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Asymptotic patterns of a structured population diffusing in a two-dimensional strip[☆]

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Abstract

In this paper, we derive a population model for the growth of a single species on a two-dimensional strip with Neumann and Robin boundary conditions. We show that the dynamics of the mature population is governed by a reaction–diffusion equation with delayed global interaction. Using the theory of asymptotic speed of spread and monotone traveling waves for monotone semiflows, we obtain the spreading speed c^*, the non-existence of traveling waves with wave speed $0 < c < c^*$, and the existence of monotone traveling waves connecting the two equilibria for $c \geq c^*$.

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1. Introduction

Aiello and Freedman [1] proposed the following system for a single species population with age stages:

$$w'(t) = \alpha e^{-\gamma r} w(t-r) - \beta w^2(t), \tag{1.1}$$

where α, β, γ and r are positive constants, w denotes the numbers of adult members of the population, and r is the time taken from birth to maturity. The first term of (1.1) represents the rate of recruitment into the adult population, and the second term represents the mortality rates of adult individuals. This system provides an alternative, and more realistic model for a single species than the logistic equation $w' = w(1-w)$. As shown in [1], all solutions of (1.1), other than the trivial solution, converge to the positive equilibrium solution w^*.

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By considering the diffusion in continuous space which is Fickian diffusion, (1.1) was generalized by AL-Omari and Gourley [2] to the following form

$$\frac{\partial w(x,t)}{\partial t} = D_m \Delta w(x,t) + \int_0^r \int_\Omega G(x,y,s) f(s) \mathrm{e}^{-\gamma s} b(w(y,t-s)) \mathrm{d}y \mathrm{d}s - d(w(x,t)) \tag{1.2}$$

subject to a Neumman boundary value condition on $\partial\Omega$, where D_m is a diffusion coefficient, d is the death function, Ω is a bounded domain, f is a probability function satisfying $\int_0^r f(s)\mathrm{d}s = 1$, G is a kernel yielded from solving the heat equation, and satisfies $\int_\Omega G(x,y,t)\mathrm{d}x = \int_\Omega G(x,y,t)\mathrm{d}y = 1$. In [2], AL-Omari and Gourley showed also the global attractivity of the positive steady state $\hat{w}$ of (1.2).

In the present article, we shall consider a similar system as in [2], which represents the population growth of a single species with age stages in a two-dimensional strip domain. In Section 2, a reaction–diffusion equation with delayed global interaction is derived for the mature population:

$$\begin{aligned} \frac{\partial w}{\partial t} &= D_m \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] - d_m w \\ &\quad + \mu \int_{\mathbb{R}} \int_0^L \Gamma(\alpha, x, z_x, y, z_y) b(w(t-r, z_x, z_y)) \mathrm{d}z_x \, \mathrm{d}z_y, \quad t > 0, (x,y) \in (0,L)\times\mathbb{R}, \\ Bw(t,x,y) &= 0, \quad t \geq 0, x = 0, L, \; y \in \mathbb{R}, \end{aligned} \tag{1.3}$$

where $b(\cdot)$ is the birth function, $Bw(t,x,y) = p(x)w(t,x,y) + \frac{\partial}{\partial n} w(t,x,y)$ is the boundary value condition including the Neumann ($p(0) = p(L) = 0$) and Robin ($p(0) \geq 0$, $p(L) \geq 0$, $[p(0)]^2 + [p(L)]^2 \neq 0$) boundary value conditions respectively. Although we only consider the case when the death function $d(w) = d_m w$, we change Ω into a unbounded strip domain $(0,L)\times\mathbb{R}$ which makes us able to discuss two very important asymptotic properties (traveling waves and spread speed) of the population system (1.3) as $t = \infty$. The boundary value conditions here are also more abundant than that in [2].

In population dynamics, two key elements to the developmental process seems to be the appearance of a traveling wave and the spread speed (or, asymptotic speed of spread). A traveling wave is a special solution which travels without any change in shape. Traveling wave solutions have been widely studied for reaction–diffusion equations [18,22,28], integral and integro-differential equations [4–6], lattice systems [25,27]. The concept of spreading speed was first introduced by Aronson and Weinberger [3] for reaction–diffusion equations, and also applied to integro-differential equations, integral equations, lattice systems and systems of recursions. See [6,11,13,16,20,21,23,24,26] and the references therein. The spreading speed is a threshold constant $c^* > 0$ which gives an important description of the long time behaviors of the population systems either for $c \in (0, c^*)$ or $c \in (c^*, \infty)$. Taking (1.3) as an example, the spreading speed c^* is a number in the sense that $\lim_{t\to\infty,|y|\geq ct} w(t,x,y) = 0$ uniformly on $x \in [0,L]$ if $c > c^*$ and the initial function is zero for y outside a bounded interval, and that $\lim_{t\to\infty,|y|\leq ct} w(t,x,y) = w^+(x)$ ($w^+(x)$ is the positive equilibrium of (1.3)) uniformly on $x \in [0,L]$ if $c \in (0,c^*)$ and the initial function is not identical to zero (see Theorem 3.2 in this article).

Recently, the theory of asymptotic speeds of spread and monotone traveling waves for monotone semiflows (discrete or continuous time) has been developed by Liang and Zhao [12] in such a way that it can be applied to various evolution equations admitting the comparison principle. For every population dynamical system admitting the comparison principle, if the solution of the initial problem exists and is unique, then all solutions form a monotone semiflow $\{Q_t\}_{t=0}^\infty$ which has an asymptotic speed of spread $c^* > 0$ under some conditions (A1)–(A5). Furthermore an estimates of c^* can be given by the linearized approach. On the other hand, the existence of traveling waves above c^* and their non-existence below c^* can be obtained with an extra condition (A6), and thus c^* is also the minimal wave speed of the system.

In this paper, we shall apply the theory mentioned above to the population model (1.3) to obtain the asymptotic speed and monotone traveling waves by imposing a sublinear condition to the birth function b. The application of this theory seems very technical and tricky. It is organized as follows. In Section 2, we give a derivation of model (1.3) and discuss the dynamical structure of the steady states. We show the existence and the global attractivity of the positive steady state by using the theory of functional differential equations in abstract space. The main results are presented in Section 3. We first investigate the existence of solutions for (1.3) and show that the system (1.3) satisfies the comparison principle, and thus all solutions of it yields a monotone semiflow $\{Q_t\}_{t=0}^{\infty}$. Furthermore, we show that $\{Q_t\}_{t=0}^{\infty}$ satisfies the assumptions (A1)–(A6) in [12] provided (P1)–(P3) hold. Therefore, we can obtain the existence of spreading speed c^*, the existence of the traveling waves and the minimal wave speed c^* by using Theorems 2.17, and 4.3–4.4 in [12] directly. The estimates of c^* is also evaluated by using two linearized systems of (1.3) and the method of asymptotic approximation. The last section is an Appendix for model derivation.

We mention here that the discussions in [2] include the global attractivity for the non-spatial problem of (1.2). As for the non-spatial problem of (1.3): $\frac{\mathrm{d}w}{\mathrm{d}t} = -d_m w(t) + \mu b(w(t-r))$, we refer the readers to Faria et al. [8], where as an application of their results, the dynamics and the bifurcation are discussed.

We must emphasize that the relevant study for age-structured population, and in particular for the reduced non-local reaction–diffusion equations with delayed interaction in a one-dimensional domain was reported in [19,21,25]. In comparison, our focus here is the asymptotic patterns of the age-structured population in a two-dimensional strip. The traveling wave connects the trivial solution to a spatially varying equilibrium, giving rise to more complicated spatially changing asymptotic patterns.

2. Model derivation and the structure of equilibria

We divide a population into juveniles and adults. We assume that age structure for mature adults is not important (i.e., vital rates are independent of age). Let $u(t, a, x, y)$ be the density of individuals with age a at a point (x, y) and time t, $r \geq 0$ be the length of juvenile period. Denote by $w(t, x, y)$ the density of mature (or adult) individuals at point (x, y) and time t. Then u is governed by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = D_j \left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right] - d_j(a) u, \quad a \in [0, r], t > 0, (x, y) \in \Omega := (0, L) \times \mathbb{R}, \\ u(t, 0, x, y) = b(w(t, x, y)), \quad t \geq -r, (x, y) \in \Omega, \\ Bu(t, a, x, y) = 0, \quad x = 0, L, \ a \in [0, r], t \geq 0, y \in \mathbb{R}; \end{cases} \tag{2.1}$$

while w satisfies

$$\begin{cases} \dfrac{\partial w}{\partial t} = D_m \left[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} \right] - d_m w + u(t, r, x, y), \quad t > 0, (x, y) \in \Omega, \\ Bw(t, x, y) = 0, \quad x = 0, L, \ t \geq 0, y \in \mathbb{R}. \end{cases} \tag{2.2}$$

Here $b(w)$ and $d_m w$ are the birth and mortality rates of mature individuals, respectively, $d_j(a)$ denotes the per capita mortality rate of juveniles at age a, D_j and D_m are the diffusion coefficients. In (2.1), $Bu(t, a, x, y) = p(x)u(t, a, x, y) + \frac{\partial}{\partial n} u(t, a, x, y)$. More precisely, letting $h_0 := -p(0)$, $h_L := p(L)$, we shall consider (2.1) subject to one of the following Neumann or Robin boundary conditions:

$$\begin{aligned} &(\mathrm{NBC}): h_0 = h_L = 0, \ \frac{\partial}{\partial x} u(t, a, 0, y) = 0, \quad \frac{\partial}{\partial x} u(t, a, L, y) = 0 \text{ for all } y \in \mathbb{R}; \\ &(\mathrm{RBC0}): h_0 = 0, h_L > 0, \ u(t, a, 0, y) = 0, \quad \frac{\partial}{\partial x} u(t, a, L, y) + h_L u(t, a, L, y) = 0 \text{ for all } y \in \mathbb{R}; \\ &(\mathrm{RBCL}): h_0 > 0, h_L = 0, \frac{\partial}{\partial x} u(t, a, 0, y) - h_0 u(t, a, 0, y) = 0, \quad u(t, a, L, y) = 0 \text{ for all } y \in \mathbb{R}; \\ &(\mathrm{RBC^*}): h_0 > 0, h_L > 0, \frac{\partial}{\partial x} u(t, a, 0, y) - h_0 u(t, a, 0, y) = 0, \\ &\qquad \frac{\partial}{\partial x} u(t, a, L, y) + h_L u(t, a, L, y) = 0 \text{ for all } y \in \mathbb{R}. \end{aligned}$$

We sometimes simply use (RBC) to include the three cases: (RBC0), (RBCL) and (RBC*). We shall use the same boundary condition for $Bw(t, x, y) = 0$.

In (2.2), $u(t, r, x, y)$ is the recruitment term, coinciding with those of maturation age r. As usual, we integrate along characteristics to derive a closed system for (2.2) involving delayed non-local terms. Let $v(\tau, a, x, y) = u(a + \tau, a, x, y)$. It follows that

$$\begin{aligned}
\frac{\partial}{\partial a}v(\tau,a,x,y) &= \left[\frac{\partial u}{\partial t}(t,a,x,y)+\frac{\partial u}{\partial a}(t,a,x,y)\right]_{t=\tau+a}\\
&= D_j\left[\frac{\partial^2 u}{\partial x^2}(a+\tau,a,x,y)+\frac{\partial^2 u}{\partial y^2}(a+\tau,a,x,y)\right]-d_j(a)u(a+\tau,a,x,y)\\
&= D_j\left[\frac{\partial^2 v}{\partial x^2}(\tau,a,x,y)+\frac{\partial^2 v}{\partial y^2}(\tau,a,x,y)\right]-d_j(a)v(\tau,a,x,y),\\
v(\tau,0,x,y) &= b(w(\tau,x,y)).
\end{aligned}$$

Regarding τ as a parameter and integrating the last equation, we get

$$v(\tau,a,x,y)=\int_{\mathbb{R}}\int_0^L \Gamma(D_j a,x,z_x,y,z_y)\mathcal{F}(a)b(w(\tau,z_x,z_y))\mathrm{d}z_x\,\mathrm{d}z_y,$$

where

$$\mathcal{F}(a)=\exp\left\{-\int_0^a d_j(s)\mathrm{d}s\right\},\ \Gamma(t,x,z_x,y,z_y)=\Gamma_1(t,x,z_x)\Gamma_2(t,y,z_y),\ \Gamma_2(t,y,z_y)=\frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(y-z_y)^2}{4t}}, \tag{2.3}$$

and $\Gamma_1(t,x,z_x)$ is the Green function of the boundary value problem

$$\begin{aligned}
&\frac{\partial W}{\partial t}-\frac{\partial^2 W}{\partial x^2}=0,\quad t>0,x\in(0,L),\\
&BW(t,x)=0,\quad t\ge 0,x=0,L.
\end{aligned}$$

Let $(\mu_n,\Phi_n(x)), n=1,2,\dots$ be the eigenvalues and eigenfunctions of $-\frac{\partial^2}{\partial x^2}$ on $(0,L)$ subject to the corresponding boundary condition with $\mu_1<\mu_2<\cdots$ (see the Appendix for details). By using the method of separation variables, we obtain that

$$\Gamma_1(t,x,z_x)=\sum_{n=1}^{\infty}\frac{1}{M_n}\mathrm{e}^{-\mu_n t}\Phi_n(z_x)\Phi_n(x),$$

where $M_n=\int_0^L\Phi_n^2(x)\mathrm{d}x$. Furthermore, let $m_n:=\int_0^L\Phi_n(\xi)\mathrm{d}\xi$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\Gamma_2(t,y,z_y)\mathrm{d}z_y=1,\quad \forall t>0,y\in\mathbb{R},\\
&\int_0^L\Gamma_1(t,x,z_x)\mathrm{d}z_x=\sum_{n=1}^{\infty}\frac{m_n}{M_n}\mathrm{e}^{-\mu_n t}\Phi_n(x)\le 1,\quad t\ge 0,x\in(0,L).
\end{aligned} \tag{2.4}$$

Since $u(t,a,x,y)=v(t-a,a,x,y)$, it follows that

$$u(t,a,x,y)=\int_{\mathbb{R}}\int_0^L\Gamma(D_j a,x,z_x,y,z_y)\mathcal{F}(a)b(w(t-a,z_x,z_y))\mathrm{d}z_x\,\mathrm{d}z_y. \tag{2.5}$$

Substituting (2.5) into (2.2), we obtain the delayed system with non-local term:

$$\begin{aligned}
&\frac{\partial w}{\partial t}=D_m\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]-d_m w\\
&\qquad +\mu\int_{\mathbb{R}}\int_0^L\Gamma(\alpha,x,z_x,y,z_y)b(w(t-r,z_x,z_y))\mathrm{d}z_x\,\mathrm{d}z_y,\quad t>0,(x,y)\in\Omega,\\
&Bw(t,x,y)=0,\quad t\ge 0,x=0,L,\ y\in\mathbb{R},
\end{aligned} \tag{2.6}$$

where $\alpha=D_j r$, $\mu=\mathcal{F}(r)$.

Let $\mathbb{R}_+=[0,\infty)$. We now formulate assumptions for the birth function b, motivated by the biological reality:

(P1) $b\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $b(0)=0$, $b'(0)>0$ and b is strictly sublinear, i.e., $b(\gamma w)>\gamma b(w)$ for $\gamma\in(0,1)$ and $w\in(0,\infty)$.

(P2) The maximum of b can be achieved at $\vartheta > 0$, b is non-decreasing on $[0, \vartheta]$ and there exists a number $N \in (0, \vartheta]$ such that $\mu b(w) < d_m w$ for all $w > N$.
(P3) $\mu b'(0)\mathrm{e}^{-\mu_1 \alpha} > d_m + D_m \mu_1$.

Remark 2.1. The birth functions $b_1(w) = pw\mathrm{e}^{-aw^q}$, $b_2(w) = \frac{pw}{1+aw^q}$ and

$$b_3(w) = \begin{cases} pw\left(1 - \dfrac{w^q}{K^q}\right), & 0 \le w \le K, \\ 0, & w > K, \end{cases}$$

with constants $p > 0, q > 0, a > 0, K > 0$ in the well-known Nicholson's blowfly model satisfy the above assumptions if the parameters are in appropriate ranges. Taking $b_2(w)$ as an example, we have $b_2(\gamma w) = \frac{\gamma p w}{1+a\gamma^q w^q} > \frac{\gamma p w}{1+aw^q} = \gamma b_2(w)$, and $b_2(w)$ is strictly sublinear. Assume that the solution of $\frac{\mu p}{d_m} - 1 = aw^q$ is N (if $\frac{\mu p}{d_m} > 1$), then $\mu b_2(w) < d_m w$ for $w > N$. Furthermore, since $b_2'(w) = \frac{p+ap(1-q)w^q}{(1+aw^q)^2}$, $b_2'(0) = p$, we can easily choose p such that (P2)–(P3) are satisfied.

Our focus here is the asymptotic patterns of (2.6), and for this purpose it is essential to first describe the structure of equilibria. Note that $w \equiv 0$ is an equilibrium of (2.6). Other equilibria for (2.6) are independent of y, and thus are equilibria of the following boundary value problem:

$$\begin{aligned} &\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \mu \int_0^L \Gamma_1(\alpha, x, z_x) b(w(t-r, z_x))\mathrm{d}z_x, \quad t > 0, x \in (0, L), \\ &Bw(t, x) = 0 \quad t \ge 0, x = 0, L. \end{aligned} \tag{2.7}$$

The linearized equation at zero solution for (2.7) is

$$\begin{aligned} &\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \mu b'(0) \int_0^L \Gamma_1(\alpha, x, z_x) w(t-r, z_x)\mathrm{d}z_x, \quad t > 0, x \in (0, L), \\ &Bw(t, x) = 0 \quad t \ge 0, x = 0, L. \end{aligned} \tag{2.8}$$

Substituting $w(t, x) = \mathrm{e}^{\lambda t} v(x)$ into (2.8), we obtain the following eigenvalue problem:

$$\begin{aligned} &\lambda v(x) = D_m \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - d_m v + \mu b'(0)\mathrm{e}^{-\lambda r} \int_0^L \Gamma_1(\alpha, x, z_x) v(z_x)\mathrm{d}z_x, \quad x \in (0, L), \\ &Bv(x) = 0, \qquad x = 0, L. \end{aligned} \tag{2.9}$$

In the following, we want to show that the principal eigenvalue λ_0 exists and to find the precise conditions for λ_0 to be positive. Substituting $v(x) = \Phi_n(x)$ with $n = 1, 2, \ldots$, into (2.9), and noting that

$$\int_0^L \Gamma_1(\alpha, x, z_x)\Phi_n(z_x)\mathrm{d}z_x = \mathrm{e}^{-\mu_n \alpha}\Phi_n(x), \tag{2.10}$$

then we obtain that the eigenvalues of (2.9) satisfy

$$\lambda = -D_m \mu_n - d_m + \mu b'(0)\mathrm{e}^{-\lambda r}\mathrm{e}^{-\mu_n \alpha}, \quad n = 1, 2, \ldots. \tag{2.11}$$

It is well-known (see [10,7]) that since $b'(0) > 0$, Eq. (2.11) with a fixed n has a real zero $\lambda_{n,1}$ and complex conjugate pair of zeros $\lambda_{n,2}, \bar{\lambda}_{n,2}, \lambda_{n,3}, \bar{\lambda}_{n,3}, \ldots$ such that $\lambda_{n,1} > \mathrm{Re}\,\lambda_{n,2} > \mathrm{Re}\,\lambda_{n,3} > \cdots$. Furthermore, we can easily show that $\lambda_{1,1} > \lambda_{2,1} > \lambda_{3,1} > \cdots$. Therefore, $\lambda_0 = \lambda_{1,1} \in \mathbb{R}$ is the principal eigenvalue of (2.9) which determines the stability of zero solution for (2.8). Let $f_n(\lambda) := \lambda + D_m \mu_n + d_m - \mu b'(0)\mathrm{e}^{-\lambda r}\mathrm{e}^{-\mu_n \alpha}$. Then we have

$$\begin{aligned} &f_n(-\infty) = -\infty < f_n(0) = D_m \mu_n + d_m - \mu b'(0)\mathrm{e}^{-\mu_n \alpha} < f_n(\infty) = \infty, \\ &f_n'(\lambda) = 1 + r\mu b'(0)\mathrm{e}^{-\lambda r}\mathrm{e}^{-\mu_n \alpha} > 0. \end{aligned}$$

We conclude that $\lambda_{n,1} > 0$ if and only if $f_n(0) < 0$. Therefore,

$$\lambda_0 > 0 \quad \text{if and only if (P3) holds.} \tag{2.12}$$

Let $\mathbb{X} = C([0, L], \mathbb{R})$ be the Banach space of all bounded and continuous functions with the supremum norm $\|\cdot\|$. Let $\mathbb{X}^+ = \{\phi \in \mathbb{X} : \phi(x) \geq 0, \forall x \in [0, L]\}$ denote the positive cone in $\mathbb{X}$, then $\mathrm{int}\mathbb{X}^+$ is non-empty under the boundary value condition (either (NBC) or (RBC), see Smith [17]). For any $\zeta \geq 0$, let $[0, \zeta]_{\mathbb{X}} = \{\phi \in \mathbb{X} : 0 \leq \phi(x) \leq \zeta, \forall x \in [0, L]\}$. We know that $\mathbb{X}$ is a Banach lattice under the partial ordering induced by $\mathbb{X}^+$.

The heat equation

$$\begin{aligned} &\frac{\partial W(t,x)}{\partial t} = D_m \frac{\partial^2 W(t,x)}{\partial x^2}, && t > 0, x \in (0, L), \\ &BW(t,x) = 0, && t \geq 0, x = 0,\ L, \\ &W(0,x) = \varphi(x), && x \in (0, L), \end{aligned}$$

has the solution

$$T(t)\varphi(x) = \int_0^L \Gamma_1(D_m t, x, z_x)\varphi(z_x)\mathrm{d}z_x, \quad t > 0, x \in [0, L], \varphi \in \mathbb{X},$$

and $T(t) : \mathbb{X} \to \mathbb{X}$ is a C_0-semigroup with $T(t)\mathbb{X}^+ \subset \mathbb{X}^+$ for all $t \geq 0$ [15].

Let $C = C([-r, 0], \mathbb{X})$ be the Banach space of continuous functions from $[-r, 0]$ to $\mathbb{X}$, $C^+ = \{\phi \in C : \phi(s) \in \mathbb{X}^+, \forall s \in [-r, 0]\}$. For any $\zeta \geq 0$, let $[0, \zeta]_C = \{\phi \in C : \phi(s) \in [0, \zeta]_{\mathbb{X}}, \forall s \in [-r, 0]\}$. Then C^+ is a closed cone of C. As usual, we identify an element $\phi \in C$ with a function from $[-r, 0] \times [0, L]$ into $\mathbb{R}$ defined by $\phi(s, x) = \phi(s)(x)$. For any function $w : [-r, a) \to \mathbb{X}$, $a > 0$, we define $w_t \in C$ with $t \in [0, a)$ by $w_t(\theta) = w(t + \theta)$ for $\theta \in [-r, 0]$.

Let $w(t)(x) = w(t, x)$. For any $\phi \in C^+$, define $F : C^+ \to \mathbb{X}$ by

$$F(\phi)(x) = -d_m\phi(0, x) + \mu \int_0^L \Gamma_1(\alpha, x, z_x)b(\phi(-r, z_x))\mathrm{d}z_x, \quad x \in [0, L]. \tag{2.13}$$

Then F is Lipschitz continuous in any bounded subset of C^+. Further, let $A = \frac{\partial^2}{\partial x^2}$, then we have the abstract setting for the initial value problem of (2.7)

$$\begin{aligned} &\frac{\mathrm{d}w}{\mathrm{d}t} = D_m A w + F(w_t), \quad t > 0, \\ &w_0 = \phi \in C^+, \end{aligned} \tag{2.14}$$

or equivalently

$$\begin{aligned} &w(t) = T(t)\phi(0) + \int_0^t T(t-s)F(w_s)\mathrm{d}s, \quad t \geq 0, \\ &w_0 = \phi \in C^+. \end{aligned} \tag{2.15}$$

Definition 2.1. A supersolution (subsolution) of (2.7) is a function $v : [-r, a) \to \mathbb{X}$ with $a > 0$ satisfying

$$v(t) \geq (\leq) T(t)v(0) + \int_0^t T(t-s)F(v_s)\mathrm{d}s, \quad t \in [0, a). \tag{2.16}$$

If v is both a supersolution and subsolution on $[0, a)$, then v is called a mild solution of (2.7).

Remark 2.2. Assume that there is a bounded and continuous $v : [-r, a) \times [0, L] \to \mathbb{R}$ with $a > 0$ such that v is in C^2 in $x \in (0, L)$, C^1 in $t \in [-r, a)$ and satisfies

$$\begin{aligned} &\frac{\partial v(t,x)}{\partial t} \geq (\leq) D_m \frac{\partial^2 v(t,x)}{\partial x^2} - d_m v(t,x) + \mu \int_0^L \Gamma_1(\alpha, x, y)b(v(t-r, y))\mathrm{d}y, \\ &\qquad t \in [-r, a),\ x \in (0, L), \\ &Bv(t,x) \geq (\leq) 0, \quad t \in [-r, a), \quad x = 0, L. \end{aligned} \tag{2.17}$$

Then, by the fact $T(t)\mathbb{X}^+ \subset \mathbb{X}^+$ for all $t \geq 0$, it follows that (2.16) holds, and hence v is a supersolution (subsolution) of (2.7) on $[0, a)$.

Theorem 2.1. *Assume that* (P1)–(P3) *hold. Then we have the following conclusions.*

(i) *For a given initial condition $\phi \in C^+$, there exists a unique non-negative solution $w(t,x) = w(t,x;\phi)$ of* (2.7) *defined on $[-r,\infty)$. Furthermore, if $\phi \in [0,N]_C$, then $w_t \in [0,N]_C$, where $w_t(\theta,x) = w(t+\theta,x)$, $(\theta,x) \in [-r,0]\times[0,L]$.*

(ii) (2.7) *admits a unique equilibrium $w^+(x)$ in $[0,N]_C$, which is globally attractive.*

Proof. For any $M > \vartheta$ and any $\phi \in [0,M]_C$, we have

$$\begin{aligned}\phi(0,x) + hF(\phi,x) &= \phi(0,x) + h\left[-d_m\phi(0,x) + \mu\int_0^L \Gamma_1(\alpha,x,y)b(\phi(-r,y))\mathrm{d}y\right]\\ &\geq \phi(0,x)(1-hd_m) \geq 0,\end{aligned}$$

when $h > 0$ is so small that $1 - hd_m > 0$. On the other hand, for a given $x \in [0,L]$ such that $\phi(0)(x) = \phi(0,x) \geq \vartheta$, we have

$$\begin{aligned}\phi(0,x) + hF(\phi)(x) &= \phi(0)(x) + h\left[-d_m\phi(0) + \mu\int_0^L \Gamma_1(\alpha,x,y)b(\phi(-r,y))\mathrm{d}y\right]\\ &\leq \phi(0,x) + h(-d_m\vartheta + \mu b(\vartheta))\\ &\leq \phi(0,x) \leq M,\end{aligned}$$

and for $x \in [0,L]$ with $\phi(0,x) < \vartheta$, we have

$$\begin{aligned}\phi(0,x) + hF(\phi)(x) &= \phi(0,x) + h\left[-d_m\phi(0,x) + \mu\int_0^L \Gamma_1(\alpha,x,y)b(\phi(-r,y))\mathrm{d}y\right]\\ &\leq \phi(0,x) + h\mu b(\vartheta) \leq \phi(0,x) + M - \vartheta \leq M,\end{aligned}$$

provided that $h > 0$ is so small that $h\mu b(\vartheta) \leq M - \vartheta$. Therefore, we always have $\phi(0) + hF(\phi) \in [0,M]_{\mathbb{X}}$. Consequently, for $M > \vartheta$, we obtain

$$\lim_{h\to 0+}\frac{1}{h}\mathrm{dist}\,(\phi(0) + hF(\phi); [0,M]_{\mathbb{X}}) = 0, \quad \forall\phi \in [0,M]_C.$$

By Corollary 4 in [14] with $K = [0,M]_C$, $S(t,s) = T(D_m(t-s))$, $B(t,\phi) = F(\phi)$, we conclude that (2.7) admits a unique mild solution $w(t,\phi)$ with $w_t(\phi) \in [0,M]_C$ for $t \in [0,\infty)$. Moreover, we have from Corollary 2.2.5 in [29] that $w(t,\phi)$ is a classical solution of (2.7) for $t > r$, and $[0,M]_C$ is an invariant subset in C^+ for (2.7).

For any $M \in [N,\vartheta]$, F is globally Lipschitz continuous in $[0,M]_C$ and F is quasi-monotone on $[0,M]_C$ in the sense that

$$\lim_{h\to 0+}\frac{1}{h}\mathrm{dist}\left([\phi_1(0) - \phi_2(0)] + h[F(\phi_1) - F(\phi_2)]; X^+\right) = 0 \tag{2.18}$$

for all $\phi_1, \phi_2 \in [0,M]_C$ with $\phi_1 \geq \phi_2$. In fact, it follows from (P2) that

$$\begin{aligned}F(\phi_1)(x) - F(\phi_2)(x) &= -d_m(\phi_1(0,x) - \phi_2(0,x)) + \mu\int_0^L \Gamma_1(\alpha,x,y)[b(\phi_1(-r,y)) - b(\phi_2(-r,y))]\mathrm{d}y\\ &\geq -d_m(\phi_1(0,x) - \phi_2(0,x)),\end{aligned}$$

hence, for any $h > 0$ with $1 > hd_m$,

$$\phi_1(0,x) - \phi_2(0,x) + h[F(\phi_1)(x) - F(\phi_2)(x)] \geq [1 - hd_m][\phi_1(0,x) - \phi_2(0,x)] \geq 0,$$

from which (2.18) follows. We note that N and 0 are supersolution and subsolution of (2.7) in view of Definition 2.1. Therefore, the existence and uniqueness of $w \in [0,N]_C$ on $[0,\infty)$ follows from Corollary 5 in [14] with $S(t,s) = T(D_m(t-s))$ for $t \geq s \geq 0$, $v^+(t) = M$, $v^-(t) = 0$, and $B(t,\phi) = F(\phi)$. This also implies that $[0,N]_C$ is an invariant subset in C^+ for (2.7). Summarizing the above discussions, we conclude that for any $\phi \in C^+$, (2.7) admits a unique solution $w(t,\phi)$ which exists on $[-r,\infty)$.

(P3) implies that the principal eigenvalue $\lambda_0 > 0$ for (2.9). The rest of the proof is similar to that in Theorem 3.2 of [30], and we omit it. ∎

In what follows, we always assume that (P1)–(P3) are satisfied. Therefore, except for the trivial solution, $w^+(x)$ is the only non-negative equilibrium of the model (2.6) in $[0, N]_C$.

3. Dynamics on a two-dimensional strip domain

Our main results will be presented in this section. Firstly, we investigate the existence and uniqueness of solutions, and show a comparison theorem for (1.3) with initial conditions. Therefore, all solutions of (1.3) forms a group of maps $\{Q_t\}_{t=0}^{\infty}$. In Section 3.2, we shall verify that $\{Q_t\}_{t=0}^{\infty}$ is a monotone and subhomogeneous semiflow. Furthermore, we show that the assumptions (A1)–(A6) for the operator Q in [12] are satisfied with $Q = Q_t$, thus by using Theorem 2.17 and Theorems 4.3–4.4 in [12], we obtain the asymptotic speed of spread and minimal wave speed c^* for $\{Q_t\}_{t=0}^{\infty}$. At the last part of Section 3.3, we give some calculation for an estimate of c^* by a linearized approach.

3.1. Existence and comparison of solutions

We have from Section 2 that $\Gamma(D_m t, x, \xi, y, \zeta)$ is the Green function of

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = D_m\left[\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}\right], & (x, y) \in \Omega = (0, L) \times \mathbb{R}, t > 0, \\ B\omega(t, x, y) = 0, & x = 0, L, \;\; y \in \mathbb{R}, t \geq 0. \end{cases} \tag{3.1}$$

Thus the solution of (3.1) with the initial condition $\omega(0, x, y) = \psi(x, y)$ is

$$\begin{aligned} \omega(t, x, y; \psi) &= \int_{\mathbb{R}} \int_0^L \Gamma(D_m t, x, \xi, y, \zeta)\psi(\xi, \zeta)\mathrm{d}\xi \mathrm{d}\zeta \\ &=: [\bar{T}(t)\psi](x, y). \end{aligned} \tag{3.2}$$

Therefore, using the linear semigroup theory of the heat equation, the solution of (2.6) with the initial value condition, is given by $w(\theta, x, y) = \phi(\theta, x, y), \theta \in [-r, 0]$ is

$$\begin{aligned} & w(t, x, y; \phi) \\ &\quad = \exp\{-d_m t\}[\bar{T}(t)\phi(0, \cdot)](x, y) \\ &\qquad + \mu \int_0^t \left\{\exp\{-d_m s\}\bar{T}(s) \cdot \left\{\int_{\mathbb{R}} \int_0^L \Gamma(\alpha, \cdot, z_x, \cdot, z_y) b(w(t - s - r, z_x, z_y))\mathrm{d}z_x\, \mathrm{d}z_y\right\}\right\}(x, y)\mathrm{d}s \\ &\quad = \mathrm{e}^{-d_m t} \int_{\mathbb{R}} \int_0^L \Gamma(D_m t, x, \xi, y, \zeta)\phi(0, \xi, \zeta)\mathrm{d}\xi \mathrm{d}\zeta + \mu \int_0^t \mathrm{e}^{-d_m s} \int_{\mathbb{R}} \int_0^L \Gamma(D_m s, x, \xi, y, \zeta) \\ &\qquad \times \left\{\int_{\mathbb{R}} \int_0^L \Gamma(\alpha, \xi, z_x, \zeta, z_y) b(w(t - s - r, z_x, z_y))\mathrm{d}z_x\, \mathrm{d}z_y\right\} \mathrm{d}\xi \mathrm{d}\zeta \mathrm{d}s, \quad t \geq 0, \\ & w(t, x, y; \phi) = \phi(t, x, y), \quad t \in [-r, 0]. \end{aligned} \tag{3.3}$$

Assume that (2.6) has a non-negative equilibrium $w^+(x)$. Let $\mathbb{Y} = BC([0, L] \times \mathbb{R}, \mathbb{R})$ be the set of all bounded and continuous functions from $[0, L] \times \mathbb{R}$ to $\mathbb{R}$. Let $\mathbb{Y}^+ = \{\phi \in \mathbb{Y} : \phi(x, y) \geq 0, \forall (x, y) \in [0, L] \times \mathbb{R}\}$, and $[0, w^+]_{\mathbb{Y}} = \{\phi \in \mathbb{Y}^+ : 0 \leq \phi(x, y) \leq w^+(x), \forall (x, y) \in [0, L] \times \mathbb{R}\}$. $\mathbb{Y}^+$ is a closed cone of $\mathbb{Y}$ under the partial ordering induced by $\mathbb{Y}^+$. We equip $\mathbb{Y}$ with a compact open topology. That is, $\phi^n \to \phi$ in $\mathbb{Y}$ if and only if that the sequence of functions $\phi^n(x, y)$ converges to $\phi(x, y)$ uniformly for (x, y) in every compact set. Moreover, we define a norm $\|\phi\|_{\mathbb{Y}}$ by

$$\|\phi\|_{\mathbb{Y}} = \sum_{k=0}^{\infty} \frac{\max\limits_{x \in [0, L], |y| \leq k} |\phi(x, y)|}{2^k}.$$

It follows that $(\mathbb{Y}, \|\phi\|_{\mathbb{Y}})$ is a normed space. Let $d(\cdot, \cdot)$ be the metric on $\mathbb{Y}$ induced by the norm $\|\phi\|_{\mathbb{Y}}$. Then $\mathbb{Y}$ is a Banach lattice, and $\bar{T}(t) : \mathbb{Y} \to \mathbb{Y}$ is a linear C_0-semigroup with $\bar{T}(t)\mathbb{Y}^+ \subset \mathbb{Y}^+$ for $t \geq 0$.

Let $C_{\mathbb{Y}} = C([-r, 0], \mathbb{Y})$ be the set of continuous functions from $[-r, 0]$ to $\mathbb{Y}$ and let $C_{\mathbb{Y}}^+ = \{\phi \in C_{\mathbb{Y}} : \phi(s) \in \mathbb{Y}^+, \forall s \in [-r, 0]\}$, and $[0, w^+]_{C_{\mathbb{Y}}} = \{\phi \in C_{\mathbb{Y}}^+ : \phi(s, \cdot) \in [0, w^+]_{\mathbb{Y}}, s \in [-r, 0]\}$. Then $C_{\mathbb{Y}}^+$ is a closed cone of $C_{\mathbb{Y}}$. For any given continuous function $w : [-r, a) \to \mathbb{Y}$, where $a > 0$, we define $w_t \in C_{\mathbb{Y}}$ with $t \in [0, a)$ by $w_t(\theta) = w(t + \theta)$ for $\theta \in [-r, 0]$.

Let $\mathcal{C}$ be the set of all bounded and continuous functions from $[-r, 0] \times [0, L] \times \mathbb{R}$ to $\mathbb{R}$. For $\phi_1, \phi_2 \in \mathcal{C}$, we write $\phi_1 \geq \phi_2 (\phi_1 \gg \phi_2)$ provided $\phi_1(\theta, x, y) \geq \phi_2(\theta, x, y)(\phi_1(\theta, x, y) > \phi_2(\theta, x, y)), \forall(\theta, x, y) \in [-r, 0] \times [0, L] \times \mathbb{R}$, and $\phi_1 > \phi_2$ provided $\phi_1 \geq \phi_2$ but $\phi_1 \neq \phi_2$. Define $\bar{\mathcal{C}} = C([-r, 0] \times [0, L], \mathbb{R})$. Then every element in $\bar{\mathcal{C}}$ can be regarded as a function in $\mathcal{C}$. Specially, we let $\mathcal{C}_{w^+} := \{u \in \mathcal{C} : w^+(x) \geq u \geq 0\}$. Since we identify an element $\phi \in C_{\mathbb{Y}}$ as a function from $[-r, 0] \times [0, L] \times \mathbb{R}$ into $\mathbb{R}$ defined by $\phi(s, x, y) = \phi(s)(x, y)$, $\phi \in \mathcal{C}$ implies $\phi \in C_{\mathbb{Y}}$ and vice versa.

For any $\phi \in C_{\mathbb{Y}}^+$, define $\bar{F} : C_{\mathbb{Y}}^+ \to \mathbb{Y}$ by

$$\bar{F}(\phi)(x, y) = -d_m\phi(0, x, y) + \mu \int_{\mathbb{R}} \int_0^L \Gamma(\alpha, x, z_x, y, z_y) b(\phi(-r, z_x, z_y)) \mathrm{d}z_x \mathrm{d}z_y. \tag{3.4}$$

Then $\bar{F}$ is Lipschitz continuous in every bounded subset of $C_{\mathbb{Y}}^+$. Furthermore, the initial value problem for (2.6) can be rewritten as

$$\begin{aligned} &w(t) = \bar{T}(t)\phi(0) + \int_0^t \bar{T}(t - s)\bar{F}(w_s)\mathrm{d}s, \quad t > 0, \\ &w(t) = \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{3.5}$$

Now we use $\mathbb{Y}$, $C_{\mathbb{Y}}$, $A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\bar{T}$, $\bar{F}$ to replace $\mathbb{X}$, $C_{\mathbb{X}}$, $A = \frac{\partial^2}{\partial x^2}$ and T, F from (2.13) to (2.16) respectively, and we define the notions of supersolution and subsolution of (2.6) similarly to Definition 2.1.

The following lemma shows that $\mathcal{C}_{w^+}$ is positively invariant for solutions of (2.6).

Lemma 3.1. *Assume that* (P1)–(P2) *hold. If $\phi \in \mathcal{C}_{w^+}$, then the solution $w(t, x, y; \phi)$ of* (2.6) *exists uniquely and satisfies $w_t(\cdot; \phi) \in \mathcal{C}_{w^+}$ for all $t > 0$, where $w_t(\cdot; \phi) = w(t + \theta, x, y; \phi)$.*

Proof. For a given $\phi \in \mathcal{C}_{w^+}$, define a set

$$\begin{aligned} S = \{&w \in C([-r, \infty) \times \bar{\Omega}, \mathbb{R}_+) | 0 \leq w(t, x, y) \leq w^+(x) \text{ for } t \in \mathbb{R}_+, (x, y) \in \bar{\Omega}; \\ &w(t, \cdot) = \phi(t, \cdot) \text{ for } t \in [-r, 0]\}, \end{aligned}$$

and a map G on S as follows:

$$G[w](t, x, y) = \begin{cases} \phi(t, x, y), \quad t \in [-r, 0], (x, y) \in \bar{\Omega}, \\ \mathrm{e}^{-d_m t} \displaystyle\int_{\mathbb{R}} \int_0^L \Gamma(D_m t, x, \xi, y, \zeta)\phi(0, \xi, \zeta)\mathrm{d}\xi \mathrm{d}\zeta + \mu \int_0^t \mathrm{e}^{-d_m s} \int_{\mathbb{R}} \int_0^L \Gamma(D_m s, x, \xi, y, \zeta) \\ \quad \times \left\{ \displaystyle\int_{\mathbb{R}} \int_0^L \Gamma(\alpha, \xi, z_x, \zeta, z_y) b(w(t - s - r, z_x, z_y)) \mathrm{d}z_x \, \mathrm{d}z_y \right\} \mathrm{d}\xi \mathrm{d}\zeta \mathrm{d}s, \\ \quad t > 0, (x, y) \in \bar{\Omega}. \end{cases}$$

Then $G[S] \subset S$. In fact, as $w^+(x)$ satisfies

$$\begin{aligned} &-D_m \frac{\mathrm{d}^2 w^+}{\mathrm{d}x^2} + d_m w^+ = \mu \int_0^L \Gamma_1(\alpha, x, z_x) b(w^+(z_x)) \mathrm{d}z_x, \quad x \in (0, L), \\ &B w^+(x) = 0, \quad x = 0, L, \end{aligned}$$

we have

$$
\begin{aligned}
0 \le\ & G[w](t,x,y) \le \mathrm{e}^{-d_m t}\int_0^L \Gamma_1(D_m t,x,\xi)\max_{y\in\mathbb{R}}\{\phi(0,\xi,y)\}\mathrm{d}\xi \\
&\quad + \mu\int_0^t \mathrm{e}^{-d_m s}\int_0^L \Gamma_1(D_m s,x,\xi)\left\{\int_0^L \Gamma_1(\alpha,\xi,z_x)b(w^+(z_x))\mathrm{d}z_x\right\}\mathrm{d}\xi\,\mathrm{d}s \\
&\le \mathrm{e}^{-d_m t}\int_0^L \Gamma_1(D_m t,x,\xi)w^+(\xi)\mathrm{d}\xi + \int_0^t \mathrm{e}^{-d_m s}\int_0^L \Gamma_1(D_m s,x,\xi)\left\{-D_m\frac{\mathrm{d}^2 w^+(\xi)}{\mathrm{d}\xi^2} + d_m w^+(\xi)\right\}\mathrm{d}\xi\,\mathrm{d}s \\
&= \mathrm{e}^{-d_m t}\int_0^L \Gamma_1(D_m t,x,\xi)w^+(\xi)\mathrm{d}\xi \\
&\quad + \int_0^t \mathrm{e}^{-d_m s}\int_0^L \left\{-D_m\frac{\partial^2\Gamma_1(D_m s,x,\xi)}{\partial\xi^2} + d_m\Gamma_1(D_m s,x,\xi)\right\}w^+(\xi)\mathrm{d}\xi\,\mathrm{d}s \\
&= \mathrm{e}^{-d_m t}\int_0^L \Gamma_1(D_m t,x,\xi)w^+(\xi)\mathrm{d}\xi \\
&\quad + \int_0^t\int_0^L \left\{-D_m\frac{\partial^2\{\mathrm{e}^{-d_m s}\Gamma_1(D_m s,x,\xi)\}}{\partial\xi^2} + d_m\{\mathrm{e}^{-d_m s}\Gamma_1(D_m s,x,\xi)\}\right\}w^+(\xi)\mathrm{d}\xi\,\mathrm{d}s \\
&= \mathrm{e}^{-d_m t}\int_0^L \Gamma_1(D_m t,x,\xi)w^+(\xi)\mathrm{d}\xi + \int_0^t\int_0^L \left\{-\frac{\partial\{\mathrm{e}^{-d_m s}\Gamma_1(D_m s,x,\xi)\}}{\partial s}\right\}w^+(\xi)\mathrm{d}\xi\,\mathrm{d}s \\
&= \mathrm{e}^{-d_m t}\int_0^L \Gamma_1(D_m t,x,\xi)w^+(\xi)\mathrm{d}\xi + \int_0^L w^+(\xi)\{\Gamma_1(0,x,\xi) - \Gamma_1(D_m t,x,\xi)\mathrm{e}^{-d_m t}\}\mathrm{d}\xi = w^+(x).
\end{aligned}
$$

Thus, $G[w] \in S$.

Note that the fixed point of G is the solution of (2.6) with the initial condition $w(t,x,y) = \phi(t,x,y)$, $t \in [-r,0]$, $(x,y) \in \bar{\Omega}$. Therefore, it suffices to show that G has a unique fixed point in S. For any $\lambda > 0$, define the norm $\|w\|_\lambda = \sup_{t\in[-r,\infty),(x,y)\in\bar{\Omega}} |w(t,x,y)|\mathrm{e}^{-\lambda t}$. Then $C([-r,\infty)\times\bar{\Omega},\mathbb{R}_+)$ is a Banach space with the norm $\|w\|_\lambda$. For any given $w_1, w_2 \in S$, let $v = w_1 - w_2$. Then $G[w_1](t,\cdot) - G[w_2](t,\cdot) = 0$ for $t \in [-r,r]$. Since b satisfies the Lipschitz condition in $[0,N]$, we may assume that $|b(w_1) - b(w_2)| \le \gamma_0|w_1 - w_2|$ for w_1, w_2 in $[0,N]$. If $t > r$, we have

$$
\begin{aligned}
&|G[w_1](t,x,y) - G[w_2](t,x,y)| \\
&\quad = \mu\int_0^t \mathrm{e}^{-d_m s}\int_{\mathbb{R}}\int_0^L \Gamma(D_m s,x,\xi,y,\zeta)\left\{\int_{\mathbb{R}}\int_0^L \Gamma(\alpha,\xi,z_x,\zeta,z_y)\right. \\
&\qquad \left.\times|b(w_1(t-s-r,z_x,z_y)) - b(w_2(t-s-r,z_x,z_y))|\mathrm{d}z_x\mathrm{d}z_y\right\}\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}s \\
&\quad \le \mu\gamma_0\int_0^t \mathrm{e}^{-d_m s}\int_{\mathbb{R}}\int_0^L \Gamma(D_m s,x,\xi,y,\zeta) \\
&\qquad \times\left\{\int_{\mathbb{R}}\int_0^L \Gamma(\alpha,\xi,z_x,\zeta,z_y)|v(t-s-r,z_x,z_y)|\mathrm{d}z_x\,\mathrm{d}z_y\right\}\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}s,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&|G[w_1](t,x,y) - G[w_2](t,x,y)|\mathrm{e}^{-\lambda t} \\
&\quad \le \mu\gamma_0\int_0^t \mathrm{e}^{-(d_m+\lambda)s-\lambda r}\int_{\mathbb{R}}\int_0^L \Gamma(D_m s,x,\xi,y,\zeta) \\
&\qquad \times\left\{\int_{\mathbb{R}}\int_0^L \Gamma(\alpha,\xi,z_x,\zeta,z_y)|v(t-s-r,z_x,z_y)|\mathrm{e}^{-\lambda(t-s-r)}\mathrm{d}z_x\mathrm{d}z_y\right\}\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}s \\
&\quad \le \mu\gamma_0\int_0^t \mathrm{e}^{-(d_m+\lambda)s-\lambda r}\|v\|_\lambda\mathrm{d}s \\
&\quad \le \frac{\mu\gamma_0}{-(d_m+\lambda)}\mathrm{e}^{-(d_m+\lambda)s}\,\Big|_0^\infty\,\|v\|_\lambda = \frac{\mu\gamma_0}{d_m+\lambda}|\|v\|_\lambda.
\end{aligned}
$$

Choosing $\lambda > 0$ large enough, we have $\frac{\mu\gamma_0}{d_m+\lambda} < 1$, and thus G is a contracting map and has a unique fixed point in S. The conclusion of the lemma follows. ■

We now establish the following comparison theorem.

Lemma 3.2. *Assume that* (P1)–(P2) *hold. Let* $\bar{w}(t,x,y)$, $\underline{w}(t,x,y) \in [0,N]$, *for* $t \in [-r,\infty)$, $(x,y) \in \Omega$, *be the supersolution and subsolution of* (2.6), *respectively. If* $\bar{w}(\theta,x,y) \geq \underline{w}(\theta,x,y)$ *for* $\theta \in [-r,0]$, *then* $\bar{w}(t,x,y) \geq \underline{w}(t,x,y)$ *for all* $t \geq 0$. *Moreover, if* $\bar{w}(\theta,x,y) \geq \underline{w}(\theta,x,y)$ *for* $\theta \in [-r,0]$ *with* $\bar{w}(0,x,y) \not\equiv \underline{w}(0,x,y)$, *then there holds*

$$\bar{w}(t,x,y) > \underline{w}(t,x,y) \quad \textit{for all}\ (t,x,y) \in (0,\infty)\times\bar{\Omega}.$$

Proof. Assume that $\bar{w}$, $\underline{w}$ are a pair of supersolution and subsolution of (2.6) with $\bar{w}(t,x,y)$, $\underline{w}(t,x,y) \in [0,N]$ for $t \in [-r,\infty)$ and $(x,y) \in \Omega$, respectively. We have from Corollary 5 in [14] and the fact $\bar{w}(\theta,x,y) \geq \underline{w}(\theta,x,y)$ for $(\theta,x,y) \in [-r,0]\times\Omega$, that the solutions of (3.5) satisfy

$$0 \leq w(t,\cdot;\underline{w}_0) \leq w(t,\cdot;\bar{w}_0) \leq N, \quad t \geq 0.$$

Again applying Corollary 5 in [14] with $[v^+(t,\cdot) = N, v^-(t,\cdot) = \underline{w}(t,\cdot)]$ and $[v^+(t,\cdot) = \bar{w}(t,\cdot), v^-(t,\cdot) = 0]$ respectively, we obtain

$$\begin{aligned}&\underline{w}(t,\cdot) \leq w(t,\cdot;\underline{w}_0) \leq N, \quad t \geq 0,\\ &0 \leq w(t,\cdot;\bar{w}_0) \leq \bar{w}(t,\cdot), \quad t \geq 0.\end{aligned}$$

Combining the above three inequalities, we have $\underline{w}(t,x,y) \leq \bar{w}(t,x,y)$ for all $(t,x,y) \in (0,\infty)\times\bar{\Omega}$.

Let $v = \bar{w} - \underline{w}$. Then we have already known that $v(t,x,y) \geq 0$ for all $(t,x,y) \in (0,\infty)\times\bar{\Omega}$. We have from the definition of supersolution and subsolution, the monotonicity of b on $[0,N]$, and the fact $\bar{T}\mathbb{Y}^+ \subset \mathbb{Y}^+$ that

$$\begin{aligned}v(t) &\geq \bar{T}(t)v(0) + \int_0^t \bar{T}(t-s)[\bar{F}(\bar{w}_s) - \bar{F}(\underline{w}_s)]\mathrm{d}s\\ &\geq \bar{T}(t)v(0) + \int_0^t \bar{T}(t-s)[d_m(\underline{w}(s) - \bar{w}(s))]\mathrm{d}s\\ &= \bar{T}(t)v(0) - d_m\int_0^t \bar{T}(t-s)v(s)\mathrm{d}s, \quad t \geq 0.\end{aligned}$$

Therefore, we have

$$v(t) \geq \bar{T}(t)v(0) - d_m\int_0^t \bar{T}(t-s)v(s)\mathrm{d}s, \quad t \geq 0.$$

Define $z(t) = \mathrm{e}^{-d_m t}\bar{T}(t)v(0)$, $t \geq 0$. Then $z(t)$ satisfies

$$z(t) = \bar{T}(t)z(0) - d_m\int_0^t \bar{T}(t-s)z(s)\mathrm{d}s, \quad t \geq 0.$$

By Proposition 3 in [14] with $v^-(t) = z(t)$, $v^+(t) = +\infty$, $S(t,s) = \bar{T}(t-s)$, $B(t,\phi) = B^-(t,\phi) = -d_m\phi(0)$, we get $v(t) \geq z(t)$ for $t \geq 0$, that is

$$v(t) \geq \mathrm{e}^{-d_m t}\bar{T}(t)v(0) = \mathrm{e}^{-d_m t}\int_{\mathbb{R}}\int_0^L \Gamma(D_m t,x,\xi,y,\zeta)v(0,\xi,\zeta)\mathrm{d}\xi\mathrm{d}\zeta \quad \text{for } t \geq 0.$$

Thus it follows that $v(t) > 0$ for $t > 0$ if $v(0,x,y) \not\equiv 0$ on Ω. ■

3.2. Monotone and subhomogeneous semiflows

In the following, we equip $\mathcal{C}$ with the compact open topology, that is, $\phi^n \to \phi$ in $\mathcal{C}$ if and only if the sequence of functions $\phi^n(\theta,x,y)$ converges to $\phi(\theta,x,y)$ uniformly for y in every compact set. Moreover, we define the norm $\|\phi\|$ by

$$\|\phi\| = \sum_{k=0}^{\infty}\frac{\max_{(\theta,x)\in[-r,0]\times[0,L],|y|\leq k}|\phi(\theta,x,y)|}{2^k},$$

and let $\rho(\phi, \psi)$ be the metric on $\mathcal{C}$ induced by the norm $\|\phi\|$. Note that $\mathcal{C}_{w^+}$ is a complete subset of $\mathcal{C}$ under this norm. We also equip $\bar{\mathcal{C}}$ with the maximum norm $\|\cdot\|$ such that $\bar{\mathcal{C}}$ is a Banach space.

Recall that a family of operators $\{\Lambda_t\}_{t=0}^{\infty}$ is said to be a semiflow on a metric space (X, ρ) with metric ρ provided Λ_t has the following properties:

(i) $\Lambda_0(v) = v, \forall v \in X$.
(ii) $\Lambda_{t_1}[\Lambda_{t_2}[v]] = \Lambda_{t_1+t_2}[v], \forall t_1, t_2 \geq 0, v \in X$.
(iii) $\Lambda(t, v) := \Lambda_t(v)$ is continuous in (t, v) on $[0, \infty) \times X$.

It is easy to see that the property (iii) holds if $\Lambda(\cdot, v)$ is continuous on $[0, +\infty)$ for each $v \in X$, and $\Lambda(t, \cdot)$ is uniformly continuous for t in any bounded intervals in the sense that for any $v_0 \in X$, bounded interval I and $\epsilon > 0$, there exists $\delta = \delta(v_0, I, \epsilon) > 0$ such that if $\rho(v, v_0) < \delta$, then $\rho(\Lambda_t[v], \Lambda_t[v_0]) < \epsilon$ for all $t \in I$.

Define a group of maps $Q_t(\phi) : \mathcal{C} \to \mathcal{C}$ as follows:

$$[Q_t(\phi)](\theta, x, y) = w_t(\theta, x, y; \phi), \ \forall \theta \in [-r, 0], (x, y) \in \bar{\Omega},$$

where $w(t, x, y; \phi)$ is the solution of (2.6) with an initial function ϕ. Then we have the following.

Theorem 3.1. *Assume that* (P1)–(P2) *hold. Then Q_t is a monotone and subhomogeneous semiflow on $\mathcal{C}_{w^+}$.*

Proof. Clearly, Q_t satisfies property (i) of semiflow. The semiflow property (ii) follows from (3.5) and the properties of $\bar{T}(t)$ (see [9,15]). Given $\phi \in \mathcal{C}_{w^+}$, it then follows from (3.5) and the semigroup theory that $Q_t(\phi) = w(t + \cdot, \cdot; \phi)$ is continuous in $t \in \mathbb{R}_+$ with respect to the compact open topology.

Let $w(t, x, y)$ and $\bar{w}(t, x, y)$ be solutions of (2.6) with the initial function $\phi(\theta, \cdot)$ and $\bar{\phi}(\theta, \cdot) \in \mathcal{C}_{w^+}$ respectively. Then we have the following

Claim. For any $\epsilon > 0$ and $t_0 > 0$, there exist $\delta > 0$ and $M > 0$ such that $|w(t, x, z) - \bar{w}(t, x, z)| < \epsilon$ for $(t, x) \in [0, t_0] \times [0, L]$ whenever $|\phi(\theta, x, y) - \bar{\phi}(\theta, x, y)| < \delta$ for $\theta \in [-r, 0]$, $(x, y) \in [0, L] \times [z - M, z + M]$ with some $z \in \mathbb{R}$.

By the spatial translation invariance of Eq. (2.6), we only need to verify the claim for the case when $z = 0$. Indeed, let $v = w - \bar{w}$. Then $v(t, x, y)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = D_m\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) - d_m v + \mu \displaystyle\int_{\mathbb{R}}\int_0^L \Gamma(\alpha, x, z_x, y, z_y) \\ \qquad \times [b(w(t-r, z_x, z_y)) - b(\bar{w}(t-r, z_x, z_y))]\mathrm{d}z_x\,\mathrm{d}z_y, \quad \forall t > 0,\ (x, y) \in \Omega, \\ Bv(t, x, y) = 0, \quad \forall t \geq 0,\ x = 0, L,\ y \in \mathbb{R}, \\ v(\theta, x, y) = \phi(\theta, x, y) - \bar{\phi}(\theta, x, y), \quad \forall \theta \in [-r, 0], (x, y) \in \Omega. \end{cases} \tag{3.6}$$

Let $\bar{b} = b(N)$. First we assume that $\phi \geq \bar{\phi}$, and thus $v = w - \bar{w} \geq 0$. First note from (2.4) that

$$\int_{\mathbb{R}} \mathrm{e}^{-\frac{(y-z_y)^2}{4\alpha}}\mathrm{d}z_y = \int_{\mathbb{R}} \mathrm{e}^{-\frac{s^2}{4\alpha}}\mathrm{d}s = \sqrt{4\alpha\pi}.$$

Choose $\bar{M} > 0$ such that

$$\int_{-\infty}^{-\bar{M}} \mathrm{e}^{-\frac{s^2}{4\alpha}}\mathrm{d}s + \int_{\bar{M}}^{\infty} \mathrm{e}^{-\frac{s^2}{4\alpha}}\mathrm{d}s < \frac{\epsilon}{4\mu\bar{b}}.$$

If $r > 0$, then for any $\epsilon > 0$ and $t_0 = r$, we can choose $\delta > 0$ such that

$$\int_{-\infty}^{-2\bar{M}} \mathrm{e}^{-\frac{(y-z_y)^2}{4\alpha}}\mathrm{d}z_y + \int_{2\bar{M}}^{\infty} \mathrm{e}^{-\frac{(y-z_y)^2}{4\alpha}}\mathrm{d}z_y < \frac{\epsilon}{4\mu\bar{b}} \quad \text{for } y \in [-\bar{M}, \bar{M}] \text{ uniformly}$$

and

$$|b(w(t-r, x, y)) - b(\bar{w}(t-r, x, y))| < \frac{\epsilon}{2\mu}, \quad \text{for } t \in [0, r], (x, y) \in [0, L] \times [-2\bar{M}, 2\bar{M}],$$

when $|\phi(\theta,x,y)-\bar{\phi}(\theta,x,y)|<\delta$ for $\theta\in[-r,0]$ and $(x,y)\in[0,L]\times[-2\bar{M},2\bar{M}]$. Therefore, we have from (2.3), (2.4) and the above conclusion that

$$\begin{aligned}
&\mu\int_{\mathbb{R}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)[b(w(t-r,z_x,z_y))-b(\bar{w}(t-r,z_x,z_y))]\mathrm{d}z_x\mathrm{d}z_y\\
&\quad=\mu\int_{-\infty}^{-2\bar{M}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)[b(w(t-r,z_x,z_y))-b(\bar{w}(t-r,z_x,z_y))]\mathrm{d}z_x\mathrm{d}z_y\\
&\qquad+\mu\int_{2\bar{M}}^{\infty}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)[b(w(t-r,z_x,z_y))-b(\bar{w}(t-r,z_x,z_y))]\mathrm{d}z_x\mathrm{d}z_y\\
&\qquad+\mu\int_{-2\bar{M}}^{2\bar{M}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)[b(w(t-r,z_x,z_y))-b(\bar{w}(t-r,z_x,z_y))]\mathrm{d}z_x\mathrm{d}z_y\\
&\quad\le 2\mu\bar{b}\left[\int_{-\infty}^{-2\bar{M}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)\mathrm{d}z_x\mathrm{d}z_y+\int_{2\bar{M}}^{\infty}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)\mathrm{d}z_x\mathrm{d}z_y\right]\\
&\qquad+\frac{\epsilon}{2}\int_{-2\bar{M}}^{2\bar{M}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)\mathrm{d}z_x\mathrm{d}z_y\\
&\quad\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\quad\text{for }(x,y)\in[0,L]\times[-\bar{M},\bar{M}]\text{ uniformly,}
\end{aligned}$$

which, together with (3.6), leads to

$$\begin{cases}
\dfrac{\partial v}{\partial t}\le D_m\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)-d_m v+\epsilon, & t\in[0,r],\ (x,y)\in(0,L)\times(-\bar{M},\bar{M}),\\
Bv(t,x,y)=0, \quad t\in[0,r],\ x=0,L,\ y\in\mathbb{R}, &\\
v(\theta,x,y)=\phi(\theta,x,y)-\bar{\phi}(\theta,x,y)\ge 0, \quad \theta\in[-r,0],\ (x,y)\in\Omega. &
\end{cases}\tag{3.7}$$

It is easy to verify that the following function

$$u(t,x,y)=\exp\{-d_m t\}\int_{\mathbb{R}}\int_0^L \Gamma(D_m t,x,\xi,y,\zeta)v(0,\xi,\zeta)\mathrm{d}\xi\mathrm{d}\zeta+\frac{\epsilon}{d_m},$$

satisfies

$$\begin{aligned}
&\frac{\partial u}{\partial t}=D_m\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-d_m u+\epsilon, \quad t\in[0,r],\ (x,y)\in(0,L)\times(-\bar{M},\bar{M}),\\
&Bu(t,x,y)\ge 0, \quad t\in[0,r],\ x=0,L,\ y\in\mathbb{R},\\
&u(0,x,y)=v(0,x,y)+\frac{\epsilon}{d_m}, \quad (x,y)\in\Omega.
\end{aligned}$$

Therefore, we have from (3.7) and the comparison theorem of linear parabolic partial differential equations that the solution of (3.6) satisfies

$$\begin{aligned}
v(t,x,y)\le&\exp\{-d_m t\}\int_{\mathbb{R}}\int_0^L \Gamma(D_m t,x,\xi,y,\zeta)v(0,\xi,\zeta)\mathrm{d}\xi\mathrm{d}\zeta\\
&+\frac{\epsilon}{d_m}\ \text{for } t\in[0,r],\ (x,y)\in[0,L]\times[-\bar{M},\bar{M}],
\end{aligned}$$

and thus

$$v(t,x,0)\le\exp\{-d_m t\}\int_{\mathbb{R}}\int_0^L \Gamma_1(D_m t,x,\xi)\Gamma_2(D_m t,0,\zeta)v(0,\xi,\zeta)\mathrm{d}\xi\mathrm{d}\zeta+\frac{\epsilon}{d_m}.$$

Note that

$$\int_{\mathbb{R}}\Gamma_2(D_m t,0,\zeta)\mathrm{d}\zeta=\frac{1}{\sqrt{4tD_m\pi}}\int_{\mathbb{R}}\mathrm{e}^{-\frac{(0-\zeta)^2}{4D_m t}}\mathrm{d}\zeta=1.$$

Similar to the above discussion, we can choose $M:=2\bar{M}>0$ and $\delta>0$ (take the larger M and the smaller δ if necessary) such that $v(t,x,0)<\frac{2\epsilon}{d_m}$ for $t\in[0,r]$ and $x\in[0,L]$, when $|\phi(\theta,x,y)-\bar{\phi}(\theta,x,y)|<\delta$ for $\theta\in[-r,0]$ and $(x,y)\in[0,L]\times[-M,M]$.

If $r > 0$, but $\phi \not\geq \bar{\phi}$ on $[-r, 0]$, we define

$$\hat{\phi} = \max\{\phi, \bar{\phi}\}, \quad \tilde{\phi} = \min\{\phi, \bar{\phi}\},$$

and assume that $\hat{w}$ and $\tilde{w}$ are solutions of (2.6), respectively, with the initial values $\hat{\phi}$ and $\tilde{\phi}$, then in view of Lemma 3.2, we have $\tilde{w} \leq w$, $\bar{w} \leq \hat{w}$. Note that

$$|w(t,x,y) - \bar{w}(t,x,y)| \leq \hat{w}(t,x,y) - \tilde{w}(t,x,y) \text{ for } (t,x,y) \in \mathbb{R}_+ \times [0,L] \times \mathbb{R}.$$

Thus, the claim is true for $t_0 = r$.

For any $t \in [mr, (m+1)r]$, we have $Q_t = Q_{t-mr} Q_{mr}$. Thus $Q_t(\cdot)$ is uniformly continuous for $t \in [mr, (m+1)r]$, which yields that $Q_t(\cdot)$ is uniformly continuous for t on any bounded interval. Therefore it follows that $Q_t(\phi)$ is continuous in $(t, \phi) \in \mathbb{R}_+ \times \mathcal{C}_{w^+}$ if $r > 0$.

It is obvious that b is Lipschitz in $[0, N]$. Suppose that the Lipschitz constant is γ_0. If $r = 0$, then we have from (3.6) that

$$\frac{\partial v}{\partial t} \leq D_m \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - d_m v + \mu\gamma_0 \int_{\mathbb{R}} \int_0^L \Gamma(\alpha, x, z_x, y, z_y) v(t, z_x, z_y) \mathrm{d}z_x \, \mathrm{d}z_y. \tag{3.8}$$

Consider the linear boundary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_m \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) - d_m u + \mu\gamma_0 \displaystyle\int_{\mathbb{R}} \int_0^L \Gamma(D_m t, x, z_x, y, z_y) u(t, z_x, z_y) \mathrm{d}z_x \, \mathrm{d}z_y, \ t \geq 0, (x,y) \in \Omega, \\ Bu(t,x,y) = 0, \ t \geq 0, \ x = 0, L, \ y \in \mathbb{R}, \\ u(0,x,y) = \phi(0,x,y) - \bar{\phi}(0,x,y). \end{cases} \tag{3.9}$$

Without loss of generality, we can assume that $\phi - \bar{\phi} \geq 0$, and then $u \geq 0$. Applying the Fourier transformation $V(t,x,\omega) = \int_{\mathbb{R}} u(t,x,y) \mathrm{e}^{-\mathrm{i}\omega y} \mathrm{d}y$ to (3.9) and using the formula

$$\int_{\mathbb{R}} \mathrm{e}^{-q^2 y^2} \mathrm{e}^{-\mathrm{i}\omega y} \mathrm{d}y = \frac{\sqrt{\pi}}{q} \mathrm{e}^{-\frac{\omega^2}{4q^2}},$$

we obtain

$$\frac{\partial V}{\partial t} = D_m \left(\frac{\partial^2 V}{\partial x^2} - \omega^2 V \right) - d_m V + \gamma_0 \mu \mathrm{e}^{-\omega^2 \alpha} \int_0^L \Gamma_1(\alpha, x, z_x) V(t, z_x, \omega) \mathrm{d}z_x. \tag{3.10}$$

Let $V(t,x,\omega) = \sum_{n=1}^{\infty} \Psi_n(t,\omega) \Phi_n(x)$, $b_n := \mu\gamma_0 \mathrm{e}^{-\mu_n \alpha}$, $a_n(\omega) := b_n \mathrm{e}^{-\omega^2 \alpha}$, where μ_n and $\Phi_n(x)$ are defined in Section 2, and $\Psi_n(t,\omega)$ are to be determined later. Now we have from (2.10) and (3.10) that

$$\begin{aligned} \frac{\partial \Psi_n}{\partial t} \Phi_n(x) &= -[D_m(\mu_n + \omega^2) + d_m] \Psi_n(t,\omega) \Phi_n(x) + \mu\gamma_0 \mathrm{e}^{-\omega^2 \alpha} \Psi_n(t,\omega) \int_0^L \Gamma_1(\alpha, x, z_x) \Phi_n(z_x) \mathrm{d}z_x \\ &= -[D_m(\mu_n + \omega^2) + d_m] \Psi_n(t,\omega) \Phi_n(x) + a_n(\omega) \Psi_n(t,\omega) \Phi_n(x) \end{aligned}$$

for $t \geq 0$ and $\omega \in \mathbb{R}$.

In what follows, we only consider the (NBC) case, since the argument for other cases are similar. We have

$$\begin{aligned} &\Psi_n(t,\omega) = \Psi_n(0,\omega) \exp\{-[D_m(\mu_n + \omega^2) + d_m - a_n(\omega)]t\}, \\ &V(t,x,\omega) = \sum_{n=1}^{\infty} \Psi_n(t,\omega) \Phi_n(x) = \sum_{n=1}^{\infty} \Psi_n(0,\omega) \Phi_n(x) \exp\{-[D_m(\mu_n + \omega^2) + d_m - a_n(\omega)]t\}, \\ &V(0,x,\omega) = \sum_{n=1}^{\infty} \Psi_n(0,\omega) \Phi_n(x). \end{aligned}$$

The last expression gives the Fourier series of $V(0,x,\omega)$. Note $\mu_1 = 0$ for (NBC). Thus we have

$$\begin{aligned} &\Psi_1(0,\omega) = \frac{1}{L} \int_0^L V(0, z_x, \omega) \mathrm{d}z_x = \frac{1}{L} \int_0^L \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega z_y} u(0, z_x, z_y) \mathrm{d}z_y \mathrm{d}z_x, \\ &\Psi_n(0,\omega) = \frac{2}{L} \int_0^L V(0, z_x, \omega) \cos(\sqrt{\mu_n} z_x) \mathrm{d}z_x = \frac{2}{L} \int_0^L \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega z_y} u(0, z_x, z_y) \cos(\sqrt{\mu_n} z_x) \mathrm{d}z_y \mathrm{d}z_x, \end{aligned}$$

where $n = 2, 3, \ldots$. Note that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega(y-z_y)-\omega^2 D_m t} d\omega = \sqrt{\frac{\pi}{D_m t}} \mathrm{e}^{-\frac{(y-z_y)^2}{4D_m t}}.$$

Therefore, by using the inverse Fourier transformation, we know that the solution of (3.9) satisfies

$$\begin{aligned}
u(t,x,y) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y\omega} V(t,x,\omega) d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y\omega} \sum_{n=1}^{\infty} \Psi_n(t,\omega)\Phi_n(x) d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega y} \sum_{n=1}^{\infty} \Psi_n(0,\omega)\Phi_n(x) \exp\{-[D_m(\mu_n+\omega^2)+d_m-a_n(\omega)]t\} d\omega \\
&= \frac{2}{2L\pi} \int_{\mathbb{R}} \sum_{n=2}^{\infty} \left\{ \mathrm{e}^{\mathrm{i}\omega y} \int_0^L \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega z_y} u(0,z_x,z_y) \cos(\sqrt{\mu_n} z_x) \mathrm{d}z_y \mathrm{d}z_x \right\} \\
&\quad \times \Phi_n(x) \exp\{-[D_m(\mu_n+\omega^2)+d_m-a_n(\omega)]t\} \mathrm{d}\omega \\
&\quad + \frac{1}{2L\pi} \int_{\mathbb{R}} \left\{ \mathrm{e}^{\mathrm{i}\omega y} \int_0^L \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega z_y} u(0,z_x,z_y) \mathrm{d}z_y \mathrm{d}z_x \right\} \exp\{-[D_m\omega^2+d_m-a_1(\omega)]t\} \mathrm{d}\omega \\
&\le \frac{1}{\pi L\sqrt{D_m t\pi}} \sum_{n=1}^{\infty} \left\{ \int_{\mathbb{R}} \int_0^L \mathrm{e}^{-\frac{(y-z_y)^2}{4D_m t}} u(0,z_x,z_y) \cos(\sqrt{\mu_n} z_x) \mathrm{d}z_x \mathrm{d}z_y \right\} \Phi_n(x) \exp\{-(D_m\mu_n+d_m-b_n)t\}
\end{aligned}$$

for $t \in [0, t_0]$. Let $m(t) = \sum_{n=1}^{\infty} \exp\{-(D_m\mu_n - b_n)t\}$, $m_{t_0} = \max_{0\le t\le t_0} m(t)$. By using the comparison theorem, we have from (3.8)–(3.9) that

$$v(t,x,0) \le \frac{\mathrm{e}^{-d_m t} m_{t_0}}{\pi L\sqrt{tD_m\pi}} \int_{\mathbb{R}} \int_0^L \mathrm{e}^{-\frac{(0-z_y)^2}{4D_m t}} v(0,z_x,z_y) \mathrm{d}z_x\, \mathrm{d}z_y, \quad \forall (t,x) \in [0,t_0] \times [0,L].$$

As argued for the case $r > 0$, we conclude that the claim holds.

By the claim above, we see that $Q_t(\phi)$ is continuous in ϕ uniformly for t in any bounded interval. It then follows that $Q_t(\phi)$ is continuous in $(t,\phi) \in \mathbb{R}_+ \times \mathcal{C}_{w^+}$ with respect to the compact open topology, i.e., Q_t satisfies property (iii) of semiflow above. Consequently, Q_t is continuous semiflow on $\mathcal{C}_{w^+}$.

Clearly, Lemma 3.2 implies that Q_t is monotone on $\mathcal{C}_{w^+}$. It remains to prove that Q_t is subhomogeneous in $\mathcal{C}_{w^+}$ in the sense that $Q_t[\gamma\phi] \ge \gamma Q_t[\phi]$ for all $\gamma \in [0,1]$ and $\phi \in \mathcal{C}_{w^+}$. In fact, let $\bar{w} = w(t,x,y;\gamma\phi)$, $\underline{w} = \gamma w(t,x,y;\phi)$. We have $\bar{w}_t, \underline{w}_t \in \mathcal{C}_{w^+}$ and

$$\begin{aligned}
&\frac{\partial \bar{w}}{\partial t} = D_m \left[\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2}\right] - d_m \bar{w} \\
&\qquad + \mu \int_{\mathbb{R}} \int_0^L \Gamma(\alpha,x,z_x,y,z_y) b(\bar{w}(t-r,z_x,z_y)) \mathrm{d}z_x \mathrm{d}z_y, \quad t>0, (x,y) \in \Omega, \\
&B\bar{w}(t,x,y) = 0, \quad t \ge 0, x = 0, L, \ y \in \mathbb{R}, \\
&\bar{w}(\theta,x,y) = \gamma\phi(\theta,x,y), \quad t \in [-r,0], (x,y) \in \Omega,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \underline{w}}{\partial t} = D_m \left[\frac{\partial^2 \underline{w}}{\partial x^2} + \frac{\partial^2 \underline{w}}{\partial y^2}\right] - d_m \underline{w} + \mu \int_{\mathbb{R}} \int_0^L \Gamma(\alpha,x,z_x,y,z_y) \gamma b(w(t-r,z_x,z_y)) \mathrm{d}z_x \mathrm{d}z_y \\
&\qquad \le D_m \left[\frac{\partial^2 \underline{w}}{\partial x^2} + \frac{\partial^2 \underline{w}}{\partial y^2}\right] - d_m \underline{w} \\
&\qquad + \mu \int_{\mathbb{R}} \int_0^L \Gamma(\alpha,x,z_x,y,z_y) b(\underline{w}(t-r,z_x,z_y)) \mathrm{d}z_x \mathrm{d}z_y, \quad t>0, (x,y) \in \Omega, \\
&B\underline{w}(t,x,y) = 0, \quad t \ge 0, x = 0, L, \ y \in \mathbb{R}, \\
&\underline{w}(\theta,x,y) = \gamma\phi(\theta,x,y), \quad t \in [-r,0], (x,y) \in \Omega.
\end{aligned}$$

Therefore, $\bar{w}$ and $\underline{w}$ are supersolution and subsolution of (2.6), respectively, with $\bar{w}(\theta,x,y) = \underline{w}(\theta,x,y)$ for $\theta \in [-r,0]$, $(x,y) \in \Omega$. Again, Lemma 3.2 yields the subhomogeneous property of Q_t in $\mathcal{C}_{w^+}$. ■

3.3. Asymptotic speed of spread and traveling waves

Define the reflection operator $\mathcal{R}$ by $\mathcal{R}[\phi](\theta, x, y) = \phi(\theta, x, -y)$. Given $z \in \mathbb{R}$, define the translation operator T_z by $T_z[\phi](\theta, x, y) = \phi(\theta, x, y - z)$.

To study the asymptotic speed of spread and traveling waves, we will apply the theorems in [12], which require some hypotheses on a map. Let $\beta \in \bar{\mathcal{C}}$ with $\beta \gg 0$ and $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$. The following hypotheses on Q are needed.

(A1) $Q[\mathcal{R}[\phi]] = \mathcal{R}[Q[\phi]]$, $T_z[Q[\phi]] = Q[T_z[\phi]]$ for $z \in \mathbb{R}$;
(A2) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology;
(A3) One of the following two properties holds:
  (a) $\{Q[\phi](\cdot, y) : \phi \in \mathcal{C}_\beta, y \in \mathbb{R}\}$ is a precompact subset of $\bar{\mathcal{C}}$;
  (b) There is a non-negative number $\zeta < r$ such that $Q[\phi](\theta, x, y) = \phi(\theta + \zeta, x, y)$ for $-r \le \theta \le -\zeta$, the operator
$$S[\phi](\theta, x, y) = \begin{cases} \phi(\theta, x, y), & -r \le \theta < -\zeta, \\ Q[\phi](\theta, x, y), & -\zeta \le \theta \le 0, \end{cases} \tag{3.11}$$
  is continuous on $\mathcal{C}_\beta$, and $\{S[\phi](\cdot, y) : \phi \in \mathcal{C}_\beta, y \in \mathbb{R}\}$ is a precompact subset of $\bar{\mathcal{C}}$;
(A4) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotone (order preserving) in the sense that $Q[\phi] \ge Q[\psi]$ whenever $\phi \ge \psi$ in $\mathcal{C}_\beta$;
(A5) $Q : \bar{\mathcal{C}}_\beta \to \bar{\mathcal{C}}_\beta$ admits exactly two fixed points 0 and β, and for any positive number ϵ, there is $\alpha \in \bar{\mathcal{C}}_\beta$ with $\|\alpha\| < \epsilon$ such that $Q[\alpha] \gg \alpha$;
(A6) One of the following two properties holds:
  (a) $Q[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$;
  (b) There is a non-negative number $\zeta < r$ such that $Q[\phi](\theta, x, y) = \phi(\theta + \zeta, x, y)$ for $-r \le \theta \le -\zeta$, the operator $S[\phi](\theta, x, y)$ defined by (3.11) is continuous on $\mathcal{C}_\beta$, and $S[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

Lemma 3.3. *Assume that* (P1)–(P3) *hold. Then for each $t > 0$, the map Q_t satisfies* (A1)–(A6) *with $\beta = w^+$.*

Proof. By (3.4), we know that Q_t satisfies (A1). By Theorem 3.1, we obtain that Q_t satisfies (A2) and (A4). We now verify that Q_t satisfies (A6). For $t \ge 0$, define

$$L(t)[\phi](\theta, x, y) = \begin{cases} \phi(t + \theta, x, y) - \phi(0, x, y), & t + \theta < 0, (x, y) \in \bar{\Omega}, \\ 0, & t + \theta \ge 0, -r \le \theta \le 0, (x, y) \in \bar{\Omega}. \end{cases}$$

Then $L(t) = 0$ for $t \ge r$. Let $S(t) = Q_t - L(t)$, $t \ge 0$. We have from the smoothness of the semigroup generated from the heat equation that Q_t satisfies (A6) (a) for $t \ge r$. If $t \in (0, r)$, let $\zeta = t$, then we have $Q_t[\phi](\theta, x, y) = \phi(\theta + t, x, y)$, $\forall \theta \in [-r, -t]$, and

$$S(t)[\phi](\theta, x, y) = \begin{cases} \phi(0, x, y), & -r \le \theta < -t, \\ Q_t[\phi](\theta, x, y), & -t \le \theta \le 0. \end{cases}$$

We obtain from the above expression that $S(t)[\phi]$ is continuous on $\mathcal{C}_{w^+}$, and we can show that $S(t)[\mathcal{C}_{w^+}]$ is precompact in $\mathcal{C}_{w^+}$ by using a method similar to Theorem 6.1 in [10]. Therefore, Q_t satisfies (A6), and this yields (A3).

Let $\hat{Q}_t$ express the restriction of Q_t on $\bar{\mathcal{C}}$. Then $\hat{Q}_t$ is the semiflow generated from the initial boundary value problem:

$$\begin{cases} \dfrac{\partial w}{\partial t} = D_m \dfrac{\partial^2 w}{\partial x^2} - d_m w + \mu \displaystyle\int_0^L \Gamma_1(\alpha, x, z_x) b(w(t - r, z_x)) \mathrm{d}z_x, & t > 0, x \in (0, L), \\ Bw(t, x) = 0, \quad t \ge 0, \ x = 0, L, \\ w(\theta, x) = \phi(\theta, x), \quad \theta \in [-r, 0], x \in [0, L]. \end{cases} \tag{3.12}$$

As discussed in Section 2, we know that the boundary value problem in (3.12) has two equilibria $w = 0$ and $w = w^+(x)$. Furthermore, similar to Lemma 3.2, we conclude that $\hat{Q}_t$ is strongly monotone in $[0, N]_{\mathbb{Y}} = \{\phi \in \mathbb{Y}^+ : 0 \le \phi \le N\}$. According to (P3), (2.12) and Theorem 2.1, $w^+(x)$ is globally attractive, and $w = 0$ is unstable. By the Dancer–Hess connecting orbit lemma (see, e.g., [32, page 39]), the semiflow $\hat{Q}_t$ admits a strongly monotone entire orbit connecting 0 and $w^+(x)$. Thus assumption (A5) holds for each map Q_t, $\forall t > 0$. ■

By Lemma 3.3 and [12, Theorems 2.11 and 2.15], it follows that the map Q_1 has an asymptotic speed of spread $c^* > 0$ in the sense that:

$$\forall c > c^*, \lim_{n\to\infty,|y|\geq nc} Q_1^n(\phi)(\theta, x, y) = 0 \text{ uniformly for } \theta \in [-r, 0],\ x \in [0, L],$$
$$\forall 0 < c < c^*, \lim_{n\to\infty,|y|\leq nc} Q_1^n(\phi)(\theta, x, y) = w^+(x) \text{ uniformly for } \theta \in [-r, 0],\ x \in [0, L],$$

where

$$\lim_{n\to\infty,|y|\geq nc} Q_1^n(\phi)(\theta, x, y) = \lim_{n\to\infty} \sup_{|y|\geq nc} Q_1^n(\phi)(\theta, x, y) = \lim_{n\to\infty} \inf_{|y|\geq nc} Q_1^n(\phi)(\theta, x, y),$$

and $\lim_{n\to\infty,|y|\leq nc} Q_1^n(\phi)(\theta, x, y)$ is defined similarly (see [21]). The following result shows that c^* is also the asymptotic speed of spread for the solutions of (2.6).

Theorem 3.2. *Assume that* (P1)–(P3) *hold. Let c^* be the asymptotic speed of spread of Q_1. Then the following statements are valid:*

(i) *If $\phi \in \mathcal{C}_{w^+}$ with $0 \leq \phi \ll w^+$, and $\phi(\cdot, x, y) = 0$ for $x \in [0, L]$ and y outside a bounded interval, then for each $c > c^*$, every solution of* (2.6) *satisfies $\lim_{t\to\infty,|y|\geq tc} w(t, x, y; \phi) = 0$ uniformly for $x \in [0, L]$;*

(ii) *If $\phi \in \mathcal{C}_{w^+}$ with $\phi(0, \cdot) \not\equiv 0$, then for any $0 < c < c^*$, every solution of* (2.6) *satisfies $\lim_{t\to\infty,|y|\leq tc} w(t, x, y; \phi) = w^+$ uniformly for $x \in [0, L]$.*

Proof. Conclusion (i) is a direct consequence from the first part of Theorem 2.17 of [12]. To obtain the conclusion (ii), we use our Lemma 3.2 combined with Theorem 2.17 of [12]. We know from Lemma 3.2 that for any $\phi \in \mathcal{C}_{w^+}$ with $\phi(0, \cdot) \not\equiv 0$, we have $w(t, x, y; \phi) \gg 0$ for $t > 0$, $(x, y) \in \bar{\Omega}$. Since Q_t is subhomogeneous, r_σ in Theorem 2.17 of [12] can be chosen to be independent of $\sigma > 0$. Let $r_\sigma = \tau$. Fixing $t_0 > 0$, we see that $w(t_0, x, y; \phi) \gg 0$ for $(x, y) \in \bar{\Omega}$. So there exists a $\sigma \in \mathbb{R}, \sigma > 0$ such that $w(t_0, x, y; \phi) \gg \sigma$ for $(x, y) \in [0, L] \times [-\tau, \tau]$. Thus we can take $w(t_0, x, y; \phi)$ as a new initial data, and use the second part of Theorem 2.17 from [12] to obtain our second conclusion. The proof is complete. ■

A traveling wave solution of (2.6) is a solution with the form $w(t, x, y) = \varphi(x, y - ct)$, where $c > 0$ is the wave speed. Let $s = y - ct$, then the profile function of traveling wave is $\varphi(x, s)$. We are concerned with the monotonic traveling waves which connects the two equilibria w^+ and 0. According to our Lemma 3.3 and [12, Theorem 4.3-4.4], we have the following theorem about the existence of traveling waves for (2.6).

Theorem 3.3. *Assume that* (P1)–(P3) *hold. Let c^* be the asymptotic speed of spread of Q_1. Then the following statements are valid.*

(i). *For each $c \geq c^*$, system* (2.6) *has a traveling wave $\varphi(x, s)$ connecting $w^+(x)$ to 0 such that $\varphi(x, s)$ is continuous and non-increasing in $s \in \mathbb{R}$;*

(ii). *For each $c \in (0, c^*)$, system* (2.6) *has no traveling wave $\varphi(x, s)$ connecting $w^+(x)$ to 0.*

Remark 3.1. Theorems 3.2 and 3.3 thus describe the fact that asymptotic speed of spread coincides with the minimal speed of monotone traveling waves.

In what follows, we need the condition $b(w) \leq b'(0)w$ for $w \in [0, N]$. We also remark that assumption P1 yields $b(w) \leq b'(0)w$ for $w \in \mathbb{R}$.

In order to compute c^*, we consider the linearized boundary value problem

$$\begin{aligned}
&\frac{\partial w}{\partial t} = D_m \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] - d_m w + \mu b'(0) \int_{\mathbb{R}} \int_0^L \Gamma(\alpha, x, z_x, y, z_y) w(t - r, z_x, z_y) \mathrm{d}z_x \mathrm{d}z_y, \\
&\qquad t > 0, (x, y) \in \Omega, \\
&Bw(t, x, y) = 0, \quad t \geq 0,\ x = 0, L,\ y \in \mathbb{R}.
\end{aligned} \tag{3.13}$$

Assume that $\{M_t\}_0^\infty$ is the linear solution map defined by (3.13). Let $w(t,x,y) = \eta(t,x)\mathrm{e}^{-\nu y}$. Note

$$\begin{aligned}
&\mu b'(0)\int_{\mathbb{R}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)w(t-r,z_x,z_y)\mathrm{d}z_x\,\mathrm{d}z_y\\
&\quad= \mu b'(0)\int_{\mathbb{R}}\int_0^L \Gamma(\alpha,x,z_x,y,z_y)\mathrm{e}^{-\nu z_y}\eta(t-r,z_x)\mathrm{d}z_x\mathrm{d}z_y\\
&\quad= \frac{\mu b'(0)}{\sqrt{4\alpha\pi}}\mathrm{e}^{-\nu y}\int_{\mathbb{R}}\int_0^L \Gamma_1(\alpha,x,z_x)\mathrm{e}^{-\frac{(y-z_y)^2}{4\alpha}+\nu(y-z_y)}\eta(t-r,z_x)\mathrm{d}z_x\mathrm{d}z_y\\
&\quad= \frac{\mu b'(0)}{\sqrt{4\alpha\pi}}\mathrm{e}^{-\nu y+\alpha\nu^2}\left\{\int_{\mathbb{R}}\mathrm{e}^{-\frac{(z-2\alpha\nu)^2}{4\alpha}}\mathrm{d}z\right\}\int_0^L \Gamma_1(\alpha,x,z_x)\eta(t-r,z_x)\mathrm{d}z_x\\
&\quad= \mu b'(0)\mathrm{e}^{-\nu y+\alpha\nu^2}\int_0^L \Gamma_1(\alpha,x,z_x)\eta(t-r,z_x)\mathrm{d}z_x.
\end{aligned}\tag{3.14}$$

We have from (3.13) and (3.14) that $\eta(t,x)$ satisfies

$$\begin{cases}
\dfrac{\partial\eta(t,x)}{\partial t} = D_m\dfrac{\partial^2\eta(t,x)}{\partial x^2} + D_m\nu^2\eta(t,x) - d_m\eta(t,x) + \mu b'(0)\mathrm{e}^{\alpha\nu^2}\displaystyle\int_0^L \Gamma_1(\alpha,x,z_x)\eta(t-r,z_x)\mathrm{d}z_x,\\
\qquad\qquad t>0,\ x\in(0,L),\\
B\eta(t,x) = 0,\quad t\ge 0,\ x=0,L.
\end{cases}\tag{3.15}$$

Substituting $\eta(t,x) = \mathrm{e}^{\lambda t}H(x)$, $\lambda>0$, $H(x)\ge 0$ into (3.15), we obtain

$$\begin{cases}
\lambda H(x) = D_m\dfrac{d^2H(x)}{\mathrm{d}x^2} + (D_m\nu^2 - d_m)H(x) + g(\nu)\mathrm{e}^{-\lambda r}\displaystyle\int_0^L \Gamma_1(\alpha,x,z_x)H(z_x)\mathrm{d}z_x,\ x\in(0,L),\\
BH(x) = 0,\ x=0,L,
\end{cases}\tag{3.16}$$

where $g(\nu) := \mu b'(0)\mathrm{e}^{\alpha\nu^2}$.

By a similar argument to (2.9), we can show that (3.16) has a real principal eigenvalue $\lambda(\nu)$ with strictly positive eigenfunction. Furthermore, we claim that $\lambda(\nu) > 0$. In fact, define $G(\lambda,\nu) := \lambda + D_m\mu_1 - D_m\nu^2 + d_m - g(\nu)\mathrm{e}^{-\lambda r}\mathrm{e}^{-\mu_1\alpha}$. We have from (P3) that

$$G(0,\nu) = D_m\mu_1 - D_m\nu^2 + d_m - g(\nu)\mathrm{e}^{-\mu_1\alpha} < 0,\qquad \frac{\partial G(\lambda,\nu)}{\partial\lambda} = 1 + rg(\nu)\mathrm{e}^{-\lambda r}\mathrm{e}^{-\mu_1\alpha} > 1,$$

which yields the conclusion that $G(\lambda,\nu)=0$ has exact one positive root $\bar\lambda(\nu)\in(0,\infty)$ with positive eigenfunction $H(x) = \Phi_1(x) > 0$. In view of the definition of principal eigenvalue, we know that $\lambda(\nu)\ge\bar\lambda(\nu)>0$. Thus $\mathrm{e}^{\lambda(\nu)t}$ is the principal eigenvalue of B_ν^t, where B_ν^t is the solution map associated with (3.15). Note that for any $\phi\in\bar{\mathcal{C}}_{w^+}$, we have

$$B_\nu^t[\phi](\theta,x) = M_t[\phi\mathrm{e}^{-\nu y}](\theta,x,0) = \mathrm{e}^{\lambda(\nu)t}\phi(\theta,x),\qquad t>0.$$

Let $t=1$. Then $\gamma(\nu) := \exp\{\lambda(\nu)\}$ is the principal eigenvalue of $B_\nu^1 =: B_\nu$. Note $\lambda(\nu)>0$ for $\nu\ge 0$. This yields $\gamma(0) = \mathrm{e}^{\lambda(0)t} > 1$ and (C7) in [12] is satisfied.

Define a function

$$\Phi(\nu) := \frac{1}{\nu}\ln\gamma(\nu) = \frac{\lambda(\nu)}{\nu}.$$

By [12, Lemma 3.8], we then have the following results.

Lemma 3.4. *The following statements are valid:*

(i) $\Phi(\nu)\to\infty$ *as* $\nu\downarrow 0$*;*
(ii) $\Phi(\nu)$ *is decreasing near* 0*;*
(iii) $\Phi'(\nu)$ *changes sign at most once on* $(0,\infty)$*;*
(iv) $\lim_{\nu\to\infty}\Phi(\nu)$ *exists, where the limits may be infinite.*

Theorem 3.4. *Assume that* (P1)–(P3) *hold. Let c^* be the asymptotic speed of spread of Q_1. Then $c^* = \inf_{\nu>0} \Phi(\nu) = \inf_{\nu>0} \frac{\lambda(\nu)}{\nu}$.*

Proof. In order to use [12, Theorem 3.10], we need to prove that $\Phi(+\infty) = +\infty$. Note that $G(\bar{\lambda}(\nu), \nu) \equiv 0$, and

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}\nu} = -\frac{\partial G(\lambda, \nu)/\partial \nu}{\partial G(\lambda, \nu)/\partial \lambda}\Big|_{\lambda=\bar{\lambda}(\nu)} = \frac{2D_m \nu + 2\alpha \nu g(\nu) \mathrm{e}^{-\bar{\lambda} r} \mathrm{e}^{-\mu_1 \alpha}}{1 + r g(\nu) \mathrm{e}^{-\bar{\lambda} r} \mathrm{e}^{-\mu_1 \alpha}} \to \infty \text{ as } \nu \to \infty.$$

Thus we obtain

$$\lim_{\nu\to\infty} \Phi(\nu) = \lim_{\nu\to\infty} \frac{\lambda(\nu)}{\nu} \geq \lim_{\nu\to\infty} \frac{\bar{\lambda}(\nu)}{\nu} = \lim_{\nu\to\infty} \frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}\nu} = +\infty.$$

Note that M_1 and B_ν satisfies (C1)–(C7) in [12] and the infimum of $\Phi(\nu)$ is attained at some finite value ν^*. Since $Q_1(\phi) \leq M_1(\phi)$ for $\phi \in \mathcal{C}_{w^*}$, [12, Theorem 3.10] implies that $c^* \leq \inf_{\nu>0} \Phi(\nu)$.

For any $\epsilon \in (0, 1)$, there is a $\delta > 0$ such that for any $\phi \in \mathcal{C}_\delta$, we have

$$b(w(t-r, x, y)) \geq (1-\epsilon) b'(0) w(t-r, x, y) \quad \text{for } (x, y) \in \Omega, t \in [0, 1].$$

Thus $w(t, x, y) = w(t, x, y; \phi)$ satisfies

$$\frac{\partial w}{\partial t} \geq D_m \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] - d_m w + (1-\epsilon)\mu b'(0) \int_{\mathbb{R}} \int_0^L \Gamma(\alpha, x, z_x, y, z_y) w(t-r, z_x, z_y) \mathrm{d}z_x \mathrm{d}z_y$$

for $t \in [0, 1]$, $(x, y) \in \Omega$.

Consider the linear system

$$\begin{aligned} &\frac{\partial w}{\partial t} = D_m \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] - d_m w + (1-\epsilon)\mu b'(0) \int_{\mathbb{R}} \int_0^L \Gamma(\alpha, x, z_x, y, z_y) w(t-r, z_x, z_y) \mathrm{d}z_x \mathrm{d}z_y, \\ &\qquad t > 0, (x, y) \in \Omega, \\ &Bw(t, x, y) = 0, \quad t \geq 0,\ x = 0, L,\ y \in \mathbb{R}. \end{aligned} \tag{3.17}$$

Let M_t^ϵ, $t \geq 0$, be the solution map associated with the above linear system. Then the comparison principle implies that $M_t^\epsilon(\phi) \leq Q_t(\phi)$, $\forall \phi \in \mathcal{C}_\delta$, $t \in [0, 1]$. In particular, $M_1^\epsilon(\phi) \leq Q_1(\phi)$, $\forall \phi \in \mathcal{C}_\delta$. We can carry out the analysis for M_t^ϵ similarly to that for M_t above, and it then follows from [12, Theorem 3.10] that $\inf_{\nu>0} \Phi_\epsilon(\nu) \leq c^*$. Thus, we have

$$\inf_{\nu>0} \Phi_\epsilon(\nu) \leq c^* \leq \inf_{\nu>0} \Phi(\nu),\ \forall \epsilon \in (0, 1).$$

Letting $\epsilon \to 0$, we obtain $c^* = \inf_{\nu>0} \Phi(\nu)$. ■

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Appendix

Consider the following initial boundary value problem

$$\begin{aligned} &W_t = W_{xx}, && t > 0,\ 0 < x < L, \\ &W(0, x) = \varphi(x), && 0 \leq x \leq L \\ &W_x(t, 0) - h_0 W(t, 0) = 0, && t > 0,\ h_0 \geq 0, \\ &W_x(t, L) + h_L W(t, L) = 0, && t > 0,\ h_L \geq 0. \end{aligned} \tag{A.1}$$

We shall use the separation of variables to solve the above problem explicitly.

Let $W(t,x) = T(t)\Phi(x)$ in the equation of (A.1), and we obtain

$$\frac{T'(t)}{T(t)} = \frac{\Phi''(x)}{\Phi(x)}, \quad t > 0,\ 0 < x < L.$$

Since the left-hand side of the above is a function of t, and the right-hand side is a function of x, there must be a constant μ such that $\mu := -\frac{T'(t)}{T(t)} = -\frac{\Phi''(x)}{\Phi(x)}$. We obtain

$$T'(t) + \mu T(t) = 0, \quad t > 0; \tag{A.2}$$

$$\begin{cases} \Phi''(x) + \mu\Phi(x) = 0, & 0 < x < L, \\ \Phi'(0) - h_0\Phi(0) = 0, & \Phi'(L) + h_L\Phi(L) = 0. \end{cases} \tag{A.3}$$

We have from [31] the following conclusions.

(i) All eigenvalues of (A.3) are non-negative real numbers.
(ii) For any given eigenvalue μ of (A.3), the corresponding eigenfunction is unique except for a constant factor.
(iii) There are countable eigenvalues of (A.3), say μ_n, $n = 1, 2, \ldots$, satisfying

$$\mu_1 < \mu_2 < \cdots < \mu_n < \cdots.$$

Furthermore, the eigenfunction $\Phi_n(x)$ corresponding to μ_n has exact $n-1$ zero points in $(0, L)$.

Assume that $\Phi_n(x)$ is the eigenfunction of (A.3) according to μ_n. Then we have $\int_0^L \Phi_n(x)\Phi_m(x)\mathrm{d}x = 0$ for $n \neq m$. We have from (A.2) that

$$T_n(t) = A_n \mathrm{e}^{-\mu_n t}, \quad A_n \text{ is any constant, } n = 1, 2, \ldots.$$

Thus the Eq. (A.1) has a series of solutions

$$W_n(t,x) = A_n \mathrm{e}^{-\mu_n t}\Phi_n(x), \quad A_n \text{ is any constant, } n = 1, 2, \ldots.$$

Let $M_n = \int_0^L \Phi_n^2(x)\mathrm{d}x$. We can show that

$$W(t,x) = \sum_{n=1}^{\infty} W_n(t,x) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\mu_n t}\Phi_n(x) = \sum_{n=1}^{\infty} \frac{1}{M_n}\int_0^L \mathrm{e}^{-\mu_n t}\varphi(\xi)\Phi_n(\xi)\Phi_n(x)\mathrm{d}\xi \tag{A.4}$$

is a classical solution of (A.1).

Now we have from the above discussion that the Green function of the BVP

$$\begin{array}{ll} W_t = W_{xx}, & t > 0,\ 0 < x < L, \\ W_x(t,0) - h_0 W(t,0) = 0, & t > 0,\ h_0 \geq 0, \\ W_x(t,L) + h_L W(t,L) = 0, & t > 0,\ h_L \geq 0, \end{array} \tag{A.5}$$

is $\Gamma_1(t,x,z_x) = \sum_{n=1}^{\infty} \frac{1}{M_n}\mathrm{e}^{-\mu_n t}\Phi_n(z_x)\Phi_n(x)$. Let $m_n := \int_0^L \Phi_n(\xi)\mathrm{d}\xi$, we have

$$\int_0^L \Gamma_1(t,x,z_x)\mathrm{d}z_x = \sum_{n=1}^{\infty} \frac{m_n}{M_n}\mathrm{e}^{-\mu_n t}\Phi_n(x) \leq 1.$$

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