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Two-parameter bifurcations in a network of two neurons with multiple delays

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Abstract

We consider a network of two coupled neurons with delayed feedback. We show that the connection topology of the network plays a fundamental role in classifying the rich dynamics and bifurcation phenomena. Regarding eigenvalues of the connection matrix as bifurcation parameters, we obtain codimension 1 bifurcations (including a fold bifurcation and a Hopf bifurcation) and codimension 2 bifurcations (including fold–Hopf bifurcations and Hopf–Hopf bifurcations). We also give concrete formulae for the normal form coefficients derived via the center manifold reduction that give detailed information about the bifurcation and stability of various bifurcated solutions. In particular, we obtain stable or unstable equilibria, periodic solutions, quasi-periodic solutions, and sphere-like surfaces of solutions. We also show how to evaluate critical normal form coefficients from the original system of delay-differential equations without computing the corresponding center manifolds.

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1. Introduction

Recent years have witnessed a growing interest in the dynamics of interacting neurons with delayed feedback, in particular, from bifurcation point of view, see [4,10,18,28–33,37–39,41] and the references therein. Much of the existing bifurcation analysis has been carried out by regarding the time delays as bifurcation parameters while the connection topology is fixed. On the other hand, realistic modeling of networks inevitably requires careful design and variation of the connection topology, and the fact that a wide range of different behaviors can be established by varying the coupling strength and structure has important implications for neural networks, since synaptic coupling can change through learning. The observation provides us the motivation of this study to regard connection topology as the bifurcation parameter while time lags are fixed. The work [17,19–21,32] certainly shows the feasibility of this approach.

We follow [24] and regard each individual neuron as a circuit with a linear resistor and a linear capacitor. With some rescaling and reparametrization, the model for an individual neuron takes the form

$$\dot{x}(t) = -x(t) + \beta f\big(x(t-\tau)\big), \tag{1.1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and $f \in C^1(\mathbb{R};\mathbb{R})$, and delayed signal transmission as a self-feedback is due to the finite switching speed of neurons. Equation (1.1) has been widely studied, and it has been shown that by varying the parameters appropriately, this fairly simple model can reproduce two fundamental states of a neuron, quiescence and periodic firing. Therefore, Eq. (1.1) can act as an oscillator for appropriate choices of τ, β, and f. More interesting dynamics can occur only when such oscillators are coupled via certain synaptic connections. For this purpose, we shall consider the following system of two neurons,

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \beta f\big(x_1(t-\tau)\big) + a_{12} f\big(x_2(t-\tau_1)\big), \\ \dot{x}_2(t) = -x_2(t) + \beta f\big(x_2(t-\tau)\big) + a_{21} f\big(x_1(t-\tau_2)\big), \end{cases} \tag{1.2}$$

where $x_1(t)$ and $x_2(t)$ denote the activations of the two neurons, τ_i, $i = 1, 2$, and τ denote the synaptic transmission delays, a_{12} and a_{21} are the synaptic coupling weights, $f : \mathbb{R} \to \mathbb{R}$ is the activation function. Throughout this paper, we always assume that $\tau_1 + \tau_2 = 2\tau > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^1-smooth function with $f(0) = 0$. Without loss of generality, we also assume that $\tau_1 \geqslant \tau_2$ and $f'(0) = 1$. We note that despite the low number of units, two-neuron networks with delay often display the same dynamical behaviors as large networks and, can thus be used as prototypes for us to understand the dynamics of large networks with delayed feedback (see, for example, the monograph of Milton [29]).

In this paper, we will use β and η as the bifurcation parameters, where $\eta = \sqrt{|a_{12}a_{21}|}$. In other words, $\mu = (\beta, \eta)$ will be varied simultaneously to generate a bifurcation curve Γ. The equilibrium 0 is asymptotically stable if all the eigenvalues of the linearized system of (1.2) at equilibrium 0, which are zeros of an analytic function of λ parametrized by μ, denoted by $\det \Delta(\mu, \lambda)$, have strictly negative real parts. If there exists a zero λ of $\det \Delta(\mu, \lambda)$ with a positive real part, then the equilibrium 0 is unstable. By varying μ, we may encounter codimension 1 bifurcations, i.e., there exists a simple zero eigenvalue or a pair of purely imaginary eigenvalues for some μ. Thus, either a fold bifurcation or Hopf bifurcation may occur in system (1.2). If we further vary μ, then extra eigenvalues can approach the imaginary axis, thus changing the dimension of the center manifold. Thus, several codimension 2 bifurcations can occur in system (1.2), where curves of codimension 1 bifurcations intersect or meet tangentially. We develop a method that evaluates critical normal form coefficients from the original system directly without computing the corresponding center manifolds. This allows us to decouple the amplitude equations and hence phase equations, and hence the phase plane analysis can be employed to explore the bifurcation diagrams.

The rest of this paper is organized as follows: In Section 2, we discuss the associated characteristic equation and obtain criteria ensuring the linear stability of the trivial solution. The existence, direction, and stability of Hopf bifurcated periodic solutions are given in Section 3. Sections 4 and 5 are devoted to the Bautin bifurcation analysis and fold bifurcation analysis, respectively. Sections 6 and 7 are devoted to codimension 2 bifurcations including fold–Hopf bifurcations and Hopf–Hopf bifurcations. The appendices contain some of the details of the calculation of some critical coefficients for the normal forms of fold–Hopf bifurcations and Hopf–Hopf bifurcations.

2. The characteristic equation

Let $\mathbf{C} = C([-\tau_1, 0], \mathbb{R}^2)$ denote the Banach space of all continuous mappings from $[-\tau_1, 0]$ into $\mathbb{R}^2$ equipped with the supremum norm $\|\phi\| = \sup_{-\tau_1 \leqslant \theta \leqslant 0} |\phi(\theta)|$ for $\phi \in \mathbf{C}$. As usual, if $\sigma \in \mathbb{R}$, $A \geqslant 0$ and $u : [\sigma - \tau_1, \sigma + A] \to \mathbb{R}^2$ is a continuous mapping, then $u_t \in \mathbf{C}$ for $t \in [\sigma, \sigma + A]$ is defined by $u_t(\theta) = u(t + \theta)$ for $-\tau_1 \leqslant \theta \leqslant 0$.

Linearizing system (1.2) at the trivial solution leads to the following linear system,

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \beta x_1(t - \tau) + a_{12} x_2(t - \tau_1), \\ \dot{x}_2(t) = -x_2(t) + a_{21} x_1(t - \tau_2) + \beta x_2(t - \tau). \end{cases} \tag{2.1}$$

Let $\eta = \sqrt{|a_{12} a_{21}|}$ and $\mu = (\beta, \eta)$, for reasons explained below. The characteristic matrix for system (2.1) is

$$\Delta(\mu, \lambda) = \begin{bmatrix} \lambda + 1 - \beta e^{-\lambda \tau} & -a_{12} e^{-\lambda \tau_1} \\ -a_{21} e^{-\lambda \tau_2} & \lambda + 1 - \beta e^{-\lambda \tau} \end{bmatrix},$$

and hence the characteristic equation is

$$\det \Delta(\mu, \lambda) = \left[\lambda + 1 - \beta e^{-\lambda \tau}\right]^2 - a_{12} a_{21} e^{-2\lambda \tau} = 0. \tag{2.2}$$

Then $\det \Delta(\mu, \lambda)$ can be decomposed as

$$\det \Delta(\mu, \lambda) = \left[\lambda + 1 - (\beta + \eta) e^{-\lambda \tau}\right]\left[\lambda + 1 - (\beta - \eta) e^{-\lambda \tau}\right] \quad \text{if } a_{12} a_{21} > 0,$$

or

$$\det \Delta(\mu, \lambda) = \left[\lambda + 1 - (\beta + i\eta) e^{-\lambda \tau}\right]\left[\lambda + 1 - (\beta - i\eta) e^{-\lambda \tau}\right] \quad \text{if } a_{12} a_{21} < 0.$$

It is well known that the trivial solution of system (1.2) is locally asymptotically stable if all roots λ of the characteristic equation (2.2) satisfy $\mathrm{Re}(\lambda) < 0$.

Observe that the connection matrix of system (1.2) is

$$\begin{bmatrix} \beta & a_{12} \\ a_{21} & \beta \end{bmatrix},$$

whose eigenvalues are $\beta \pm \eta$ (if $a_{12}a_{21} > 0$) or $\beta \pm i\eta$ (if $a_{12}a_{21} < 0$). It follows that it is natural to choose β and η as the bifurcation parameters. As a result, it is necessary to determine when the infinitesimal generator $\mathcal{A}(\mu)$ of the C^0-semigroup generated by the linear system (2.1) has eigenvalues lying on the imaginary axis. For this purpose, we first consider

$$P_z(\lambda) = \lambda + 1 - ze^{-\lambda\tau}, \tag{2.3}$$

where $z \in \mathbb{C}$.

Define a curve Σ with the following parametric equations,

$$\begin{cases} u(t) = \cos\tau t - t\sin\tau t, \\ v(t) = t\cos\tau t + \sin\tau t, \end{cases} \quad t \in \mathbb{R}. \tag{2.4}$$

It is easy to see that the curve Σ is symmetric about the u-axis. Let $\theta(t) = v(t)/u(t)$. Then $\theta'(t) = u^{-2}(t)[1 + \tau + \tau t^2] > 0$ for all $t \in \mathbb{R}$ such that $u(t) \neq 0$. This implies that, as t increases, the corresponding point $(u(t), v(t))$ on the curve Σ moves anticlockwise around the origin. Moreover, it follows from $u^2(t) + v^2(t) = 1 + t^2$ that $\Sigma^+ = \{(u(t), v(t))\colon t \in \mathbb{R}^+\}$ is simple, i.e., it cannot intersect with itself. Let $\{t_n\}_{n=0}^{+\infty}$ be the monotonic increasing sequence of the non-negative zeros of $v(t)$, and $c_n = u(t_n)$ for all $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Obviously, we have $t_0 = 0$ and $t_n \in ((2n-1)\pi/(2\tau), n\pi/\tau)$ for all $n \in \mathbb{N}$. Therefore, the curve Σ intersects with the u-axis at $(c_n, 0)$, $n \in \mathbb{N}_0$. It follows from the anticlockwise property of the curve Σ that $(-1)^n c_n > 0$ for all $n \in \mathbb{N}_0$. In addition, we have $|c_n| = \sqrt{1 + t_n^2}$, which implies that $c_n = (-1)^n\sqrt{1 + t_n^2}$ for $n \in \mathbb{N}_0$ and $\{|c_n|\}_{n\in\mathbb{N}_0}$ is an increasing sequence. In particular, $c_0 = 1$ and $c_1 = \sec\tau t_1 < -1$. Moreover, we claim that

$$(-1)^n v'(t_n) > 0 \quad \text{and} \quad (-1)^n u'(t_n) \geqslant 0 \quad \text{for } n \in \mathbb{N}_0. \tag{2.5}$$

Equality in the second formula of (2.5) holds if and only if $n = 0$. In fact, we can check that $v'(t_n) \neq 0$ when $v(t_n) = 0$. This, combined with the anticlockwise property of the curve Σ, gives the first inequality in (2.5). From $u^2(t) + v^2(t) = 1 + t^2$, we have $u'(t)u(t) + v'(t)v(t) = t$ for $t \in \mathbb{R}^+$. Particularly, $u'(t_n)c_n = t_n$ for all $n \in \mathbb{N}_0$. This, combined with $(-1)^n c_n > 0$ for $n \in \mathbb{N}_0$, immediately implies the second inequality in (2.5). This proves the claim. Finally, $u^2(t) + v^2(t) = 1 + t^2 \geqslant 1$ also implies that the curve is not inside the unit circle and it has only one intersection point $(1, 0)$ with the unit circle.

For each $n \in \mathbb{N}_0$, let $\Sigma_n = \{(u(t), v(t))\colon t \in [-t_{n+1}, -t_n] \cup [t_n, t_{n+1}]\}$, which is a closed curve with $(0, 0)$ inside. The curve Σ is schematically illustrated in Fig. 1. In the sequel, we will identify Σ with $\{u(t) + iv(t)\colon t \in \mathbb{R}\} \subset \mathbb{C}$. The following lemma will play an important role in analyzing the distributions of the roots of (2.2).

Lemma 2.1. *Consider $P_z(\lambda)$ defined in* (2.3) *with $z \in \mathbb{C}$. Then the following statements are true*:

(i) *$P_z(\lambda)$ has a purely imaginary zero if and only if $z \in \Sigma$. Moreover, if $z = u(\theta) + iv(\theta)$ then the purely imaginary zero is $i\theta$ except that there is a pair of conjugate purely imaginary zeros $\pm it_n$ if $z = c_n$ for $n \in \mathbb{N}$.*

(ii) *For each fixed $z_0 = u(\theta_0) + iv(\theta_0) \in \Sigma$, there exists an open δ-neighborhood of z_0 in the complex plane, denoted by $B(z_0, \delta)$, and an analytical function $\lambda : B(z_0, \delta) \to \mathbb{C}$ such that*

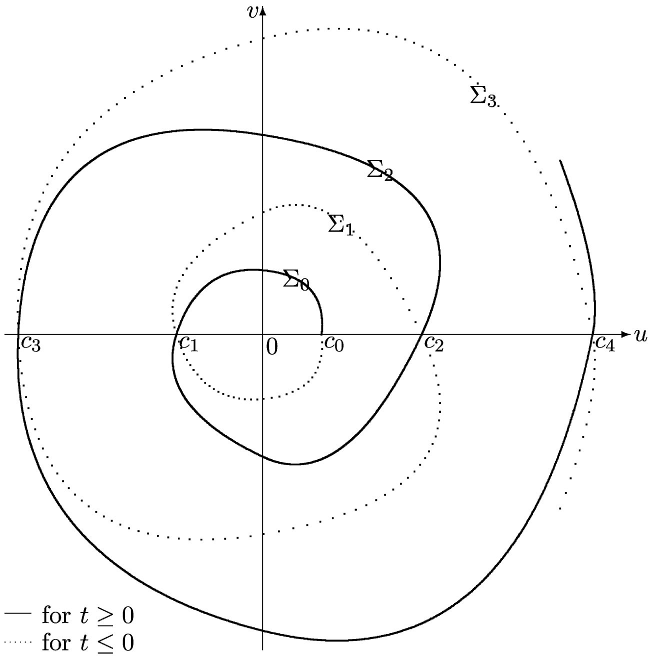

Fig. 1. The parametric curve Σ.

$\lambda(z_0) = i\theta_0$ and $\lambda(z)$ is a zero of $P_z(\lambda)$ for all $z \in B(z_0, \delta)$. Moreover, along the vector $\vartheta(\xi) = (v'(\theta_0), -u'(\theta_0))M(\xi)$, the directional derivative of $\mathrm{Re}\{\lambda(z)\}$ *at z_0 is positive, where $\xi \in (-\pi/2, \pi/2)$ and*

$$M(\xi) = \begin{bmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{bmatrix}.$$

(iii) *$P_z(\lambda)$ has only zeros with strictly negative real parts if and only if z is inside the curve Σ_0; exactly $j \in \mathbb{N}$ zeros with positive real parts if z is between Σ_{j-1} and Σ_j. In particular, if $z \in \Sigma_0$, $P_z(\lambda)$ has either a simple real zero* 0 (*if $z = 1$*) *or a simple purely imaginary zero* (*if* $\mathrm{Im}(z) \neq 0$), *or a pair of simple purely imaginary zeros* (*if $z = c_1$*), *and all other zeros have strictly negative real parts.*

Proof. (i) $P_z(\lambda)$ has a purely imaginary zero, say $\lambda = i\theta$, if and only if $e^{i\tau\theta}(1 + i\theta) = z$, which is equivalent to $z \in \Sigma$ by separating the real and imaginary parts of $e^{i\tau\theta}(1 + i\theta)$.

(ii) Note that $P_{z_0}(i\theta_0) = 0$ and $i\theta_0$ is a simple zero of $P_{z_0}(\lambda)$. The existence of δ and the mapping λ follow from the Implicit Function Theorem. Moreover, $\lambda(z)$ is analytic with respect to z. Thus,

$$\lambda'(z) = \frac{\partial}{\partial a}\mathrm{Re}\{\lambda(z)\} + i\frac{\partial}{\partial a}\mathrm{Im}\{\lambda(z)\} = \frac{\partial}{\partial b}\mathrm{Im}\{\lambda(z)\} - i\frac{\partial}{\partial b}\mathrm{Re}\{\lambda(z)\},$$

where $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. On the other hand, differentiating $P_z(\lambda) = 0$ with respect to z and using $P_{z_0}(i\theta_0) = 0$ and $|u(\theta_0) + iv(\theta_0)| = \sqrt{1+\theta_0^2}$, we have

$$\lambda'(z_0) = \varepsilon_1(\theta_0)\big\{u(\theta_0)\varepsilon_2(\theta_0) + \theta_0 v(\theta_0) + i\big[\theta_0 u(\theta_0) - v(\theta_0)\varepsilon_2(\theta_0)\big]\big\},$$

where $\varepsilon_1(\theta) = [(1+\tau)^2 + (\tau\theta)^2]^{-1}(1+\theta^2)^{-1}$ and $\varepsilon_2(\theta) = 1+\tau+\tau\theta^2$. It follows that

$$\begin{aligned}\nabla \mathrm{Re}\big\{\lambda(z_0)\big\} &= \left(\frac{\partial}{\partial a}\mathrm{Re}\big\{\lambda(z_0)\big\}, \frac{\partial}{\partial b}\mathrm{Re}\big\{\lambda(z_0)\big\}\right)^T \\ &= \varepsilon_1(\theta_0)\big(u(\theta_0)\varepsilon_2(\theta_0) + \theta_0 v(\theta_0), v(\theta_0)\varepsilon_2(\theta_0) - \theta_0 u(\theta_0)\big)^T.\end{aligned}$$

Thus, for a given $\xi \in (-\pi/2, \pi/2)$, the directional derivative along the vector $\vartheta(\xi)$ at z_0 is

$$\begin{aligned}\frac{d}{d\vartheta(\xi)}\mathrm{Re}\big\{\lambda(z_0)\big\} &= \varepsilon_3(\theta_0)\big(v'(\theta_0), -u'(\theta_0)\big)M(\xi)\nabla \mathrm{Re}\big\{\lambda(z_0)\big\} \\ &= \varepsilon_1(\theta_0)\varepsilon_3(\theta_0)\big(\varepsilon_2^2(\theta_0) + \theta_0^2\big)\cos\xi > 0,\end{aligned}$$

where $\varepsilon_3(\theta_0) = |\vartheta(\xi)|^{-1} = 1/\sqrt{(1+\tau)^2 + \tau^2\theta_0^2}$.

(iii) Note that $P_0(\lambda)$ has exactly one zero -1, which obviously has a negative real part. Since zeros of $P_z(\lambda)$ depend continuously on z, there exists a region Ω_0 containing $z = 0$ such that for $z \in \Omega_0$, all zeros of $P_z(\lambda)$ have negative real parts. Moreover, as z varies and passes through the boundary $\partial\Omega_0$, only one (or two if z is real) zero point of $P_z(\lambda)$ varies from a complex number with a negative real part to a purely imaginary number and then to a complex number with a positive real part. By (i), $\partial\Omega_0 = \Sigma_0$. Therefore, $P_z(\lambda)$ has only zeros with negative real parts if z is inside the curve Σ_0.

If z is a real number between Σ_{j-1} and Σ_j, then one can easily show that $P_z(\lambda)$ has exactly j zeros with positive real parts (see, for example, the discussion in Chen and Wu [6]). This, combined with (i) and the continuous dependence of zeros of $P_z(\lambda)$ on z, completes the proof. □

For the sake of convenience, we introduce the following notations:

$$\begin{aligned}\Omega_j^{\pm} &= \big\{(\beta,\eta):\ \beta \pm \eta = c_j,\ \eta > 0\big\}, \quad j \in \mathbb{N}_0, \\ \Sigma_j^{+} &= \big\{(\beta,\eta) \in \Sigma_j:\ \eta > 0\big\}, \quad j \in \mathbb{N}_0, \\ \Sigma^{+} &= \bigcup_{j\in\mathbb{N}_0} \Sigma_j^{+}, \\ \mathbb{R}^{2+} &= \big\{(\beta,\eta) \in \mathbb{R}^2:\ \eta > 0\big\}.\end{aligned}$$

We call $\Omega_j^{\pm}$, $j \in \mathbb{N}_0$, critical lines.

Corollary 2.2. *Suppose $a_{12}a_{21} > 0$. Then the following statements hold.*

(i) *All zeros of* $\det \Delta(\mu,\lambda)$ *have negative real parts if and only if* $c_1 < \beta - \eta < \beta + \eta < 1$.

(ii) *If and only if* $\mu = (\beta, \eta) \in \Omega_n^+ \cup \Omega_n^-$ *for some* $n \in \mathbb{N}$, $\det \Delta(\mu, \lambda)$ *has a pair of simple conjugate purely imaginary zeros, which are* $\pm i t_n$.

(iii) *If and only if* $\mu = (\beta, \eta) \in \Omega_0^+ \cup \Omega_0^-$, $\det \Delta(\mu, \lambda)$ *has a simple zero* $\lambda = 0$. *Moreover, if* $c_1 < \beta - \eta < \beta + \eta = 1$, *then all zeros but* $\lambda = 0$ *of* $\det \Delta(\mu, \lambda)$ *have strictly negative real parts.*

(iv) *For each fixed* $\mu_0 = (\beta_0, \eta_0) \in \Omega_n^+$ *(or* Ω_n^-*), there exist an open* δ*-neighborhood* $B(\mu_0, \delta)$ *of* μ_0 *and a smooth function* $\lambda : B(\mu_0, \delta) \to \mathbb{C}$ *such that* $\lambda(\mu_0) = i t_n$ *(or* $-i t_n$*) and* $\lambda(\mu)$ *is a zero of* $\det \Delta(\mu, \lambda)$ *for all* $\mu \in B(\mu_0, \delta)$. *Moreover, the directional derivative* $\frac{d}{d\vartheta(\xi)} \operatorname{Re}\{\lambda(\mu_0)\} > 0$, *where* $\vartheta(\xi) = (-1)^n(\cos\xi - \sin\xi, \cos\xi + \sin\xi)$ *(or* $(-1)^n(\cos\xi + \sin\xi, -\cos\xi + \sin\xi)$*) with* $\xi \in (-\pi/2, \pi/2)$.

Proof. By Lemma 2.1, it suffices to prove $\frac{d}{d\vartheta(\xi)} \operatorname{Re}\{\lambda(\mu_0)\} > 0$. If $\mu_0 = (\beta_0, \eta_0) \in \Omega_n^+$, then

$$\frac{\partial}{\partial \beta} \operatorname{Re}\big\{\lambda(\mu_0)\big\} = \frac{\partial}{\partial \eta} \operatorname{Re}\big\{\lambda(\mu_0)\big\} = c_n \varepsilon_1(t_n)\varepsilon_2(t_n),$$

where the functions $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are those defined in the proof of Lemma 2.1(ii). Hence,

$$\frac{d}{d\vartheta(\xi)} \operatorname{Re}\big\{\lambda(\mu_0)\big\} = \frac{\sqrt{2}}{2} \vartheta(\xi) \nabla \operatorname{Re}\big\{\lambda(\mu_0)\big\} = \sqrt{2}|c_n| \varepsilon_1(t_n)\varepsilon_2(t_n) \cos\xi > 0,$$

where $\vartheta(\xi) = (-1)^n(\cos\xi - \sin\xi, \cos\xi + \sin\xi)$ and $\xi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now, if $\mu_0 = (\beta_0, \eta_0) \in \Omega_n^-$, then

$$\frac{\partial}{\partial \beta} \operatorname{Re}\big\{\lambda(\mu_0)\big\} = -\frac{\partial}{\partial \eta} \operatorname{Re}\big\{\lambda(\mu_0)\big\} = c_n \varepsilon_1(t_n)\varepsilon_2(t_n).$$

Hence,

$$\frac{d}{d\vartheta(\xi)} \operatorname{Re}\big\{\lambda(\mu_0)\big\} = \frac{\sqrt{2}}{2} \vartheta(\xi) \nabla \operatorname{Re}\big\{\lambda(\mu_0)\big\} = \sqrt{2}|c_n| \varepsilon_1(t_n)\varepsilon_2(t_n) \cos\xi > 0,$$

where $\vartheta(\xi) = (-1)^n(\cos\xi + \sin\xi, -\cos\xi + \sin\xi)$ and $\xi \in (-\pi/2, \pi/2)$. □

Similarly, we can prove

Corollary 2.3. *Suppose* $a_{12}a_{21} < 0$. *Then the following statements are true.*

(i) $\det \Delta(\mu, \lambda)$ *has a purely imaginary zero if and only if* $\mu = (\beta, \eta) \in \Sigma^+$. *The purely imaginary zero is given by* $i\theta$, *where* θ *satisfies* $u(\theta) = \beta$ *and* $v(\theta) = \eta$.

(ii) *All zeros of* $\det \Delta(\mu, \lambda)$ *have strictly negative real parts if and only if* $\mu = (\beta, \eta) \in \mathbb{R}^{2+}$ *is inside the curve* Σ_0.

(iii) *For each fixed* $\mu_0 = (\beta_0, \eta_0) \in \Sigma^+$, *there exist an open* δ*-neighborhood* $B(\mu_0, \delta)$ *of* μ_0 *and a smooth function* $\lambda : B(\mu_0, \delta) \to \mathbb{C}$ *such that* $\lambda(\mu_0) = i\theta_0$ *and* $\lambda(\mu)$ *is a zero of* $\det \Delta(\mu, \lambda)$ *for all* $\mu \in B(\mu_0, \delta)$, *where* θ_0 *satisfies* $u(\theta_0) = \beta_0$ *and* $v(\theta_0) = \eta_0$. *Moreover, the directional derivative* $\frac{d}{d\vartheta(\xi)} \operatorname{Re}\{\lambda(\mu_0)\} > 0$, *where* $\vartheta(\xi) = (v'(\theta_0), -u'(\theta_0))M(\xi)$ *with* $\xi \in (-\pi/2, \pi/2)$.

(iv) *If* $\mu = (\beta, \eta) \in \Sigma_0^+$, *then* $\det \Delta(\mu, \lambda)$ *has a simple purely imaginary zero and all other zeros have strictly negative real parts.*

Based on Lemmas 2.2 and 2.3, we have the following results about the stability of the trivial solution of system (1.2).

Corollary 2.4. *The trivial solution of system* (1.2) *is asymptotically stable if and only if one of the following two conditions holds.*

(i) $a_{12}a_{21} > 0$ *and* $c_1 < \beta - \eta < \beta + \eta < 1$.
(ii) $a_{12}a_{21} < 0$ *and* (β, η) *is inside the curve* Σ_0.

Note that there are stability switches at the trivial solution as $\mu = (\beta, \eta) \in \mathbb{R}^{2+}$ crosses the critical lines $\Omega_j^{\pm}$ or the curves Σ_j^+, $j \in \mathbb{N}_0$. Hence, a bifurcation from a stationary point to periodic solutions occurs. In the following sections, we consider two kinds of bifurcations: codimension one and codimension two bifurcations.

3. Hopf bifurcation

Hopf bifurcation occurs possibly at $\mu_0 = (\beta_0, \eta_0) \in \mathbb{R}^{2+}$ where the infinitesimal generator $\mathcal{A}(\mu)$ has a pair of purely imaginary eigenvalues, which is true if either $\mu_0 \in \bigcup_{j \geqslant 1}(\Omega_j^+ \cup \Omega_j^-)$ when $a_{12}a_{21} > 0$, or $\mu_0 \in \Sigma^+$ when $a_{12}a_{21} < 0$.

In this section, we focus on the codimension one Hopf bifurcation, i.e., the infinitesimal generator $\mathcal{A}(\mu)$ has only one pair of purely imaginary eigenvalues at $\mu_0 = (\beta_0, \eta_0) \in \mathbb{R}^{2+}$. Therefore, throughout this section, we always assume that either (i) there exists $j \in \mathbb{N}$ such that $\mu_0 = (\beta_0, \eta_0) \in \Omega_j^+ \setminus \{\bigcup_{s \geqslant 0} \Omega_s^-\}$ or $\Omega_j^- \setminus \{\bigcup_{s \geqslant 0} \Omega_s^+\}$ when $a_{12}a_{21} > 0$, or (ii) $\mu_0 = (\beta_0, \eta_0) \in \Sigma^+$ when $a_{12}a_{21} < 0$. Then $\mathcal{A}(\mu)$ has exactly a pair of simple conjugate purely imaginary eigenvalues $\pm i\omega_0$ at $\mu_0 = (\beta_0, \eta_0)$, where $\omega_0 = t_j$ if $a_{12}a_{21} > 0$, while ω_0 satisfies $u(\omega_0) = \beta_0$ and $v(\omega_0) = \eta_0$ if $a_{12}a_{21} < 0$.

In order to study the Hopf bifurcation, we need the reduced system on the center manifold associated with the pair of conjugate purely imaginary solutions $\pm i\omega_0$ of the characteristic equation (2.2). With the reduced system, we can determine bifurcation direction and the stability of the bifurcated periodic solutions. For this purpose, we further assume that

(H1) $f \in C^3(\mathbb{R}, \mathbb{R})$ and $xf(x) \neq 0$ when $x \neq 0$.

Let $X(t) = (x_1(t), x_2(t))^T$ and $X_t(\theta) = X(t + \theta)$ for $\theta \in [-\tau_1, 0]$. Let BC be the set of all functions from $[-\tau, 0]$ to $\mathbb{R}^2$ which are uniformly continuous on $[-\tau, 0)$ and may have a possible jump discontinuity at 0. Thus, we can rewrite system (1.2) as

$$\dot{X}_t = \mathcal{A}(\mu)X_t + \mathcal{R}(\mu)X_t, \tag{3.1}$$

where the infinitesimal generator $\mathcal{A}(\mu) : C^1([-\tau_1, 0], \mathbb{R}^2) \to BC$ is given by

$$\mathcal{A}(\mu)\varphi(\theta) = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-\tau_1, 0), \\ \begin{bmatrix} -\varphi_1(0)+\beta\varphi_1(-\tau)+a_{12}\varphi_2(-\tau_1) \\ -\varphi_2(0)+\beta\varphi_2(-\tau)+a_{21}\varphi_1(-\tau_2) \end{bmatrix}, & \theta = 0, \end{cases}$$

and the nonlinear operator $\mathcal{R}(\mu):\mathbf{C}\to BC$ is

$$\mathcal{R}(\mu)\varphi=\frac{1}{2}f''(0)\mathcal{B}(\mu)(\varphi,\varphi)+\frac{1}{6}f'''(0)\mathcal{C}(\mu)(\varphi,\varphi,\varphi)+\cdots,$$

with $\mathcal{B}(\mu):\mathbf{C}\times\mathbf{C}\to BC$ and $\mathcal{C}(\mu):\mathbf{C}\times\mathbf{C}\times\mathbf{C}\to BC$ being defined as

$$\begin{aligned}
\mathcal{B}(\mu)(\varphi,\psi)(\theta)&=0,\quad \theta\in[-\tau_1,0),\\
\mathcal{B}(\mu)(\varphi,\psi)(0)&=\begin{bmatrix}\beta\varphi_1(-\tau)\psi_1(-\tau)+a_{12}\varphi_2(-\tau_1)\psi_2(-\tau_1)\\ a_{21}\varphi_1(-\tau_2)\psi_1(-\tau_2)+\beta\varphi_2(-\tau)\psi_2(-\tau)\end{bmatrix},\\
\mathcal{C}(\mu)(\varphi,\psi,\phi)(\theta)&=0,\quad \theta\in[-\tau_1,0),\\
\mathcal{C}(\mu)(\varphi,\psi,\phi)(0)&=\begin{bmatrix}\beta\varphi_1(-\tau)\psi_(-\tau)\phi_1(-\tau)+a_{12}\varphi_2(-\tau_1)\psi_2(-\tau_1)\phi_2(-\tau_1)\\ a_{21}\varphi_1(-\tau_2)\psi_1(-\tau_2)\phi_1(-\tau_2)+\beta\varphi_2(-\tau)\psi_2(-\tau)\phi_2(-\tau)\end{bmatrix},
\end{aligned}$$

for $\varphi,\psi,\phi\in\mathbf{C}$. By the Riesz representation theorem, there exists a matrix $\Xi(\theta,\mu)$ whose components are functions of bounded variation in $\theta\in[-\tau_1,0]$ such that

$$\big(\mathcal{A}(\mu)\varphi\big)(0)=\int_{-\tau_1}^{0}d\Xi(\theta,\mu)\,\varphi(\theta)\quad\text{for }\varphi\in\mathbf{C}.$$

It is easy to check that

$$q(\theta)=(1,d)^Te^{i\omega_0\theta},\quad \theta\in[-\tau_1,0],$$

where $d=(1+i\omega_0-\beta e^{-i\omega_0\tau})e^{i\omega_0\tau_1}/a_{12}$, is an eigenvector of $\mathcal{A}(\mu_0)$ associated with the eigenvalue $i\omega_0$. The adjoint operator $\mathcal{A}^*(\mu_0)$ is given by

$$\big(\mathcal{A}^*(\mu_0)\psi\big)(\xi)=\begin{cases}-\frac{d\psi}{d\xi}, & \text{if }\xi\in(0,\tau_1],\\ \int_{-\tau_1}^{0}\psi(-t)\,d\Xi(t,\mu_0), & \text{if }\xi=0.\end{cases}$$

For convenience in computation, we shall allow functions with range in $\mathbb{C}^2$ instead of in $\mathbb{R}^2$. Thus, the domains of $\mathcal{A}(\mu_0)$ and $\mathcal{A}^*(\mu_0)$ are $C^1([-\tau_1,0],\mathbb{C}^2)$ and $C^1([0,\tau_1],\mathbb{C}^{2*})$, respectively, where $\mathbb{C}^{2*}$ is the space of 2-dimensional complex row vectors. It follows that $-i\omega_0$ is an eigenvalue of $\mathcal{A}^*(\mu_0)$ and

$$\mathcal{A}^*(\mu_0)p(\xi)=-i\omega_0p(\xi)$$

for some nonzero row-vector function $p(\xi)$, $\xi\in[0,\tau_1]$.

In order to construct coordinates to describe the center manifold $\mathcal{C}_\mu$ near the origin, we use the bilinear form (see Hale and Verduyn Lunel [22] and Diekmann et al. [7])

$$\langle\psi,\varphi\rangle_\mu=\overline{\psi}(0)\varphi(0)-\int_{\theta=-\tau_1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\,d\Xi(\theta,\mu)\,\varphi(\xi)\,d\xi$$

for $\psi \in C([0,\tau_1],\mathbb{C}^{2*})$ and $\varphi \in C([-\tau_1,0],\mathbb{C}^2)$. In particular, let $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mu_0}$. Then, as usual,

$$\langle \psi, \mathcal{A}(\mu_0)\varphi\rangle = \langle \mathcal{A}^*(\mu_0)\psi, \varphi\rangle \quad \text{for } (\varphi,\psi) \in \mathrm{Dom}(\mathcal{A}(\mu_0)) \times \mathrm{Dom}(\mathcal{A}^*(\mu_0)).$$

We normalize q and p so that

$$\langle p, q\rangle = 1 \quad \text{and} \quad \langle p, \bar{q}\rangle = 0.$$

By direct computation, we obtain that

$$p(\xi) = \overline{D}(\bar{d}, 1)e^{i\omega_0\xi}, \quad \xi \in [0,\tau_1],$$

where $D = \{2d[1+\tau(1+i\omega_0)]\}^{-1}$. For each $X \in \mathrm{Dom}(\mathcal{A}(\mu))$ with sufficiently small $\|\mu-\mu_0\|$, we associate it with the pair (z,w), where $z = \langle p, X\rangle$ and $w = X - zq - \bar{z}\bar{q} = X - 2\,\mathrm{Re}\{zq\}$. For a solution X_t of (3.1) at μ with sufficiently small $\|\mu-\mu_0\|$, we define $z(t) = \langle p, X_t\rangle$ and $w(z,\bar{z},\mu) = X_t - 2\,\mathrm{Re}\{z(t)q\}$. In fact, z and $\bar{z}$ are local coordinates for $\mathcal{C}_\mu$ in the directions of p and $\bar{p}$. Note that w is real if X_t is, since we shall deal with real solutions only. It is easy to see that $\langle p, w\rangle = 0$.

Now, for solutions $X_t \in \mathcal{C}_\mu$ of (3.1), $\langle p, \dot{X}_t\rangle = \langle p, \mathcal{A}(\mu)X_t + \mathcal{R}(\mu)X_t\rangle$. Then, on the center manifold $\mathcal{C}_\mu$ with sufficiently small $\|\mu-\mu_0\|$, we have

$$\dot{z}(t) = i\omega_0 z + g(z,\bar{z},\mu), \tag{3.2}$$

where the smooth functions $g(z,\bar{z},\mu) = -i\omega_0 z + \langle p, \mathcal{A}(\mu)(w(z,\bar{z},\mu) + 2\,\mathrm{Re}\{zq\})\rangle + \langle p, \mathcal{R}(\mu)(w(z,\bar{z},\mu) + 2\,\mathrm{Re}\{zq\})\rangle$. Let

$$g(z,\bar{z},\mu) = \sum_{s+k\geqslant 1} \frac{1}{s!k!} g_{sk}(\mu) z^s \bar{z}^k \quad \text{and} \quad w(z,\bar{z},\mu) = \sum_{s+k\geqslant 1} \frac{1}{s!k!} w_{sk}(\mu) z^s \bar{z}^k.$$

Similar to the computation in [18], Eq. (3.2) can be transformed by an invertible parameter-dependent change of complex coordinates into an equation with only cubic term:

$$\dot{z} = \lambda^*(\mu)z + \frac{1}{2}e_1(\mu)z^2\bar{z} + O(|z|^4), \tag{3.3}$$

where $\lambda^*(\mu) = i\omega_0 + (\mu-\mu_0)\nabla\lambda(\mu_0) + O(\|\mu-\mu_0\|^2)$, $\lambda(\mu)$ is a smooth function defined implicitly in Corollaries 2.2(iv) and 2.3(iii) and satisfies $\lambda(\mu_0) = i\omega_0$, and

$$e_1(\mu_0) = \frac{i}{\omega_0}\left[g_{20}(\mu_0)g_{11}(\mu_0) - 2\left|g_{11}(\mu_0)\right|^2 - \frac{1}{3}\left|g_{02}(\mu_0)\right|^2\right] + g_{21}(\mu_0).$$

Let $z = re^{i\theta}$. Then we can rewrite the normal form (3.3) as

$$\begin{aligned}\dot{r} &= r(\mu-\mu_0)\nabla\,\mathrm{Re}\{\lambda(\mu_0)\} + \frac{1}{2}r^3\,\mathrm{Re}\{e_1(\mu_0)\} + \text{h.o.t.},\\ \dot{\theta} &= \omega_0 + (\mu-\mu_0)\nabla\,\mathrm{Im}\{\lambda(\mu_0)\} + \frac{1}{2}r^2\,\mathrm{Im}\{e_1(\mu_0)\} + \text{h.o.t.}\end{aligned} \tag{3.4}$$

Theorem 3.1. *Assume that* $\mathbf{U} \triangleq \mathrm{Re}\{e_1(\mu_0)\}\nabla\,\mathrm{Re}\{\lambda(\mu_0)\} \neq 0$. *If* ϑ *is a row vector such that* $\vartheta\mathbf{U} \neq 0$ *then system* (3.2) *has a branch of bifurcated periodic solutions for* $\mu = \mu_0 + s\vartheta$ *with* s *satisfying* $s\vartheta\mathbf{U} < 0$. *The bifurcated periodic solutions are orbitally stable* (*respectively, unstable*) *if* $\mathrm{Re}\{e_1(\mu_0)\} < 0$ (*respectively*, > 0). *Moreover, the periods of the bifurcated periodic solutions are* $> \frac{2\pi}{\omega_0}$ (*respectively*, $< \frac{2\pi}{\omega_0}$) *if* $T_\vartheta < 0$ (*respectively*, > 0), *where*

$$T_\vartheta = \mathrm{Im}\big\{e_1(\mu_0)\big\} - \frac{\vartheta\nabla\mathrm{Im}\{\lambda(\mu_0)\}}{\vartheta\nabla\mathrm{Re}\{\lambda(\mu_0)\}}\,\mathrm{Re}\big\{e_1(\mu_0)\big\}.$$

Proof. We consider the following truncated system of (3.4),

$$\begin{aligned}\dot r &= rs\vartheta\nabla\,\mathrm{Re}\big\{\lambda(\mu_0)\big\} + \frac{1}{2}r^3\,\mathrm{Re}\big\{e_1(\mu_0)\big\},\\ \dot\theta &= \omega_0 + s\vartheta\nabla\,\mathrm{Im}\big\{\lambda(\mu_0)\big\} + \frac{1}{2}r^2\,\mathrm{Im}\big\{e_1(\mu_0)\big\},\end{aligned} \tag{3.5}$$

where $s\vartheta = \mu - \mu_0$. System (3.5) exhibits the same local bifurcation in a small neighborhood of the origin with sufficiently small s. We first consider the amplitude equation,

$$\dot r = rs\vartheta\nabla\,\mathrm{Re}\big\{\lambda(\mu_0)\big\} + \frac{1}{2}r^3\,\mathrm{Re}\big\{e_1(\mu_0)\big\}, \tag{3.6}$$

as it is decoupled from θ. Equation (3.6) always has the trivial equilibrium $r_0 = 0$. Other equilibria r of (3.6) satisfy $2s\vartheta\nabla\,\mathrm{Re}\{\lambda(\mu_0)\} + r^2\,\mathrm{Re}\{e_1(\mu_0)\} = 0$, which has exactly one positive solution

$$r_1 = \sqrt{\frac{-2s\vartheta\nabla\,\mathrm{Re}\{\lambda(\mu_0)\}}{\mathrm{Re}\{e_1(\mu_0)\}}} \tag{3.7}$$

if and only if $s\vartheta\mathbf{U} < 0$. Obviously, $r_1 \to 0$ as $s \to 0$ or equivalently $\mu \to \mu_0$. This implies that system (3.2) has a branch of periodic solutions bifurcated from the origin and exists for $\mu = \mu_0 + s\vartheta$ with $s\vartheta\mathbf{U} < 0$.

Note that the eigenvalue of the linearized operator of the right-hand side of (3.6) at $r = r_1$ is $r_1^2\,\mathrm{Re}\{e_1(\mu_0)\}$. As the stability of the bifurcated periodic solutions is the same as that of r_1, it follows that the bifurcated periodic solutions are stable if $\mathrm{Re}\{e_1(\mu_0)\} < 0$ and unstable otherwise.

Finally, we consider the phase equation for the bifurcated periodic solution of (3.5) corresponding to the equilibrium r_1. Namely,

$$\dot\theta = \omega_0 + s\vartheta\nabla\,\mathrm{Im}\big\{\lambda(\mu_0)\big\} + \frac{1}{2}r_1^2\,\mathrm{Im}\big\{e_1(\mu_0)\big\}. \tag{3.8}$$

It follows from (3.7) that

$$s = -\frac{r_1^2\,\mathrm{Re}\{e_1(\mu_0)\}}{2\vartheta\nabla\,\mathrm{Re}\{\lambda(\mu_0)\}}.$$

Substituting the above expression of s into Eq. (3.8) yields

$$\dot{\theta} = \omega_0 + \frac{1}{2} r_1^2 T_\vartheta .$$

Then the remaining conclusion of the theorem follows immediately. □

Now apply Theorem 3.1 to Hopf bifurcation analysis for (1.2) under the following additional assumption:

(H2) $f''(0) = 0$ and $f'''(0) \neq 0$.

Then we have $g_{20} = g_{11} = g_{02} = 0$ and

$$e_1(\mu_0) = g_{21}(\mu_0) = f'''(0)\langle p, \mathcal{C}(q, q, \bar{q})\rangle = N f'''(0)\big[\varepsilon_2(\omega_0) + \omega_0 i\big],$$

where $N = \frac{1}{2}(|d|^2 + 1)[(1+\tau)^2 + \tau^2\omega_0^2]^{-1}$ and $\varepsilon_2(\cdot)$ is the function defined in the proof of Lemma 2.1(ii). It follows that

$$\operatorname{Re}\big\{e_1(\mu_0)\big\} = N\varepsilon_2(\omega_0) f'''(0) \quad \text{and} \quad \operatorname{Im}\big\{e_1(\mu_0)\big\} = N\omega_0 f'''(0).$$

Therefore, if $a_{12}a_{21} < 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Sigma^+$, then it follows from the proof of Lemma 2.1(ii) that

$$\mathbf{U} = f'''(0)\varepsilon_1(\omega_0) N \varepsilon_2(\omega_0)\big(\beta_0\varepsilon_2(\omega_0) + \omega_0\eta_0, \eta_0\varepsilon_2(\omega_0) - \omega_0\beta_0\big)^T . \tag{3.9}$$

On the other hand, if $a_{12}a_{21} > 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Omega_j^+ \setminus [\bigcup_{s\geqslant 0} \Omega_s^-]$ or $\Omega_j^- \setminus [\bigcup_{s\geqslant 0} \Omega_s^+]$ for some $j \in \mathbb{N}$, then $\omega_0 = t_j$ and it follows from the proof of Corollary 2.2(iv) that

$$\mathbf{U} = f'''(0)\varepsilon_1(t_j) N \varepsilon_2^2(t_j) c_j (1, \pm 1)^T . \tag{3.10}$$

Corollary 3.2. *Assume that* $a_{12}a_{21} > 0$ *and that* (H1) *and* (H2) *hold. Then at* $\mu = \mu_0 \in \Omega_j^+ \setminus [\bigcup_{s\geqslant 0} \Omega_s^-]$ *or* $\Omega_j^- \setminus [\bigcup_{s\geqslant 0} \Omega_s^+]$ *for some* $j \in \mathbb{N}$, *system* (1.2) *undergoes a Hopf bifurcation. Namely, in every neighborhood of* $(0, \mu_0)$ *there exists a branch of periodic solutions, which approach the trivial solution as* $\mu \to \mu_0$. *Their period* $\omega(\mu)$ *satisfies* $\omega(\mu) \to 2\pi/t_j$ *as* $\mu \to \mu_0$. *Furthermore,* $\omega(\mu) \in [2\tau/j, 4\tau/(2j-1)]$. *The direction of the Hopf bifurcation and stability of the bifurcated periodic solutions are determined by* $\operatorname{sign}\{f'''(0)\}$. *More precisely*:

(i) *If* $f'''(0) < 0$ (*respectively,* > 0)*, then each branch of the bifurcated periodic solutions exists for* $\mu = \mu_0 + s\vartheta^{\pm}(\xi)$ *with* $s > 0$ (*respectively,* < 0) *and* $\xi \in (-\pi/2, \pi/2)$, *and the bifurcated periodic solutions have the same stability as the trivial solution had before the bifurcation* (*respectively, is unstable*)*, where* $\vartheta^{\pm}(\xi) = (-1)^j(1, \pm 1)M(\xi)$ *and the matrix function* $M(\xi)$ *is defined in Lemma* 2.1(ii).
(ii) *If* $[t_j - \varepsilon_2(t_j)\tan\xi] f'''(0) < 0$ (*respectively,* > 0) *then the period of the bifurcated periodic solutions is* $> \frac{2\pi}{t_j}$ (*respectively,* $< \frac{2\pi}{t_j}$).

Proof. It follows from the expression of **U** in (3.10) that, for $\xi \in (-\pi/2, \pi/2)$,

$$\vartheta(\xi)\mathbf{U} = 2f'''(0)\varepsilon_1(t_j)N\varepsilon_2^2(t_j)|c_j|\cos\xi.$$

Moreover, we have

$$\mathrm{Re}\{e_1(\mu_0)\} = N\varepsilon_2(t_j)f'''(0) \quad \text{and} \quad T_{\vartheta(\xi)} = \left[t_j - \varepsilon_2(t_j)\tan\xi\right]Nf'''(0).$$

Now we can apply Theorem 3.1 directly to complete the proof. □

Corollary 3.3. *Assume that* $a_{12}a_{21} < 0$ *and that* (H1) *and* (H2) *hold. Then at* $\mu_0 = (\beta_0, \eta_0) \in \Sigma^+$, *system* (1.2) *undergoes a Hopf bifurcation. Namely, in every neighborhood of* $(0, \mu_0)$ *there exists a branch of periodic solutions, which approach the trivial solution as* $\mu \to \mu_0$. *Their period* $\omega(\mu)$ *satisfies* $\omega(\mu) \to 2\pi/\omega_0$ *as* $\mu \to \mu_0$, *where* $\omega_0 \in \mathbb{R}$ *satisfies* $u(\omega_0) = \beta_0$ *and* $v(\omega_0) = \eta_0$. *The direction of the Hopf bifurcation and stability of the bifurcating periodic solutions are determined by* $\mathrm{sign}\{f'''(0)\}$. *More precisely*:

- (i) *If* $f'''(0) < 0$ (*respectively,* > 0), *then each branch of the bifurcating periodic solutions exists for* $\mu = \mu_0 + s\vartheta^{\pm}(\xi)$ *with* $s > 0$ (*respectively,* < 0) *and* $\xi \in (-\pi/2, \pi/2)$, *and the bifurcating periodic solution has the same stability as the trivial solution had before the bifurcation* (*respectively, is unstable*), *where* $\vartheta(\xi) = (v'(\theta_0), -u'(\theta_0))M(\xi)$ *and the matrix function* $M(\xi)$ *is defined in Lemma* 2.1(ii).
- (ii) *If* $[\omega_0 - \varepsilon_2(\omega_0)\tan\xi]f'''(0) < 0$ (*respectively,* > 0) *then the period of the bifurcated periodic solutions are* $> \frac{2\pi}{\omega_0}$ (*respectively,* $< \frac{2\pi}{\omega_0}$).

Proof. It follows from the expression of **U** in (3.9) that

$$\vartheta^{\pm}(\xi)\mathbf{U} = 2f'''(0)N\varepsilon_2(\omega_0)\cos\xi$$

for $\xi \in (-\pi/2, \pi/2)$. Moreover,

$$T_{\vartheta^{\pm}(\xi)} = \left[\omega_0 - \varepsilon_2(\omega_0)\tan\xi\right]Nf'''(0).$$

Again, applying Theorem 3.1 immediately completes the proof. □

Remark 3.4. In Corollaries 3.2 and 3.3, the stability of the bifurcated periodic solutions also depends on the nonexistence of unstable manifolds containing the trivial solutions. If there exists an unstable manifold containing the trivial solution, even though the periodic solution on the center manifold in the neighborhood of the trivial solution is stable, the bifurcated periodic solution is still unstable. Therefore, under assumptions of Corollary 3.2, only if $f'''(0) < 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Omega_1^-$ and $c_1 < \beta_0 + \eta_0 < 1$, are the bifurcated periodic solutions stable. Under assumptions of Corollary 3.3, only if $f'''(0) < 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Sigma_0^+$, are the bifurcated periodic solutions stable.

4. Bautin bifurcation

Throughout this section, as in Section 3, we still assume that either there exists $j \in \mathbb{N}$ such that $\mu_0 = (\beta_0, \eta_0) \in \Omega_j^+ \setminus \{\bigcup_{s \geqslant 0} \Omega_s^-\}$ or $\Omega_j^- \setminus \{\bigcup_{s \geqslant 0} \Omega_s^+\}$ when $a_{12}a_{21} > 0$ or $\mu_0 = (\beta_0, \eta_0) \in \Sigma^+$ when $a_{12}a_{21} < 0$. Similar to the analysis of Hopf bifurcation in Section 3, we can perform an invertible parameter-dependent change of complex coordinates to obtain (3.3) for all sufficiently small $\|\mu - \mu_0\|$. If $\operatorname{Re}\{e_1(\mu_0)\} \neq 0$ then we have obtained some results about the codimension one Hopf bifurcation in Corollaries 3.2 and 3.3. However, if $\operatorname{Re}\{e_1(\mu_0)\} = 0$ then the Bautin bifurcation occurs (see Kuznetsov [27]) and we have to further perform an invertible parameter-dependent change of complex coordinates smoothly depending on μ to annihilate all the fourth-order terms in (3.2) with the coefficient of the resonant cubic term $e_1(\mu)$ untouched while changing the coefficients of the fifth- and higher-order terms. Finally, we can remove all the fifth-order terms except the resonant one shown in $z|z|^4$ and obtain the following normal form,

$$\dot{z} = \lambda^*(\mu)z + \frac{1}{2}e_1(\mu)z|z|^2 + \frac{1}{12}e_2(\mu)z|z|^4 + O\big(|z|^6\big), \tag{4.1}$$

where $\lambda^*(\mu)$ and $e_1(\mu)$ are the same as those in Section 3 satisfying $\operatorname{Re}\{e_1(\mu_0)\} = 0$. According to the center manifold theory, if there exists an unstable manifold containing the trivial solution of system (1.2), then all bifurcated periodic solutions are unstable. If there exists no unstable manifold containing the trivial solution, then the bifurcated periodic solutions of system (1.2) have the same stability as the corresponding periodic solutions of (4.1).

Let $z = re^{i\theta}$. Then we can rewrite the normal form (4.1) as

$$\begin{aligned} \dot{r} &= \alpha_1 r + \alpha_2 r^3 + Lr^5 + \text{h.o.t.}, \\ \dot{\theta} &= \omega_0 + (\mu - \mu_0)\nabla \operatorname{Im}\big\{\lambda(\mu_0)\big\} + \frac{1}{2}r^2 \operatorname{Im}\big\{e_1(\mu_0)\big\} + \text{h.o.t.}, \end{aligned} \tag{4.2}$$

where $\alpha_1(\mu) = (\mu - \mu_0)\nabla \operatorname{Re}\lambda(\mu_0)$, $\alpha_2(\mu) = \frac{1}{2}\operatorname{Re}\{e_1(\mu)\}$ and $L = \frac{1}{12}\operatorname{Re}\{e_2(\mu_0)\}$. It is easy to see that $\alpha = (\alpha_1, \alpha_2)$ satisfies $\alpha(\mu_0) = 0$. Assume that the map $\mu \mapsto \alpha$ is regular at μ_0, i.e., the Jacobian matrix $\partial(\alpha_1, \alpha_2)/\partial(\beta, \eta)$ is nonsingular at μ_0. In other words, the map $\mu \mapsto \alpha$ is locally invertible in a sufficiently small neighborhood of μ_0. Thus, we can use α instead of μ to give a complete bifurcation diagram of (4.2), i.e., existence, multiplicity, and stability of bifurcated periodic solutions. By a direct computation, we have

$$\begin{aligned} 12L = {} & \operatorname{Re}\{g_{32}\} + \frac{1}{\omega_0}\operatorname{Im}\left\{g_{20}\bar{g}_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12}\right\} \\ & + \frac{1}{\omega_0^2}\operatorname{Re}\left\{g_{20}\left[\bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30}\right) + \bar{g}_{02}g_{03}\right]\right\} \\ & + \frac{1}{\omega_0^2}\operatorname{Re}\left\{g_{11}\left[\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12}\right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30}\right]\right\} \\ & + \frac{3}{\omega_0^2}\operatorname{Im}\{g_{20}g_{11}\}\operatorname{Im}\{g_{21}\} + \frac{1}{\omega_0^3}\operatorname{Im}\big\{g_{11}\bar{g}_{02}\big[\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\big]\big\} \\ & + \frac{1}{\omega_0^3}\operatorname{Im}\{g_{11}g_{20}\}\big[3\operatorname{Re}\{g_{11}g_{20}\} - 2|g_{02}|^2\big], \end{aligned}$$

where all the g_{kl}'s are evaluated at μ_0. In deriving this formula, we have taken the equation $\mathrm{Re}\{e_1(\mu_0)\} = 0$ (or, equivalently, $\omega_0 \mathrm{Re}\{g_{21}\} = \mathrm{Im}\{g_{20}g_{11}\}$) into account.

The first equation in (4.2) is uncoupled from the second one. The equation for θ describes a rotation around (0, 0) with almost constant angular velocity $\dot{\theta} \approx \omega_0$ for $\|\mu - \mu_0\|$ small. Thus, to understand the bifurcations in (4.2), it suffices to study the scalar equation for r, that is,

$$\dot{r} = \alpha_1 r + \alpha_2 r^3 + L r^5 + O(r^6). \tag{4.3}$$

Rescaling $r = |L|^{-1/4}\sqrt{\rho}$ and defining the parameters $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2/\sqrt{|L|}$, we obtain the following truncated equation for ρ without the $O(\rho^4)$ terms,

$$\dot{\rho} = 2\rho(\gamma_1 + \gamma_2\rho - \rho^2) \quad \text{if } L < 0 \tag{4.4}$$

and

$$\dot{\rho} = 2\rho(\gamma_1 + \gamma_2\rho + \rho^2) \quad \text{if } L > 0. \tag{4.5}$$

It follows that the trivial equilibrium $\rho = 0$ of (4.4) or (4.5) corresponds to the equilibrium $z = 0$ of (4.1), and the existence and stability of positive equilibria of (4.4) or (4.5) determine the existence and stability of periodic solutions of (4.1) and hence of the original system (1.2). In the remaining part of this section, we depict the complete bifurcation diagrams of (4.4) and (4.5) on the (γ_1, γ_2)-parameter plane.

We first consider (4.4). Positive equilibria of (4.4) satisfy $\gamma_1 + \gamma_2\rho - \rho^2 = 0$, which can have zero, one, or two positive solutions. These solutions branch from the trivial one along the line l_1 on the (γ_1, γ_2)-parameter plane and collide and disappear at the half-parabola l_2 (see Fig. 2(a)), where

$$l_1 : \gamma_1 = 0 \quad \text{and} \quad l_2 : \gamma_2^2 + 4\gamma_1 = 0 \quad \text{with } \gamma_2 > 0.$$

The detail is summarized below.

(1) In the region $\mathcal{D}_{11} = \{(\gamma_1, \gamma_2)\colon\ \gamma_2^2 + 4\gamma_1 < 0 \text{ or } \gamma_1 < 0 \text{ and } \gamma_2 < 0\}$, (4.4) has no positive equilibria. Thus, the equilibrium $\rho = 0$ is globally asymptotically stable, which means that system (4.1) has no periodic solutions in a sufficiently small neighborhood of the stable equilibrium $z = 0$.

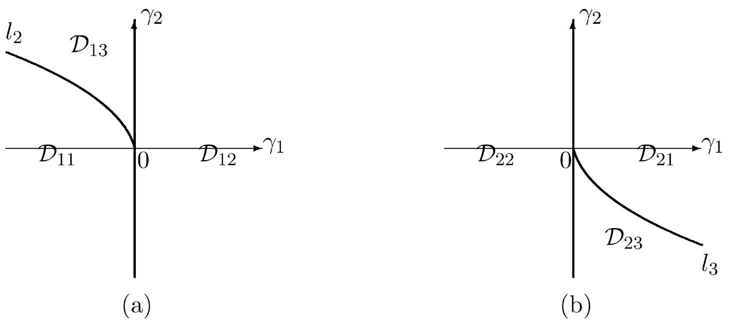

Fig. 2. Bifurcation sets for (4.4) and (4.5).

(2) In the region $\mathcal{D}_{12} = \{(\gamma_1, \gamma_2)\colon\ \gamma_1 > 0\}$, (4.4) has only one positive equilibrium, which is stable. This means that system (4.1) has exactly one stable periodic solution in a sufficiently small neighborhood of the unstable equilibrium $z = 0$.
(3) In the region $\mathcal{D}_{13} = \{(\gamma_1, \gamma_2)\colon\ \gamma_1 < 0,\ \gamma_2 > 0, \text{ and } \gamma_2^2 + 4\gamma_1 > 0\}$, (4.4) has two positive equilibria, one stable and the other unstable. This means that system (4.1) has one stable periodic solution and one unstable periodic solution in a sufficiently small neighborhood of the stable equilibrium $z = 0$.

Therefore, on the (γ_1, γ_2)-parameter plane, the line l_1 and the half-parabola l_2 are bifurcation curves. The bifurcation scenario is explained below.

(i) On the (γ_1, γ_2)-parameter plane, if the point (γ_1, γ_2) crosses the line l_1 from region $\mathcal{D}_{11}$ to region $\mathcal{D}_{12}$, then (4.1) undergoes a Hopf bifurcation and a stable limit cycle bifurcates from $z = 0$.
(ii) On the (γ_1, γ_2)-parameter plane, if the point (γ_1, γ_2) crosses the line l_1 from region $\mathcal{D}_{12}$ to region $\mathcal{D}_{13}$, then (4.1) undergoes a Hopf bifurcation and an unstable limit cycle bifurcates from $z = 0$.
(iii) On the (γ_1, γ_2)-parameter plane, if the point (γ_1, γ_2) crosses the line l_2 from region $\mathcal{D}_{13}$ to region $\mathcal{D}_{11}$, then limit cycles of (4.1) undergo a fold bifurcation, i.e., the two limit cycles collide and then disappear.

Now we come to the complete bifurcation diagram of (4.5). Positive equilibria of (4.5) satisfy $\gamma_1 + \gamma_2\rho + \rho^2 = 0$, which can have zero, one, or two positive solutions. These solutions branch from the trivial one along the line l_1 on the (γ_1, γ_2)-parameter plane and collide and disappear at the half-parabola l_3: $\gamma_2^2 - 4\gamma_1 = 0$ and $\gamma_2 < 0$ (see Fig. 2(b)). We have the following conclusions.

(1) In the region $\mathcal{D}_{21} = \{(\gamma_1, \gamma_2)\colon\ \gamma_2^2 - 4\gamma_1 < 0 \text{ or } \gamma_1 > 0 \text{ and } \gamma_2 > 0\}$, (4.5) has no positive equilibrium. Thus, the equilibrium $\rho = 0$ is unstable. This means that system (4.1) has no periodic solutions in a sufficiently small neighborhood of the unstable equilibrium $z = 0$.
(2) In the region $\mathcal{D}_{22} = \{(\gamma_1, \gamma_2)\colon\ \gamma_1 < 0\}$, (4.5) has only one positive equilibrium, which is unstable. This means that system (4.1) has exactly one unstable periodic solution in a sufficiently small neighborhood of the stable equilibrium $z = 0$.
(3) In the region $\mathcal{D}_{23} = \{(\gamma_1, \gamma_2)\colon\ \gamma_1 > 0,\ \gamma_2 < 0, \text{ and } \gamma_2^2 - 4\gamma_1 > 0\}$, (4.4) has two positive equilibria, one stable and the other unstable. This means that system (4.1) has one stable periodic solution and one unstable periodic solution in a sufficiently small neighborhood of the unstable equilibrium $z = 0$.

Therefore, on the parameter plane (γ_1, γ_2), the line l_1 and the half-parabola l_3 are bifurcation curves. More specifically, we have

(i) on the (γ_1, γ_2)-parameter plane, if the point (γ_1, γ_2) crosses the line l_1 from region $\mathcal{D}_{21}$ to region $\mathcal{D}_{22}$, then (4.1) undergoes a Hopf bifurcation and an unstable limit cycle bifurcates from $z = 0$;
(ii) on the (γ_1, γ_2)-parameter plane, if the point (γ_1, γ_2) crosses the line l_1 from region $\mathcal{D}_{22}$ to region $\mathcal{D}_{23}$, then (4.1) undergoes a Hopf bifurcation and a stable limit cycle bifurcates from $z = 0$;

(iii) on the (γ_1, γ_2)-parameter plane, if the point (γ_1, γ_2) crosses the line l_3 from region $\mathcal{D}_{23}$ to region $\mathcal{D}_{21}$, then limit cycles of (4.1) undergo a fold bifurcation, i.e., the two limit cycles collide and then disappear.

5. Fold bifurcation

When $a_{12}a_{21} > 0$, a fold bifurcation may occur at $\mu_0 = (\beta_0, \eta_0) \in \{\Omega_0^+ \setminus [\bigcup_{s\geqslant 0} \Omega_s^-]\} \cup \{\Omega_0^- \setminus [\bigcup_{s\geqslant 0} \Omega_s^+]\}$, where the infinitesimal generator $\mathcal{A}(\mu_0)$ may have a simple eigenvalue zero. In this section, we only consider the first case because the second one can be discussed analogously. Therefore, throughout this section, we always assume that

(H3) $a_{12}a_{21} > 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Omega_0^+ \setminus [\bigcup_{s\geqslant 0} \Omega_s^-]$.

Under assumption (H3), $\beta_0 + \eta_0 = 1$ and $\beta_0 - \eta_0 \neq c_n$ for all $n \in \mathbb{N}_0$. It follows from Corollary 2.2 that $\mathcal{A}(\mu)$ has a simple real eigenvalue $\lambda(\mu)$ for all sufficiently small $\|\mu - \mu_0\|$, where λ is a smooth function of μ such that $\lambda(\mu_0) = 0$. Thus, $\mathcal{A}(\mu)$ has a real eigenvector $\mathcal{Q}(\mu) \in \mathbf{C}$, smoothly dependent on the parameter and corresponding to the eigenvalues $\lambda(\mu)$:

$$\mathcal{A}(\mu)\mathcal{Q}(\mu) = \lambda(\mu)\mathcal{Q}(\mu).$$

Moreover, $\lambda(\mu)$ is also an eigenvalue of the adjoint operator $\mathcal{A}^*(\mu)$ with adjoint eigenvector defined by

$$\mathcal{A}^*(\mu)\mathcal{P}(\mu) = \lambda(\mu)\mathcal{P}(\mu).$$

We normalize the eigenvectors such that $\langle \mathcal{P}(\mu), \mathcal{Q}(\mu)\rangle_\mu = 1$ for all sufficiently small $\|\mu - \mu_0\|$. For simplicity, in the sequel, let $q = \mathcal{Q}(\mu_0)$ and $p = \mathcal{P}(\mu_0)$. In fact, we can choose $\mathcal{P}(\mu)$ and $\mathcal{Q}(\mu)$ such that $q(\theta) = (1, d_0)^T$ for $\theta \in [-\tau_1, 0]$, and $p(\xi) = D_0(d_0, 1)$ for $\xi \in [0, \tau_1]$, where $d_0 = \eta_0/a_{12}$ and $D_0 = [2d_0(1+\tau)]^{-1}$. Obviously, $\langle p, q\rangle = 1$.

We associate each $X \in \mathrm{Dom}(\mathcal{A}(\mu))$ with the pair (x, w), where $x = \langle p, X\rangle$ and $w = X - xq$. For a solution X_t of (3.1) at μ, we define $x(t) = \langle p, X_t\rangle$ and $w(x, \mu) = X_t - xq$. It is easy to see that $\langle p, w\rangle = 0$. Now, for solutions $X_t \in \mathcal{C}_\mu$ of (3.1), $\langle p, \dot{X}_t\rangle = \langle p, \mathcal{A}(\mu)X_t + \mathcal{R}(\mu)X_t\rangle$. Then, on the center manifold $\mathcal{C}_\mu$, we have

$$\dot{x}(t) = g(x, \mu), \tag{5.1}$$

where the real smooth function g is given as follows,

$$g(x, \mu) = \big\langle p, \mathcal{A}(\mu)\big(w(x, \mu) + xq\big) + \mathcal{R}(\mu)\big(w(x, \mu) + xq\big)\big\rangle = \sum_{j\geqslant 1} \frac{1}{j!} g_j(\mu) x^j.$$

By a direct computation, we have

$$\begin{aligned}
g_1(\mu) &= (\mu - \mu_0)\nabla\lambda(\mu_0) + O\big(\|\mu - \mu_0\|^2\big),\\
g_2(\mu_0) &= f''(0)\big\langle p, \mathcal{B}(q, q)\big\rangle = \frac{1}{2} f''(0)(1+\tau)^{-1}(1+d_0),\\
g_3(\mu_0) &= \big\langle p, f'''(0)\mathcal{C}(q, q, q) + 3f''(0)\mathcal{B}(w_2, q)\big\rangle.
\end{aligned}$$

We still need to compute $w_2(\theta)$ for $\theta \in [-\tau_1, 0)$. Indeed, on the center manifold $\mathcal{C}_0$, at $\mu = \mu_0$, $\dot{w} = \dot{X}_t - \dot{x}q = \mathcal{A}(\mu_0)w + \mathcal{R}(\mu_0)(xq_0 + w) - g(x, \mu_0)q_0$. We rewrite this as

$$\dot{w} = \mathcal{A}(\mu_0)w + h(x), \tag{5.2}$$

where $h(x)(\theta) = -g(x, \mu_0)q(\theta)$ for $\theta \in [-\tau_1, 0)$, and $h(x)(0) = \mathcal{R}(\mu_0)(xq + w)(0) - g(x, \mu_0)q(0)$. Therefore, $h_2(\theta) = -g_{20}q(\theta)$ for $\theta \in [-\tau_1, 0)$.

Let $w(x, \mu_0) = \frac{1}{2}w_2x^2 + \frac{1}{6}w_3x^3 + \cdots$ and $h(x) = \frac{1}{2}h_2x^2 + \frac{1}{6}h_3x^3 + \cdots$. Comparing the coefficients, we obtain $(\mathcal{A}(\mu_0)w_2)(\theta) = -h_2(\theta)$. Thus, for $\theta \in [-\tau_1, 0)$,

$$\dot{w}_2(\theta) = g_2(\mu_0)q(\theta).$$

Solving for $w_2(\theta)$, we obtain $w_2(\theta) = g_2(\mu_0)\theta q(\theta) + E_0$, where E_0 is a 2-dimensional vector and it can be determined by the equation $h(x)(0) = \mathcal{R}(\mu_0)(xq + w)(0) - g(x, \mu_0)q(0)$. Namely,

$$h_2(0) = -g_2(\mu_0)q(0) + f''(0)\big(\beta + a_{12}d_0^2, a_{21} + \beta d_0^2\big)^T.$$

This, together with (5.2), implies that

$$\begin{aligned}
-w_2^1(0) + \beta w_2^1(-\tau) + a_{12}w_2^2(-\tau_1) &= g_2(\mu_0) - f''(0)\big(\beta + a_{12}d_0^2\big),\\
-w_2^2(0) + a_{21}w_2^1(-\tau_2) + \beta w_2^2(-\tau) &= g_2(\mu_0)d_0 - f''(0)\big(a_{21} + \beta d_0^2\big),
\end{aligned}$$

where w_2^1 and w_2^2 are the components of w_2. Substituting $w_2(\theta) = g_2(\mu_0)\theta q(\theta) + E_0$ into the above equation and noticing that $\Delta(\mu_0, 0)q_0(0) = 0$, we have

$$E_0 = e_0(1, -d_0)^T \quad \text{and} \quad e_0 = \left[\frac{(1 + d_0)(\tau_2 - \tau_1)}{8(1 + \tau)} - \frac{(d_0 - 1)(1 - 2\eta_0)}{4\eta_0}\right]f''(0).$$

If $f''(0) \neq 0$, then the truncated form of (5.1) is

$$\dot{x} = (\mu - \mu_0)\nabla\lambda(\mu_0)x + \frac{1}{2}g_2(\mu_0)x^2, \tag{5.3}$$

where $\nabla\lambda(\mu) = (\frac{\partial\lambda}{\partial\beta}, \frac{\partial\lambda}{\partial\eta})^T$. It follows from the proof of Corollary 2.2 that

$$\nabla\lambda(\mu_0) = (1 + \tau)^{-1}(1, 1)^T.$$

Therefore, (5.3) is equivalent to the following equation,

$$(1 + \tau)\dot{x} = (\beta + \eta - 1)x + \frac{1}{4}f''(0)(1 + d_0)x^2. \tag{5.4}$$

It is easy to see that (5.4) has two equilibria: $x_1 = 0$ and $x_2 = 4(1 - \beta - \eta)/[(1 + d_0)f''(0)]$. Moreover, if $\beta + \eta > 1$ then x_1 is unstable and x_2 is stable. If $\beta + \eta < 1$, then x_1 is stable and x_2 is unstable. These two equilibria coalesce at $\mu = \mu_0$. Thus, we obtain a transcritical bifurcation of equilibria of system (1.2).

Theorem 5.1. *Under assumptions* (H1) *and* (H3), *if* $f''(0) \neq 0$, *then system* (1.2) *undergoes a transcritical bifurcation near* μ_0. *Namely, besides the trivial solution, system* (1.2) *has a nonzero equilibrium, which continuously depends on* μ *for all sufficiently small* $\|\mu - \mu_0\|$. *Moreover, this nonzero equilibrium is stable if* $\beta + \eta > 1$ *and* $\mu_0 = (\beta_0, \eta_0)$ *satisfies* $c_1 < \beta_0 - \eta_0 < \beta_0 + \eta_0 = 1$, *and unstable otherwise.*

In what follows, we investigate the case where assumption (H2) holds. Then the truncated form of (5.1) is

$$(1+\tau)\dot{x} = (\beta + \eta - 1)x + \frac{1}{6} g_3(\mu_0) x^3, \tag{5.5}$$

where $g_3(\mu_0) = \langle p_0, f'''(0)\mathcal{C}(q_0, q_0, q_0)\rangle = \frac{1}{2} f'''(0)(1 + d_0^2)/(1+\tau)$. It is easy to see that (5.4) has only one equilibrium $x_1 = 0$ if $(\beta + \eta - 1) f'''(0) > 0$, and otherwise, three equilibria $x_1 = 0$ and $x_{2,3} = \pm\sqrt{6(1 - \beta - \eta)/g_3(\mu_0)}$. Namely, there exists a pitchfork bifurcation at $\mu = \mu_0$. More precisely, if $f'''(0) < 0$, then, for $\beta + \eta < 1$, (5.5) has the stable equilibrium x_1; for $\beta + \eta > 1$, x_1 is still an equilibrium, but two new equilibria x_2 and x_3 appear. In this process, x_1 becomes unstable for $\beta + \eta > 1$ while the other two equilibria are stable. Therefore, we have the following conclusion:

Theorem 5.2. *Under assumptions* (H1)–(H3), *system* (1.2) *undergoes a pitchfork bifurcation near* μ_0. *More precisely, we have the following statements.*

(i) *If* $f'''(0) < 0$, *two nontrivial equilibria exist for* μ *with* $\beta + \eta > 1$ (*which are stable if* $c_1 < \beta_0 - \eta_0 < 1$ *and unstable otherwise*) *and only the trivial equilibrium continues existing for* $\beta + \eta < 1$. *Moreover, the two nontrivial equilibria coalesce into zero as* μ *goes to* μ_0.
(ii) *If* $f'''(0) > 0$, *two nontrivial equilibria exist for* μ *with* $\beta + \eta < 1$ (*which are unstable*) *and only the trivial equilibrium continues existing for* $\beta + \eta > 1$. *Moreover, the two nontrivial equilibria coalesce into zero as* μ *goes to* μ_0.

6. Fold–Hopf bifurcation

Assume that $a_{12}a_{21} > 0$. If $\mu_0 = (\beta_0, \eta_0) \in \Omega_0^+ \cap \Omega_n^-$ or $\mu_0 = (\beta_0, \eta_0) \in \Omega_0^- \cap \Omega_n^+$ for some $n \in \mathbb{N}$, then zero and $\pm i t_n$ are simple eigenvalues of the infinitesimal generator $\mathcal{A}(\mu_0)$. Thus, a fold–Hopf bifurcation, which is a type of codimension two bifurcations, may occur. Again, we only consider the first case because the second one can be discussed analogously. Therefore, throughout this section, we always assume that

(H4) $a_{12}a_{21} > 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Omega_0^+ \cap \Omega_n^-$ for some $n \in \mathbb{N}$.

It follows from Corollary 2.2 that $\mathcal{A}(\mu)$ has simple eigenvalues $\lambda_0(\mu)$, $\lambda_1(\mu)$, and $\overline{\lambda_1(\mu)}$ for all sufficiently small $\|\mu - \mu_0\|$, where the real function $\lambda_0(\mu)$ and the complex function $\lambda_1(\mu)$ respectively satisfy

$$\lambda_0(\mu_0) = 0 \quad \text{and} \quad \lambda_1(\mu_0) = i t_n. \tag{6.1}$$

Thus, $\mathcal{A}(\mu)$ has two eigenvectors $\mathcal{Q}_0(\mu) \in \mathbf{C}$ and $\mathcal{Q}_1(\mu) \in \mathbf{C}$, smoothly dependent on the parameter and corresponding to the eigenvalues $\lambda_0(\mu)$ and $\lambda_1(\mu)$, respectively:

$$\mathcal{A}(\mu)\mathcal{Q}_0(\mu) = \lambda_0(\mu)\mathcal{Q}_0(\mu), \qquad \mathcal{A}(\mu)\mathcal{Q}_1(\mu) = \lambda_1(\mu)\mathcal{Q}_1(\mu).$$

Moreover, $\lambda_0(\mu)$ and $\overline{\lambda_1(\mu)}$ are also eigenvalues of the adjoint operator $\mathcal{A}^*(\mu)$ with adjoint eigenvectors defined by

$$\mathcal{A}^*(\mu)\mathcal{P}_0(\mu) = \lambda_0(\mu)\mathcal{P}_0(\mu), \qquad \mathcal{A}^*(\mu)\mathcal{P}_1(\mu) = \overline{\lambda_1(\mu)}\mathcal{P}_1(\mu).$$

We normalize the eigenvectors such that $\langle \mathcal{P}_j(\mu), \mathcal{Q}_k(\mu)\rangle_\mu = \delta_{jk}$ for all sufficiently small $\|\mu - \mu_0\|$, where δ_{jk} is the Kronecker delta. For simplicity, in the sequel, let $p_j = \mathcal{P}_j(\mu_0)$ and $q_j = \mathcal{Q}_j(\mu_0)$, $j = 0, 1$. In view of (6.1), we can choose

$$q_0(\theta) = (1, d_0)^T, \qquad q_1(\theta) = (1, d_1)^T e^{it_n\theta}, \quad \theta \in [-\tau_1, 0],$$

and

$$p_0(\xi) = D_0(d_0, 1), \qquad p_1(\xi) = \overline{D}_1(\bar{d}_1, 1)e^{it_n\xi}, \quad \xi \in [0, \tau_1],$$

where $d_0 = \eta_0/a_{12}$ and $d_1 = -\eta_0 e^{it_n(\tau_1-\tau)}/a_{12}$, $D_0 = [2d_0(1+\tau)]^{-1}$, and $D_1 = \{2d_1[1 + \tau(1+it_n)]\}^{-1}$.

We again associate each $X \in \mathrm{Dom}(\mathcal{A}(\mu))$ with the triple (x, z, w), where $x = \langle p_0, X\rangle$, $z = \langle p_1, X\rangle$, and $w = X - xq_0 - zq_1 - \bar{z}\bar{q}_1 = X - xq_0 - 2\,\mathrm{Re}\{zq_1\}$. For a solution X_t of (3.1) at μ, we define $x(t) = \langle p_0, X_t\rangle$, $z(t) = \langle p_1, X_t\rangle$, and $w(x, z, \bar{z}, \mu) = X_t - xq_0 - 2\,\mathrm{Re}\{z(t)q_1\}$. In fact, x, z, and $\bar{z}$ are local coordinates for $\mathcal{C}_\mu$ in the directions of p_0, p_1 and $\overline{p}_1$. It is easy to see that $\langle p_0, w\rangle = 0$ and $\langle p_1, w\rangle = 0$. Now, for solutions $X_t \in \mathcal{C}_\mu$ of (3.1), $\langle p_j, \dot{X}_t\rangle = \langle p_j, \mathcal{A}(\mu)X_t + \mathcal{R}(\mu)X_t\rangle$, $j = 0, 1$. Then, on the center manifold $\mathcal{C}_\mu$, we have

$$\begin{aligned} \dot{x}(t) &= g(x, z, \bar{z}, \mu), \\ \dot{z}(t) &= it_n z(t) + h(x, z, \bar{z}, \mu), \end{aligned} \tag{6.2}$$

where the smooth functions g and h are given as follows,

$$\begin{aligned} g(x, z, \bar{z}, \mu) &= \big\langle p_0, \mathcal{A}(\mu)\big(w(x, z, \bar{z}, \mu) + xq_0 + 2\,\mathrm{Re}\{zq_1\}\big)\big\rangle \\ &\quad + \big\langle p_0, \mathcal{R}(\mu)\big(w(x, z, \bar{z}, \mu) + xq_0 + 2\,\mathrm{Re}\{zq_1\}\big)\big\rangle, \\ h(x, z, \bar{z}, \mu) &= \big\langle p_1, \mathcal{A}(\mu)\big(w(x, z, \bar{z}, \mu) + xq_0 + 2\,\mathrm{Re}\{zq_1\}\big)\big\rangle - it_n z \\ &\quad + \big\langle p_1, \mathcal{R}(\mu)\big(w(x, z, \bar{z}, \mu) + xq_0 + 2\,\mathrm{Re}\{zq_1\}\big)\big\rangle. \end{aligned}$$

According to the center manifold theory, if there exists an unstable manifold containing the trivial solution of system (1.2), then all bifurcated equilibria, periodic solutions, and invariant tori (quasi-periodic solutions) are unstable. If there exists no unstable manifold containing the trivial solution, then the bifurcated equilibria, periodic solutions, or invariant tori of system (1.2) have the same stability as the corresponding solution of (6.2) has.

System (6.2) is a system of ordinary differential equations and has a fold–Hopf singularity at $\mu = \mu_0$. The unfoldings of this singularity have been studied by several authors [1,5,8,11,12, 14–16,27,34–36]. Here we present a different description of the singularity as well as the normal form for the unfoldings of this singularity.

We leave the detailed derivations of normal form of (6.2) to Appendix A. Thus, under the nondegeneracy conditions that $g_{200}(\mu_0)g_{011}(\mu_0) \neq 0$, system (6.2) can be simplified to

$$\begin{aligned}\dot{y} &= e_{10}(\mu)y + e_{11}(\mu)y^2 + e_{12}(\mu)|\nu|^2 + O\big(\big\|(y,\nu,\bar{\nu})\big\|^4\big),\\ \dot{\nu} &= it_n\nu + e_{20}(\mu)\nu + e_{21}(\mu)y\nu + e_{22}(\mu)y^2\nu + O\big(\big\|(y,\nu,\bar{\nu})\big\|^4\big),\end{aligned} \tag{6.3}$$

where $e_{1j}(\mu)$ $(j = 0, 1, 2)$ and $e_{22}(\mu)$ are smooth real-valued functions, while e_{2j} $(j = 0, 1)$ are smooth complex-valued functions.

Let $\nu = re^{i\xi}$. Then the above equations can be rewritten as

$$\begin{aligned}\dot{y} &= (\mu-\mu_0)\nabla\lambda_0(\mu_0)y + e_{11}(\mu_0)y^2 + e_{12}(\mu_0)r^2 + O\big(\big|(y,r)\big|^4\big),\\ \dot{r} &= (\mu-\mu_0)\nabla\operatorname{Re}\big\{\lambda_1(\mu_0)\big\}r + a_1yr + a_2y^2r + O\big(\big|(y,r)\big|^4\big),\\ \dot{\xi} &= t_n + (\mu-\mu_0)\nabla\operatorname{Im}\big\{\lambda_1(\mu_0)\big\} + b_1y + O\big(\big|(y,r)\big|^3\big),\end{aligned} \tag{6.4}$$

where $a_1 + ib_1 = e_{21}(\mu_0)$ and $a_2 = e_{22}(\mu_0)$. Let $\alpha_1(\mu) = (\mu-\mu_0)\nabla\lambda_0(\mu_0)$ and $\alpha_2(\mu) = (\mu - \mu_0)\nabla\operatorname{Re}\{\lambda_1(\mu_0)\}$, i.e.,

$$\alpha_1 = (\beta+\eta-1)/(1+\tau), \qquad \alpha_2 = (\beta-\eta-c_n)c_n\varepsilon_1(t_n)\varepsilon_2(t_n).$$

It follows from the proof of Corollary 2.2 that

$$\left|\frac{\partial(\alpha_1,\alpha_2)}{\partial(\beta,\eta)}\right|_{\mu=\mu_0} = -2(1+\tau)^{-1}c_n\varepsilon_1(t_n)\varepsilon_2(t_n) \neq 0.$$

This means that the mapping $\mu \to (\alpha_1, \alpha_2)$ is regular at μ_0. Thus, system (6.4) can be rewritten as

$$\begin{aligned}\dot{y} &= \alpha_1y + e_{11}(\mu_0)y^2 + e_{12}(\mu_0)r^2 + O\big(\big|(y,r)\big|^4\big),\\ \dot{r} &= \alpha_2r + a_1yr + a_2y^2r + O\big(\big|(y,r)\big|^4\big),\\ \dot{\xi} &= t_n + (\mu-\mu_0)\nabla\operatorname{Im}\big\{\lambda(\mu_0)\big\} + b_1y + O\big(\big|(y,r)\big|^3\big).\end{aligned} \tag{6.5}$$

The first two equations of (6.5) are decoupled from the third one. The equation for ξ describes a rotation around the y-axis with almost constant angular velocity $\dot{\xi} \approx t_n$ for y and $\|\mu - \mu_0\|$ small. Thus, to understand the bifurcations in (6.5), we only need to study the planar system for (y, r) with $r \geqslant 0$:

$$\begin{aligned}\dot{y} &= \alpha_1y + e_{11}(\mu_0)y^2 + e_{12}(\mu_0)r^2 + O\big(\big|(y,r)\big|^4\big),\\ \dot{r} &= \alpha_2r + a_1yr + a_2y^2r + O\big(\big|(y,r)\big|^4\big).\end{aligned} \tag{6.6}$$

This system is often called an amplitude system.

If considered in the whole (y, r)-plane, system (6.6) is $\mathbb{Z}_2$-symmetric, since the reflection $r \to -r$ leaves it invariant. Using the following linear scalings of the variables and time,

$$y = \frac{e_{11}(\mu_0)}{a_2} y^*, \qquad r = \sqrt{\frac{|e_{11}^3(\mu_0)|\rho}{|e_{12}(\mu_0)|a_2^2}}, \qquad t = \frac{a_2}{e_{11}^2(\mu_0)} t^*, \tag{6.7}$$

and then dropping $*$, we obtain the following truncated system of (6.6) without the $O(|(y, \rho^2)|^4)$ terms,

$$\begin{aligned} \dot{y} &= \gamma_1 y + y^2 + \varepsilon\rho, \\ \dot{\rho} &= 2\rho\big(\gamma_2 + \gamma_3 y + y^2\big), \end{aligned} \tag{6.8}$$

where $\varepsilon = \operatorname{sign}\{e_{12}(\mu_0)\}$, $\gamma_1 = a_2\alpha_1 e_{11}^{-2}(\mu_0)$, $\gamma_2 = a_2\alpha_2 e_{11}^{-2}(\mu_0)$, and $\gamma_3 = a_1 e_{11}^{-1}(\mu_0)$. Since $e_{11}(\mu_0)e_{12}(\mu_0) \neq 0$ due to $g_{200}(\mu_0)g_{011}(\mu_0) \neq 0$, we have only to assume that $a_2 = e_{22}(\mu_0) \neq 0$ for the rescaling (6.7) to be valid. Notice that t^* has the same direction as t only if $a_2 > 0$. We should keep this in mind when interpreting stability results. Using the Implicit Function Theorem, we can show that, for sufficient small $\|(\gamma_1, \gamma_2)\|$, system (6.8) exhibits the same local bifurcations in a small neighborhood of the origin in the phase plane as (6.6) does with sufficiently small $\|\mu - \mu_0\|$. In fact, an equilibrium $(\tilde{y}, 0)$ of (6.8) corresponds to an equilibrium of (6.3); an equilibrium $(\tilde{y}, \tilde{\rho})$ of (6.8) with $\tilde{\rho} > 0$ corresponds to a limit cycle of (6.3); a limit cycle of (6.8) corresponds to an invariant torus of (6.3); a heteroclinic orbit corresponds to a sphere-like surface of (6.3). Moreover, they have the same (respectively, contrary) stability if $a_2 > 0$ (respectively, $a_2 < 0$).

In the remaining part of this section, we assume that $\varepsilon = 1$ because the case where $\varepsilon = -1$ can be discussed similarly. Thus, we shall describe the complete bifurcation diagram of

$$\begin{aligned} \dot{y} &= \gamma_1 y + y^2 + \rho, \\ \dot{\rho} &= 2\rho\big(\gamma_2 + \gamma_3 y + y^2\big), \end{aligned} \tag{6.9}$$

which has a rich dynamics. Nevertheless, we only consider the case where $\|(\gamma_1, \gamma_2)\|$ is sufficiently small because of the local equivalence between systems (6.9) and (6.2).

It is easy to see that system (6.9) always has two equilibria: $E_1 = (0, 0)$ and $E_2 = (-\gamma_1, 0)$, and that there always exists one orbit connecting E_1 and E_2 due to the symmetry that the y-axis is always invariant. Other equilibria (y, ρ) of (6.9) with $\rho > 0$ satisfy

$$\gamma_1 y + y^2 + \rho = 0 \quad \text{and} \quad \gamma_2 + \gamma_3 y + y^2 = 0, \tag{6.10}$$

which can have zero, one, two solutions in the interior of the quadrants with $\rho > 0$. Moreover, if (y, ρ) is an equilibrium of (6.9), then its stability is determined by the signs of the two eigenvalues of the following matrix

$$\begin{bmatrix} \gamma_1 + 2y & 1 \\ 2\rho(\gamma_3 + 2y) & 2(\gamma_2 + \gamma_3 y + y^2) \end{bmatrix}.$$

Since we only consider the dynamics of (6.9) with γ_1 and γ_2 sufficiently close to 0, we can require the parameters (γ_1, γ_2) be in $\mathcal{J} = \{(\gamma_1, \gamma_2)\colon |\gamma_1| < \frac{1}{2}|\gamma_3| \text{ and } |\gamma_2| < \frac{1}{4}\gamma_3^2\}$. Thus, the second equation of (6.10) has two solutions y_1 and y_2 with $y_1 < y_2$. Moreover, $y_1 < y_2 < 0$ if $\gamma_2 > 0$ and $y_1 < 0 < y_2$ if $\gamma_2 < 0$. Next, we determine the signs of $\rho_j = -y_j^2 - \gamma_1 y_j$, $j = 1, 2$, because we only consider the equilibrium (y, ρ) of (6.9) with $\rho \geqslant 0$.

We first consider the case where $\gamma_3 > 0$. We divide the region $\mathcal{J}$ into six parts:

$$\mathcal{J}_{11} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 < 0 \text{ and } \gamma_2 > 0\},$$
$$\mathcal{J}_{12} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 < 0 \text{ and } \gamma_2 < 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 > 0\},$$
$$\mathcal{J}_{13} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 < 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 < 0\},$$
$$\mathcal{J}_{14} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 > 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 > 0\},$$
$$\mathcal{J}_{15} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 > 0 \text{ and } \gamma_2 > 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 < 0\},$$
$$\mathcal{J}_{16} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 > 0 \text{ and } \gamma_2 < 0\}.$$

These regions are illustrated in Fig. 3(a), where the bold curve l_4 represents the parabola $\gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 = 0$.

Lemma 6.1. *Suppose $\gamma_3 > 0$. Then in the interior of the quadrants of the (y, ρ)-plane with $\rho > 0$, system* (6.10) *has no solution* (*respectively, one solution (y_2, ρ_2) with $y_2 > 0$, one solution (y_2, ρ_2) with $y_2 < 0$*) *for parameters (γ_1, γ_2) in $\mathcal{J} \setminus (\mathcal{J}_{12} \cup \mathcal{J}_{15})$* (*respectively, $\mathcal{J}_{12}$, $\mathcal{J}_{15}$*).

Proof. We distinguish two cases:

Case 1: $\gamma_1 < 0$. Then $y^2 + \gamma_1 y$ is negative if $0 < y < -\gamma_1$ and positive otherwise. If $(\gamma_1, \gamma_2) \in \mathcal{J}_{11}$, then $y_1 < y_2 < 0$ and hence $\rho_j = -y_j^2 - \gamma_1 y_1 < 0$, $j = 1, 2$. If $(\gamma_1, \gamma_2) \in \mathcal{J}_{12}$, then $y_1 < 0 < y_2 < -\gamma_1$ and hence $\rho_1 < 0$ and $\rho_2 > 0$. If $(\gamma_1, \gamma_2) \in \mathcal{J}_{13}$, then $y_1 < 0 < -\gamma_1 < y_2$ and hence $\rho_j = -y_j^2 - \gamma_1 y_1 < 0$, $j = 1, 2$.

Case 2: $\gamma_1 > 0$. Then $y^2 + \gamma_1 y$ is negative if $-\gamma_1 < y < 0$ and positive otherwise. If $(\gamma_1, \gamma_2) \in \mathcal{J}_{14}$, then $y_1 < y_2 < -\gamma_1 < 0$ and hence $\rho_j < 0$, $j = 1, 2$. If $(\gamma_1, \gamma_2) \in \mathcal{J}_{15}$, then $y_1 < -\gamma_1 < y_2 < 0$ and hence $\rho_1 < 0$ and $\rho_2 > 0$. If $(\gamma_1, \gamma_2) \in \mathcal{J}_{16}$, then $y_1 < -\gamma_1 < 0 < y_2$ and hence $\rho_1 < 0$ and $\rho_2 < 0$. The proof is complete. □

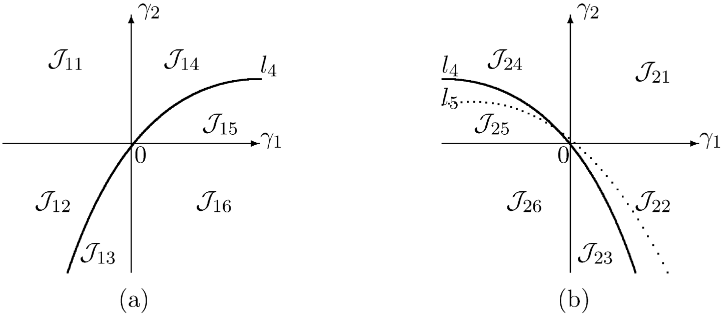

Fig. 3. Bifurcation sets for (6.9).

Moreover, for parameters $(\gamma_1, \gamma_2) \in \mathcal{J}_{12} \cup \mathcal{J}_{15}$, the characteristic polynomial of (6.9) at the equilibrium $E_3 = (y_2, \rho_2)$ is

$$\varsigma^2 - (\gamma_1 + 2y_2)\varsigma - 2\rho_2(\gamma_3 + 2y_2) = 0.$$

The two eigenvalues $\varsigma_{1,2}$ satisfy $\varsigma_1\varsigma_2 = -2\rho_2(\gamma_3 + 2y_2)$. For $(\gamma_1, \gamma_2) \in \mathcal{J}_{12}$, it follows from the proof of Lemma 6.1 that $y_2 > 0$, and so $\varsigma_1\varsigma_2 < 0$. For $(\gamma_1, \gamma_2) \in \mathcal{J}_{15}$, it follows from the proof of Lemma 6.1 that $-\frac{1}{2}\gamma_3 < -\gamma_1 < y_2 < 0$, and hence $\varsigma_1\varsigma_2 < 0$. Thus, we obtain the following:

Proposition 6.2. *Suppose $\gamma_3 > 0$. Then, in the quadrants of the (y, ρ)-plane with $\rho \geqslant 0$, we have the following information on the equilibria of system* (6.9)*.*

(i) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{11}$, where E_1 is a saddle and E_2 is a source.*
(ii) *There are three equilibria E_1, E_2, and E_3 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{12}$, where E_1 is a sink, E_2 is a source, $E_3 = (y_2, \rho_2)$ satisfying $y_2 > 0$ and $\rho_2 > 0$ is a saddle.*
(iii) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{13}$, where E_1 is a sink and E_2 is a saddle.*
(iv) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{14}$, where E_1 is a source and E_2 is a saddle.*
(v) *There are three equilibria E_1, E_2, and E_3 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{15}$, where E_1 is a source, E_2 is a sink, $E_3 = (y_2, \rho_2)$ satisfying $y_2 < 0$ and $\rho_2 > 0$ is a saddle.*
(vi) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{16}$, where E_1 is a saddle and E_2 is a sink.*

The following theorem follows immediately from the equivalence mentioned before.

Theorem 6.3. *Suppose $\gamma_3 > 0$. Then a semi-stable limit cycle of* (6.2) *appears as (γ_1, γ_2) crosses the negative γ_1-axis from $\mathcal{J}_{11}$ to $\mathcal{J}_{12}$, which is always present for $(\gamma_1, \gamma_2) \in \mathcal{J}_{12}$, and then disappears as (γ_1, γ_2) crosses the parabola l_4 from $\mathcal{J}_{12}$ to $\mathcal{J}_{13}$. Similarly, a semi-stable limit cycle of* (6.2) *appears as (γ_1, γ_2) crosses the parabola l_4 from $\mathcal{J}_{14}$ to $\mathcal{J}_{15}$, which is always present for $(\gamma_1, \gamma_2) \in \mathcal{J}_{15}$, and then disappears as (γ_1, γ_2) crosses the positive γ_1-axis from $\mathcal{J}_{15}$ to $\mathcal{J}_{16}$.*

Now, we consider the case where $\gamma_3 < 0$. Again, we divide the region $\mathcal{J}$ into six parts:

$$\mathcal{J}_{21} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 > 0 \text{ and } \gamma_2 > 0\},$$
$$\mathcal{J}_{22} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 > 0 \text{ and } \gamma_2 < 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 > 0\},$$
$$\mathcal{J}_{23} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 > 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 < 0\},$$
$$\mathcal{J}_{24} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 < 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 > 0\},$$
$$\mathcal{J}_{25} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 < 0 \text{ and } \gamma_2 > 0 \text{ and } \gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 < 0\},$$
$$\mathcal{J}_{26} = \{(\gamma_1, \gamma_2) \in \mathcal{J}\colon \gamma_1 < 0 \text{ and } \gamma_2 < 0\}.$$

These regions are illustrated in Fig. 3(b), where the bold curve l_4 and the dot curve l_5 represent the parabolas $\gamma_1^2 - \gamma_1\gamma_3 + \gamma_2 = 0$ and $\gamma_1^2 - 2\gamma_1\gamma_3 + 4\gamma_2 = 0$, respectively. Similarly, we have

Lemma 6.4. *Suppose $\gamma_3 < 0$. Then, in the interior of the quadrants of the (y, ρ)-plane with $\rho > 0$, system* (6.10) *has no solution* (*respectively, one solution* (y_1, ρ_1) *with* $y_1 < 0$, *one solution* (y_1, ρ_1) *with* $y_1 > 0$) *for parameters* (γ_1, γ_2) *in* $\mathcal{J} \setminus (\mathcal{J}_{22} \cup \mathcal{J}_{25})$ (*respectively,* $\mathcal{J}_{22}$, $\mathcal{J}_{25}$).

Moreover, for parameters $(\gamma_1, \gamma_2) \in \mathcal{J}_{22} \cup \mathcal{J}_{25}$, the characteristic polynomial of (6.9) at the equilibrium $E_4 = (y_1, \rho_1)$ is

$$\varsigma^2 - (\gamma_1 + 2y_1)\varsigma - 2\rho_1(\gamma_3 + 2y_1) = 0.$$

The two eigenvalues $\varsigma_{1,2}$ satisfy $\varsigma_1\varsigma_2 = -2\rho_1(\gamma_3 + 2y_1)$, which can be shown to be positive. Then, we need to consider the sign of $\varsigma_1 + \varsigma_2$ in order to discuss the stability of the equilibrium E_4. In fact

$$\varsigma_1 + \varsigma_2 = \gamma_1 + 2y_1 = \gamma_1 - \gamma_3 - \sqrt{\gamma_3^2 - 4\gamma_2}.$$

It follows from $2|\gamma_1| < |\gamma_3|$ and $\gamma_3 < 0$ that $\gamma_1 - \gamma_3 > 0$ and hence

$$\begin{aligned} \operatorname{sign}(\varsigma_1 + \varsigma_2) &= \operatorname{sign}\{(\gamma_1 - \gamma_3)^2 - \gamma_3^2 + 4\gamma_2\} \\ &= \operatorname{sign}\{\gamma_1^2 - 2\gamma_1\gamma_3 + 4\gamma_2\}. \end{aligned}$$

Let

$$\mathcal{J}^+ = \{(\gamma_1, \gamma_2):\ \gamma_1^2 - 2\gamma_1\gamma_3 + 4\gamma_2 > 0\}$$

and

$$\mathcal{J}^- = \{(\gamma_1, \gamma_2):\ \gamma_1^2 - 2\gamma_1\gamma_3 + 4\gamma_2 < 0\}.$$

Then we have the following:

Lemma 6.5. *Suppose $\gamma_3 < 0$. For parameters $(\gamma_1, \gamma_2) \in \mathcal{J}_{22} \cup \mathcal{J}_{25}$, besides equilibria E_1 and E_2, system* (6.10) *has a third equilibrium E_4, which is a sink if $(\gamma_1, \gamma_2) \in \mathcal{J}^- \cap (\mathcal{J}_{22} \cup \mathcal{J}_{25})$ and is a source if $(\gamma_1, \gamma_2) \in \mathcal{J}^+ \cap (\mathcal{J}_{22} \cup \mathcal{J}_{25})$.*

Proposition 6.6. *Suppose $\gamma_3 < 0$. Then, in the quadrants of the (y, ρ)-plane with $\rho \geqslant 0$, we have the following information about equilibria of system* (6.9).

(i) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{21}$, where E_1 is a source and E_2 is a saddle.*
(ii) *There are three equilibria E_1, E_2, and E_4 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{22}$, where E_1 and E_2 are saddles, and $E_4 = (y_1, \rho_1)$ satisfying $y_1 < 0$ and $\rho_1 > 0$ is a sink if $\gamma_1^2 - 2\gamma_1\gamma_3 + 4\gamma_2 < 0$ and is a source otherwise. Namely, in the region $\mathcal{J}_{22}$, as (γ_1, γ_2) crosses the parabola l_5 from the region $\mathcal{J}_{22} \cap \mathcal{J}^+$ to the region $\mathcal{J}_{22} \cap \mathcal{J}^-$, the equilibrium E_4 gains stability and hence system* (6.9) *undergoes a Hopf bifurcation and a stable limit cycle appears*; *as (γ_1, γ_2) varies further, this limit cycle can approach a heteroclinic cycle formed by the separatrices of the two saddles E_1 and E_2, i.e., its period tends to infinity and the cycle disappears.*

(iii) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{23}$, where E_1 is a saddle and E_2 is a sink.*
(iv) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{24}$, where E_1 is a saddle and E_2 is a source.*
(v) *There are three equilibria E_1, E_2, and E_4 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{25}$, where E_1 and E_2 are saddles, and $E_4 = (y_1, \rho_1)$ satisfying $y_1 > 0$ and $\rho_1 > 0$ is a sink if $\gamma_1^2 - 2\gamma_1\gamma_3 + 4\gamma_2 < 0$ and is a source otherwise. Namely, in the region $\mathcal{J}_{25}$, as (γ_1, γ_2) crosses the parabola l_5 from the region $\mathcal{J}_{25} \cap \mathcal{J}^+$ to the region $\mathcal{J}_{25} \cap \mathcal{J}^-$, equilibrium E_4 gains stability and hence system* (6.9) *undergoes a Hopf bifurcation and a stable limit cycle appears*; *as (γ_1, γ_2) varies further, this limit cycle can approach a heteroclinic cycle formed by the separatrices of the two saddles E_1 and E_2, i.e., its period tends to infinity and the cycle disappears.*
(vi) *There are two equilibria E_1 and E_2 for $(\gamma_1, \gamma_2) \in \mathcal{J}_{26}$, where E_1 is a sink and E_2 is a saddle.*

Theorem 6.7. *Suppose that $\gamma_3 < 0$. Then the following statements are true.*

(i) *An unstable limit cycle $\mathcal{O}_1$ of* (6.3) *appears as (γ_1, γ_2) crosses the positive γ_1-axis from $\mathcal{J}_{21}$ to $\mathcal{J}_{22}$. As (γ_1, γ_2) crosses the parabola l_5 from $\mathcal{J}_{22} \cap \mathcal{J}^+$ to $\mathcal{J}_{22} \cap \mathcal{J}^-$, this limit cycle $\mathcal{O}_1$ becomes stable and generates an unstable torus $\mathcal{T}_1$. Under further variation of the parameter (γ_1, γ_2) in $\mathcal{J}_{22} \cap \mathcal{J}^-$, this torus $\mathcal{T}_1$ degenerates to a sphere-like surface $\mathcal{S}_1$ and then disappears. As (γ_1, γ_2) crosses the parabola l_4 from $\mathcal{J}_{22} \cap \mathcal{J}^-$ to $\mathcal{J}_{23}$, the stable limit circle $\mathcal{O}_1$ disappears.*
(ii) *An unstable limit cycle $\mathcal{O}_2$ of* (6.3) *appears as (γ_1, γ_2) crosses the parabola l_4 from $\mathcal{J}_{24}$ to $\mathcal{J}_{25}$. As (γ_1, γ_2) crosses the parabola l_5 from $\mathcal{J}_{25} \cap \mathcal{J}^+$ to $\mathcal{J}_{25} \cap \mathcal{J}^-$, this limit cycle $\mathcal{O}_2$ becomes stable and generates an unstable torus $\mathcal{T}_2$. Under further variation of the parameter (γ_1, γ_2) in $\mathcal{J}_{25} \cap \mathcal{J}^-$, this torus $\mathcal{T}_2$ degenerates to a sphere-like surface $\mathcal{S}_2$ and then disappears. As (γ_1, γ_2) crosses the negative γ_1-axis from $\mathcal{J}_{25} \cap \mathcal{J}^-$ to $\mathcal{J}_{26}$, the stable limit circle $\mathcal{O}_2$ disappears.*

7. Hopf–Hopf bifurcation

In this section, we consider another type of codimension two bifurcation: Hopf–Hopf bifurcation, which may occur when the infinitesimal generator $\mathcal{A}(\mu)$ has two pairs of purely imaginary eigenvalues at some μ. This is the case if $a_{12}a_{21} > 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Omega_n^+ \cap \Omega_m^-$ for some $n, m \in \mathbb{N}$, $m \neq n$, where the two pairs of purely imaginary eigenvalues are $\pm it_n$ and $\pm it_m$. Therefore, throughout this section, we always assume that

(H5) $a_{12}a_{21} > 0$ and $\mu_0 = (\beta_0, \eta_0) \in \Omega_n^+ \cap \Omega_m^-$ for $n, m \in \mathbb{N}$ such that $c_n > c_m$.

Under assumption (H5), $\beta_0 = \frac{1}{2}(c_n + c_m)$ and $\eta_0 = \frac{1}{2}(c_n - c_m)$. It is easy to see that there exists an unstable manifold containing the trivial solution, and hence all bifurcated periodic solutions and invariant cycles are unstable. Let $\omega_1 = t_n$ and $\omega_2 = t_m$. The following result indicates that ω_1 and ω_2 are not in low order resonance.

Lemma 7.1. $k\omega_1 - s\omega_2 \neq 0$ *for all integers k and s such that* $0 < |k| + |s| \leqslant 4$.

Proof. Since $\eta_0 > 0$, we have $\omega_1 \neq \omega_2$. Without loss of generality, we assume that $\omega_1 < \omega_2$. It suffices to exclude the resonances of orders 2 and 3, i.e., ω_2/ω_1 cannot be 2 or 3. In fact, if

$\omega_2 = 2\omega_1$, then it follows from $\tan\tau\omega_j = -\omega_j$ $(j = 1, 2)$ that

$$2\omega_1 = \omega_2 = -\tan(2\tau\omega_1) = -\frac{2\tan(\tau\omega_1)}{1-\tan^2(\tau\omega_1)} = \frac{2\omega_1}{1-\omega_1^2}$$

which is impossible since $\omega_1 > 0$. If $\omega_2 = 3\omega_1$, then, similarly,

$$3\omega_1 = \omega_2 = -\tan(3\tau\omega_1) = \frac{\tan^3(\tau\omega_1) - 3\tan(\tau\omega_1)}{1-3\tan^2(\tau\omega_1)} = \frac{3\omega_1 - \omega_1^3}{1-3\omega_1^2},$$

which implies $\omega_1 = 0$, a contradiction. □

Since the eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$ of the infinitesimal generator $\mathcal{A}(\mu_0)$ are simple, it follows from Corollary 2.2 that $\mathcal{A}(\mu)$ has simple eigenvalues $\lambda_1(\mu)$, $\overline{\lambda_1(\mu)}$, $\lambda_2(\mu)$, and $\overline{\lambda_2(\mu)}$ for all sufficiently small $\|\mu-\mu_0\|$, where $\lambda_1(\mu)$ and $\lambda_2(\mu)$ satisfy $\lambda_j(\mu_0) = i\omega_j$, $j = 1, 2$. Then, $\mathcal{A}(\mu)$ has two eigenvectors $\mathcal{Q}_1(\mu) \in \mathbf{C}$ and $\mathcal{Q}_2(\mu) \in \mathbf{C}$, smoothly dependent on the parameter and corresponding to the eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$, respectively:

$$\mathcal{A}(\mu)\mathcal{Q}_j(\mu) = \lambda_j(\mu)\mathcal{Q}_j(\mu), \quad j = 1, 2.$$

Moreover, $\overline{\lambda_j(\mu)}$, $j = 1, 2$, are also eigenvalues of the adjoint operator $\mathcal{A}^*(\mu)$ with adjoint eigenvectors defined by

$$\mathcal{A}^*(\mu)\mathcal{P}_j(\mu) = \overline{\lambda_j(\mu)}\mathcal{P}_j(\mu), \quad j = 1, 2.$$

The eigenvectors will be normalized such that

$$\langle \mathcal{P}_j(\mu), \mathcal{Q}_k(\mu)\rangle_\mu = \delta_{jk}$$

for all sufficiently small $\|\mu-\mu_0\|$. For simplicity, let $p_j = \mathcal{P}_j(\mu_0)$ and $q_j = \mathcal{Q}_j(\mu_0)$, $j = 1, 2$. In fact, we can choose

$$q_j(\theta) = (1, d_j)^T e^{i\omega_j\theta}, \quad \theta \in [-\tau_1, 0], \; j = 1, 2,$$

and

$$p_j(\xi) = \overline{D}_j(\bar{d}_j, 1)e^{i\omega_j\xi}, \quad \xi \in [0, \tau_1], \; j = 1, 2,$$

where $d_j = (-1)^{j-1}\eta_0 e^{i\omega_j(\tau_1-\tau)}/a_{12}$ and $D_j = \{2d_j[1+\tau(1+i\omega_j)]\}^{-1}$.

We associate each $X \in \mathrm{Dom}(\mathcal{A}(\mu))$ with $(z_1, \bar{z}_1, z_2, \bar{z}_2, w)$, where $z_j = \langle p_j, X\rangle$, $j = 1, 2$, and $w = X - z_1q_1 - \bar{z}_1\bar{q}_1 - z_2q_2 - \bar{z}_2\bar{q}_2 = X - 2\,\mathrm{Re}\{z_1q_1 + z_2q_2\}$. For a solution X_t of (3.1) at μ, we define $z_j(t) = \langle p_j, X_t\rangle$, $j = 1, 2$, and $w(z, \mu) = X_t - 2\,\mathrm{Re}\{z_1(t)q_1 + z_2(t)q_2\}$, where $z = (z_1, z_2) \in \mathbb{C}^2$. In fact, z_j and $\bar{z}_j$ are local coordinates for $\mathcal{C}_\mu$ in the directions of p_j and $\overline{p}_j$, $j = 1, 2$. It is easy to see that $\langle p_j, w\rangle = 0$. Now, for solutions $X_t \in \mathcal{C}_\mu$ of (3.1), $\langle p_j, \dot{X}_t\rangle = \langle p_j, \mathcal{A}(\mu)X_t + \mathcal{R}(\mu)X_t\rangle$, $j = 1, 2$. Then, on the center manifold $\mathcal{C}_\mu$, we have

$$\dot{z}(t) = \Lambda z(t) + g(z, \mu), \tag{7.1}$$

where $\Lambda = \operatorname{diag}(i\omega_1, i\omega_2)$, and $g = (g^1, g^2)$ with the smooth functions g^j given by

$$g^j(z,\mu) = \langle p_j, \mathcal{A}(\mu)\big(w(z,\mu) + 2\operatorname{Re}\{z_1 q_1 + z_2 q_2\}\big)\rangle - i\omega_j z_j \\ + \langle p_j, \mathcal{R}(\mu)\big(w(z,\mu) + 2\operatorname{Re}\{z_1 q_1 + z_2 q_2\}\big)\rangle$$

for $(z,\mu) \in \mathbb{C}^2 \times \mathbb{R}^2$. Let

$$g^j(z,\mu) = \sum_{l+s+r+k \geqslant 1} \frac{1}{l!s!r!k!} g^j_{lsrk}(\mu) z_1^l \bar{z}_1^s z_2^r \bar{z}_2^k.$$

Elphick et al. [9] showed that there is a normal form

$$\dot{v} = \Lambda v + \mathcal{G}(v,\mu), \tag{7.2}$$

which commutes with $e^{\Lambda\theta}$, where the action of $e^{\Lambda\theta}$ on $\mathbb{C}^2$ is given by

$$e^{\Lambda\theta} \cdot (z_1, z_2) = \big(e^{i\omega_1\theta} z_1, e^{i\omega_2\theta} z_2\big), \quad \theta \in \mathbb{R},\ z = (z_1, z_2) \in \mathbb{C}^2.$$

The following lemma shows that the group $\{e^{\Lambda\theta}: \theta \in \mathbb{R}\}$ is either the torus group $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ or the cyclic group $\mathbb{S}^1$.

Lemma 7.2. *(See [40].)*

(i) *Suppose ω_2/ω_1 is irrational, that is, $k\omega_1 - s\omega_2 \neq 0$ for any nonzero integers k and s, then the normal form, computed up to any finite order, is equivariant with respect to $\mathbb{T}^2$. Namely, the normal form for the nonresonant Hopf–Hopf bifurcation is of the form*

$$\dot{z}_j = i\omega_j z_j + z_j P_j\big(|z_1|^2, |z_2|^2, \mu\big), \quad j = 1, 2, \tag{7.3}$$

where P_1 and P_2 are complex polynomials in $|z_1|^2$, $|z_2|^2$, and μ.

(ii) *Suppose ω_2/ω_1 is rational, that is, $k\omega_1 - s\omega_2 = 0$ for some nonzero integers k and s, with k and s relatively prime, then the normal form, computed up to any finite order, is equivariant with respect to $\mathbb{S}^1$. Namely, the normal form takes the following form*

$$\begin{aligned}\dot{z}_1 &= i\omega_1 z_1 + z_1 P_1\big(|z_1|^2, |z_2|^2, z_1^k \bar{z}_2^s, \mu\big) + \bar{z}_1^{k-1} z_2^s P_2\big(|z_1|^2, |z_2|^2, \bar{z}_1^k z_2^s, \mu\big),\\ \dot{z}_2 &= i\omega_2 z_2 + z_2 Q_1\big(|z_1|^2, |z_2|^2, \bar{z}_1^k z_2^s, \mu\big) + z_1^k z_2^{s-1} Q_2\big(|z_1|^2, |z_2|^2, z_1^k \bar{z}_2^s, \mu\big),\end{aligned} \tag{7.4}$$

where P_1, P_2, Q_1, and Q_2 are complex polynomials in their arguments.

In view of Lemmas 7.1 and 7.2, by a near identity transformation

$$z = v + \psi(v, \bar{v}), \quad \psi = O\big(|v|^2\big), \quad v = (v_1, v_2) \in \mathbb{C}^2, \tag{7.5}$$

system (7.1) can be simplified to (7.2), where ψ is at least second order in ν_1, $\bar{\nu}_1$, ν_2, and $\bar{\nu}_2$, and some resonant cubic terms of (7.2) can be *removed* under certain nondegeneracy conditions. Thus, $\mathcal{G}$ in the normal form (7.2) takes the form

$$\mathcal{G}(\nu,\mu)=\begin{pmatrix} e_{10}(\mu)\nu_1+e_{11}(\mu)|\nu_1|^2\nu_1+e_{12}(\mu)|\nu_2|^2\nu_1+O(|\nu|^5) \\ e_{20}(\mu)\nu_2+e_{21}(\mu)|\nu_2|^2\nu_2+e_{22}(\mu)|\nu_1|^2\nu_2+O(|\nu|^5) \end{pmatrix}.$$

In Appendix B, we derived the concrete expressions for $e_{jk}(\mu_0)$, $j,k=1,2$, under the nondegeneracy conditions that $A_{jk}=\mathrm{Re}\{e_{jk}(\mu_0)\}\neq 0$ for all $j,k=1,2$. Moreover, it is easy to see that $e_{10}(\mu)=(\mu-\mu_0)\nabla\lambda_1(\mu_0)+O(\|\mu-\mu_0\|^2)$ and $e_{20}(\mu)=(\mu-\mu_0)\nabla\lambda_2(\mu_0)+O(\|\mu-\mu_0\|^2)$.

Let $\nu_1=r_1e^{i\xi_1}$ and $\nu_2=r_2e^{i\xi_2}$. Then Eq. (7.2) can be rewritten as

$$\begin{aligned}
\dot r_1&=\alpha_1 r_1+A_{11}r_1^3+A_{12}r_1r_2^2+O\big(\|(r_1,r_2)\|^5\big),\\
\dot r_2&=\alpha_2 r_2+A_{21}r_2^3+A_{22}r_2r_1^2+O\big(\|(r_1,r_2)\|^5\big),\\
\dot\xi_1&=\omega_1+(\mu-\mu_0)\nabla\,\mathrm{Im}\big\{\lambda_1(\mu_0)\big\}+B_{11}r_1^2+B_{12}r_2^2+O\big(\|(r_1,r_2)\|^4\big),\\
\dot\xi_2&=\omega_2+(\mu-\mu_0)\nabla\,\mathrm{Im}\big\{\lambda_2(\mu_0)\big\}+B_{21}r_2^2+B_{22}r_1^2+O\big(\|(r_1,r_2)\|^4\big),
\end{aligned}\tag{7.6}$$

where $\alpha_1=(\mu-\mu_0)\nabla\,\mathrm{Re}\{\lambda_1(\mu_0)\}$, $\alpha_2=(\mu-\mu_0)\nabla\,\mathrm{Re}\{\lambda_2(\mu_0)\}$, and $B_{jk}=\mathrm{Im}\{e_{jk}(\mu_0)\}$, $j,k=1,2$. From the proof of Corollary 2.2, we know

$$\alpha_1=(\beta+\eta-c_n)c_n\varepsilon_1(\omega_1)\varepsilon_2(\omega_1),\qquad \alpha_2=(\beta-\eta-c_m)c_m\varepsilon_1(\omega_2)\varepsilon_2(\omega_2).$$

Obviously,

$$\left|\frac{\partial(\alpha_1,\alpha_2)}{\partial(\beta,\eta)}\right|_{\mu=\mu_0}=-2c_nc_m\varepsilon_1(\omega_1)\varepsilon_2(\omega_1)\varepsilon_1(\omega_2)\varepsilon_2(\omega_2)\neq 0.$$

This means that the mapping $\mu\to(\alpha_1,\alpha_2)$ is regular at μ_0, and hence we can unfold this degenerate case by varying α_1 and α_2 in a full neighborhood of $(0,0)$. Such unfoldings are studied by several authors (see, for example, [2,15,16,23,26,36]). However, our analysis here is the first complete investigation of the double Hopf bifurcation as it occurs in a system of delay-differential equations, and the relationship between unfolded flows on a four-dimensional center manifold and the original system of delay-differential equations: previous investigations only considered simpler bifurcations of a scalar delay-differential equation (to name a few, see [2,3, 11]).

Note that, up to $O(\|(r_1,r_2)\|^4)$, the amplitude and phase variables of (7.6) decouple. As a result, the bifurcation and asymptotic behavior of solutions of (1.2) under assumption (H5) can be studied via the two-dimensional amplitude equations alone. That is, we consider the following truncated system of amplitude equations in (7.6):

$$\begin{aligned}
\dot r_1&=\alpha_1 r_1+A_{11}r_1^3+A_{12}r_1r_2^2,\\
\dot r_2&=\alpha_2 r_2+A_{21}r_2^3+A_{22}r_2r_1^2.
\end{aligned}\tag{7.7}$$

The relation between equilibria of (7.7) and bifurcations of (7.6) is as follows.

(i) If (7.7) has an asymptotically stable (respectively, unstable) equilibrium $(\tilde{r}_1, 0)$ (respectively, $(0, \tilde{r}_2)$) on either axis, then (7.6) has an asymptotically stable (respectively, unstable) periodic orbit of frequency $\omega_1 + O(\sqrt{|\alpha_1|})$ (respectively, $\omega_2 + O(\sqrt{|\alpha_2|})$) and hence system (1.2) has a periodic solution in the neighborhood of the origin, which is unstable because of the existence of unstable manifolds of (1.2) containing the origin.

(ii) If (7.7) has an asymptotically stable (respectively, unstable) equilibrium $(\tilde{r}_1, \tilde{r}_2)$ in the interior of the positive quadrant, then (7.6) has an asymptotically stable (respectively, unstable) two-dimensional invariant torus, i.e., (7.6) has quasi-periodic solutions and hence system (1.2) has an unstable quasi-periodic solution in the neighborhood of the origin.

(iii) If (7.7) has an asymptotically stable (respectively, unstable) limit cycle in the interior of the positive quadrant, then (7.6) has an asymptotically stable (respectively, unstable) three-dimensional invariant torus and hence system (1.2) has an unstable three-dimensional invariant torus in the neighborhood of the origin.

From the above, we see that sufficiently close to the Hopf–Hopf bifurcation points μ_0, system (1.2) will exhibit either periodic or quasi-periodic motions. Thus, if we can find combinations of parameters α_j and A_{ij} $(i, j = 1, 2)$ which yield stable equilibria $(\tilde{r}_1, \tilde{r}_2)$ with $\tilde{r}_1\tilde{r}_2 \neq 0$, we can conclude that the stable quasi-periodic motions should occur for the corresponding parameter values of system (1.2). Therefore, from now on, we concentrate on describing the behavior of the coupled amplitude equation (7.7) in the (α_1, α_2)-parameter plane. The mode interaction equations (7.7) have been investigated by many researchers. See, for example, Guckenheimer and Holmes [16, Section 7.5]. Here, for the sake of completeness, we shall employ some techniques from the classical work of Guckenheimer and Holmes [16] (including rescaling in time and variables) to investigate the qualitative behavior of the mode interaction equations (7.7) in the parameter ranges of interest. We discuss these case by case.

Firstly, we consider the case where $A_{11} < 0$ and $A_{21} < 0$. Introducing new phase variables and rescaling time in (7.7) according to

$$r_1^* = \sqrt{|A_{11}|}r_1, \qquad r_2^* = \sqrt{|A_{21}|}r_2, \qquad t^* = 2t, \tag{7.8}$$

and then dropping $*$ yield

$$\begin{aligned} \dot{r}_1 &= \alpha_1 r_1 - r_1^3 - \theta r_1 r_2^2, \\ \dot{r}_2 &= \alpha_2 r_2 - r_2^3 - \Delta r_2 r_1^2, \end{aligned} \tag{7.9}$$

where $\theta = A_{12}/A_{21}$ and $\Delta = A_{22}/A_{11}$. Notice that the r_1- and r_2-axes are invariant lines for the flow of (7.9). If $(\tilde{r}_1, \tilde{r}_2)$ is an equilibrium of (7.9), then its stability is determined by the two eigenvalues of the following characteristic matrix of (7.9) at $(\tilde{r}_1, \tilde{r}_2)$:

$$\mathcal{M}(\tilde{r}_1, \tilde{r}_2) = \begin{bmatrix} \alpha_1 - 3\tilde{r}_1^2 - \theta\tilde{r}_2^2 & -2\theta\tilde{r}_1\tilde{r}_2 \\ -2\Delta\tilde{r}_1\tilde{r}_2 & \alpha_2 - 3\tilde{r}_2^2 - \Delta\tilde{r}_1^2 \end{bmatrix}.$$

Obviously,

$$\det\mathcal{M}(\tilde{r}_1, \tilde{r}_2) = 4(1 - \theta\Delta)\tilde{r}_1^2\tilde{r}_2^2 \quad \text{and} \quad \operatorname{tr}\mathcal{M}(\tilde{r}_1, \tilde{r}_2) = -2(\tilde{r}_1^2 + \tilde{r}_2^2).$$

Simple linear analysis reveals the following results about equilibria of (7.9):

(i) $(r_1, r_2) = (0, 0)$ is always an equilibrium. It is a stable sink if $\max\{\alpha_1, \alpha_2\} < 0$, a saddle if $\alpha_1\alpha_2 < 0$, and an unstable source if $\min\{\alpha_1, \alpha_2\} > 0$.
(ii) $(r_1, r_2) = (\sqrt{\alpha_1}, 0)$ is an equilibrium if $\alpha_1 > 0$. If, in addition, $\Delta\alpha_1 > \alpha_2$, then it is a sink; otherwise it is a saddle.
(iii) $(r_1, r_2) = (0, \sqrt{\alpha_2})$ is an equilibrium if $\alpha_2 > 0$. If, in addition, $\theta\alpha_2 > \alpha_1$, then it is a sink; otherwise it is a saddle.
(iv) $(r_1, r_2) = (\sqrt{[\alpha_1 - \theta\alpha_2]/[1 - \theta\Delta]}, \sqrt{[\alpha_2 - \Delta\alpha_1]/[1 - \theta\Delta]})$ is an equilibrium if both radicand are positive. It is a saddle if $\theta\Delta > 1$ and a sink if $\theta\Delta < 1$.

Therefore, we deduce that bifurcations to the pure modes $(\sqrt{\alpha_1}, 0)$ and $(0, \sqrt{\alpha_2})$ occur on the lines $\alpha_1 = 0$ and $\alpha_2 = 0$, whereas bifurcations to the mixed mode occur on the line $\alpha_1 = \theta\alpha_2$ and $\alpha_2 = \Delta\alpha_1$ if they exist. In addition, we need check that no closed orbits (or limit cycles) can occur. Since the r_1- and r_2-axes are invariant, any such closed orbit would have to lie in the interior of the positive quadrant and must enclosed at least one equilibrium with Poincaré index equal to 1.

If $\theta\Delta > 1$ and $\alpha_1 - \theta\alpha_2 < 0$ and $\alpha_2 - \Delta\alpha_1 < 0$, then system (7.9) has an equilibrium $(\tilde{r}_1, \tilde{r}_2)$ with $\tilde{r}_1\tilde{r}_2 \neq 0$. Recall that $(\tilde{r}_1, \tilde{r}_2)$ is a saddle with the Poincaré index equal to -1. We immediately see that no closed orbit can occur around $(\tilde{r}_1, \tilde{r}_2)$. If $\theta\Delta < 1$ and $\alpha_1 - \theta\alpha_2 > 0$ and $\alpha_2 - \Delta\alpha_1 > 0$, then system (7.9) has an equilibrium $(\tilde{r}_1, \tilde{r}_2)$ with $\tilde{r}_1\tilde{r}_2 \neq 0$, which is a sink. In what follows, we distinguish several cases to conclude that no closed orbits can occur around the sink $(\tilde{r}_1, \tilde{r}_2)$ when $\theta\Delta < 1$ and (α_1, α_2) is in the sector $\mathcal{E} = \{(\alpha_1, \alpha_2)\colon \alpha_1 - \theta\alpha_2 > 0$ and $\alpha_2 - \Delta\alpha_1 > 0\}$.

Case 1: $\theta > 0$ and $\Delta > 0$. We follow a directional arc $\vec{l}_1$ crossing the line $\alpha_1 = \theta\alpha_2 > 0$ and then passing through the sector $\mathcal{D}$ and finally crossing the line $\alpha_2 = \Delta\alpha_2 > 0$. When $(\alpha_1, \alpha_2) \in \vec{l}_1$ crosses the line $\alpha_1 = \theta\alpha_2 > 0$, the sink $(0, \sqrt{\alpha_2})$ becomes a saddle and a sink $(\tilde{r}_1, \tilde{r}_2)$ bifurcates from $(0, \sqrt{\alpha_2})$ and the unstable separatrix of the saddle $(0, \sqrt{\alpha_2})$ limits in this bifurcated sink $(\tilde{r}_1, \tilde{r}_2)$. Thus, after bifurcation there is no closed orbit around this sink. The only way where the closed orbit can appear in the positive quadrant is by Hopf bifurcation from $(\tilde{r}_1, \tilde{r}_2)$. But this is impossible because $(\tilde{r}_1, \tilde{r}_2)$ remains stable for all $(\alpha_1, \alpha_2) \in \mathcal{E}$.

Case 2: $\theta > 0 > \Delta$. Similar arguments as those in Case 1 show that there is no closed orbit in the positive quadrant when (α_1, α_2) is in the sector $0 < \alpha_2 < \alpha_1/\theta$. In order to rule out the existence of closed orbits in the positive quadrant when (α_1, α_2) is in the sector $\Delta\alpha_1 < \alpha_2 < 0$, we follow another directional arc $\vec{l}_2$ crossing the line $\alpha_2 = \Delta\alpha_1$ and then passing through the sector $\Delta\alpha_1 < \alpha_2 < 0$. When $(\alpha_1, \alpha_2) \in \vec{l}_2$ crosses the line $\alpha_2 = \Delta\alpha_1$, the sink $(\sqrt{\alpha_1}, 0)$ becomes a saddle and a sink $(\tilde{r}_1, \tilde{r}_2)$ bifurcates from $(\sqrt{\alpha_1}, 0)$ and the unstable separatrix of the saddle $(\sqrt{\alpha_1}, 0)$ limits in this bifurcated sink $(\tilde{r}_1, \tilde{r}_2)$. Thus, after bifurcation there is no closed orbit around this sink. Similarly, no Hopf bifurcation can occur from $(\tilde{r}_1, \tilde{r}_2)$ as it remains stable for all $(\alpha_1, \alpha_2) \in \mathcal{E}$.

Case 3: $\theta < 0 < \Delta$. Similar arguments as those in Case 1 tell us that there is no closed orbit in the positive quadrant when (α_1, α_2) is in the sector $\theta\alpha_2 < \alpha_1 < 0$; while arguments as those in Case 2 produce that there is no closed orbit in the positive quadrant when (α_1, α_2) is in the sector $0 < \alpha_1 < \alpha_2/\Delta$.

Case 4: $\theta < 0$ and $\Delta < 0$. The discussion is similar to that in Case 1 and hence is omitted.

In summary, we have proved the following

Theorem 7.3. *No closed orbit of system* (7.9) *can occur around the mixed mode* $(\tilde{r}_1, \tilde{r}_2)$.

Secondly, for the case where $A_{11} > 0$ and $A_{21} > 0$, we introduce new phase variables and rescaling time in (7.7) according to

$$r_1^* = \sqrt{|A_{11}|}r_1, \qquad r_2^* = \sqrt{|A_{21}|}r_2, \qquad t^* = -2t. \tag{7.10}$$

After dropping $*$, we obtain

$$\begin{aligned} \dot r_1 &= -\alpha_1 r_1 - r_1^3 - \theta r_1 r_2^2, \\ \dot r_2 &= -\alpha_2 r_2 - r_2^3 - \Delta r_2 r_1^2, \end{aligned} \tag{7.11}$$

where θ and Δ are the same as before. System (7.11) is quite similar to (7.9) and hence similar arguments can be employed. We omit the detail here.

Thirdly, for the case where $A_{11} > 0$ and $A_{21} < 0$, we introduce new phase variables and rescaling time in (7.7) as (7.8). After dropping $*$, we obtain

$$\begin{aligned} \dot r_1 &= \alpha_1 r_1 + r_1^3 - \theta r_1 r_2^2, \\ \dot r_2 &= \alpha_2 r_2 - r_2^3 + \Delta r_2 r_1^2, \end{aligned} \tag{7.12}$$

where θ and Δ are the same as before. Again, notice that the r_1- and r_2-axes are invariant lines for the flow of (7.12). If $(\tilde r_1, \tilde r_2)$ is an equilibrium of (7.12), then its stability is determined by the two eigenvalues of the following characteristic matrix of (7.12) at $(\tilde r_1, \tilde r_2)$:

$$\mathcal{N}(\tilde r_1, \tilde r_2) = \begin{bmatrix} \alpha_1 + 3\tilde r_1^2 - \theta \tilde r_2^2 & -2\theta \tilde r_1 \tilde r_2 \\ 2\Delta \tilde r_1 \tilde r_2 & \alpha_2 - 3\tilde r_2^2 + \Delta \tilde r_1^2 \end{bmatrix}.$$

Obviously,

$$\det \mathcal{N}(\tilde r_1, \tilde r_2) = 4(\theta\Delta - 1)\tilde r_1^2 \tilde r_2^2 \quad \text{and} \quad \operatorname{tr} \mathcal{N}(\tilde r_1, \tilde r_2) = 2\big(\tilde r_1^2 - \tilde r_2^2\big).$$

Simple linear analysis produces the following results:

(i) $(r_1, r_2) = (0, 0)$ is always an equilibrium. It is a stable sink if $\max\{\alpha_1, \alpha_2\} < 0$, a saddle if $\alpha_1\alpha_2 < 0$, and an unstable source if $\min\{\alpha_1, \alpha_2\} > 0$.
(ii) $(r_1, r_2) = (\sqrt{-\alpha_1}, 0)$ is an equilibrium if $\alpha_1 < 0$. If, in addition, $\Delta\alpha_1 < \alpha_2$, then it is a source; otherwise it is a saddle.
(iii) $(r_1, r_2) = (0, \sqrt{\alpha_2})$ is an equilibrium if $\alpha_2 > 0$. If, in addition, $\theta\alpha_2 > \alpha_1$, then it is a sink; otherwise it is a saddle.
(iv) $(r_1, r_2) = (\sqrt{[\alpha_1 - \theta\alpha_2]/[\theta\Delta - 1]}, \sqrt{[\Delta\alpha_1 - \alpha_2]/[\theta\Delta - 1]})$ is an equilibrium if both radicands are positive. If $\theta\Delta < 1$ then it is a saddle; if $\theta\Delta > 1$ and $\tilde r_1 > \tilde r_2$ then it is a source; if $\theta\Delta > 1$ and $\tilde r_1 < \tilde r_2$ then it is a sink.

It follows from the above results that bifurcations to the pure modes $(\sqrt{-\alpha_1}, 0)$ and $(0, \sqrt{\alpha_2})$ occur on the lines $\alpha_1 = 0$ and $\alpha_2 = 0$, whereas bifurcations to the mixed modes occur on the lines $\alpha_1 = \theta\alpha_2$ and $\alpha_2 = \Delta\alpha_1$ if they exist. Since the r_1- and r_2-axes are invariant, any such closed orbit would have to lie in the interior of the positive quadrant and must enclosed at least one equilibrium with Poincaré index equal to 1. If $\theta\Delta < 1$ and $\alpha_1 - \theta\alpha_2 < 0$ and $\alpha_2 - \Delta\alpha_1 > 0$, then system (7.12) has an equilibrium $(\tilde{r}_1, \tilde{r}_2)$ with $\tilde{r}_1\tilde{r}_2 \neq 0$, which is a saddle with the Poincaré index equal to -1. We immediately conclude that

Theorem 7.4. *Assume that $\theta\Delta < 1$ and $\alpha_1 - \theta\alpha_2 < 0$ and $\alpha_2 - \Delta\alpha_1 > 0$. Then no closed orbit of system* (7.12) *can occur around* $(\tilde{r}_1, \tilde{r}_2)$.

If $\theta\Delta > 1$ and (α_1, α_2) is in the sector $\mathcal{I} = \{(\alpha_1, \alpha_2)\colon \alpha_1 - \theta\alpha_2 > 0 \text{ and } \alpha_2 - \Delta\alpha_1 < 0\}$, then system (7.12) has an equilibrium $(\tilde{r}_1, \tilde{r}_2)$ with $\tilde{r}_1\tilde{r}_2 \neq 0$. It follows from the expressions for $\tilde{r}_1$ and $\tilde{r}_2$ that $\operatorname{sign}(\tilde{r}_1 - \tilde{r}_2) = \operatorname{sign}(1-\theta)\operatorname{sign}\{\alpha_2 - \chi\alpha_1\}$, where $\chi = (1-\Delta)/(\theta - 1)$. Furthermore, if $\theta > 1$ then $\chi < 1/\theta$ and $\chi < \Delta$; if $\theta < 1$, then $\chi > 1/\theta$ and $\chi > \Delta$. Therefore, we have the following observations:

Lemma 7.5. *If $\Delta > 1/\theta > 0$ and $(\alpha_1, \alpha_2) \in \mathcal{I}$, then system* (7.12) *has a mixed mode* $(\tilde{r}_1, \tilde{r}_2)$. *Moreover, it is a sink* (*respectively, source*) *if* (α_1, α_2) *is in the sector* $\mathcal{I}_1$ (*respectively,* $\mathcal{I}_2$), *where*

$$\mathcal{I}_1 = \begin{cases} \{(\alpha_1, \alpha_2)\colon\ \chi\alpha_1 < \alpha_2 < \alpha_1/\theta\} & \text{if } \theta > 1, \\ \{(\alpha_1, \alpha_2)\colon\ \alpha_2 < \chi\alpha_1 \text{ and } \alpha_2 < \alpha_1/\theta\} & \text{if } \theta < 1, \end{cases}$$
$$\mathcal{I}_2 = \begin{cases} \{(\alpha_1, \alpha_2)\colon\ \alpha_2 < \chi\alpha_1 \text{ and } \alpha_2 < \Delta\alpha_1\} & \text{if } \theta > 1, \\ \{(\alpha_1, \alpha_2)\colon\ \chi\alpha_1 < \alpha_2 < \Delta\alpha_1\} & \text{if } \theta < 1. \end{cases}$$

Lemma 7.6. *If $\Delta < 1/\theta < 0$ and $(\alpha_1, \alpha_2) \in \mathcal{I}$, then system* (7.12) *has a mixed mode* $(\tilde{r}_1, \tilde{r}_2)$. *Moreover, it is a sink* (*respectively, source*) *if* (α_1, α_2) *is in the sector* $\mathcal{I}_3$ (*respectively,* $\mathcal{I}_4$), *where*

$$\mathcal{I}_3 = \{(\alpha_1, \alpha_2)\colon\ \alpha_1/\theta < \alpha_2 < \chi\alpha_1\},$$
$$\mathcal{I}_4 = \{(\alpha_1, \alpha_2)\colon\ \chi\alpha_1 < \alpha_2 < \Delta\alpha_1\}.$$

The following result describes the phase portrait of (7.12).

Theorem 7.7. *Assume $\theta\Delta > 1$. Then, for some points $(\alpha_1, \alpha_2) \in \mathcal{I}$, system* (7.12) *has closed orbits surrounding the mixed mode* $(\tilde{r}_1, \tilde{r}_2)$.

Proof. Here, we only consider the case where $\theta > 1 > \Delta > 1/\theta > 0$, because other cases can be handled similarly. If $\theta\Delta > 1$ and $\theta > 1$, then $(\alpha_1, \alpha_2) \in \mathcal{I}$, system (7.12) has a mixed mode $(\tilde{r}_1, \tilde{r}_2)$. We follow a directional arc in the (α_1, α_2)-parameter plane, which starts from a point in the sector $\alpha_1/\theta < \alpha_2 < \Delta\alpha_1$, then crosses the line $\alpha_1 = \theta\alpha_2 > 0$ into the sector $\mathcal{I}_1$, and finally successively crosses the line $\alpha_2 = \chi\alpha_1 > 0$ and the positive α_1-axis. When the point (α_1, α_2) is in the sector $\alpha_1/\theta < \alpha_2 < \Delta\alpha_1$, system (7.12) has a source $(0, 0)$ and a sink $(0, \sqrt{\alpha_2})$. As (α_1, α_2) crosses the line $\alpha_1 = \theta\alpha_2 > 0$, a mixed mode $(\tilde{r}_1, \tilde{r}_2)$ (which is a sink) bifurcates from $(0, \sqrt{\alpha_2})$ and the unstable separatrix of the saddle $(0, \sqrt{\alpha_2})$ limits in the newly bifurcated mixed mode. Thus, immediately after bifurcation no closed orbit can surround the mixed mode. However, as (α_1, α_2) crosses the line $\alpha_2 = \chi\alpha_1 > 0$, the mixed mode $(\tilde{r}_1, \tilde{r}_2)$ losses its stability and hence system (7.12) undergoes a Hopf bifurcation, i.e., a stable closed orbit appears in the positive quadrant. Moreover, as (α_1, α_2) crosses the positive α_1-axis, the pure mode $(0, \sqrt{\alpha_2})$ collides with $(0, 0)$ and disappears. □

Theorem 7.7 implies that crossing the line $\alpha_2 = \chi\alpha_1$ in the sector $\mathcal{I}$ results in the branching of a three-dimensional torus from the two-dimensional torus of system (7.7). Under certain conditions on the higher-order terms of (7.7) (see [16,27]), this three-dimensional torus may have up to three frequencies and hence chaotic motions are even possible near such a Hopf–Hopf bifurcation point.

Finally, for the case where $A_{11} < 0$ and $A_{21} > 0$, we can obtain the reparametrized equation of the form (7.12) by reversing time and hence the detail is omitted.

To conclude this section, we relate the above results to the dynamical behavior of (1.2) under assumptions (H1), (H2), and (H5). By a direct computation,

$$\begin{aligned}
e_{11}(\mu_0) &= \frac{1}{4} f'''(0)\big(|d_1|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_1^2\big]^{-1}\big[\varepsilon_2(\omega_1)+i\big],\\
e_{12}(\mu_0) &= \frac{1}{2} f'''(0)\big(|d_2|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_1^2\big]^{-1}\big[\varepsilon_2(\omega_1)+i\big],\\
e_{21}(\mu_0) &= \frac{1}{4} f'''(0)\big(|d_2|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_2^2\big]^{-1}\big[\varepsilon_2(\omega_2)+i\big],\\
e_{22}(\mu_0) &= \frac{1}{2} f'''(0)\big(|d_1|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_2^2\big]^{-1}\big[\varepsilon_2(\omega_2)+i\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_{11} &= \frac{1}{4} f'''(0)\big(|d_1|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_1^2\big]^{-1}\varepsilon_2(\omega_1),\\
A_{12} &= \frac{1}{2} f'''(0)\big(|d_2|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_1^2\big]^{-1}\varepsilon_2(\omega_1),\\
A_{21} &= \frac{1}{4} f'''(0)\big(|d_2|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_2^2\big]^{-1}\varepsilon_2(\omega_2),\\
A_{22} &= \frac{1}{2} f'''(0)\big(|d_1|^2+1\big)\big[(1+\tau)^2+\tau^2\omega_2^2\big]^{-1}\varepsilon_2(\omega_2).
\end{aligned}$$

Observe that all A_{ij}'s and B_{ij}'s have the same sign as that of $f'''(0)$. Here, we only consider the case where $f'''(0) < 0$ because in the case where $f'''(0) > 0$ we can obtain the same amplitude equation (7.7) by reversing the time and replacing α_j by $-\alpha_j$ for $j = 1, 2$. Under the assumption that $f'''(0) < 0$, $A_{ij} < 0$ and $B_{ij} < 0$ for all i, j. By changing variables in (7.8), we can obtain (7.7) with $\theta = 2(|d_2|^2+1)/(|d_1|^2+1)$ and $\Delta = 2(|d_1|^2+1)/(|d_2|^2+1)$. Note that $\theta\Delta = 4 > 1$. Then the mixed mode exists for (α_1, α_2) in the region III of Fig. 4.

It follows from the existence of an unstable manifold containing the trivial solution that all bifurcated periodic solutions and invariant tori are unstable. For sufficiently small $\|(\alpha_1, \alpha_2)\|$, we obtain the following results about the local dynamical behaviors of system (1.2) near the Hopf–Hopf bifurcation point μ_0.

Theorem 7.8. *In addition to the assumptions* (H1), (H2), *and* (H5), *assume that* $f'''(0) < 0$. *Then on the* (α_1, α_2)*-parameter plane depicted in Fig.* 4, *anticlockwise, there is a cycle with sufficiently small radius.*

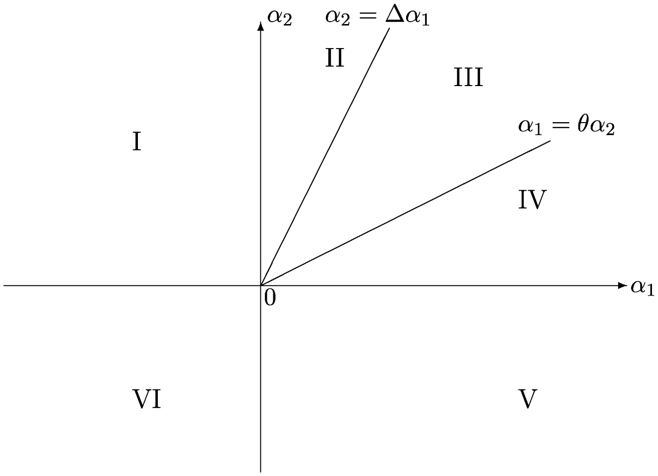

Fig. 4. Bifurcation sets for the amplitude equation (7.7).

(i) *Crossing the negative α_1-axis from below, an unstable periodic solution $\mathcal{O}_1$ appears by Hopf bifurcation from the trivial solution. This periodic solution $\mathcal{O}_1$ is persistent for* (α_1, α_2) *in regions* I, II, III, *and* IV.

(ii) *Crossing the positive α_2-axis from left, another unstable periodic solution $\mathcal{O}_2$ appears by Hopf bifurcation from the trivial solution. This new bifurcated periodic solution $\mathcal{O}_2$ is persistent for* (α_1, α_2) *in regions* II, III, IV, *and* V.

(iii) *Crossing the line $\alpha_2 = \Delta\alpha_1 > 0$ from above, an unstable invariant torus $\mathcal{T}$ appears by Neimark–Sacker bifurcation from the periodic solution $\mathcal{O}_1$. This invariant torus $\mathcal{T}$ is persistent for* (α_1, α_2) *in region* III.

(iv) *Crossing the line $\alpha_1 = \theta\alpha_2 > 0$ from above, the unstable invariant torus $\mathcal{T}$ collides with the periodic solution $\mathcal{O}_2$ and then disappears.*

(v) *Crossing the positive α_1-axis from above, the periodic solution $\mathcal{O}_1$ collides with the trivial solution and then disappears.*

(vi) *Crossing the negative α_2-axis from right, the periodic solution $\mathcal{O}_2$ collides with the trivial solution and then disappears.*

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Appendix A. Derivation of normal form (6.3)

The purpose of this appendix is to derive the concrete expressions for e_{jk}, $j,k=1,2$. Let

$$g(x,z,\bar{z},\mu)=\sum_{j+s+k\geqslant 1}\frac{1}{j!s!k!}g_{jsk}(\mu)x^j z^s\bar{z}^k,$$
$$h(x,z,\bar{z},\mu)=\sum_{j+s+k\geqslant 1}\frac{1}{j!s!k!}h_{jsk}(\mu)x^j z^s\bar{z}^k.$$

Since g must be real, we have $g_{jsk}(\mu)=\overline{g_{jks}(\mu)}$. Therefore, g_{jsk} is real for $s=k$.

When $\mu=\mu_0$, system (6.2) takes the form of

$$\begin{aligned}\dot{x}(t)&=g(x,z,\mu_0),\\ \dot{z}(t)&=it_n z(t)+h(x,z,\mu_0).\end{aligned}\tag{A.1}$$

In addition, we have, at $\mu=\mu_0$,

$$\begin{aligned}\dot{w}&=\dot{X}_t-\dot{x}q_0(\mu_0)-\dot{z}q_1(\mu_0)-\dot{\bar{z}}\overline{q_1(\mu_0)}\\ &=\mathcal{A}(\mu_0)w-g(x,z,\bar{z},\mu_0)q_0(\mu_0)-2\operatorname{Re}\{h(x,z,\bar{z},\mu_0)q_1(\mu_0)\}\\ &\quad+\mathcal{R}(\mu_0)\big(w+xq_0(\mu_0)+zq_1(\mu_0)+\overline{zq_1(\mu_0)}\big).\end{aligned}$$

Lemma A.1. *Assume that $g_{200}(\mu_0)\neq 0$. Then there is a locally defined smooth, invertible variable transformation, smoothly depending on μ, which for all sufficiently small $\|\mu-\mu_0\|$ reduces* (6.2) *into the following form*:

$$\begin{aligned}\dot{y}&=G_{100}(\mu)y+\frac{1}{2}G_{200}(\mu)y^2+G_{011}(\mu)|\nu|^2+\frac{1}{6}G_{300}y^3\\ &\quad+G_{111}(\mu)y|\nu|^2+O\big(\|(y,\nu,\bar{\nu})\|^4\big),\\ \dot{\nu}&=H_{010}(\mu)\nu+H_{110}(\mu)y\nu+\frac{1}{2}H_{210}(\mu)y^2\nu+\frac{1}{2}H_{021}(\mu)\nu|\nu|^2\\ &\quad+O\big(\|(y,\nu,\bar{\nu})\|^4\big),\end{aligned}\tag{A.2}$$

where $y\in\mathbb{R}$, $\nu\in\mathbb{C}$, $\|(y,\nu,\bar{\nu})\|^2=|y|^2+|\nu|^2$, G_{jkl} are real-valued smooth functions, while H_{jkl} are complex-valued smooth functions. Moreover, $G_{100}(\mu_0)=0$, $H_{010}(\mu_0)=it_n$, and $G_{200}(\mu_0)=g_{200}(\mu_0)$, $G_{011}(\mu_0)=g_{011}(\mu_0)$, $H_{110}(\mu_0)=h_{110}(\mu_0)$, and

$$\begin{aligned}G_{300}(\mu_0)&=g_{300}(0)-\frac{6}{t_n}\operatorname{Im}\{g_{110}(\mu_0)h_{200}(\mu_0)\},\\ G_{110}(\mu_0)&=g_{111}(0)-\frac{1}{t_n}\big[2\operatorname{Im}\{g_{110}(\mu_0)h_{011}(\mu_0)\}+\operatorname{Im}\{g_{020}(\mu_0)h_{101}(\mu_0)\}\big],\\ H_{210}(\mu_0)&=h_{210}(\mu_0)+\frac{i}{t_n}h_{200}(\mu_0)\big[h_{020}(\mu_0)-2g_{110}(\mu_0)\big]\\ &\quad-\frac{i}{t_n}\big[\big|h_{101}(\mu_0)\big|^2+h_{011}\bar{h}_{200}(\mu_0)\big],\end{aligned}$$

$$H_{021}(\mu_0) = h_{021}(\mu_0) - \frac{i}{3t_n}\left|h_{002}(\mu_0)\right|^2 + \frac{i}{2t_n}\left[2h_{011}(\mu_0)h_{020}(\mu_0) - g_{020}(\mu_0)h_{101}(\mu_0) - 4\left|h_{011}(\mu_0)\right|^2\right].$$

Proof. Consider the action of $\mathbb{S}^1$ on $\mathbb{R} \times \mathbb{C}$ given by

$$\theta \cdot (x, z) = \left(x, ze^{-it_n\theta}\right), \quad \theta \in \mathbb{S}^1,\ (x, z) \in \mathbb{R} \times \mathbb{C}.$$

According to normal form theorem in [25], by a near identity transformation

$$x = y + V(y, \nu, \bar{\nu}, \mu), \qquad z = \nu + W(y, \nu, \bar{\nu}, \mu), \tag{A.3}$$

with

$$V(y, \nu, \bar{\nu}, \mu) = \sum_{j+s+k\geqslant 1} \frac{1}{j!s!k!} V_{jsk}(\mu) y^j \nu^s \bar{\nu}^k,$$
$$W(y, \nu, \bar{\nu}, \mu) = \sum_{j+s+k\geqslant 1} \frac{1}{j!s!k!} W_{jsk}(\mu) y^j \nu^s \bar{\nu}^k,$$

system (6.2) can be simplified to

$$\begin{aligned} \dot{y} &= G(y, \nu, \mu), \\ \dot{\nu} &= it_n\nu + H(y, \nu, \mu), \end{aligned} \tag{A.4}$$

where V and W are at least second order in y, ν, and $\bar{\nu}$, (G, H) is $\mathbb{S}^1$-equivariant, i.e., for all $t \in \mathbb{R}$,

$$G\left(y, \nu e^{-it_n t}, \bar{\nu} e^{it_n t}, \mu\right) = G(y, \nu, \bar{\nu}, \mu),$$
$$H\left(y, \nu e^{-it_n t}, \bar{\nu} e^{it_n t}, \mu\right) = e^{-it_n t} H(y, \nu, \bar{\nu}, \mu).$$

Therefore, there exist real smooth functions $Q_1, Q_2, Q_3 : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$G(y, \nu, \bar{\nu}, \mu) = Q_3\left(y, |\nu|^2, \mu\right), \qquad H(y, \nu, \bar{\nu}, \mu) = \nu Q_1\left(y, |\nu|^2, \mu\right) + i\nu Q_2\left(y, |\nu|^2, \mu\right).$$

Namely, system (A.4) takes the form of (A.2). Let us first prove the lemma for $\mu = \mu_0$. We can reduce (A.1) into the form (A.2) with $\mu = \mu_0$ by performing a nonlinear invertible transformation (A.3) with $\mu = \mu_0$ and

$$V_{020}(\mu_0) = -\frac{g_{020}}{2it_n}, \qquad V_{002}(\mu_0) = \frac{g_{002}}{2it_n}, \qquad V_{110}(\mu_0) = -\frac{g_{110}}{it_n},$$
$$V_{101}(\mu_0) = \frac{g_{101}}{it_n}, \qquad W_{200}(\mu_0) = \frac{h_{200}}{it_n}, \qquad W_{020}(\mu_0) = -\frac{h_{020}}{it_n},$$
$$W_{101}(\mu_0) = \frac{h_{101}}{2it_n}, \qquad W_{002}(\mu_0) = \frac{h_{002}}{3it_n}, \qquad W_{011}(\mu_0) = \frac{h_{011}}{it_n},$$

where all the g_{jkl} and h_{jkl} have to be evaluated at μ_0. These coefficients are selected exactly in order to annihilate all the quadratic terms in the resulting system except those present in (A.2). Then, one can eliminate all nonresonant order-three terms without changing the coefficients in front of the resonant ones displayed in (A.2). To prove the lemma for $\mu \neq \mu_0$ with small $\|\mu - \mu_0\|$, we have to perform a parameter-dependent transformation (A.3) coinciding with $W_{jkl}(\mu_0)$ given above. To prove that it is possible to select the parameter-dependent coefficients of (A.3) to eliminate all linear and quadratic terms except those shown in (A.2), for all small $\|\mu - \mu_0\|$, one has to apply the Implicit Function Theorem as well as the assumption (H4) and condition of Lemma A.1. □

Making a nonlinear time reparametrization in (A.4) and performing an extra variable transformation allows one to simplify the system further. According to the paper by Gavrilov [13], we have the following lemma, which shows that all but one resonant cubic term can be *removed* under certain nondegeneracy conditions.

Lemma A.2. *Assume that* $G_{200}(\mu_0)G_{011}(\mu_0) \neq 0$. *Then, system* (A.4) *is locally smoothly orbitally equivalent near the origin to system* (6.3) *with* $e_{1j}(\mu)$ $(j = 0, 1, 2)$ *and* $e_{22}(\mu)$ *being smooth real-valued functions, while* e_{2j} $(j = 1, 2)$ *being smooth complex-valued functions. Moreover,* $e_{10}(\mu_0) = e_{20}(\mu_0) = 0$, *and*

$$\begin{aligned}
e_{11}(\mu_0) &= \frac{1}{2}G_{200}(\mu_0), \qquad e_{12}(\mu_0) = G_{011}(\mu_0),\\
e_{21}(\mu_0) &= H_{110}(\mu_0) - it_n\frac{G_{300}(\mu_0)}{3G_{200}(\mu_0)},\\
e_{22}(\mu_0) &= \frac{1}{2}\operatorname{Re}\left[H_{210}(\mu_0) + \frac{H_{021}(\mu_0)G_{200}(\mu_0)}{2G_{011}(\mu_0)}\right]\\
&\quad + \frac{1}{2}\operatorname{Re}\left[H_{110}(\mu_0)\left(\frac{\operatorname{Re} H_{021}(\mu_0)}{G_{011}(\mu_0)} - \frac{G_{300}(\mu_0)}{G_{200}(\mu_0)} + \frac{G_{111}(\mu_0)}{G_{011}(\mu_0)}\right)\right].
\end{aligned}$$

Proof. As stated before, we start with $\mu = \mu_0$. Make the following time reparametrization in (A.4):

$$t = \left(1 + \sigma_1 y + \sigma_2|v|^2\right)s \tag{A.5}$$

with the constant $\sigma_1, \sigma_2 \in \mathbb{R}$ to be determined later. Simultaneously introduce new variables, u and z, via the invertible transformation

$$u = y + \frac{1}{2}\sigma_3 y^2, \qquad z = v + \sigma_4 yv, \tag{A.6}$$

where $\sigma_3 \in \mathbb{R}$ and $\sigma_4 \in \mathbb{C}$ to be determined later too. In the new variables and time, system (A.4) takes the form of

$$\begin{aligned}
\dot u &= e_{11}(\mu_0)u^2 + e_{12}(\mu_0)|z|^2 + O\left(\left\|(u, z, \bar z)\right\|^4\right),\\
\dot z &= it_n z + e_{21}(\mu_0)uz + e_{22}(\mu_0)u^2 z + O\left(\left\|(u, z, \bar z)\right\|^4\right).
\end{aligned} \tag{A.7}$$

Set

$$\sigma_1 = -\frac{G_{300}(\mu_0)}{3G_{200}(\mu_0)}, \quad \sigma_3 = 2\,\mathrm{Re}(\sigma_4) - \sigma_1 - \frac{G_{111}(\mu_0)}{G_{011}(\mu_0)}, \quad \sigma_4 = -\frac{2it_n\sigma_2 + H_{021}(\mu_0)}{2G_{011}(\mu_0)},$$

and then use the free parameter σ_2 to annihilate the imaginary part of the coefficient of the u^2z-term so that the coefficient $e_{22}(\mu_0)$ is a real number given in the lemma statement. A similar construction can be carried out for small $\|\mu - \mu_0\|$ with the help of the Implicit Function Theorem if one considers σ_1 and σ_2 in (A.5) as functions of μ and replaces (A.6) by

$$u = y + \sigma_5(\mu)y + \frac{1}{2}\sigma_3(\mu_0)y^2, \qquad z = \nu + \sigma_4(\mu_0)y\nu,$$

for smooth function σ_j $(j = 3, 4, 5)$ satisfying $\sigma_j(\mu_0) = \sigma_j$ $(j = 3, 4)$ and $\sigma_5(\mu_0) = 0$. □

Appendix B. Derivation of normal form (7.2)

In this appendix, we derive the concrete expressions for $e_{jk}(\mu_0)$, $j, k = 1, 2$. Recall that, at $\mu = \mu_0$, for each $j = 1, 2$,

$$\begin{aligned} \dot{z}_j(t) &= \langle p_j, \dot{u}_t\rangle = \langle p_j, \mathcal{A}(\mu_0)X_t\rangle + \langle p_j, \mathcal{R}(\mu_0)X_t\rangle \\ &= i\omega_j z_j(t) + \langle p_j(\mu_0), \mathcal{R}(\mu_0)X_t\rangle. \end{aligned}$$

Thus, $g^j(z, \mu_0) = \langle p_j, \mathcal{R}(\mu_0)X_t\rangle$. In addition, we have, at $\mu = \mu_0$,

$$\begin{aligned} \dot{w} &= \dot{X}_t - \dot{z}_1 q_1 - \dot{\bar{z}}_1\overline{q_1} - \dot{z}_2 q_2 - \dot{\bar{z}}_2\overline{q_2} \\ &= \mathcal{A}(\mu_0)w - 2\,\mathrm{Re}\{g^1(z, \mu_0)q_1 + g^2(z, \mu_0)q_2\} \\ &\quad + \mathcal{R}(\mu_0)(w + z_1q_1 + \overline{z_1q_1} + z_2q_2 + \overline{z_2q_2}). \end{aligned}$$

The normal form of (7.2) becomes

$$\dot{\nu} = \Lambda\nu + \mathcal{G}(\nu, \mu_0). \tag{B.1}$$

Substituting (7.5) into (7.1) and using (B.1), we obtain

$$\begin{aligned} &i\omega_1[\psi_{\nu_1}\nu_1 - \psi_{\overline{\nu_1}}\overline{\nu_1}] + i\omega_2[\psi_{\nu_2}\nu_2 - \psi_{\overline{\nu_2}}\overline{\nu_2}] - \Lambda(\mu_0)\psi \\ &\quad = g(\nu + \psi, \mu_0) - \mathcal{G}(\nu, \mu_0) - \psi_{\nu_1}\mathcal{G}^1(\nu, \mu_0) \\ &\qquad - \psi_{\overline{\nu_1}}\overline{\mathcal{G}^1}(\nu, \mu_0) - \psi_{\nu_2}\mathcal{G}^2(\nu, \mu_0) - \psi_{\overline{\nu_2}}\overline{\mathcal{G}^2}(\nu, \mu_0), \end{aligned} \tag{B.2}$$

where subscripts denote partial differentiations. We now express ψ as a Taylor series with $\psi_{jksl} = \partial\psi^{j+i+s+l}/\partial\nu_1^j\partial\overline{\nu_1}^k\partial\nu_2^s\partial\overline{\nu_2}^l$:

$$\psi(\nu, \bar{\nu}) = \sum_{2\leqslant j+k+s+l\leqslant 3} \psi_{jksl}\frac{\nu_1^j\overline{\nu_1}^k\nu_2^s\overline{\nu_2}^l}{j!k!s!l!} + O\big(|\nu|^4\big).$$

Next, using the facts that the normal form $\mathcal{G}_1(z,\mu_0) = e_{11}|\nu_1|^2\nu_1 + e_{12}|\nu_2|^2\nu_1 + O(|\nu|^5)$ and $\mathcal{G}_2(z,\mu_0) = e_{21}|\nu_2|^2\nu_2 + e_{22}|\nu_1|^2\nu_2 + O(|\nu|^5)$, and substituting the above equations into (B.2), we obtain

$$\begin{aligned}
&\big[2i\omega_1 \mathrm{I}_2 - \Lambda(\mu_0)\big]\psi_{2000}\nu_1^2 - \big[2i\omega_1 \mathrm{I}_2 + \Lambda(\mu_0)\big]\psi_{0200}\overline{\nu_1}^2 \\
&\quad + \big[2i\omega_2 \mathrm{I}_2 - \Lambda(\mu_0)\big]\psi_{0020}\nu_2^2 - \big[2i\omega_2 \mathrm{I}_2 + \Lambda(\mu_0)\big]\psi_{0002}\overline{\nu_2}^2 \\
&\quad + 2\big[i(\omega_1+\omega_2) - \Lambda(\mu_0)\big]\psi_{1010}\nu_1\nu_2 + 2\big[i(\omega_1-\omega_2) - \Lambda(\mu_0)\big]\psi_{1001}\nu_1\overline{\nu_2} \\
&\quad - 2\big[i(\omega_1-\omega_2) + \Lambda(\mu_0)\big]\psi_{0110}\overline{\nu_1}\nu_2 - 2\big[i(\omega_1+\omega_2) + \Lambda(\mu_0)\big]\psi_{0101}\overline{\nu_1}\overline{\nu_2} \\
&\quad - 2\Lambda(\mu_0)\psi_{0011}\nu_2\overline{\nu_2} - 2\Lambda(\mu_0)\psi_{1100}\nu_1\overline{\nu_1} \\
&= 2\sum_{j+k+s+l=2} (j!k!s!l!)^{-1} h_{jksl}\nu_1^j\overline{\nu_1}^k\nu_2^s\overline{\nu_2}^l + O\big(|\nu|^3\big).
\end{aligned}$$

Equating coefficients yields the leading terms in the transform:

$$\begin{aligned}
\big[2i\omega_1\mathrm{I}_2 - \Lambda(\mu_0)\big]\psi_{2000} &= g_{2000}, & -\big[2i\omega_1\mathrm{I}_2 + \Lambda(\mu_0)\big]\psi_{0200} &= g_{0200}, \\
-\Lambda(\mu_0)\psi_{0011} &= g_{0011}, & -2\Lambda(\mu_0)\psi_{1100} &= g_{1100}, \\
\big[2i\omega_2\mathrm{I}_2 - \Lambda(\mu_0)\big]\psi_{0020} &= g_{0020}, & -\big[2i\omega_2\mathrm{I}_2 + \Lambda(\mu_0)\big]\psi_{0002} &= g_{0002}, \\
\big[i(\omega_1+\omega_2) - \Lambda(\mu_0)\big]\psi_{1010} &= g_{1010}, & \big[i(\omega_1-\omega_2) - \Lambda(\mu_0)\big]\psi_{1001} &= g_{1001}, \\
-\big[i(\omega_1-\omega_2) + \Lambda(\mu_0)\big]\psi_{0110} &= g_{0110}, & -\big[i(\omega_1+\omega_2) + \Lambda(\mu_0)\big]\psi_{0101} &= g_{0101}, \\
\big[2i\omega_1\mathrm{I}_2 - \Lambda(\mu_0)\big]\psi_{2000} &= g_{2000}, & -\big[2i\omega_1\mathrm{I}_2 + \Lambda(\mu_0)\big]\psi_{0200} &= g_{0200}, \\
-\Lambda(\mu_0)\psi_{0011} &= g_{0011}, & -2\Lambda(\mu_0)\psi_{1100} &= g_{1100}, \\
\big[2i\omega_2\mathrm{I}_2 - \Lambda(\mu_0)\big]\psi_{0020} &= g_{0020}, & -\big[2i\omega_2\mathrm{I}_2 + \Lambda(\mu_0)\big]\psi_{0002} &= g_{0002}, \\
\big[i(\omega_1+\omega_2) - \Lambda(\mu_0)\big]\psi_{1010} &= g_{1010}, & \big[i(\omega_1-\omega_2) - \Lambda(\mu_0)\big]\psi_{1001} &= g_{1001}, \\
-\big[i(\omega_1-\omega_2) + \Lambda(\mu_0)\big]\psi_{0110} &= g_{0110}, & -\big[i(\omega_1+\omega_2) + \Lambda(\mu_0)\big]\psi_{0101} &= g_{0101}.
\end{aligned}$$

We now carry out the expansion to higher order and equate the coefficients of the normal form terms $(|\nu_1|^2\nu_1, |\nu_2|^2\nu_2)^T$ and $(|\nu_2|^2\nu_1, |\nu_1|^2\nu_2)^T$. It is easy to see that for these two terms, some coefficients on the right-hand of (B.2) vanish identically. For the terms $|\nu_1|^2\nu_1$, we have

$$\begin{aligned}
0 = g_{2000}^1\psi_{1100}^1 + \frac{1}{2}g_{0200}^1\overline{\psi_{0200}^1} + g_{1100}^1\bigg[\overline{\psi_{1100}^1} + \frac{1}{2}\psi_{2000}\bigg] + \frac{1}{2}g_{2100}^1 \\
+ g_{1010}^1\psi_{1100}^2 + g_{1001}^1\overline{\psi_{1100}^2} + \frac{1}{2}g_{0110}^1\psi_{2000}^2 + \frac{1}{2}g_{0101}^1\psi_{0200}^2 - e_{11}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}i(\omega_1-\omega_2)\psi_{2100}^2 = g_{2000}^2\psi_{1100}^1 + \frac{1}{2}g_{0200}^2\overline{\psi_{0200}^1} + g_{1100}^2\bigg[\overline{\psi_{1100}^1} + \frac{1}{2}\psi_{2000}\bigg] + \frac{1}{2}g_{2100}^2 \\
+ g_{1010}^2\psi_{1100}^2 + g_{1001}^2\overline{\psi_{1100}^2} + \frac{1}{2}g_{0110}^2\psi_{2000}^2 + \frac{1}{2}g_{0101}^2\psi_{0200}^2.
\end{aligned}$$

For the terms $|\nu_2|^2\nu_1$, we have

$$\begin{aligned}0 &= g^1_{2000}\psi^1_{0011} + g^1_{0020}\psi^2_{1001} + g^1_{0002}\overline{\psi^2_{0101}} + g^1_{1100}\overline{\psi^1_{0011}} + g^1_{1010}\big[\psi^2_{0011} + \psi^1_{1001}\big]\\ &\quad + g^1_{1001}\overline{\psi^2_{0011}} + g^1_{0110}\overline{\psi^1_{0110}} + g^1_{0101}\overline{\psi^1_{0101}} + g^1_{0011}\big[\overline{\psi^2_{0110}} + \psi^2_{1010}\big] - e_{12}\end{aligned}$$

and

$$\begin{aligned}i(\omega_1-\omega_2)\psi^2_{1011} &= g^2_{2000}\psi^1_{0011} + g^2_{0020}\psi^2_{1001} + g^2_{0002}\overline{\psi^2_{0101}} + g^2_{1100}\overline{\psi^1_{0011}}\\ &\quad + g^2_{1010}\big[\psi^2_{0011} + \psi^1_{1001}\big] + g^2_{1001}\overline{\psi^2_{0011}} + g^2_{0110}\overline{\psi^1_{0110}}\\ &\quad + g^2_{0101}\overline{\psi^1_{0101}} + g^2_{0011}\big[\overline{\psi^2_{0110}} + \psi^2_{1010}\big].\end{aligned}$$

For the terms $|\nu_2|^2\nu_2$, we have

$$\begin{aligned}0 &= g^1_{2000}\psi^1_{0011} + g^1_{0020}\psi^2_{1001} + g^1_{0002}\overline{\psi^2_{0101}} + g^1_{1100}\overline{\psi^1_{0011}} + g^1_{1010}\big[\psi^2_{0011} + \psi^1_{1001}\big]\\ &\quad + g^1_{1001}\overline{\psi^2_{0011}} + g^1_{0110}\overline{\psi^1_{0110}} + g^1_{0101}\overline{\psi^1_{0101}} + g^1_{0011}\big[\overline{\psi^2_{0110}} + \psi^2_{1010}\big] - e_{12}\end{aligned}$$

and

$$\begin{aligned}0 &= g^2_{0020}\psi^1_{0011} + \frac{1}{2}g^2_{0002}\overline{\psi^1_{0002}} + g^2_{0011}\left[\overline{\psi^1_{0011}} + \frac{1}{2}\psi_{0020}\right] + \frac{1}{2}g^2_{0021}\\ &\quad + g^2_{1010}\psi^2_{0011} + g^2_{0110}\overline{\psi^2_{0011}} + \frac{1}{2}g^2_{1001}\psi^2_{0020} + \frac{1}{2}g^2_{0101}\psi^2_{0002} - e_{21}.\end{aligned}$$

For the terms $|\nu_1|^2\nu_2$, we have

$$\begin{aligned}i(\omega_2-\omega_1)\psi^2_{1110} &= g^1_{0020}\psi^1_{1100} + g^1_{2000}\psi^2_{0110} + g^1_{0200}\overline{\psi^2_{0101}} + g^1_{0011}\overline{\psi^1_{1100}}\\ &\quad + g^1_{1010}\big[\psi^2_{1100} + \psi^1_{0110}\big] + g^1_{0110}\overline{\psi^2_{1100}} + g^1_{1001}\overline{\psi^1_{1001}}\\ &\quad + g^1_{0101}\overline{\psi^1_{0101}} + g^1_{1100}\big[\overline{\psi^2_{1001}} + \psi^2_{1010}\big]\end{aligned}$$

and

$$\begin{aligned}0 &= g^2_{0020}\psi^1_{1100} + g^2_{2000}\psi^2_{0110} + g^2_{0200}\overline{\psi^2_{0101}} + g^2_{0011}\overline{\psi^1_{1100}} + g^2_{1010}\big[\psi^2_{1100} + \psi^1_{0110}\big]\\ &\quad + g^2_{0110}\overline{\psi^2_{1100}} + g^2_{1001}\overline{\psi^1_{1001}} + g^2_{0101}\overline{\psi^1_{0101}} + g^2_{1100}\big[\overline{\psi^2_{1001}} + \psi^2_{1010}\big] - e_{22}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}e_{11} &= \frac{1}{2}g^1_{2100} + \frac{i}{2\omega_1}g^1_{1100}g^1_{2000} + \frac{i}{\omega_2}\big(g^1_{1010}g^2_{1100} - g^1_{1001}\overline{g^2_{1100}}\big) - \frac{i}{4\omega_1+2\omega_2}g^1_{0101}\overline{g^2_{0200}}\\ &\quad - \frac{i}{4\omega_1-2\omega_2}g^1_{0110}g^2_{2000} - \frac{i}{\omega_1}\big|g^1_{1100}\big|^2 - \frac{i}{6\omega_1}\big|g^1_{0200}\big|^2,\end{aligned}$$

$$
\begin{aligned}
e_{12} = {} & g^1_{1011} + \frac{i}{\omega_2}\left(g^1_{1010}g^2_{0011} - g^1_{1001}\overline{g^2_{0011}}\right) \\
& + \frac{i}{\omega_1}\left(g^1_{2000}g^1_{0011} - g^1_{1100}\overline{g^1_{0011}} - g^1_{0011}\overline{g^2_{0110}} - g^1_{0011}g^1_{1010}\right) - \frac{i}{\omega_1 + 2\omega_2}g^1_{0002}\overline{g^2_{0101}} \\
& - \frac{i}{\omega_1 - 2\omega_2}g_{0020}g^2_{1001} - \frac{i}{2\omega_1 - \omega_2}\left|g^1_{0110}\right|^2 - \frac{i}{2\omega_1 + \omega_2}\left|g^1_{0101}\right|^2, \\
e_{21} = {} & \frac{1}{2}g^2_{0021} + \frac{i}{2\omega_2}g^2_{0011}g^2_{0020} + \frac{i}{\omega_1}\left(g^1_{0011}g^2_{1010} - \overline{g^1_{0011}}g^2_{0110}\right) - \frac{i}{4\omega_2 - 2\omega_1}g^1_{0020}g^2_{1001} \\
& - \frac{i}{4\omega_2 + 2\omega_1}\overline{g^1_{0002}}g^2_{0101} - \frac{i}{\omega_2}\left|g^2_{0011}\right|^2 - \frac{i}{6\omega_2}\left|g^2_{0002}\right|^2, \\
e_{22} = {} & g^2_{1110} + \frac{i}{\omega_1}\left(g^1_{1100}g^2_{1010} - \overline{g^1_{1100}}g^2_{0110}\right) \\
& + \frac{i}{\omega_2}\left(g^2_{0020}g^2_{1100} - g^2_{0011}\overline{g^2_{1100}} - g^1_{1010}g^2_{1100} - \overline{g^1_{1001}}g^2_{1100}\right) - \frac{i}{2\omega_1 + \omega_2}\overline{g^1_{0101}}g^2_{0200} \\
& + \frac{i}{2\omega_1 - \omega_2}g^1_{0110}g^2_{2000} + \frac{i}{\omega_1 - 2\omega_2}\left|g^2_{1001}\right|^2 - \frac{i}{\omega_1 + 2\omega_2}\left|g^2_{0101}\right|^2.
\end{aligned}
$$

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