Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback

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The dynamics generated by the delay differential equation $\dot{x}(t) = -\mu x(t) + f(x(t-\tau))$ with unimodal feedback is studied. The existence of the global attractor is shown and bounds of the attractor are given. We find attractive invariant intervals and give sufficient conditions that guarantee that all solutions enter the domain where f' is negative with respect to a positive equilibrium, so the results for delayed monotone feedback can be applied to describe the asymptotic behaviour of solutions. In particular, the existence of heteroclinic orbits from the trivial equilibrium to a periodic orbit oscillating around the positive equilibrium is established. Numerical examples using Nicholson's blowflies equation and the Mackey–Glass equation are provided to illustrate the main results.

Keywords: delay differential equation; stability; global attractivity; unimodal feedback; heteroclinic orbits; delay-induced chaos

1. Introduction

The delay differential equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-\tau)), \quad \mu > 0, \tag{1.1}$$

has been widely investigated in the literature. The global dynamics, structure of the global attractor, existence and properties of periodic orbits have been discussed in detail for the monotone positive and the monotone negative feedback cases (see Walther (1995), Krisztin *et al*. (1999), Krisztin (2000), Krisztin & Walther (2001) and Aschwanden *et al*. (2006) and references thereof). Some Poincaré–Bendixson type theorems were proved in Mallet-Paret & Sell (1996), showing that chaotic behaviour is not possible in the case of monotone feedback.

On the other hand, for the so-called unimodal feedback (when $f(\xi)$ has exactly one extremum and changes the monotonicity at only one point), time delay may lead to complex dynamics, as shown in Lani-Wayda (1999) and many

others. Well-known model equations including the form (1.1) with unimodal feedback can be found, e.g. the Nicholson's blowflies equation (Smith 1995), where $f(\xi) = a\xi \exp(-b\xi)$ or the Mackey–Glass equation (Mackey & Glass 1977), where $f(\xi) = a\xi/(1 + \xi^n)$, the latter is also a nice example of delay-induced chaotic behaviour.

Our goal here is to initiate a systematic study of equation (1.1) with the unimodal feedback in the most challenging case where the unique positive equilibrium is larger than the critical point of the feedback f. Therefore, full understanding of the asymptotic behaviour of a trajectory requires description of the transition process of this trajectory from the domain where f' is positive to the domain where f' is negative; as such the approach to be developed here can be regarded as a 'domain-decomposition method' for the study of dynamics of such a simple-looking equation with nonlinearity that changes its monotonicity multiple times. A natural and important first step is to determine when such a system admits an invariant closed interval (with respect to the pointwise ordering in the usual phase space) that attracts every non-trivial trajectory, and that belongs to the domain where f' is negative. This then enables us to apply the powerful theory of delay differential equations with negative feedback such as the Poincaré–Bendixson theorem of Mallet-Paret and Sell. It should be emphasized that the existence of a non-trivial invariant interval is shown to exist for a very large parameter range, but this interval falls into the domain where f' is negative only under further restrictions: our numerical examples provided at the end of this paper clearly show that without these further restrictions chaotic behaviours can be easily observed.

The paper is organized as follows. We formulate the problem and present the basic definitions and notations of the background theory in §2. In §3, we prove the existence of an attractive invariant interval. This gives bounds for the global attractor of the generated semiflow. We are concerned with the stability properties of equilibria in §4. Many papers are devoted to the global attractivity of the positive equilibrium; the equivalence of the local and the global stability is still an open and interesting problem (for an overview and related results, see Liz *et al.* (2002) and references therein). The existence of heteroclinic orbits from the trivial equilibrium to a periodic orbit oscillating around the positive equilibrium is shown in §5 and finally §6 provides a collection of some examples and numerical simulations based on the well-known Nicholson's blowflies equation and the Mackey–Glass equation to illustrate the main results.

2. Preliminaries

We call the delay feedback function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ unimodal, if

$f(\xi) \geq 0$ for all $\xi \geq 0$, $f(0) = 0$, and there is a unique $\xi_0 > 0$

such that $f'(\xi) > 0$ if $0 \leq \xi < \xi_0, f'(\xi_0) = 0$ and $f'(\xi) < 0$ if $\xi > \xi_0$, (U)

moreover $f''(\xi) < 0$ if $0 \leq \xi \leq \xi_0$ and $\lim_{\xi \to \infty} f(\xi) = 0$.
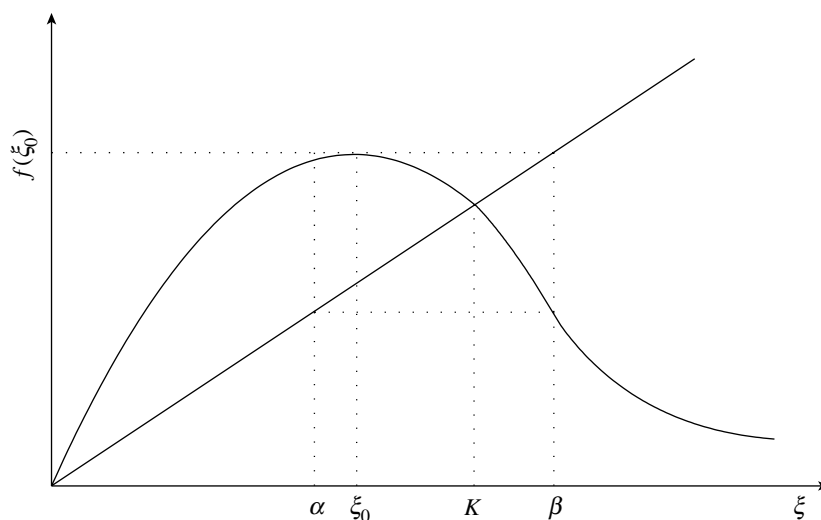
Figure 1. Profile of a unimodal feedback, the straight line is $\mu\xi$.

A schematic of such a function is given in figure 1. It is easy to check that the functions $f(\xi) = a\xi \exp(-b\xi)$ and $f(\xi) = a\xi/(1 + \xi^n)$ are unimodal. We study equation (1.1), where $\mu, \tau > 0$ and $f(\xi)$ satisfies (U).

Let $C = C([-\tau, 0], \mathbb{R})$ denote the Banach space of continuous functions $\phi:[-\tau, 0]$ with the norm given by

$$||\phi|| = \max_{-\tau \le s \le 0} |\phi(s)|.$$

The Banach space C contains the cone

$$C_+ = \{\phi \in C : \phi(s) \ge 0, \ -\tau \le s \le 0\},$$

which generates various order relations on the space C, denoted by usual notations such as $<$, $\le$ and $\ll$. In particular, $\phi \le \psi$ holds in C if and only if $\phi(s) \le \psi(s)$ for all $s \in [-\tau, 0]$; $\phi < \psi$ if and only if $\phi \le \psi$ and $\phi \ne \psi$; $\phi \ll \psi$ if and only if $\phi(s) < \psi(s)$ for all $s \in [-\tau, 0]$. Thus, we can define the order intervals $[\phi, \psi] := \{\zeta \in C : \phi \le \zeta \le \psi\}$ if $\phi \le \psi$ and $(\phi, \psi) := \{\zeta \in C : \phi \ll \zeta \ll \psi\}$ if $\phi \ll \psi$.

Every $\phi \in C_+$ determines a unique continuous function $x = x^\phi \colon [-\tau, \infty) \to \mathbb{R}$, which is differentiable on $(0, \infty)$, satisfies (1.1) for all $t > 0$, and $x(s) = \phi(s)$ for all $s \in [-\tau, 0]$. It is easy to see that if (U) is satisfied then the cone C_+ is positively invariant, i.e. a solution $x^\phi(t)$ with non-negative initial function ϕ remains non-negative for all $t \ge 0$. Hereafter, by a solution of (1.1) we always mean a non-negative solution. The segment $x_t \in C$ of a solution is defined by the relation $x_t(s) = x(t+s)$, where $s \in [-\tau, 0]$ and $t \ge 0$. Then, $x_0 = \phi$ and $x_t(0) = x(t)$. The family of maps

$$\Phi : [0, \infty) \times C_+ \ni (t, \phi) \to x_t(\phi) := \Phi_t(\phi) \in C_+,$$

defines a continuous semiflow on C_+. Sometimes, we use the notation $x^\phi(t)$ for a solution with initial function $\phi \in C_+$, thus x_t^ϕ denotes the segment of such a solution. We shall also use the functional $\varrho : C \to \mathbb{R}$ given by

$$\varrho(\phi) := -\mu\phi(0) + f(\phi(-1)),$$

for any $\phi \in C$. So equation (1.1) can be written as $x'(t) = \varrho(x_t)$. For a given $\phi \in C_+$, the ω−limit set of ϕ is defined as

$$\omega(\phi) = \{\psi \in C_+ : \text{there is a sequence } t_n, \ n \in N, \text{ such that } t_n \to \infty \text{ and}$$

$$\Phi_{t_n}(\phi) \to \psi \text{ as } n \to \infty\}.$$

For any $\xi \in \mathbb{R}$, we write ξ_* for the element of C satisfying $\xi_*(s) = \xi$ for all $s \in [-\tau, 0]$. The set of equilibria of the semiflow generated by (1.1) is given by

$$E = \{\xi_* \in C_+ : \xi \in \mathbb{R} \text{ and } \mu\xi = f(\xi)\}.$$

Obviously $0_* \in E$. In addition, there exists at most, one positive equilibrium. The positive equilibrium K_*, if exists, is called the carrying capacity.

The semiflow Φ is said to be monotone if (Smith 1995)

$$\Phi_t(\phi) \leq \Phi_t(\psi) \quad \text{whenever} \quad \phi \leq \psi \quad \text{and} \quad t \geq 0.$$

If $f'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, then (1.1) defines a monotone semiflow for which the maps $\Phi_t : C_+ \to C_+$, $t \geq 0$, are injective. However, this injectivity property in the unimodal case is no longer true, as the following shows.

Proposition 2.1. *The time-τ map $\Phi_\tau : C_+ \to C_+$ is not injective.*

Proof. Let $\hat{f}$ be the restriction of f to the interval $[0, \xi_0]$. Then, $\hat{f}$ is a bijection between the intervals $[0, \xi_0]$ and $[0, f(\xi_0)]$. Define the map $g : [\xi_0, \infty) \to [0, \xi_0]$ by $\xi \to \hat{f}^{-1}(f(\xi))$. Fix a $\delta > 0$. Consider the functions

$$\phi(s) := \xi_0 - \delta(s), \quad \psi(s) := g(\xi_0 - \delta_s), \quad s \in [-\tau, 0].$$

Clearly $\phi, \psi \in C_+$, $\phi \neq \psi$ but $\phi(0) = \xi_0 = \psi(0)$, moreover $f(\phi(s)) = f(\psi(s))$ for all $s \in [-\tau, 0]$. By the variation-of-constants formula, for any $t \in [0, \tau]$, we have

$$x^\phi(t) = e^{-\mu t}\phi(0) + \int_0^t e^{-\mu(t-u)} f(\phi(u-\tau)) du = x^\psi(t),$$

that means $\Phi_\tau(\phi) = \Phi_\tau(\psi)$.   ∎

As a consequence, there is no backward uniqueness for the semiflow. In what follows we shall use the following numbers (figure 1):

$$\beta := \frac{f(\xi_0)}{\mu} \in \mathbb{R}, \qquad \alpha := \frac{f(\beta)}{\mu} = \frac{f\left(\frac{f(\xi_0)}{\mu}\right)}{\mu} \in \mathbb{R}.$$

The numbers α and β play a crucial role in characterizing the nonlinear dynamics of equation (1.1).

3. Invariant and attractive intervals

We distinguish three cases, as depicted in figure 2. Case A corresponds to the condition $f'(0) \leq \mu$. We are in case B, if $f'(0) > \mu$, but $K \leq \xi_0$. Case C represents the most interesting situation where $K > \xi_0$. In the absence of delay, when $\tau = 0$, (1.1) reduces to the ordinary differential equation $x'(t) = -\mu x(t) + f(x(t))$, for which all solutions converge to 0 if $f'(0) \leq \mu$, and to K if $f'(0) > \mu$.
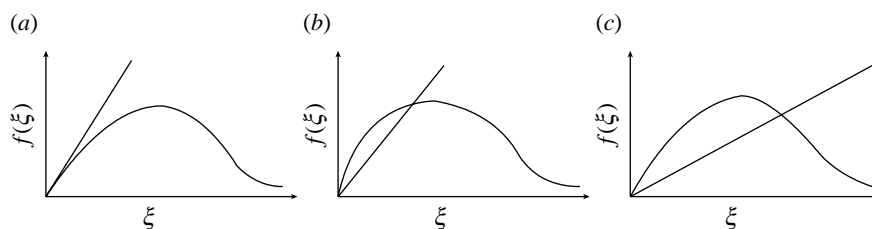
Figure 2. Three different situations between the instant friction $\mu\xi$ and the delayed feedback $f(\xi)$: (*a*) case A, (*b*) case B and (*c*) case C.

Analogous results to propositions 3.1 and 3.2 below are contained in Smith (1995) for the special case of the Nicholson's blowflies equation, where we deal with general unimodal feedback.

Proposition 3.1. *If $f'(0) \leq \mu$, then for all $\tau > 0$ the equilibrium 0_* is globally attractive.*

Proof. If 0 is the unique non-negative equilibrium, then $\mu\xi_0 > f(\xi_0) \geq f(\xi)$ for any $\xi \geq 0$. First note that if $x(t)$ is a solution and $x(t_0) \leq \xi_0$ for some $t_0 > 0$, then $x(t) \leq \xi_0$ for all $t > t_0$. Otherwise, there exists a $t_1 \geq t_0$ such that $x(t_1) = \xi_0$ and $x'(t_1) \geq 0$. But $x'(t_1) = -\mu\xi_0 + f(x(t_1 - \tau)) < 0$, a contradiction. This implies the positive invariance of $[0_*, \xi_{0*}]$.

Now we show that all positive solutions eventually enter this interval. Suppose the contrary, i.e. that there exist a $t_2 > 0$ and a solution $x(t)$ with $x(t) > \xi_0$ for all $t > t_2$. Let $\Delta_0 := \mu\xi_0 - f(\xi_0) > 0$. It follows that $x'(t) < -\Delta_0$ for all $t > t_2 + \tau$ and hence $x(t) \to -\infty$ as $t \to \infty$, contradicting to $x(t) > \xi_0$ for all $t > t_2$.

The positive invariance of the order interval $[0_*, \xi_{0*}]$ implies $x_t^{\xi_{0*}} \leq \xi_{0*}$ for all $t \geq 0$. In this interval, the semiflow is monotone; hence we have $x_{t+u}^{\xi_{0*}} \leq x_u^{\xi_{0*}}$, for $t, u > 0$. Equivalently, $0_* \leq x_t^{\xi_{0*}} \leq x_u^{\xi_{0*}} \leq \xi_{0*}$ whenever $0 \leq u \leq t$. Evaluating at $s = 0$, this yields $0 \leq x^{\xi_{0*}}(t) \leq x^{\xi_{0*}}(u) \leq \xi_0$, that is $x^{\xi_{0*}}(t)$ is monotonically decreasing. We obtain that $\lim_{t\to\infty} x^{\xi_{0*}}(t) \to w$ for some $w \in [0, \xi_0]$, hence the ω-limit set of $x_t^{\xi_{0*}}$ is the singleton $\{w_*\}$ that must be an equilibrium. The only equilibrium is 0_*, thus $x^{\xi_{0*}}(t)$ converges to 0. Any arbitrary solution $x(t)$ enters the positively invariant interval $[0_*, \xi_{0*}]$, thus the monotonicity of the semiflow and the standard comparison argument assure that $x(t)$ converges to 0. ∎

Proposition 3.2. *If $f'(0) > \mu$, but $K \leq \xi_0$, then for all $\tau > 0$ the equilibrium K_* is globally attractive.*

Proof. Similar argument to that for proposition 3.1 shows that every solution enters $[0_*, \xi_{0*}]$, which is positively invariant, and the semiflow is monotone restricted to this interval. Hence, $x^{\xi_{0*}}(t)$ is monotonically decreasing and converges to an equilibrium, but now we have two equilibria, 0_* and K_*. If $\varepsilon \in (0, K)$, then $f(\xi) > \mu\varepsilon$ for all $\xi \in [\varepsilon, \xi_0]$ and $[\varepsilon_*, \xi_{0*}]$ is also positively invariant, because $\varrho(x(t)) = -\mu\varepsilon + f(x(t - \tau)) > 0$ whenever $x_t \in [\varepsilon_*, \xi_{0*}]$ and $x(t) = \varepsilon$. This implies that $x^{\xi_{0*}}(t)$ must converge to K, because K_* is the only equilibrium in $[\varepsilon_*, \xi_{0*}]$. Analogously we can deduce that $x^{\varepsilon_*}(t)$ is increasing and converges to K. Note that if $\phi > 0_*$, then there is a t_0 such that $x_t^\phi \gg 0_*$ for all $t > t_0$. It follows that if $\phi \neq 0_*$, then there exists an $\varepsilon \in (0, K)$ such that $\xi_{0*} \geq x_t^\phi \geq \varepsilon_*$ for some t, thus $x^\phi(t)$ is forced to converge to K. ∎

In the remaining part of this section, we assume that we are in the case C, namely $K > \xi_0$. In this situation $\varrho(\xi_{0*}) > 0$ and the interval $[0_*, \xi_{0*}]$ is no longer positively invariant.

Proposition 3.3. *The inequality $\alpha < K < \beta$ holds.*

Proof. In case C we have $f(K) = \mu K$ and $K > \xi_0$, which yields $f(K) < f(\xi_0)$, thus $\beta = (f(\xi_0)/\mu) > (f(K)/\mu) = K$. Furthermore, $\beta > K$ implies $f(\beta) < f(K)$, therefore $\alpha = (f(\beta)/\mu) < (f(K)/\mu) = K$, and the proposition is proved. ∎

Accordingly, α_* and β_* define an order interval $J := [\alpha_*, \beta_*] \subset C_+$.

Proposition 3.4. *The interval J is positively invariant.*

Proof. Suppose the contrary. Then there is a $\phi \in J$ and a corresponding solution $x(t) = x^\phi(t)$ such that there exists a $T \geq 0$ with either (i) $x_T \in J$, $x(T) = \alpha$, $x'(T) < 0$ or (ii) $x_T \in J$, $x(T) = \beta$, $x'(T) > 0$.

In the case of (i) it follows that $\varrho(x_T) < 0$, but $\varrho(x_T) = -\mu\alpha + f(x(T-\tau)) \geq 0$, a contradiction. Similarly, for the case of (ii) it follows that $\varrho(x_T) > 0$, but $\varrho(x_T) = -\mu\alpha + f(x(T-\tau)) \leq 0$, a contradiction. Hence, $J = [\alpha_*, \beta_*]$ is positively invariant. ∎

Theorem 3.5. *J attracts every solution, or equivalently, if x is a solution, then $\liminf_{t \to \infty} x(t) \geq \alpha$ and $\limsup_{t \to \infty} x(t) \leq \beta$.*

Proof. First we show that every solution eventually enters $[0, \beta]$ and remains there. Note that it is impossible that a solution crosses the level $\beta + \varepsilon$ for any $\varepsilon > 0$ from below. Otherwise, for some T, $x(T) = \beta + \varepsilon > (f(\xi_0)/\mu)$ and $0 \leq x'(T) = -\mu(\beta + \varepsilon) + f(x(T-\tau)) < 0$, a contradiction. It is also impossible that $x(t) \geq \beta$ for all $t > T_0$ with some T_0, because in this case

$$x'(t) = -\mu x(t) + f(x(t-\tau)) < -\mu\beta + f(\beta) < 0 \text{ for all } t > T_0 + \tau,$$

hence $x(t) \to \infty$ as $t \to \infty$, contradicting to $x(t) \geq \beta$ for all $t > T_0$. We have thus proved that $\limsup_{t \to \infty} x(t) \leq \beta$.

Now suppose that $j = \liminf_{t \to \infty} x(t) < \alpha$ for a solution x. Then $j < K$, $f(j) > \mu j$ and there is an $\varepsilon > 0$ such that $\Delta := M - \mu(j + 3\varepsilon) > 0$ and $j + 3\varepsilon < \alpha$, where $I := [j - \varepsilon, j + 3\varepsilon] \subset \mathbb{R}$ and $M := \min\{f(\xi) | \xi \in I\}$. Since j is the lim inf, there exists a T_0 such that $x(t) > j - \varepsilon$ for all $t \geq T_0$. Let $I_* := [(j - \varepsilon)_*, (j + 3\varepsilon)_*] \subset C_+$. Note that $x_t \in I_*$ implies $x'(t) \geq \Delta$. We define two sets,

$$V := \{t | t \geq T_0 \text{ and } x(t) < j + \varepsilon\}, \qquad W := \{t | t \geq T_0 \text{ and } x(t) > j + 3\varepsilon\}.$$

If W is bounded, then there is a T_1 such that $x(t) \in I$ for all $t > T_1$, but then $x_t \in I_*$ and $x'(t) \geq \Delta$ for all $t > T_1 + \tau$, implying $x(t) \to \infty$ as $t \to \infty$, a contradiction, because $\limsup_{t \to \infty} x(t) \leq \beta$. Thus W is unbounded. On the other hand, V is unbounded too, because j is the lim inf. We can conclude that the solution 'oscillates' between V and W, and there exists a sequence $s_n \to \infty$, $n \in \mathbb{N}$ such that $s_n > T_0 + \tau$, $x(s_n) < j + 2\varepsilon$ and $x'(s_n) < 0$. But

$$0 > x'(s_n) = -\mu x(s_n) + f(x(s_n - \tau)) > -\mu(j + 2\varepsilon) + f(x(s_n - \tau)),$$

implies $f(x(s_n - \tau)) < \mu(j + 2\varepsilon)$. There are two distinct values ξ_1 and ξ_2 such that $f(\xi_1) = f(\xi_2) = \mu(j + 2\varepsilon)$ and $\xi_1 < \xi_0 < \xi_2$. By the unimodal property of f, we obtain that $x(s_n - \tau) < \xi_1$ or $x(s_n - \tau) > \xi_2$. We have $M > \mu(j + 3\varepsilon) > \mu(j + 2\varepsilon) = f(\xi_1)$,

hence $\xi_1 \notin I$, thus $\xi_1 < j - \varepsilon$. From $s_n - \tau > T_0$, we get $x(s_n - \tau) > j - \varepsilon$ and we obtain that $x(s_n - \tau) < \xi_1$ is impossible and $x(s_n - \tau) > \xi_2$ holds. But on the other hand, $f(\xi_2) = \mu(j + 2\varepsilon) < \mu\alpha = f(\beta)$, implying $x(s_n - \tau) > \xi_2 > \beta$ for all $n \in \mathbb{N}$. This means that $\limsup_{t \to \infty} x(t) > \beta$.

We have thus arrived to a contradiction. Therefore, $\liminf_{t \to \infty} x(t) \geq \alpha$. ∎

Theorem 3.6. *There exists a global attractor $\mathcal{A}$ of the semiflow Φ, i.e. a non-empty compact set $\mathcal{A} \subset C_+$, which is invariant in the sense that $\Phi_t(\mathcal{A}) = \mathcal{A}$ for all $t \geq 0$ and attracts bounded sets. Moreover, $\mathcal{A} \subset J$.*

Proof. Proposition 3.4 guarantees that the interval J is a bounded set that attracts each non-zero point of C_+, therefore the semiflow is point dissipative. Applying theorem 3.4.8 of Hale (1988), if there is a $t_1 \geq 0$ such that the operators Φ_t are completely continuous for $t \geq t_1$ and Φ is point dissipative, then there exists a global attractor $\mathcal{A}$. It is easy to check using the Arzelà–Ascoli theorem that the maps Φ_t are completely continuous (compact) for any $t \geq \tau$. Clearly $\mathcal{A} \subset J$, hence theorem 3.5 provides estimates for the global attractor. ∎

Under the additional condition

$$\alpha = \frac{f\left(\frac{f(\xi_0)}{\mu}\right)}{\mu} > \xi_0. \tag{L}$$

$[\alpha, \beta] \subset [\xi_0, \infty]$, where $f'(\xi) < 0$ and every solution, independently of τ, eventually enters this domain where f is monotone (decreasing). Thus, it is natural for us to apply the comprehensive results of monotone-delayed feedback theory to describe the dynamics. The condition (L) is fulfilled for a wide range of parameters, but it is also not satisfied in many situations, see §6 for examples of both. Numerical experiments (to be discussed below) suggest that if τ is large enough, then there may exist solutions $x(t)$ with $\liminf_{t \to \infty} x(t) \geq \alpha$ and $\limsup_{t \to \infty} x(t)$ arbitrarily close to α and β. Therefore, if (L) does not hold and the delay is large, we cannot expect solutions to enter and remain in a domain where the feedback f is monotone.

On the other hand, the following result shows that condition (L) can always be satisfied for some carefully chosen μ.

Proposition 3.7. *For any f satisfying (U), there exists a non-empty interval $H \subset \mathbb{R}$, such that the condition (L) is fulfilled for all $\mu \in H$.*

Proof. Suppose that a unimodal f is given. Then the condition (L) is equivalent with

$$F(\mu) := f\left(\frac{f(\xi_0)}{\mu}\right) - \mu\xi_0 > 0.$$

Let $\mu_0 = (f(\xi_0)/\xi_0)$. Then $\mu = \mu_0$ corresponds to the situation $\xi_0 = K$, and the case C $K > \xi_0$ is characterized by $\mu < \mu_0$. Note that $F(\mu_0) = 0$. The derivative

$$F'(\mu) = f'\left(\frac{f(\xi_0)}{\mu}\right)\left(-\frac{f(\xi_0)}{\mu^2}\right) - \xi_0,$$

at $\mu = \mu_0$ is given by

$$F'(\mu_0) = f'(\xi_0)\left(-\frac{\xi_0^2}{f(\xi_0)}\right) - \xi_0 = -\xi_0 < 0,$$

thus there is a $\mu^* < \mu_0$ such that $F(\mu)$ is positive and the condition (L) is satisfied if $\mu \in (\mu^*, \mu_0)$. ∎

The following result shows that, in the absence of condition (L), we can get similar results at least for small delays. Define a map $\Pi \colon (0, K) \to [K, \infty]$ as follows:

$$\Pi(\xi) := f^{-1}(\mu\xi),$$

where f^{-1} is well defined if we restrict f to $[K, \infty)$. Clearly $\Pi(K) = K$, Π is continuous and $\Pi(\xi) - \xi$ is monotonically decreasing on $(0, K]$.

Theorem 3.8. *If*

$$\tau < \tau^* := \frac{\Pi(\xi_0) - \xi_0}{f(\xi_0) - f\left(\frac{f(\xi_0)}{\mu}\right)}, \tag{$L\tau$}$$

then every solution enters the domain where f' is negative.

Proof. Remark that K is between ξ_0 and $\Pi(\xi_0)$, hence $\Pi(\xi_0) - \xi_0 > 0$ and the condition $(L\tau)$ is meaningful. Using the arguments of theorem 3.5 we obtain that if for a solution $x(t)$, $j = \liminf_{t\to\infty} x(t) < \xi_1$ for some ξ_1, then there is an $s_n \to \infty$ such that $x(s_n) < \xi_1$, but $x(s_n - \tau) > \Pi(\xi_1)$. For any $\varepsilon > 0$, there exists a T_0 such that $x(t) \in [\alpha - \varepsilon, \beta]$ for all $t > T_0$, and there exists an $n_0 \in \mathbb{N}$ such that $s_n - \tau > T_0$ for all $n \geq n_0$. Then for any $n \geq n_0$, we have

$$\Pi(\xi_1) - (\xi_1) < x(s_n - \tau) - x(s_n) \leq \int_{s_n - \tau}^{s_n} |x'(u)| \mathrm{d}u$$

$$= \int_{s_n - \tau}^{s_n} |-\mu x(u) + f(x(u - \tau))| \mathrm{d}u \leq \tau(f(\xi_0) - \mu(\alpha - \varepsilon)). \tag{3.1}$$

Letting $\varepsilon \to 0$, we obtain that for any ξ_1 if $\tau < (\Pi(\xi_1) - \xi_1 / f(\xi_0) - \mu\alpha)$, then $\liminf_{x(t)} \geq \xi_1$ for all solutions. Taking particularly $\xi_1 = \xi_0$, we arrive to the condition formulated in the theorem. This means if τ is small enough, every solution eventually enters the domain where f' is negative. ∎

We note that propositions 3.1, 3.2, 3.4 and theorem 3.5 can be derived using theorems 2.1–2.3 of Ivanov & Sharkovsky (1992). All these results are independent of the delay. However, our approach is different and can be applied to obtain delay-dependent results as illustrated in theorem 3.8.

4. Stability of equilibria

It becomes natural now that we should consider the stability properties of the equilibria. It is clear from §3 that in case A, 0 is stable and there is no positive equilibrium, and in case B, 0 is unstable and K is stable. In case C, 0 is unstable as well, therefore the interesting question is the local stability of K in case C. Let $x(t)$ be a solution and $y(t) := x(t) - K$. Then we have

$$y'(t) = -\mu(y(t) + K) + f(y(t - \tau) + K),$$

the corresponding linear variational equation is

$$z'(t) = -\mu z(t) + f'(K)z(t - \tau).$$

The operators $D\Phi_t(K_*)$, $t \geq 0$ form a strongly continuous semigroup, the spectrum of its generator consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equations

$$\lambda + \mu - f'(K)e^{-\tau\lambda} = 0.$$

Our primary interest is the sign of the real part of the roots whereas this determines the local stability. Normalizing with $w = \tau\lambda$, the real parts of λ and w have the same sign and we obtain

$$w = -\tau\mu + \tau f'(K)e^{-w}. \tag{4.1}$$

Detailed analysis of characteristic equations of this form can be found in Diekmann *et al.* (1995), ch. XI, where one can find the stability chart corresponding to (4.1) and the number of roots in the right half of the complex plane can be given in terms of the parameters of (4.1). As we can see, this depends on the delay τ as well. Let us briefly summarize these findings. When $\mu \geq 0$, $\tau > 0$ and $f'(K) < 0$, the roots of (4.1) are in conjugated pairs, and a finite number of them have positive real parts. Every root has negative real part and the equilibrium K is locally stable if $\mu > -f'(K)$. The interesting case is $\mu < -f'(K)$, when K is stable only for small delays, and as τ increases, a series of Hopf bifurcations occurs and periodic solutions arise from the equilibrium K. The critical delays when the bifurcation occurs are given by

$$\tau_k = \frac{\cos^{-1}\left(\frac{\mu}{f'(K)}\right) + 2k\pi}{\sqrt{f'(K)^2 - \mu^2}}.$$

Thus, when $\mu < -f'(K)$, K is stable if $\tau < \tau_0$ and unstable if $\tau > \tau_0$. The bifurcation of the Nicholson's blowflies equation was studied in Wei & Li (2005), and it was shown that there is a supercritical bifurcation at $\tau = \tau_0$ and under some additional technical conditions, an application of a global Hopf bifurcation theorem of Wu (1998) ensures that the emerged periodic solution is always present when $\tau > \tau_0$. Similar bifurcation analysis was done for the Mackey–Glass equation in Song *et al.* (2004). We note that the direction of the Hopf bifurcation depends also on the third-order derivative of f, but here exist explicit algorithms that can be used to determine the direction of the bifurcation in particular cases.

There are many interesting studies for the global attractivity of the positive equilibrium with either specific or general nonlinearity f, and several sufficient conditions were given to guarantee the global attractivity of K. Smith formulated the conjecture that the local asymptotic stability of the positive equilibrium implies its global asymptotic stability for the Nicholson's blowflies equation; unfortunately this remains an open and very interesting question. In Győri & Trofimchuk (1999), the negative Schwarzian derivative has been used to prove global attractivity, other interesting results and a very nice overview of the problem can be found in Liz *et al.* (2002, 2003), where an application of the negative Schwarzian approach yields the global stability of the positive equilibrium for Mackey–Glass type equations near the boundary of the local stability region.

An interesting situation arises naturally when there is a Hopf bifurcation of stable periodic solutions bifurcated from the positive equilibrium. Should the global dynamics be characterized by the transition from the trivial equilibrium to this periodic orbit? This question will be addressed in §§5 and 6.

5. Heteroclinic orbits from zero to a periodic orbit

In this section, we show the existence of heteroclinic orbits. Assume that (L) or $(L\tau)$ is satisfied and every solution eventually enters the domain where f' is negative. Define two new modified nonlinearities $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ as

$$f_1(\xi) := \begin{cases} f(\xi) & \text{if } \xi \geq \xi_1 \\ f(\xi_0) + (f(\xi_1) - f(\xi_0))\exp(\xi - \xi_1) & \text{if } \xi < \xi_1, \end{cases}$$

with some $\xi_1 > \xi_0$ (to be determined later) and

$$f_2(\xi) := \begin{cases} f(\xi) & \text{if } \xi \geq 0 \\ -f(-\xi) & \text{if } \xi < 0. \end{cases}$$

The delay differential equations

$$y'(t) = -\mu y(t) + f_i(y(t-\tau)), \quad i = 1, 2, \tag{5.1}$$

generate semiflows Φ^1 and Φ^2 on the whole space C.

Theorem 5.1. *For any $\phi \in C_+$, $\omega(\phi)$ is $\{K_*\}$ or a periodic orbit oscillating about K.*

Proof. By the assumptions, for any solution x^ϕ of Φ, $j = \liminf_{t \to \infty} x^\phi(t) > \xi_0$. Fix a $\xi_1 \in (\xi_0, j)$. It is easy to check that $\liminf_{\xi \to \infty} f_1(\xi) = f(\xi_0)$ and $f_1(\xi)$ is monotonically decreasing. Hence the corresponding semiflow Φ^1 is generated by a scalar delay differential equation with delayed negative feedback, and we can apply the Poincaré–Bendixson type theorem 10.1 of Walther (1995) to conclude that the ω-limit set of a solution of Φ^1 is the positive equilibrium K_* or a periodic orbit oscillating about K. There is a T_0 such that $x(t) > \xi_1$ for all $t > T_0$. The functions f and f_1 coincide on $[\xi_1, \infty)$, thus we have $\Phi_s(x_t) = \Phi_s^1(x_t)$ whenever $t > T_0 + \tau$ and $s \geq 0$. We obtain that the ω-limit set of ϕ with respect to Φ is the positive equilibrium K_* or a periodic orbit oscillating about K as well. ∎

Theorem 5.2. *Assuming (L) or $(L\tau)$, there exists a heteroclinic orbit x_t, which connects 0_* with K_* or with a periodic orbit oscillating about K.*

Proof. Consider (5.1) with $i = 2$. Since $f_2(0) = 0$, 0_* is an equilibrium of Φ^2. The linearization about the 0 solution gives

$$z'(t) = -\mu z(t) + f_2'(0)z(t-\tau),$$

with the corresponding characteristic equation

$$\lambda = -\mu + f_2'(0)e^{-\lambda\tau}.$$

In case C, we have $f_2'(0) = f'(0) > \mu$ and there is one leading real eigenvalue $\lambda_0 > 0$. Furthermore, the other roots form a sequence of complex conjugate pairs $(\lambda_j, \bar{\lambda}_j)$ with $\operatorname{Re} \lambda_{j+1} < \operatorname{Re} \lambda_j < \lambda_0$ for all integers $j \geq 1$. There is a $\gamma > 0$ such that $\lambda_0 > \gamma > \operatorname{Re} \lambda_j$ for all $j \geq 1$. The corresponding eigenfunction is given by $\chi_0(t) := e^{\lambda_0 t}, t \in [-\tau, 0]$. The phase space C can be decomposed as $C = X_0 \oplus X_1$, where the function $\chi_0 \in C$ spans the linear eigenspace $X_0 := \{c\chi_0 : c \in \mathbb{R}\}$. There exist open neighbourhoods N_0, N_1 in X_0, X_1,

respectively, and a C^1-map $w: N_0 \to X_1$ with range in N_1 and $w(0) = 0$, $Dw(0) = 0$, so that the γ-unstable set of the equilibrium 0_*, namely

$$\mathcal{W}_0(0) := \{\phi \in N_0 + N_1 : \text{there is a trajectory } y_t, t \in \mathbb{R} \text{ with } y_0 = \phi,$$

$$y_t \in N_0 + N_1 \text{ when } t \leq 0 \text{ and } y(t)e^{-\gamma t} \to 0 \text{ as } t \to -\infty\},$$

coincides with the graph

$$\mathcal{W} := \{\phi + w(\phi) : \phi \in N_0\}.$$

(For details see Krisztin *et al.* (1999)).

For any $\phi \in N_0$, there is a $c \in \mathbb{R}$ such that $\phi = c\chi_0$. We have $||\chi_0|| = 1$ and $\chi_0(t) \geq e^{-\lambda_0 \tau} > 0$ for all $t \in [-\tau, 0]$. It follows from $Dw(0) = 0$ that

$$\lim_{||\phi|| \to 0} \frac{||w(\phi)||}{||\phi||} = 0, \phi \in N_0,$$

which means

$$\lim_{c \to 0} \frac{||w(c\chi_0)||}{|c|} = 0.$$

Therefore, there exists a $c_0 > 0$ such that $(||w(c\chi_0)||/|c|) < (e^{-\lambda \tau}/2)$ whenever $c \in (0, c_0)$, or, equivalently $||w(c\chi_0)|| < (c/2)e^{-\lambda \tau}$. Then

$$\min\{c\chi_0(t) + w(c\chi_0)(t) : t \in [-\tau, 0], c \in (0, c_0)\} \geq ce^{-\lambda \tau} - \frac{c}{2}e^{-\lambda \tau} = \frac{c}{2}e^{-\lambda \tau} > 0,$$

thus

$$\phi_c := c\chi_0 + w(c\chi_0) \in \mathcal{W} \cap C_+ \text{ for all } c \in (0, c_0).$$

The unstable set $\mathcal{W}_0(0)$ intersects the positive cone, for any function ϕ_c of this intersection, there is a complete solution $y(t) : \mathbb{R} \to \mathbb{R}_0^+$ such that $y_0 = \phi_c$ and $y_t \to 0_*$ as $t \to -\infty$. We also know that $\omega(\phi_c)$ is K_* or a periodic solution oscillating about K, hence the trajectory $\{y_t^{\phi_c} : t \in \mathbb{R}\}$ is a heteroclinic orbit for the semiflow Φ^2. By the negative invariance of $\mathcal{W}_0(0)$, the trajectory is completely in the cone C_+, where Φ^2 and Φ coincide, hence we have proved the existence of the heteroclinic orbit for the semiflow Φ. ∎

We conclude this section with a general remark that as long as (L) or $(L\tau)$ holds, we have the situation of a monotone negative-delayed feedback with respect to K in a positively invariant domain where f' is negative and into which every solution enters, all the rich results for monotone negative-delayed feedback are then applicable.

6. Examples and simulations

In this section, we discuss several examples and numerical experiments with Mackey–Glass and Nicholson's blowflies equations in different scenarios: whether the condition (L) or $(L\tau)$ is satisfied or not, with different delays and frictions. Our purpose is to illustrate the sharpness of the results in the previous sections, particularly §§3 and 5. The numerical values of different quantities given below are mostly approximations up to the third digit, and the plotted solutions always start from constant initial data. We always plot two distinct solutions for the sake of comparison.
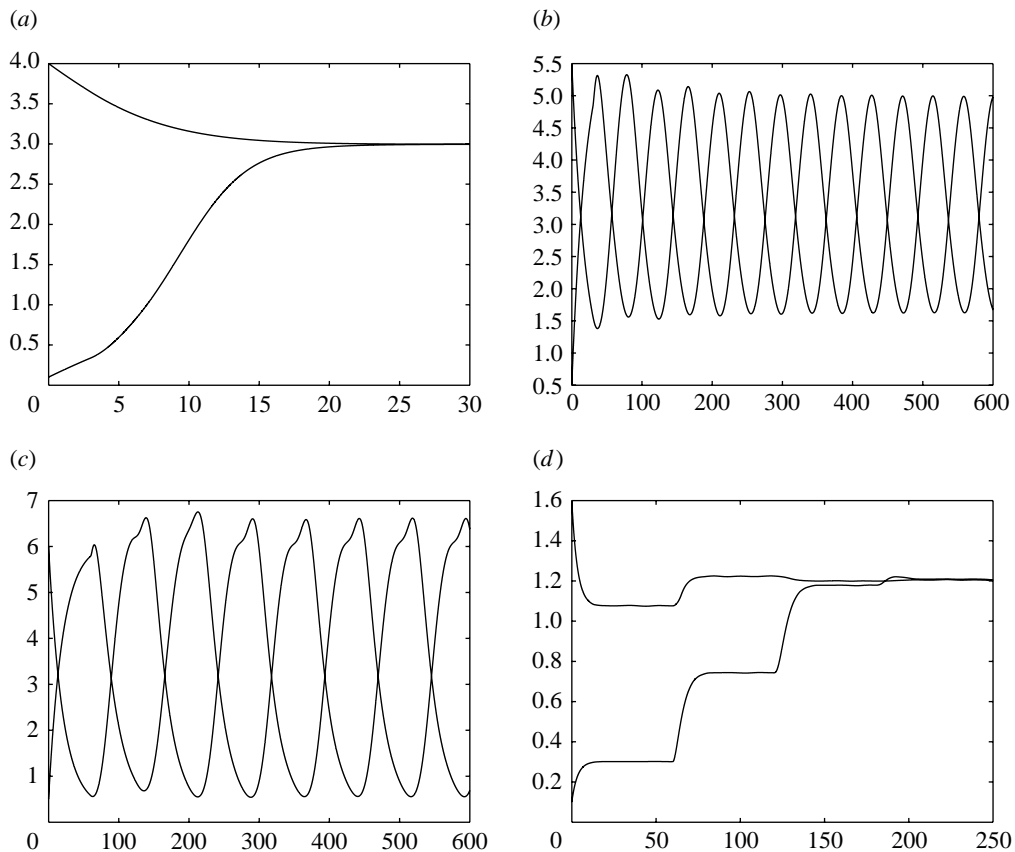
(*a*)

(*b*)

(*c*)

(*d*)

Figure 3. Nicholson equation with different parameters. (*a*) $\tau=3$, $\mu=0.05$, (*b*) $\tau=30$, $\mu=0.05$, (*c*) $\tau=60$, $\mu=0.05$ and (*d*) $\tau=60$, $\mu=0.3$.

Our first example is the following Nicholson's blowflies equation

$$x'(t) = -\mu x(t) + x(t-\tau)\exp(-x(t-\tau)).$$

For this nonlinearity, $\xi_0=1$ and $f(\xi_0)=0.368$. Non-trivial solutions in cases A and B are always convergent to an equilibrium as described by propositions 3.1 and 3.2. So, we focus on the case C. To be in the case C, $\mu < \mu_0 := (f(\xi_0)/\xi_0) = 0.368$ must hold. It is easy to check numerically that the condition (L) is not satisfied for $\mu < \mu^* = 0.105$ and satisfied for any $\mu \in (\mu^*, \mu_0)$, with some μ^* predicted by proposition 3.7.

We chose $\mu=0.05$. Then $\alpha=0.094$, $\beta=7.358$ and (L) is not satisfied. We have $K=2.996$, $f'(K)=-0.1$ and the first bifurcation point is $\tau_0=24.184$, where K_* undergoes a supercritical Hopf bifurcation and a stable periodic orbit emerges. For small delays, the solutions converge to the positive equilibrium. Increasing the delay, we observe the convergence to periodic solutions. As we increase the delay further, we see slower oscillations with greater amplitudes and we observe that the lim inf and lim sup of the solutions get closer and closer to α and β. See figure 3*a*,*b* and *c*, where the delay is 3, 30 and 60, respectively.
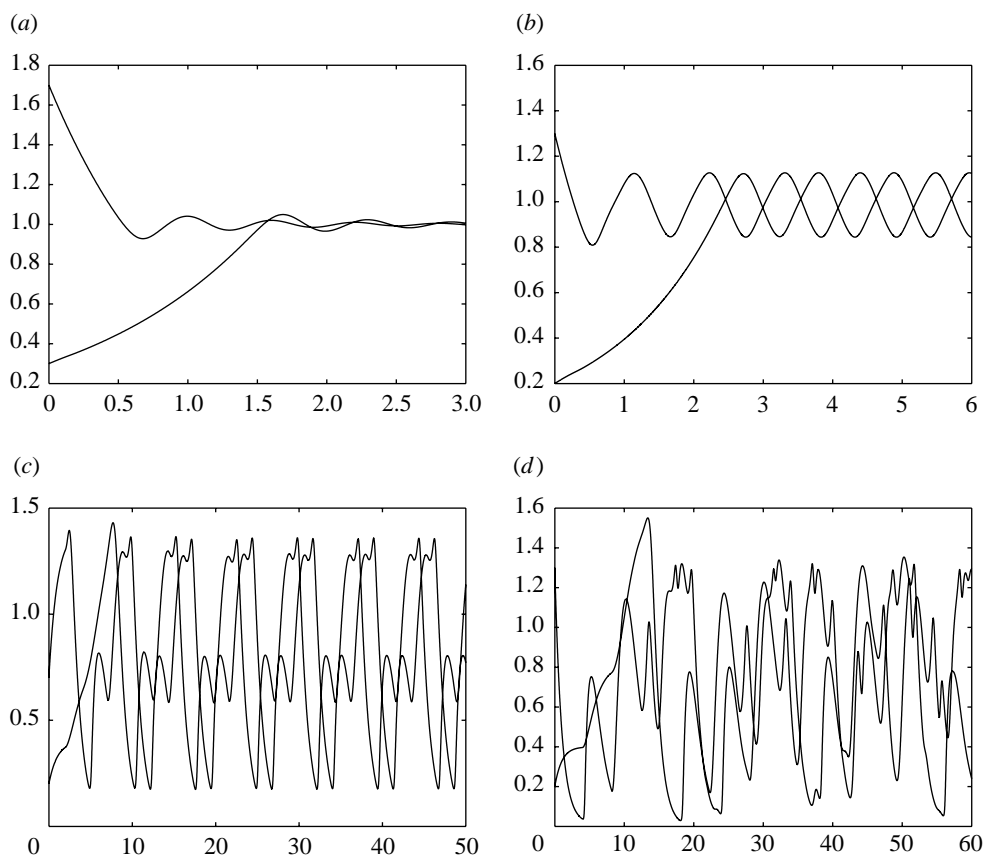
Figure 4. Mackey–Glass equation in the absence of the condition (L). (a) $\tau=0.15$, $\mu=1$, (b) $\tau=0.3$, $\mu=1$, (c) $\tau=2$, $\mu=1$ and (d) $\tau=5$, $\mu=1$.

Increasing μ, we see that α is increasing and β is decreasing, thus the attractive interval J is slowly shrinking. Passing μ^*, α becomes greater than ξ_0 and the condition (L) holds. In particular, if we set $\mu=0.3$, we then have $\beta=1.23$ and $\alpha=1.12>1$, (L) is satisfied. We have $K=1.204$ and $J=[1.199, 1.226]$. In this case, all solutions are attracted by the tiny interval $[1.199, 1.226]$ regardless of the size of the delay, therefore only very small (amplitude) oscillations are possible. Practically this is like a global convergence, regardless of τ.

Next we consider the Mackey–Glass equation

$$x'(t) = -\mu x(t) + \frac{2x(t-\tau)}{1 + x(t-\tau)^{20}}.$$

Now $\xi_0=0.863$ and $f(\xi_0)=1.64$. First let $\mu=1$, this yields that the positive equilibrium is $K=1$, $f'(K)=-9$, $\alpha\ll0.1$ and $\beta=1.64$, we can see that (L) is not satisfied, the first bifurcation point is $\tau_0=0.188$. For small delays, the solutions converge to 1 very quickly (figure 4a), then after $\tau=0.188$, where the first Hopf bifurcation occurs, we can see the convergence to a periodic solution (figure 4b). Increasing the delay further, the solutions become more irregular, but still asymptotically periodic with several extrema within one period (figure 4c). Finally,
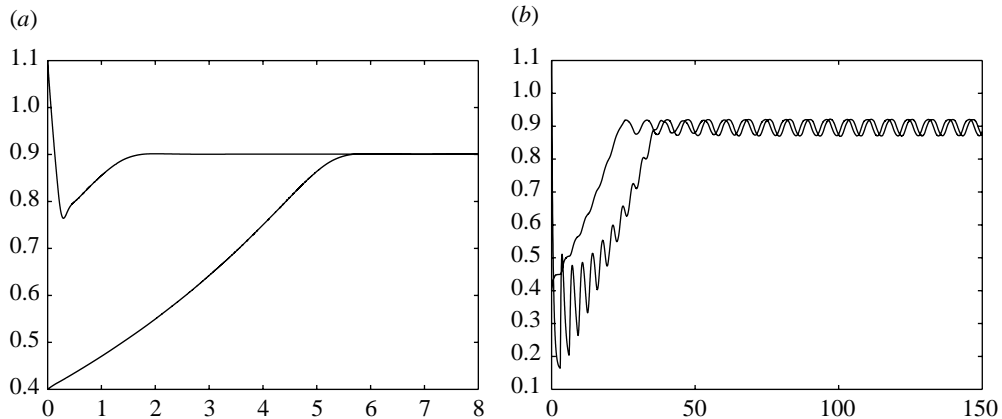
(a)

(b)

Figure 5. Mackey–Glass equation when (L) is fulfilled. (a) $\tau = 0.2$, $\mu = 1.78$ and (b) $\tau = 3$, $\mu = 1.78$.

we observe apparently aperiodic behaviours (figure 4d) where the delay is still not very large. We remark that even in the chaotic case, the lim sup and lim inf of solutions tend to increase as the delay increases, getting closer and closer to α and β.

One can check that (L) is satisfied if $\mu > \mu^* = 1.774$. To stay in case C, we need $\mu < \mu_0 = 1.9$. Setting $\mu = 1.78$, we obtain $K = 0.901$, $f'(K) = -2.136$, $\alpha = 0.867 > \xi_0$ and $\beta = 0.921$. Since $|f'(K)| > \mu$, K undergoes a series of Hopf bifurcations as the delay increases, but chaotic behaviour is not possible for any delay, since (L) holds. As figure 5 shows, we have quick convergence to the equilibrium for small delays, and for greater delays periodic oscillations appear, bounded by $[\alpha, \beta]$. When $\tau = 3$, the lim inf and the lim sup of the periodic solution are indistinguishable from α and β. The picture does not change much for larger delays.

In the previous examples, τ^* was typically smaller than τ_0, thus condition $(L\tau)$ does not play much of a role. However, it is easy to construct examples when $(L\tau)$ becomes more important. Suppose that f and μ satisfies $(L\tau)$. We construct a new nonlinearity f_3 such that $f_3(\xi) = f(\xi)$ whenever $\xi \notin (K - \varepsilon, K + \varepsilon)$, $\varepsilon > 0$, moreover $f_3(K) = f(K)$, and f_3 is smooth and monotonically decreasing in $[K - \varepsilon, K + \varepsilon]$. If ε is sufficiently small, then it does not affect τ^* and such a modification is possible with arbitrarily large $|f_3'(K)|$. By $\tau_k < (2k + 1)\pi / \sqrt{f_3'(K)^2 - \mu^2}$, choosing $|f_3'(K)|$ sufficiently big, τ^* is greater than an arbitrary bifurcation point. Therefore, the dynamics cannot be complicated even after a series of Hopf bifurcations; we cannot create chaos by a small local perturbation (which preserves the unimodal property) of f, if $(L\tau)$ holds.

To conclude this section, we recall that the easily checkable condition (L) is indeed satisfied in some particular situations and this excludes chaos even for the Mackey–Glass equation for certain parameters. We note that sometimes $[\alpha, \beta]$ is a tiny interval, as such our attractivity theorem is especially useful for applications. In particular cases, $[\alpha, \beta]$ seems to be a very sharp bound for the global attractor in the sense that both the Nicholson and the Mackey–Glass equations may exhibit solutions with their lim inf and lim sup close to α and β. Some results of Mallet-Paret & Nussbaum (1989) and Ivanov & Sharkovsky (1992) seem to suggest that the smallest interval which contains the global attractor for all τ may be smaller than $[\alpha, \beta]$, and in the limit case $\alpha = \xi_0$ these two intervals coincide.

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