\alpha x}$ with $\beta_0 = 0.062$ and $\alpha = 2$.
(b) Graph of $\beta_N(x) = (\beta_0 + \beta_1 x)\mathrm{e}^{-\alpha x}$ with $\beta_0 = 0.062$, $\beta_1 = 1$ and $\alpha = 2$

When $\beta(x)$ is a monotonically decreasing function such as $\beta_M(x)$, $\mathscr{R}_0 > 1$ implies that (2.3) has a unique endemic equilibrium $x^* \in (0,1)$ with $B_2 < 0$. When $\beta(x)$ is non-monotonic such as $\beta_N(x)$, however, (2.3) may have multiple endemic equilibria. In particular, backward bifurcations can occur.

2.2 Backward bifurcation

To illustrate the possibility of backward bifurcations, we consider

$$\beta(x) = \beta_N(x) = (\beta_0 + \beta_1 x)\mathrm{e}^{-\alpha x}. \tag{2.16}$$

We note that

$$[(1-x)\beta_N(x)]' = -\mathrm{e}^{-\alpha x}[(\alpha + 1 - \alpha x)(\beta_0 + \beta_1 x) - \beta_1(1-x)]. \tag{2.17}$$

Therefore, if $\beta_0(1+\alpha) < \beta_1$, then $[(1-x)\beta_N(x)]' = 0$ has a unique solution $\overline{x} \in (0,1)$ and $[(1-x)\beta_N(x)]$ achieves the maximum on $[0,1]$ at $\overline{x}$. That is,

$$\beta^* = (\beta_0 + \beta_1\overline{x})\mathrm{e}^{-\alpha\overline{x}}(1-\overline{x}).$$

We already noted that the equilibrium $x = 0$ is locally asymptotically stable if $\gamma > \beta_0$, and unstable if $\gamma < \beta_0$. Note also that endemic equilibrium $x^* \in (0,1)$

exists only when $\gamma < \beta^*$. It is therefore natural to consider the cases where $\gamma < \beta_0$ and where $\beta_0 < \gamma < \beta^*$.

Theorem 2.2 *Consider $\beta(x) = \beta_N(x)$ and assume $\beta_0(1+\alpha) < \beta_1$. Then in the case where $\tau = 0$, we have*

(i) *if $0 < \gamma < \beta_0$, then (2.3) has a unique endemic equilibrium $x^* \in (0,1)$ which is asymptotically stable;*

(ii) *if $\beta_0 < \gamma < \beta^*$, then (2.3) has two endemic equilibrium $x = x_* \in (0, \overline{x})$ and $x = x^* \in (\overline{x}, 1)$. The equilibrium x_* is unstable and the equilibrium x^* is stable.*

Proof We refer the readers to Fig. 2 for schematic illustrations of various numbers involved in the proof.

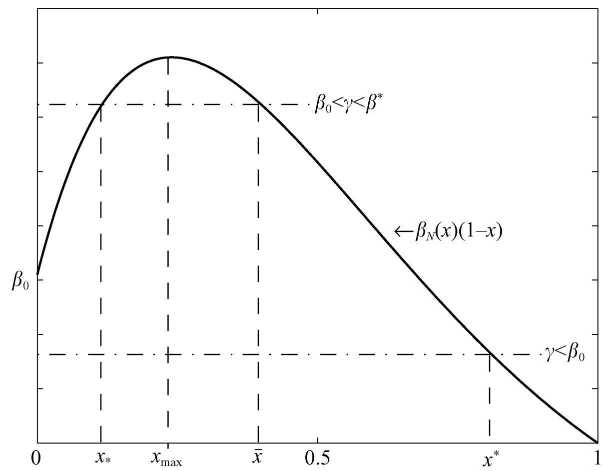


Fig. 2 Relationship between $\beta(x)(1-x)$ and γ, whose intersection in $(0,1)$ gives rise to endemic equilibria

(i) Let $0 < \gamma < \beta_0$. Observe that $x^* \in (\overline{x}, 1)$. If $\beta'(x^*) \leqslant 0$, then $B_2 \leqslant 0$ and the solution of (2.14) is asymptotically stable, and so is the equilibrium x^*. Consider the case when $\beta'(x^*) > 0$. For $x \in (\overline{x}, x^*]$, since $\beta(x)(1-x)$ is decreasing on $(\overline{x}, x^*]$, we have $\beta(x) \geqslant \gamma/(1-x)$ with equality holds only at $x = x^*$. This implies $\beta'(x^*) \leqslant \gamma/(1-x^*)^2$. We claim $\beta'(x^*) < \gamma/(1-x^*)^2$. In fact, if $\beta'(x^*) = \gamma/(1-x^*)^2$, then we have $\beta'(x^*)(1-x^*) = \beta(x^*)$. In other words, x^* is a root of equation (2.16) in $(0,1)$. Thus $x^* = \overline{x}$, a contradiction. Hence $\beta'(x^*) < \gamma/(1-x^*)^2$. Noting that $\gamma = \beta(x^*)(1-x^*)$, we obtain

$$\beta'(x^*)(1-x^*) < \beta(x^*).$$

That is, $B_2 < B_1$. Thus the equilibrium $x = x^*$ is asymptotically stable.

(ii) Let $\beta_0 < \gamma < \beta^*$. For $x \in [x_*, \overline{x})$, we observe that $\beta(x) \geqslant \gamma/(1-x)$ with equality holds only at $x = x_*$. This implies $\beta'(x_*) \geqslant \gamma/(1-x_*)^2$. Using the same argument as in part (i), we can show that $\beta'(x_*) > \gamma/(1-x_*)^2$. Therefore, since $\gamma = \beta(x_*)(1-x_*)$, we obtain

$$\beta'(x_*)(1-x_*) > \beta(x_*).$$

That is, $B_2 > B_1$. Thus the equilibrium $x = x_*$ is unstable.

Next, we show the equilibrium $x = x^*$ is stable. If $\beta'(x^*) \leqslant 0$, then we get

$$B_2 = \beta'(x^*)(1-x^*) \leqslant 0.$$

If $\beta'(x^*) > 0$, then we can use the fact that $\beta(x) \geqslant \gamma/(1-x)$ for $x \in (\overline{x}, x^*]$ with equality only at $x = x^*$ to show that $\beta'(x^*)(1-x^*) < \beta(x^*)$. Thus the equilibrium $x = x^*$ is asymptotically stable. This completes the proof. □

Based on Theorem 2.2, we conclude that equation (2.1) with $\tau = 0$ undergoes a forward transcritical bifurcation at $x = 0$ when $\gamma = \beta_0$, and a backward bifurcation at $x = \overline{x}$ when $\gamma = \beta^*$. This hysteresis-like bifurcation is illustrated in Figure 3.

Note that this is the case where $\tau = 0$. Increasing the size of the delay may generate sustained oscillation as the next section shows.

3 Hopf bifurcation analysis

3.1 Existence of Hopf bifurcations

Let x^* be an endemic equilibrium of (2.3). Then the linearized system of (2.3) at x^* is

$$x'(t) = -B_1 x(t) - B_2 x(t-\tau), \tag{3.1}$$

where

$$B_1 = \beta(x^*)x^*, \quad B_2 = -\beta'(x^*)(1-x^*)x^*.$$

Recall that $B_1 > 0$. The associated characteristic equation becomes

$$\lambda = -B_1 - B_2 \mathrm{e}^{-\lambda\tau}. \tag{3.2}$$

If $\tau = 0$, the stability of x^* is described in Theorem 2.2. To consider the occurrence of a local Hopf bifurcation at x^* by increasing τ, we assume that $\lambda = \pm \mathrm{i}\omega$ with $\omega > 0$ are roots of equation (3.2). Then

$$\mathrm{i}\omega = -B_1 - B_2 \mathrm{e}^{-\mathrm{i}\omega\tau}.$$

Separating the real and imaginary parts gives

$$B_1 = -B_2 \cos\omega\tau, \quad \omega = B_2 \sin\omega\tau,$$

which leads to

$$\tau_k = \frac{1}{\sqrt{B_2^2 - B_1^2}} \Big[\arctan\Big(-\sqrt{B_2^2 - B_1^2}\,/B_1\Big) + (k+1)\pi\Big], \quad k = 0, 1, \ldots,$$

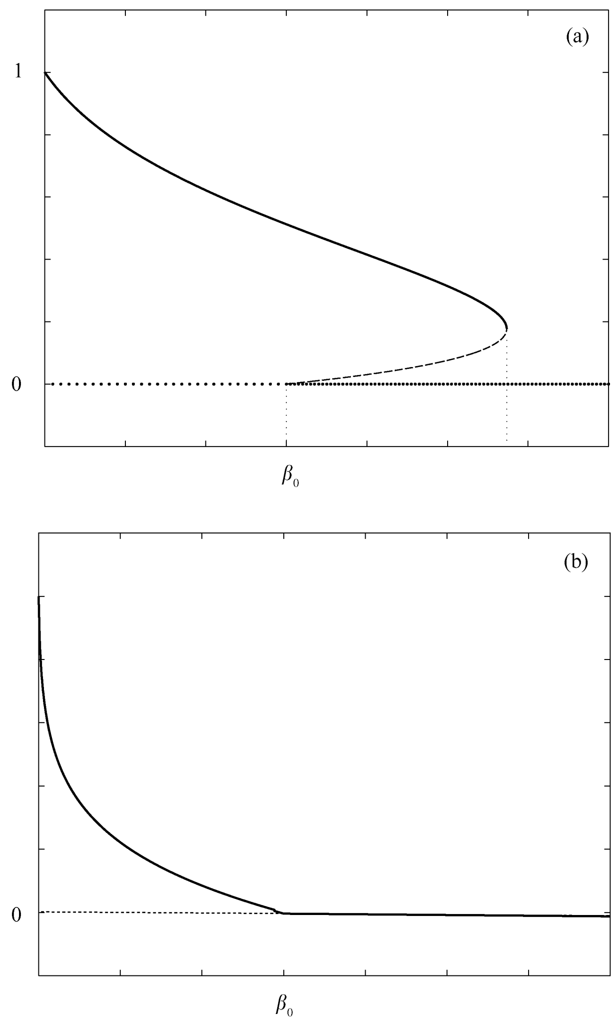


Fig. 3 (a) Hysteresis-like bifurcation diagram for $\beta_1 > (1+\alpha)\beta_0$ with $\beta_0 = 0.06,\ \alpha = 3$ and $\beta_1 = 1$. For comparison, we also depict here the bifurcation diagram (b) for $\beta_1 \leqslant (1+\alpha)\beta_0$ with $\beta_0 = 0.3,\ \alpha = 5$ and $\beta_1 = 0.4$. In both pictures, dotted lines represent unstable equilibria

$$\omega_0 = \sqrt{B_2^2 - B_1^2},$$

provided $|B_1| < |B_2|$, or equivalently $\gamma > \beta^2(x^*)/|\beta'(x^*)|$.

To check the transversality condition, we obtain from (3.2), by regarding λ as a smooth function of τ in the neighborhood of τ_k, the following:

$$\lambda' = B_2(\lambda + \lambda'\tau)\mathrm{e}^{-\lambda\tau}.$$

Using

$$\lambda(\tau_k) = \mathrm{i}\omega_0 = -B_1 - B_2\mathrm{e}^{-\mathrm{i}\omega_0\tau_k},$$

we obtain

$$\begin{aligned}\lambda'(\tau_k) &= B_2\mathrm{e}^{-\mathrm{i}\omega_0\tau_k}(\mathrm{i}\omega_0 + \lambda'(\tau_k)\tau_k)\\ &= (-B_1 - \mathrm{i}\omega_0)(\mathrm{i}\omega_0 + \lambda'(\tau_k)\tau_k).\end{aligned}$$

Therefore,

$$\lambda'(\tau_k) = \frac{\omega_0^2 - \mathrm{i}B_1\omega_0}{1 + B_1\tau_k + \mathrm{i}B_1\omega_0\tau_k},$$

and hence

$$\mathrm{Re}\lambda'(\tau_k) = \frac{\omega_0^2}{(1 + B_1\tau_k)^2 + \omega_0^2\tau_k^2} > 0.$$

To verify the nonresonance condition, i.e., equation (3.2) has no root $\lambda(\tau_k) = \pm\mathrm{i}\omega_0 n$ with integer $n \geqslant 2$, we assume by way of contradiction that $\mathrm{i}\omega_0 n (n \geqslant 2)$ is a root of equation (3.2). Then

$$\mathrm{i}\omega_0 n = -B_1 - B_2\mathrm{e}^{-\mathrm{i}\omega_0 n\tau_k},$$

which implies that

$$B_1 = -B_2\cos(\omega_0 n\tau_k), \quad \omega_0 n = B_2\sin(\omega_0 n\tau_k).$$

Solving for ω_0 gives us

$$\omega_0 = \frac{\sqrt{B_2^2 - B_1^2}}{n^2} \leqslant \frac{\sqrt{B_2^2 - B_1^2}}{4} = \frac{\omega_0}{4},$$

a contradiction.

We can now state the following existence result.

Theorem 3.1 *When*

$$\beta^* > \gamma > \frac{\beta^2(x^*)}{|\beta'(x^*)|},$$

equation (3.2) *has a pair of simple imaginary roots* $\pm\mathrm{i}\omega_0$ *at* $\tau = \tau_0$. *If* $\tau \in [0, \tau_0)$, *then all roots of* (3.2) *have negative real parts; if* $\tau = \tau_0$, *then all roots of* (3.2) *except* $\pm\mathrm{i}\omega_0$ *have negative real parts. In particular, the equilibrium* $x = x^*$ *is asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$; *and equation* (2.1) *undergoes a Hopf bifurcation at* $x = x^*$ *when* $\tau = \tau_0$.

3.2 Stability and direction of bifurcation

In the previous subsection, we obtained the conditions under which system (2.1) undergoes a Hopf bifurcation of periodic solutions from the endemic steady state at the critical values of $\tau = \tau_0$. Whether such a nonlinear oscillation can be observed in real epidemiological data depends on the stability of bifurcated periodic solutions. Determining the stability of bifurcated periodic solutions requires a very lengthy calculation, as shown in this subsection. The method to be used is a standard application of the normal form and center manifold theory.

Letting

$$u = x - x^*, \quad \overline{x}(t) = u(\tau t), \quad \tau = \tau_0 + \mu,$$

and dropping the bars for notational simplification, we can transfer system (2.3) into the FDE in $C = C([-1, 0], \mathbb{R})$ as follows:

$$x'(t) = L_\mu(x_t) + f(\mu, x_t), \tag{3.3}$$

where $x(t) \in \mathbb{R},\ x_t(\theta) = x(t + \theta)$, and $L_\mu\colon\ C \to \mathbb{R}$ is given by

$$L_\mu \phi = (\tau_0 + \mu)(-B_1\phi(0) - B_2\phi(-1)), \quad \phi \in C, \tag{3.4}$$

and $f\colon\ \mathbb{R} \times C \to \mathbb{R}$ is the high order term. The linear part can be written as

$$L_\mu \phi = \int_{-1}^{0} \mathrm{d}\eta(\theta, \mu)\phi(\theta),$$

where

$$\eta(\theta, \mu) = (\tau_0 + \mu)(-B_1\delta(\theta) - B_2\delta(\theta + 1)),$$

and δ is the Dirac delta function.

For $\phi \in C^1([-1, 0], \mathbb{R})$, define

$$(A(\mu)\phi)(\theta) = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & -1 \leqslant \theta < 0, \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(s, \mu)\phi(s) \equiv L_\mu\phi, & \theta = 0 \end{cases}$$

and

$$(R(\mu)\phi)(\theta) = \begin{cases} 0, & -1 \leqslant \theta < 0, \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Thus system (3.3) is equivalent to

$$x_t' = A(\mu)(x_t) + R(\mu)(x_t). \tag{3.5}$$

The adjoint operator of $A(\mu)$ is defined as

$$(A^*(0)\psi)(\zeta) = \begin{cases} -\dfrac{\mathrm{d}\psi(\zeta)}{\mathrm{d}\zeta}, & 0 < \zeta \leqslant 1, \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(s, 0)\psi(-s), & \zeta = 0, \end{cases}$$

where $\psi \in C^1([0, 1], \mathbb{R})$. For $\phi \in C^1([-1, 0], \mathbb{R})$ and $\psi \in C^1([0, 1], \mathbb{R})$, we can define the bilinear inner product by

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi, \tag{3.6}$$

where $\eta(\theta) = \eta(\theta, 0)$. From the previous discussion, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Suppose that $q(\theta) = e^{i\theta\omega_0\tau_0}q(0)$ is an eigenfunction of $A(0)$ associated with $i\omega_0\tau_0$. Then

$$A(0)q(\theta) = i\omega_0\tau_0 q(\theta).$$

It follows from the definition of $A(0)$ that

$$\tau_0(i\omega_0 - (-B_1 - B_2e^{-i\omega_0\tau_0}))q(0) = 0.$$

Thus, $q(0)$ can be any non-zero constant vector.

Since $i\omega_0\tau_0$ is an eigenvalue for $A(0)$, $-i\omega_0\tau_0$ is an eigenvalue for $A^*(0)$. Suppose that $q^*(\zeta) = e^{i\zeta\omega_0\tau_0}q^*(0)$ is an eigenfunction of $A^*(0)$ corresponding to $-i\omega_0\tau_0$, we have

$$A^*(0)q^*(\zeta) = -i\omega_0\tau_0 q^*(\zeta).$$

Let $q(0) = 1$. We use (3.6) to calculate

$$\begin{aligned}
\langle q^*, q\rangle &= \overline{q^*}(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{q^*}(0)e^{-i(\xi-\theta)\omega_0\tau_0}d\eta(\theta)e^{i\xi\omega_0\tau_0}d\xi \\
&= \overline{q^*}(0)\left(1 - \int_{-1}^{0}\theta e^{i\theta\omega_0\tau_0}d\eta(\theta)\right) \\
&= \overline{q^*}(0)(1 + \tau_0 B_2 e^{-i\omega_0\tau_0}).
\end{aligned}$$

Thus, we can choose

$$q^*(0) = \frac{1}{1 + \tau_0 B_2 e^{i\omega_0\tau_0}},$$

so that $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

A center manifold $\widetilde{C}$ at $\mu = 0$ is a locally invariant, attracting two-dimensional manifold in C described as follows. If we define

$$z(t) = \langle q^*, x_t\rangle, \quad W(t,\theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}, \tag{3.7}$$

where x_t is a solution of (3.5), then on the center manifold $\widetilde{C}$ we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta)$$

and

$$W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.8}$$

As z and $\overline{z}$ are local coordinates for the center manifold $\widetilde{C}$, we note that W is real if x_t is real. We consider only real solutions. For the solution $x_t \in \widetilde{C}$, since $\mu = 0$, we have

$$\begin{aligned}
z'(t) &= i\omega_0\tau_0 z + \overline{q^*}(0)f(0, W(z, \overline{z}, 0) + 2\mathrm{Re}\{zq(0)\}) \\
&= i\omega_0\tau_0 z + \overline{q^*}(0)f_0(z, \overline{z}).
\end{aligned}$$

This can be rewritten as

$$z'(t) = \mathrm{i}\omega_0\tau_0 z + g(z, \overline{z}),$$

with

$$g(z, \overline{z}) = \overline{q^*}(0) f_0(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.9}$$

Noticing that

$$x_t(\theta) = W(t,\theta) + zq(\theta) + \overline{zq}(\theta), \quad q(\theta) = \mathrm{e}^{\mathrm{i}\theta\omega_0\tau_0},$$

we have

$$x_t(0) = z + \overline{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\overline{z} + W_{02}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),$$

$$\begin{aligned} x_t(-1) = {} & \mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z + \mathrm{e}^{\mathrm{i}\omega_0\tau_0}\overline{z} + W_{20}(-1)\frac{z^2}{2} \\ & + W_{11}(-1)z\overline{z} + W_{02}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3). \end{aligned}$$

Consequently, the Taylor expansion of $f(\mu, x_t)$ in (3.3) can be expressed in terms of z and $\overline{z}$ as follows:

$$\begin{aligned} f_0(z,\overline{z}) = {} & f(0, x_t) \\ = {} & \tau_0\Big([-\beta(x_0)]x_t^2(0) + [\beta'(x_0)(1-2x_0)]x_t(0)x_t(-1) \\ & + \Big[\frac{1}{2}\beta^{(2)}(x_0)(1-x_0)x_0\Big]x_t^2(-1) + [-\beta'(x_0)]x_t^2(0)x_t(-1)) \\ & + \Big[\frac{1}{2}\beta^{(2)}(x_0)(1-2x_0)\Big]x_t(0)x_t^2(-1) \\ & + \Big[\frac{1}{6}\beta^{(3)}(x_0)(1-x_0)x_0\Big]x_t^3(-1)\Big) + \cdots \\ = {} & \tau_0\Big\{f_{20}\Big(z + \overline{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\overline{z} + W_{02}(0)\frac{\overline{z}^2}{2}\Big)^2 \\ & + f_{11}\Big(z + \overline{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\overline{z} + W_{02}(0)\frac{\overline{z}^2}{2}\Big) \\ & \times\Big(\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z + \mathrm{e}^{\mathrm{i}\omega_0\tau_0}\overline{z} + W_{20}(-1)\frac{z^2}{2} + W_{11}(-1)z\overline{z} + W_{02}(-1)\frac{\overline{z}^2}{2}\Big) \\ & + f_{02}\Big(\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z + \mathrm{e}^{\mathrm{i}\omega_0\tau_0}\overline{z} + W_{20}(-1)\frac{z^2}{2} \\ & + W_{11}(-1)z\overline{z} + W_{02}(-1)\frac{\overline{z}^2}{2}\Big)^2 \\ & + f_{21}\Big(z + \overline{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\overline{z} + W_{02}(0)\frac{\overline{z}^2}{2}\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&\times \Big(z + \overline{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\overline{z} + W_{02}(0)\frac{\overline{z}^2}{2}\Big) \\
&+ f_{12}\Big(z + \overline{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\overline{z} + W_{02}(0)\frac{\overline{z}^2}{2}\Big) \\
&\times \Big(\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z + \mathrm{e}^{\mathrm{i}\omega_0\tau_0}\overline{z} + W_{20}(-1)\frac{z^2}{2} \\
&+ W_{11}(-1)z\overline{z} + W_{02}(-1)\frac{\overline{z}^2}{2}\Big)^2 \\
&+ f_{03}\Big(\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}z + \mathrm{e}^{\mathrm{i}\omega_0\tau_0}\overline{z} + W_{20}(-1)\frac{z^2}{2} \\
&+ W_{11}(-1)z\overline{z} + W_{02}(-1)\frac{\overline{z}^2}{2}\Big)^3\Big\} + \cdots .
\end{aligned}
$$

From equation (3.9), we can compare the corresponding coefficients to obtain

$$
\begin{aligned}
g_{20} &= 2\overline{q^*}(0)\tau_0(f_{20} + f_{11}\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + f_{02}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}), \\
g_{11} &= \overline{q^*}(0)\tau_0(2f_{20} + f_{11}(\mathrm{e}^{\mathrm{i}\omega_0\tau_0} + \mathrm{e}^{-\mathrm{i}\omega_0\tau_0}) + 2f_{02}), \\
g_{02} &= 2\overline{q^*}(0)\tau_0(f_{20} + f_{11}\mathrm{e}^{\mathrm{i}\omega_0\tau_0} + f_{02}\mathrm{e}^{2\mathrm{i}\omega_0\tau_0}),
\end{aligned}
$$

$$
\begin{aligned}
g_{21} = 2\overline{q^*}(0)\tau_0\Big\{ & f_{20}(W_{20}(0) + 2W_{11}(0)) \\
& + f_{11}\Big(\frac{\mathrm{e}^{\mathrm{i}\omega_0\tau_0}}{2}W_{20}(0) + \frac{1}{2}W_{20}(-1) + \mathrm{e}^{-\mathrm{i}\omega_0\tau_0}W_{11}(0) + W_{11}(-1)\Big) \\
& + f_{02}(\mathrm{e}^{\mathrm{i}\omega_0\tau_0}W_{20}(-1) + 2\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}W_{11}(-1)) \\
& + 3f_{21} + f_{12}(2 + \mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}) + 3f_{03}\mathrm{e}^{-\mathrm{i}\omega_0\tau_0}\Big\}.
\end{aligned}
$$

As $W_{20}(\theta)$ and $W_{11}(\theta)$ are involved in g_{21}, we have to find W_{20} and W_{11} first.

From (3.5) and (3.7), we have

$$
\begin{aligned}
(W')(\theta) &= (x_t' - z'q - \overline{z}'\overline{q})(\theta) \\
&= \begin{cases} AW - 2\mathrm{Re}\{\overline{q^*}(0)f_0q(\theta)\}, & -1 \leqslant \theta < 0, \\ AW - 2\mathrm{Re}\{\overline{q^*}(0)f_0q(0)\} + f_0, & \theta = 0. \end{cases}
\end{aligned}
\tag{3.10}
$$

That is,

$$W' = AW + H(z, \overline{z}, \theta),$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots . \tag{3.11}$$

On the other hand, on the center manifold $\widetilde{C}$,

$$W' = W_z z' + W_{\overline{z}}\overline{z}'.$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$(2\mathrm{i}\omega_0\tau_0 - A(0))W_{20}(\theta) = H_{20}(\theta), \quad -A(0)W_{11}(\theta) = H_{11}(\theta), \quad \ldots. \tag{3.12}$$

From (3.10), we know that for $\theta \in [-1, 0)$,

$$H(z, \overline{z}, \theta) = -\overline{q^*}(0)f_0 q(\theta) \ - q^*(0)\overline{f}_0\overline{q}(\theta) = -gq(\theta) - \overline{g}\,\overline{q}(\theta).$$

Comparing the coefficients with (3.11) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \quad H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta), \quad \ldots. \tag{3.13}$$

From (3.12), (3.13) and the definition of A, it follows that

$$W_{20}'(\theta) = 2\mathrm{i}\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta).$$

As $q(\theta) = \mathrm{e}^{-\mathrm{i}\theta\omega_0\tau_0}$, we conclude

$$W_{20}(\theta) = \frac{\mathrm{i}}{\omega_0\tau_0}g_{20}\mathrm{e}^{\mathrm{i}\theta\omega_0\tau_0} + \frac{\mathrm{i}}{3\omega_0\tau_0}\overline{g}_{02}\mathrm{e}^{-\mathrm{i}\theta\omega_0\tau_0} + E_1\mathrm{e}^{2\mathrm{i}\theta\omega_0\tau_0},$$

where E_1 is a constant vector. Similarly, we can obtain

$$W_{11}(\theta) = -\frac{\mathrm{i}}{\omega_0\tau_0}g_{11}\mathrm{e}^{\mathrm{i}\theta\omega_0\tau_0} + \frac{\mathrm{i}}{\omega_0\tau_0}\overline{g}_{11}\mathrm{e}^{-\mathrm{i}\theta\omega_0\tau_0} + E_2,$$

where E_2 is a constant vector.

We now seek appropriate E_1 and E_2. From the definition of A and (3.12), when $\theta = 0$, we have

$$\int_{-1}^{0} \mathrm{d}\eta(s)W_{20}(s) = 2\mathrm{i}\omega_0\tau_0 W_{20}(0) - H_{20}(0). \tag{3.14}$$

On the other hand, when $\theta = 0$, equations (3.10) and (3.11) give

$$H_{20}(0) = -g_{20} - \overline{g}_{02} + \tau_0(f_{20} + f_{11}\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + f_{02}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}).$$

Substituting this and

$$W_{20}(0) = \frac{\mathrm{i}}{\omega_0\tau_0}g_{20} + \frac{\mathrm{i}}{3\omega_0\tau_0}\overline{g}_{02} + E_1$$

into (3.14) yields

$$\begin{aligned}
&\int_{-1}^{0} \mathrm{d}\eta(s)W_{20}(s)\\
&= \int_{-1}^{0}\Big(\frac{\mathrm{i}}{\omega_0\tau_0}g_{20}\mathrm{e}^{\mathrm{i}s\omega_0\tau_0} + \frac{\mathrm{i}}{3\omega_0\tau_0}\overline{g}_{02}\mathrm{e}^{-\mathrm{i}s\omega_0\tau_0} + E_1\mathrm{e}^{2\mathrm{i}s\omega_0\tau_0}\Big)\mathrm{d}\eta(s)\\
&= -g_{20} + \frac{1}{3}\overline{g}_{02} + 2\mathrm{i}\omega_0\tau_0 E_1 - \tau_0(f_{20} + f_{11}\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + f_{02}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}).
\end{aligned}$$

Observing

$$\int_{-1}^{0} \frac{\mathrm{i}}{\omega_0\tau_0} g_{20} \mathrm{e}^{\mathrm{i}s\omega_0\tau_0} \mathrm{d}\eta(s) = \frac{\mathrm{i}}{\omega_0\tau_0} g_{20} \mathrm{i}\omega_0\tau_0 = -g_{20},$$

and

$$\int_{-1}^{0} \frac{\mathrm{i}}{3\omega_0\tau_0} \overline{g}_{02} \mathrm{e}^{-\mathrm{i}s\omega_0\tau_0} \mathrm{d}\eta(s) = \frac{\mathrm{i}}{3\omega_0\tau_0} \overline{g}_{02}(-\mathrm{i}\omega_0\tau_0) = \frac{1}{3}\overline{g}_{02},$$

we obtain

$$\int_{-1}^{0} E_1 \mathrm{e}^{2\mathrm{i}s\omega_0\tau_0} \mathrm{d}\eta(s) = 2\mathrm{i}\omega_0\tau_0 E_1 - \tau_0(f_{20} + f_{11}\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + f_{02}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}),$$

which leads to

$$E_1 = \frac{f_{20} + f_{11}\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + f_{02}\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}}{2\mathrm{i}\omega_0 + B_1 + B_2\mathrm{e}^{-2\mathrm{i}\omega_0\tau_0}}.$$

Using the same argument, we get

$$E_2 = \frac{2f_{20} + f_{11}(\mathrm{e}^{\mathrm{i}\omega_0\tau_0} + \mathrm{e}^{-\mathrm{i}\omega_0\tau_0}) + 2f_{02}}{B_1 + B_2}.$$

We thus complete the calculation of $W_{20}(\theta)$ and $W_{11}(\theta)$, and hence of g_{21}. From the information of g_{20}, g_{11}, g_{02} and g_{21}, we obtain

$$C_1(0) = \frac{\mathrm{i}}{2\omega_0\tau_0}\Big(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big) + \frac{g_{21}}{2}. \tag{3.15}$$

Set

$$\mu_2 = -\frac{\mathrm{Re}C_1(0)}{\mathrm{Re}\lambda'(\tau_0)}, \quad \beta_2 = 2\mathrm{Re}C_1(0).$$

By the general results for the direction and stability of a Hopf bifurcation [6,11], we see that the sign of β_2 determines the direction of Hopf bifurcations, and the sign of μ_2 determines the stability of the bifurcated periodic solution.

Noting $\mathrm{Re}\lambda'(\tau_0) > 0$, we obtain the following result.

Theorem 3.2 *Let*

$$\beta^* > \gamma > \frac{\beta^2(x^*)}{|\beta'(x^*)|}$$

and $C_1(0)$ *be given by* (3.15). *Then*

(i) *the Hopf bifurcation occurs as* τ *crosses* τ_0 *to the right if* $\mathrm{Re}C_1(0) < 0$, *and to the left if* $\mathrm{Re}C_1(0) > 0$;

(ii) *the bifurcated periodic solution is stable if* $\mathrm{Re}C_1(0) < 0$, *and unstable if* $\mathrm{Re}C_1(0) > 0$;

(iii) *the period* T *and amplitude* R *of the bifurcated periodic solution are given by*

$$T(\mu) = \frac{2\pi}{\omega_0\tau_0} + o(|\mu|), \quad R(\mu) = \sqrt{-\frac{2\mathrm{Re}\lambda'(\tau_0)\mu}{\mathrm{Re}C_1(0)}} + o(|\mu|).$$

4 Numerical simulations

Theorem 3.2 provides an explicit algorithm for determining the stability and direction of the Hopf bifurcation at $x = x^*$ near $\tau = \tau_0$. This facilitates the construction of numerical examples to illustrate nonlinear oscillations due to the time delay in the nonlinear incidence rate. This is illustrated for both β_M (Fig. 4 (a)) and β_N (Fig. 4 (b)).

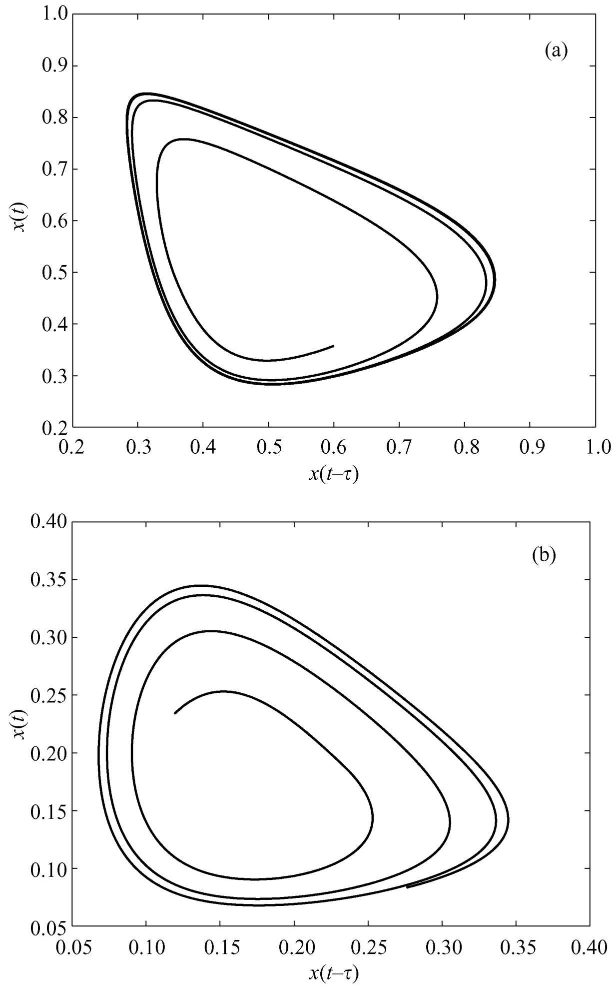
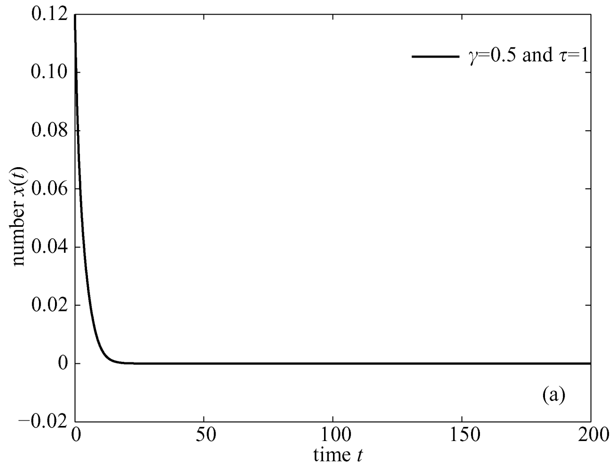
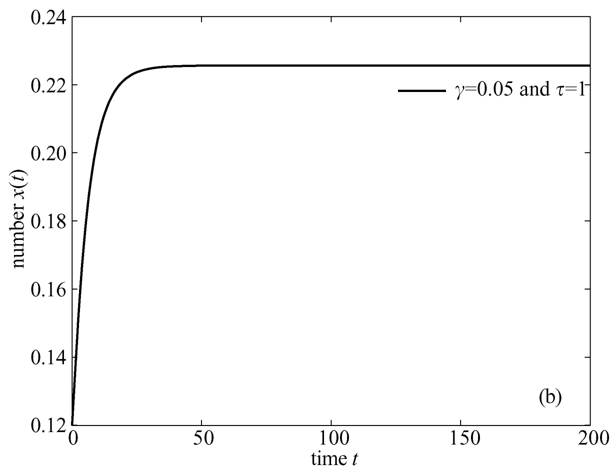


Fig. 4 Occurrence of stable periodic solutions from a unique endemic equilibrium when delay in the incidence rate is increased to a certain value. (a) $\beta(x) = \beta_M(x)$ with $\beta_0 = 8$, $\alpha = 8$ and $\gamma = 0.1$, where $x^* = 0.4687$ and $\tau_0 = 4.9619$; (b) $\beta(x) = \beta_N(x)$ with $\beta_0 = 0.1$, $\beta_1 = 8$, $\alpha = 15$ and $\gamma = 0.1$, where $x^* = 0.1648$ and $\tau_0 = 11.1060$

Figure 5 illustrates the bifurcation processes from the disease free equilibrium (Fig. 5 (a) to (b)) to a unique endemic equilibrium by decreasing γ, from a unique endemic equilibrium to two endemic equilibria with one being stable by increasing γ again (backward bifurcation) (Fig. 5 (b) to (c)), and from a stable endemic equilibrium to a stable periodic oscillation by increasing the time delay τ (Hopf bifurcation) (Fig. 5 (c) to (d)).





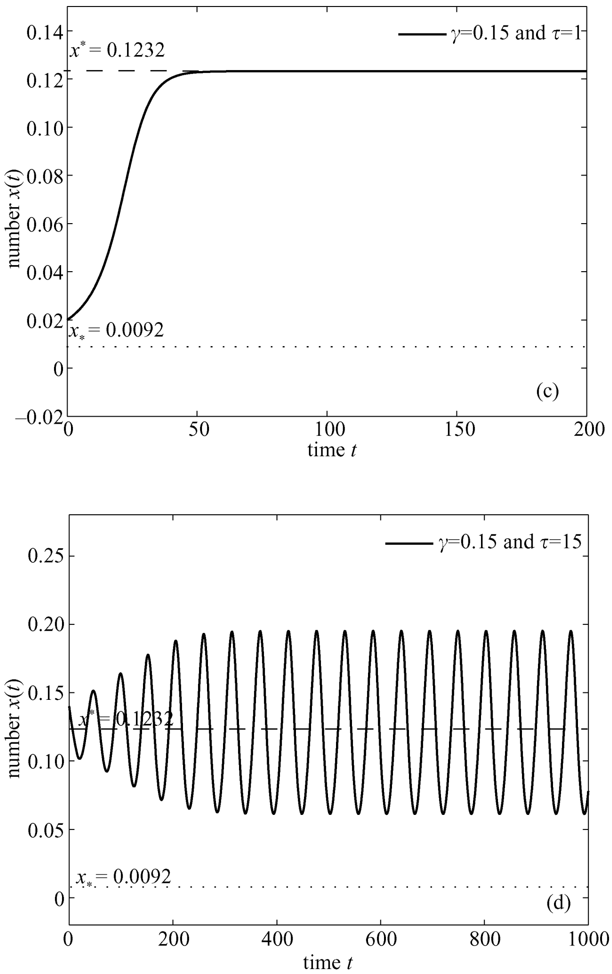


Fig. 5 The bifurcation processes from the disease free equilibrium ((a) to (b)) to a unique endemic equilibrium by decreasing γ, from a unique endemic equilibrium to two endemic equilibria with one being stable by increasing γ again (backward bifurcation) ((b) to (c)), and from a stable endemic equilibrium to a stable periodic oscillation by increasing the time delay τ (Hopf bifurcation) ((c) to (d)). In the figure, $\beta(x) = \beta_N(x)$ with $\beta_0 = 0.1$, $\beta_1 = 8$, $\alpha = 15$. (a) $\gamma = 0.5$ and $\tau = 1$; (b) $\gamma = 0.05$ and $\tau = 1$; (c) $\gamma = 0.15$ and $\tau = 1$; (d) $\gamma = 0.15$ and $\tau = 15$

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