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A small twist theorem and boundedness of solutions for semilinear Duffing equations at resonance☆

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Abstract

We obtain a new variant of Moser's small twist theorem and apply this new version to investigate the boundedness of solutions for the following semilinear Duffing equation

$$\ddot{x} + n^2 x + g(x) = p(t),$$

where p is a 2π-periodic smooth function and $\lim_{|x|\to\infty} x^{-1} g(x) = 0$. We obtain some sharp sufficient conditions for the boundedness of *all* solutions to the above equation at resonance. Unlike many existing results in the literature where the function g is required to be a bounded function with asymptotic limits, our main results here allow g be unbounded or oscillatory without asymptotic limits.

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1. Introduction and main results

In this paper, we consider the boundedness of solutions to the Duffing equation

$$\ddot{x} + \bar{g}(x) = p(t), \quad p(t + 2\pi) \equiv p(t), \tag{1.0}$$

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in the semilinear case, to be specified later. This equation, as one of the simplest nontrivial conservative systems, has been widely investigated and many results have been obtained for the existence and multiplicity of periodic solutions by various methods, such as the critical point theory, the phase plane technique and the continuation method based on a degree theory. We refer the reader to [5,19] and the references therein. We also refer the reader to [1,2] for the general theory. The serious work on boundedness of solutions to (1.0) dated back to at least 1966, when Littlewood [8] asked whether or not all the solutions of (1.0) are bounded for all time, i.e., whether there are resonances that might cause the amplitude of the oscillations to increase without bound. This problem was also raised and studied by Markus [12] and Moser [16].

The first result on the boundedness of solutions in the *superlinear* case (i.e., $x^{-1}\bar{g}(x) \to +\infty$ as $|x| \to +\infty$) was due to Morris [14]. On the basis of Moser's twist theorem [15], Morris proved that every solution of (1.0) is bounded when $\bar{g}(x) = 2x^3$ and p is a piecewise continuous function, here and in what follows, a solution $x(t)$ is bounded if it exists for all $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}}(|x(t)| + |x'(t)|) < +\infty.$$

Morris's result was later improved by several authors (see [3,7] and references there) for a large class of (1.0) in the superlinear case.

However, in the *semilinear* case, where

$$0 < \liminf_{|x|\to+\infty} x^{-1}\bar{g}(x) \le \limsup_{|x|\to+\infty} x^{-1}\bar{g}(x) < +\infty,$$

the situation is quite different and the study of the boundedness is delicate and difficult. To the best of our knowledge, so far very little has been achieved, and the main difficulty lies in the well-known phenomenon of linear resonance. See [9,10,17,18]. To be more precise, we consider

$$\ddot{x} + n^2 x + g(x) = p(t), \tag{1.1}$$

where $n \in \mathbb{N}$, $p(t + 2\pi) = p(t)$ and $\lim_{|x|\to\infty} x^{-1} g(x) = 0$.

When g is piecewise linear and given by

$$g(x) = \begin{cases} L, & \text{if } x \ge 1,\\ Lx, & \text{if } |x| < 1,\\ -L, & \text{if } x \le -1,\end{cases}$$

and p is 2π-periodic and of class C^5, Ortega [18] proved that every solution of (1.1) is bounded if

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t\right| < \frac{2L}{\pi}.$$

By using a variant of Moser's twist theorem due to Ortega [18], Liu [10] obtained a similar result for the boundedness of solutions to (1.1) with smooth g and p. More precisely, Liu proved the following[1]

Theorem 0. *Suppose that $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$ and $g \in C^6(\mathbb{R})$, and assume that the limits*

$$g(+\infty) := \lim_{x\to+\infty} g(x), \quad g(-\infty) := \lim_{x\to-\infty} g(x),$$

[1] A short proof can also be found in M. Kunze, Remarks on boundedness in semilinear oscillators, In Nonlinear Analysis and its Applications to Differential Equations, Birkhauser, 2001.

are finite and

$$\lim_{|x|\to+\infty} x^6 g^{(6)}(x) = 0.$$

Then every solution of (1.1) *is bounded provided*

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| < 2|g(+\infty) - g(-\infty)|.$$

To state and compare other boundedness results with our main contributions here, we note that Massera's theorem [13] shows a close connection between the existence of periodic solutions and the boundedness of solutions of (1.1). We also note that the auxiliary equation

$$\ddot{x} + n^2 x + g(x) = 0 \tag{1.2}$$

is equivalent to the following planar autonomous Hamiltonian system

$$\dot{x} = -ny, \quad \dot{y} = nx + \frac{1}{n}g(x) \tag{1.3}$$

with the Hamiltonian function

$$H_0(x, y) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}G(x),$$

where $G(x) = \int_0^x g(s)\mathrm{d}s$. For $h > 0$, we denote by $\tau(h)$ the least positive period of the orbit $\Gamma_h : H_0(x, y) = h$ of the system (1.3), and we set

$$\Gamma(h) := \sqrt{h}\left(\tau(h) - \frac{2\pi}{n}\right). \tag{1.4}$$

The asymptotic behaviors of the time map $\tau(h)$ play an important role in some recent work regarding the existence and multiplicity of periodic solutions of (1.1). In particular, in [5,19] it was shown that if g is continuous and

$$\limsup_{h\to+\infty} \Gamma(h) = +\infty, \quad \liminf_{h\to+\infty} \Gamma(h) = -\infty,$$

then (1.1) has infinitely many 2π-periodic solutions. Moreover, in [5] it was shown that if g is Lipschitz continuous and

$$\limsup_{h\to+\infty} |\Gamma(h)| = +\infty,$$

then (1.1) has at least one 2π-periodic solution. Therefore, it is natural to ask whether or not (1.1) admits a 2π-periodic solution under the condition

$$\limsup_{h\to+\infty} |\Gamma(h)| < +\infty.$$

A positive answer in some cases is given in [11]. Our main result, stated below, shows that some sharp sufficient conditions for the boundedness of solutions of (1.1) can also be obtained in terms of the behaviors at infinity of the time map $\tau(h)$.

Theorem 1. *Assume that* $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$ *and* $g \in C^7(\mathbb{R})$. *In addition, suppose that*

$$\lim_{|x|\to+\infty} x^{k-1} g^{(k)}(x) = 0, \quad 0 \le k \le 6 \tag{1.5}$$

and

$$\lim_{h\to+\infty} h\Gamma'(h) = 0, \quad \limsup_{h\to+\infty} |h^k \Gamma^{(k)}(h)| < +\infty, \quad 2 \le k \le 7. \tag{1.6}$$

Then under the condition

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| < \sqrt{2}n^{5/2} \limsup_{h\to+\infty} |\Gamma(h)| < +\infty, \tag{1.7}$$

every solution of (1.1) *is bounded.*

In the above result, we implicitly state that $\Gamma(h)$ is of class C^7. This is true, since $g \in C^7(\mathbb{R})$, by using the expression of $\tau(h)$ given in the main body of the paper. We shall also note that in Theorem 1, the condition $\lim_{h\to+\infty} h\Gamma'(h) = 0$ may by replaced by a weaker condition:

$$\limsup_{h\to+\infty} |h\Gamma'(h)| < \varepsilon_0,$$

where $\varepsilon_0 > 0$ is small and can be determined explicitly using the quantities of $\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|$ and $\limsup_{h\to+\infty} |\Gamma(h)|$.

As consequences of Theorem 1, we shall have the following

Theorem 2. *Assume that $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$ and $g \in C^8(\mathbb{R})$. In addition, suppose that the function g satisfies the following conditions:*

$$\lim_{|x|\to+\infty} |x|^{k-1/2} g^{(k)}(x) = 0, \quad 0 \le k \le 8, \tag{1.8}$$

$$\lim_{\rho\to+\infty} \int_0^{2\pi} \rho g'(\rho\cos\theta)\cos^2\theta \mathrm{d}\theta = 0, \tag{1.9}$$

$$\limsup_{\rho\to+\infty} \left|\int_0^{2\pi} \rho^k g^{(k)}(\rho\cos\theta)\cos^{k+1}\theta \mathrm{d}\theta\right| < +\infty, \quad 2 \le k \le 7. \tag{1.10}$$

Then every solution of (1.1) *is bounded provided*

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| < \limsup_{\rho\to+\infty} \left|\int_0^{2\pi} g(\rho\cos\theta)\cos\theta \mathrm{d}\theta\right| < +\infty. \tag{1.11}$$

Theorem 3. *Assume that $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$ and $g \in C^7(\mathbb{R})$. In addition, suppose that the function g satisfies the following conditions:*

$$\lim_{x\to+\infty} xg'(x) = \lim_{x\to-\infty} xg'(x), \quad \limsup_{|x|\to+\infty} |x^k g^{(k)}(x)|, \quad 2 \le k \le 7 \tag{1.12}$$

are finite and

$$\lim_{|x|\to+\infty} |x|^{-1/2} g(x) = 0. \tag{1.13}$$

Then every solution of (1.1) *is bounded provided* (1.11) *holds.*

Remark. Clearly, if $g(+\infty) = \lim_{x\to+\infty} g(x)$ and $g(-\infty) = \lim_{x\to-\infty} g(x)$ exist and are finite, then it follows from the dominated convergence theorem that the inequality (1.7) or (1.11) reduces to

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| < 2|g(+\infty) - g(-\infty)|. \tag{1.14}$$

Therefore, our Theorem 1 implies Theorem 0 as a special case. Note, however, in our results, g is allowed to be unbounded and oscillatory without asymptotic limits. We also note that if, in addition, $g(-\infty) \le g(x) \le g(+\infty)$ or $g(+\infty) \le g(x) \le g(-\infty)$ for all $x \in \mathbb{R}$, then it is well known (see [6]) that (1.14) is a sufficient and necessary condition for the existence of 2π-periodic solutions of (1.1). So it follows from Massera's theorem [13] that (1.14) is also a sufficient and necessary condition for the boundedness of solutions of (1.1). In this sense, we say that inequalities (1.7) and (1.11) are sharp conditions for the boundedness of solutions of (1.1).

Remark. We must emphasize that in contrast with the case for most of the known results regarding the boundedness of all the solutions of (1.1) that impose conditions of the boundedness and non-oscillation with asymptotic limits on the function g, i.e.,

$$\sup_{x\in\mathbb{R}} |g(x)| < +\infty, \quad xg(x) > 0 \quad (\text{or } xg(x) < 0), \quad \text{for } |x| \ge M,$$

in our main results, the function g may be unbounded and oscillatory without asymptotic limits. In particular, we note that if $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$, then from Theorem 2 and the dominated convergence theorem, it follows that provided

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| < 2\pi,$$

then every solution of the following three equations is bounded:

$$\ddot{x} + n^2 x + \arctan x + a\sin\ln(1+x^2) = p(t),$$
$$\ddot{x} + n^2 x + \arctan x + \ln(1+x^2)\cdot\sin\ln(1+x^2) = p(t),$$
$$\ddot{x} + n^2 x + \arctan x + \ln(1+x^2) = p(t).$$

Remark. It was shown in [6] that if $p \in C(\mathbb{R}/2\pi\mathbb{Z})$, $g \in C(\mathbb{R})$ is bounded and if

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| < 2\max\left\{\liminf_{x\to+\infty} g(x) - \limsup_{x\to-\infty} g(x), \liminf_{x\to-\infty} g(x) - \limsup_{x\to+\infty} g(x)\right\}, \tag{1.15}$$

then Eq. (1.1) has at least one 2π-periodic solution. Moreover, if g is not constant and if

$$\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right| \ge 2\left(\sup_{x\in\mathbb{R}} g(x) - \inf_{x\in\mathbb{R}} g(x)\right), \tag{1.16}$$

then Eq. (1.1) has no 2π-periodic solution. The following result shows that similar results hold true for boundedness of solutions:

Corollary. *Assume that $p \in C^7(\mathbb{R}/2\pi\mathbb{Z})$ and $g \in C^7(\mathbb{R})$ is bounded. In addition, suppose that the quantities:*

$$\lim_{x\to+\infty} xg'(x) = \lim_{x\to-\infty} xg'(x), \quad \limsup_{|x|\to+\infty} |x^k g^{(k)}(x)|, \quad 2 \le k \le 7$$

are finite. Then

(i) *every solution of* (1.1) *is bounded if* (1.15) *holds;*
(ii) *every solution of* (1.1) *is unbounded if* $g(x)$ *is not constant and* (1.16) *holds.*

This is an immediate consequence of Theorem 3 and the Massera's theorem [13], since by using the Arzela–Ascoli theorem and the dominated convergence theorem, we have

$$\limsup_{\rho\to+\infty}\int_0^{2\pi} g(\rho\cos\theta)\cos\theta \mathrm{d}\theta \geq 2\left(\liminf_{x\to+\infty} g(x) - \limsup_{x\to-\infty} g(x)\right),$$

and

$$\liminf_{\rho\to+\infty}\int_0^{2\pi} g(\rho\cos\theta)\cos\theta \mathrm{d}\theta \leq 2\left(\limsup_{x\to+\infty} g(x) - \liminf_{x\to-\infty} g(x)\right).$$

We conclude this introduction with a short sketch of the proof. As usual (see [7,9,10,18]), we will first use standard transformations to rewrite (1.1) as a perturbation of an integrable Hamiltonian system out of a large disc $D = \{(x, x') \in \mathbb{R}^2 : x^2 + x'^2 \leq r^2\}$ in the (x, x')-plane. The Poincaré map of the transformed system is close to a so-called twist map in $\mathbb{R}^2 \setminus D$. Then we obtain a variant of Moser's twist theorem that guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x, x')-plane. Every such invariant curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in the interior and leads to a bound for these solutions.

The rest of this paper is organized as follows. In Sections 2 and 3, we give some technical lemmas which will be employed in the proof of our main theorems. Section 4 is devoted to the proof of Theorem 1. Finally, we will prove that if (1.8)–(1.10) or (1.12) and (1.13) hold, then (1.6) also holds. We then conclude with the proofs of Theorem 2 and Theorem 3.

2. Global existence and canonical transformations

Throughout this paper, we will denote by $C > 1$ a universal positive constant and by $\varepsilon(h)$ a universal non-negative function satisfying $\lim_{h\to+\infty} \varepsilon(h) = 0$. Throughout this section, we assume that the hypotheses of Theorem 1 hold.

By introducing a new variable y as $y = -n^{-1}\dot{x}$, (1.1) is changed into the following planar Hamiltonian system

$$\dot{x} = -ny, \quad \dot{y} = nx + \frac{1}{n}g(x) - \frac{1}{n}p(t) \tag{2.1}$$

with the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}G(x) - \frac{1}{n}xp(t),$$

and (1.2) is changed into the Hamiltonian system (1.3) with the Hamiltonian function

$$H_0(x, y) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}G(x),$$

where $G(x) = \int_0^x g(s)\mathrm{d}s$.

First, we deal with the global existence of solutions.

Lemma 2.1. *For any* $(x_0, y_0) \in \mathbb{R}^2$ *and* $t_0 \in \mathbb{R}$, *the unique solution*

$$(x(t), y(t)) = (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$$

of (2.1) *satisfying* $(x(t_0), y(t_0)) = (x_0, y_0)$ *exists on the whole* t*-axis.*

Proof. Since

$$\lim_{|x|\to\infty} \frac{G(x)}{x^2} = \lim_{|x|\to\infty} \frac{g(x)}{2x} = 0,$$

there exists an $M_0 > 0$ such that

$$|G(x)| \le \frac{1}{4} n^2 x^2, \quad \text{for } |x| \ge M_0.$$

Let

$$\bar{M} = \max_{|x|\le M_0} |G(x)| + 1,$$

and

$$F(t) = \frac{1}{2} n(x^2(t) + y^2(t)) + \frac{1}{n} G(x(t)) + \bar{M}.$$

Then an argument similar to that used in the proof of Lemma 3.1 in [10] can be used to show that

$$\frac{1}{4} n(x^2(t) + y^2(t)) \le F(t) \le F(t_0) \cdot \mathrm{e}^{E|t-t_0|}.$$

Therefore, the solution $(x(t), y(t))$ exists on the whole t-axis and the proof is complete. □

Under the standard symplectic transformation $(r, \theta) \mapsto (x, y)$ with $r > 0$ and $\theta (\mathrm{mod}\ 2\pi)$, given by

$$x = \sqrt{2r} \cos\theta, \quad y = \sqrt{2r} \sin\theta, \tag{2.2}$$

systems (2.1) and (1.3) are transformed into the following Hamiltonian systems

$$\dot{r} = -\frac{\partial}{\partial\theta} h(r, \theta, t), \quad \dot{\theta} = \frac{\partial}{\partial r} h(r, \theta, t) \tag{2.3}$$

and

$$\dot{r} = -\frac{\partial}{\partial\theta} h_0(r, \theta), \quad \dot{\theta} = \frac{\partial}{\partial r} h_0(r, \theta), \tag{2.4}$$

respectively, where

$$h(r, \theta, t) = H(x, y, t) = nr + \frac{1}{n} G(\sqrt{2r} \cos\theta) - \frac{1}{n} \sqrt{2r}\, p(t) \cos\theta, \tag{2.5}$$

and

$$h_0(r, \theta) = H_0(x, y) = nr + \frac{1}{n} G(\sqrt{2r} \cos\theta). \tag{2.6}$$

Observe that

$$r\mathrm{d}\theta - h\mathrm{d}t = -(h\mathrm{d}t - r\mathrm{d}\theta), \quad r\mathrm{d}\theta - h_0\mathrm{d}t = -(h_0\mathrm{d}t - r\mathrm{d}\theta).$$

This means that if one can solve for $r = r(h, t, \theta)$ from (2.5) and solve for $r_0 = r_0(h, \theta)$ from (2.6), then

$$\frac{\mathrm{d}h}{\mathrm{d}\theta} = -\frac{\partial}{\partial t} r(h, t, \theta), \quad \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\partial}{\partial h} r(h, t, \theta) \tag{2.7}$$

and

$$\frac{\mathrm{d}h}{\mathrm{d}\theta} = -\frac{\partial}{\partial t} r_0(h, \theta) = 0, \quad \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\partial}{\partial h} r_0(h, \theta) \tag{2.8}$$

are two Hamiltonian systems with Hamiltonian functions $r = r(h, t, \theta)$ and $r_0 = r_0(h, \theta)$, respectively. Now the action, angle and time variables are h, t and θ respectively. This trick has been used by several authors (see [7,9,10]).

It follows from (2.6) and (2.8) that

$$\begin{aligned}\tau(h) &= \int_0^{2\pi} \frac{\partial}{\partial h} r_0(h, \theta) \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{\frac{\partial}{\partial r} h_0(r_0, \theta)} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{n + n^{-1}(2r_0)^{-1/2} g(\sqrt{2r_0}\cos\theta)\cos\theta},\end{aligned} \tag{2.9}$$

where $r_0 = r_0(h, \theta)$.

From (2.6), we see that

$$\begin{aligned}\lim_{r\to+\infty} \frac{h_0}{r} &= \lim_{r\to+\infty} \left[n + \frac{1}{nr} G(\sqrt{2r}\cos\theta) \right] \\ &= \lim_{r\to+\infty} \left[n + \frac{1}{n\sqrt{2r}} g(\sqrt{2r}\cos\theta)\cos\theta \right] = n\end{aligned}$$

and

$$\frac{\partial h_0}{\partial r} = n + \frac{1}{n\sqrt{2r}} g(\sqrt{2r}\cos\theta)\cos\theta > 0,$$

for $r \gg 1$. By the implicit function theorem, we know that there exists a function $R_1 = R_1(h, \theta)$ of class C^8 with $|R_1(h, \theta)| \le \epsilon(h)h$ such that

$$r_0(h, \theta) = n^{-1}h + R_1(h, \theta)$$

solves the equation

$$h = h(r_0, \theta) = nr_0 + \frac{1}{n} G(\sqrt{2r_0}\cos\theta).$$

In a similar way, we can show that there is a function $R_2 = R_2(h, t, \theta)$ of class C^7 with $|R_2(h, t, \theta)| \le \epsilon(h)h$ such that

$$r(h, t, \theta) = n^{-1}h + R_2(h, t, \theta)$$

satisfies

$$h = h(r, t, \theta) = nr + \frac{1}{n} G(\sqrt{2r}\cos\theta) - \frac{1}{n}\sqrt{2r}\, p(t)\cos\theta.$$

Lemma 2.2. *For any smooth function* $I = I(h,t,\theta)$ *with* $|I(h,t,\theta)| \le \varepsilon(h)h$, *let* $u = u(h,t,\theta) = n^{-1}h + I(h,t,\theta)$, *then*

$$\lim_{h\to+\infty} (2u)^{\frac{k-1}{2}} g^{(k)}(\sqrt{2u}\cos\theta)\cos^k\theta = 0, \tag{2.10}$$

for $0 \le k \le 6$, *uniformly for* t *and* θ.

Proof. Since

$$\lim_{|x|\to\infty} x^{k-1} g^{(k)}(x) = 0, \quad 0 \le k \le 6,$$

for any $\varepsilon > 0$, there exist constants $M_k > 0$ such that

$$|x^{k-1} g^{(k)}(x)| < \varepsilon, \quad \text{for } |x| > M_k.$$

Since $\lim_{h\to\infty} u(h,t,\theta) = +\infty$ uniformly for t and $\theta \in [0, 2\pi]$, there exist positive numbers $N_k > 0$ such that for $h > N_k$ and any $\theta \in [0, 2\pi]$,

$$\frac{1}{\sqrt{2u}} \max_{|x|\le M_k} |x^k g^{(k)}(x)| < \varepsilon.$$

Therefore, for $|\sqrt{2u}\cos\theta| > M_k$, we have

$$\left|(2u)^{\frac{k-1}{2}} g^{(k)}(\sqrt{2u}\cos\theta)\cos^k\theta\right| \le \left|(\sqrt{2u})^{k-1}\cos^{k-1}\theta g^{(k)}(\sqrt{2u}\cos\theta)\right| < \varepsilon,$$

and, for $|\sqrt{2u}\cos\theta| \le M_k$, we have

$$\left|(2u)^{\frac{k-1}{2}} g^{(k)}(\sqrt{2u}\cos\theta)\cos^k\theta\right| \le \frac{1}{\sqrt{2u}} \max_{|x|\le M_k} |x^k g^{(k)}(x)| < \varepsilon.$$

This completes the proof. □

Lemma 2.3. *Let* $r_1(h,t,\theta) = r(h,t,\theta) - r_0(h,\theta)$, *then*

$$r_1(h,t,\theta) = \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta + R(h,t,\theta), \tag{2.11}$$

and

$$\left|\frac{\partial^{k+m}}{\partial h^k \partial t^m} R(h,t,\theta)\right| \le \varepsilon(h)\cdot h^{-k+1/2} \tag{2.12}$$

for $k + m \le 6$, *uniformly for* t *and* θ.

We defer the proof to Section 3.

By Lemma 2.3, we have

$$r(h,t,\theta) = r_0(h,\theta) + \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta + R(h,t,\theta).$$

Now system (2.7) can be written in the form

$$\begin{cases} \dfrac{\mathrm{d}h}{\mathrm{d}\theta} = -\sqrt{2}n^{-5/2}h^{1/2}p'(t)\cos\theta - \dfrac{\partial}{\partial t}R(h,t,\theta), \\ \dfrac{\mathrm{d}t}{\mathrm{d}\theta} = \dfrac{\partial}{\partial h}r_0(h,\theta) + \dfrac{\sqrt{2}}{2}n^{-5/2}h^{-1/2}p(t)\cos\theta + \dfrac{\partial}{\partial h}R(h,t,\theta). \end{cases} \tag{2.13}$$

Lemma 2.4. *There is a canonical transformation Ψ of the form*

$$\Psi : h = \rho, \quad t = \tau + T(\rho, \theta)$$

with $T(\rho, \theta + 2\pi) = T(\rho, \theta)$ such that the transformed system of (2.13) is of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}\theta} = -\frac{\partial}{\partial \tau}\tilde{r}(\rho, \tau, \theta), \quad \frac{\mathrm{d}\tau}{\mathrm{d}\theta} = \frac{\partial}{\partial \rho}\tilde{r}(\rho, \tau, \theta), \tag{2.14}$$

where

$$\tilde{r}(\rho, \tau, \theta) = J(\rho) + \sqrt{2}n^{-5/2}\rho^{1/2}p(\tau)\cos\theta + \tilde{R}(\rho, \tau, \theta),$$
$$J(\rho) = \frac{1}{2\pi}\int_0^{2\pi} r_0(\rho, \theta)\mathrm{d}\theta.$$

For the new perturbation $\tilde{R}$, we have

$$\left|\frac{\partial^{k+m}}{\partial \rho^k \partial \tau^m}\tilde{R}(\rho, \tau, \theta)\right| \le \varepsilon(\rho) \cdot \rho^{-k+1/2} \tag{2.15}$$

for $k + m \le 6$, uniformly for τ and θ.

Proof. The transformation Ψ is defined implicitly by

$$\Psi : \rho = h + \frac{\partial}{\partial \tau}S(h, \tau, \theta), \quad t = \tau + \frac{\partial}{\partial h}S(h, \tau, \theta),$$

where the generating function $S = S(h, \tau, \theta)$ will be determined later.

Under this canonical transformation Ψ, (2.13) is transformed into the system

$$\frac{\mathrm{d}\rho}{\mathrm{d}\theta} = -\frac{\partial}{\partial \tau}\tilde{r}(\rho, \tau, \theta), \quad \frac{\mathrm{d}\tau}{\mathrm{d}\theta} = \frac{\partial}{\partial \rho}\tilde{r}(\rho, \tau, \theta),$$

where the Hamiltonian function $\tilde{r}$ is of the form

$$\tilde{r} = r_0(h, \theta) + \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta + R(h, t, \theta) - \frac{\partial S}{\partial \theta}.$$

Now we choose

$$S = \int_0^{\theta} [r_0(h, s) - J(h)]\mathrm{d}s. \tag{2.16}$$

Obviously, S does not depend on τ and is 2π-periodic in θ. Hence, $\rho = h$. Let

$$T(h, \theta) = \frac{\partial S}{\partial h}. \tag{2.17}$$

Then the canonical transformation Ψ is of the form

$$h = \rho, \quad t = \tau + T(\rho, \theta).$$

Let

$$\begin{aligned}\tilde{R}(\rho, \tau, \theta) = {} & R(\rho, \tau + T(\rho, \theta), \theta) \\ & + \sqrt{2}n^{-5/2}\rho^{1/2}\cos\theta \int_0^1 p'(\tau + \mu T(\rho, \theta))T(\rho, \theta)\mathrm{d}\mu.\end{aligned} \tag{2.18}$$

Then the transformed Hamiltonian function $\tilde{r}$ is of the form

$$\tilde{r}(\rho, \tau, \theta) = J(\rho) + \sqrt{2} n^{-5/2} \rho^{1/2} p(\tau) \cos\theta + \tilde{R}(\rho, \tau, \theta).$$

Now we prove (2.15). By a direct computation, we know that $(\partial^{k+m}/\partial\rho^k \partial\tau^m)(R(\rho, \tau + T(\rho, \theta), \theta))$ is a sum of the following terms

$$\frac{\partial^{p+q+m}}{\partial h^p \partial t^{m+q}} R(\rho, \tau + T(\rho, \theta), \theta) \cdot \prod_{s=1}^{q} \frac{\partial^{j_s}}{\partial \rho^{j_s}} T(\rho, \theta), \tag{2.19}$$

where

$$p + q \le k, \quad j_1, j_2, \ldots, j_q \ge 0, \quad \sum_{s=1}^{q} j_s = k - p.$$

Lemma 2.5. *Let $T(\rho, \theta)$ be given by (2.17), then*

$$\left| \frac{\partial^k}{\partial \rho^k} T(\rho, \theta) \right| \le \varepsilon(\rho) \rho^{-k}, \quad 0 \le k \le 6. \tag{2.20}$$

Proof. Let

$$q_h(\theta) = q(h, \theta) = n + n^{-1} (2h)^{-1/2} g(\sqrt{2h} \cos\theta) \cos\theta,$$
$$q_h^{(k)}(\theta) = \frac{\partial^k}{\partial h^k} q(h, \theta).$$

Then it is not difficult to show that for $1 \le k \le 6$

$$q_h^{(k)}(\theta) = n^{-1} \sum_{j=0}^{k} c_{k,j} (2h)^{-k + \frac{j-1}{2}} g^{(j)}(\sqrt{2h} \cos\theta) \cos^{j+1}\theta,$$

where

$$c_{0,0} = 1, \quad c_{k,0} = c_{k-1,0} \cdot (-2k + 1), \quad c_{k,k} = 1,$$
$$c_{k,j} = c_{k-1,j-1} + c_{k-1,j} \cdot (-2k + 1 + j), \quad 1 \le j \le k - 1.$$

Let $r_0 = r_0(\rho, \theta)$. It follows from (2.6) that

$$\frac{\partial}{\partial \rho} r_0(\rho, \theta) = \left[\frac{\partial}{\partial r} h_0(r, \theta)|_{r=r_0} \right]^{-1} = [q_{r_0}(\theta)]^{-1}.$$

By a direct computation, we see that $(\partial^k/\partial\rho^k)[q_{r_0}(\theta)]^{-1}$ is a sum of the following terms

$$[q_{r_0}(\theta)]^{-\tilde{p}} \cdot \prod_{s=1}^{k} [q_{r_0}^{(s)}(\theta)]^{\tilde{j}_s}, \tag{2.21}$$

where

$$k + 1 \le \tilde{p} \le 2k + 1, \quad \tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_k \ge 0, \quad \sum_{s=1}^{k} s \cdot \tilde{j}_s = k, \quad \sum_{s=1}^{k} \tilde{j}_s = \tilde{p} - k - 1.$$

Since $r_0 = r_0(\rho, \theta) = n^{-1} h + R_1(\rho, \theta)$, $|R_1(\rho, \theta)| \le \varepsilon(\rho)\rho$, by Lemma 2.2, we have

$$|q_{r_0}^{(s)}(\theta)| \le \varepsilon(\rho) \rho^{-s}, \quad 1 \le s \le 6,$$

which together with (2.21) yields

$$\left|\frac{\partial^k}{\partial\rho^k}[q_{r_0}(\theta)]^{-1}\right| \le \prod_{s=1}^{k}\varepsilon(\rho)\rho^{-s\cdot\tilde{j}_s}$$
$$\le \varepsilon(\rho)\rho^{-\sum_{s=1}^{k}s\cdot\tilde{j}_s} = \varepsilon(\rho)\rho^{-k}. \tag{2.22}$$

It follows from (2.9) and (2.16) that

$$\begin{aligned}T(\rho,\theta) = \frac{\partial}{\partial\rho}S(\rho,\theta) &= \frac{\partial}{\partial\rho}\int_0^\theta [r_0(\rho,s) - J(\rho)]\mathrm{d}s\\ &= \int_0^\theta \left[\frac{\partial}{\partial\rho}r_0(\rho,s) - J'(\rho)\right]\mathrm{d}s\\ &= \int_0^\theta \left[(q_{r_0}(s))^{-1} - \frac{1}{2\pi}\tau(\rho)\right]\mathrm{d}s\\ &= \int_0^\theta [q_{r_0}(s)]^{-1}\mathrm{d}s - \frac{\theta}{2\pi}\int_0^{2\pi}[q_{r_0}(s)]^{-1}\mathrm{d}s,\end{aligned}$$

which, together with (2.22), implies that

$$\left|\frac{\partial^k}{\partial\rho^k}T(\rho,\theta)\right| \le \varepsilon(\rho)\rho^{-k}, \quad 1\le k\le 6.$$

Moreover, by the dominated convergence theorem, we have

$$\lim_{\rho\to+\infty} T(\rho,\theta) = 0.$$

Thus the proof is complete. □

Therefore, by (2.19) and (2.20) and Lemma 2.3, it follows that for $k+m\le 6$,

$$\begin{aligned}\left|\frac{\partial^{k+m}}{\partial\rho^k\partial\tau^m}R(\rho,\tau+T(\rho,\theta),\theta)\right| &\le \varepsilon(\rho)\rho^{-p+1/2}\cdot\prod_{s=1}^{q}\varepsilon(\rho)\rho^{-j_s}\\ &\le \varepsilon(\rho)\rho^{-p+1/2}\cdot\rho^{-\sum_{s=1}^{q}j_s} = \varepsilon(\rho)\rho^{-k+1/2}.\end{aligned}$$

Similarly, we can prove that

$$\left|\frac{\partial^{k+m}}{\partial\rho^k\partial\tau^m}\left(\sqrt{2}n^{-5/2}\rho^{1/2}\cos\theta\int_0^1 p'(\tau+\mu T(\rho,\theta))T(\rho,\theta)\mathrm{d}\mu\right)\right| \le \varepsilon(\rho)\rho^{-k+1/2}.$$

Therefore, (2.15) holds. This completes the proof of Lemma 2.4. □

Let $\theta = n\vartheta$, then system (2.14) is transformed into the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}\vartheta} = -\frac{\partial}{\partial\tau}\tilde{H}(\rho,\tau,\vartheta), \quad \frac{\mathrm{d}\tau}{\mathrm{d}\vartheta} = \frac{\partial}{\partial\rho}\tilde{H}(\rho,\tau,\vartheta), \tag{2.23}$$

where

$$\tilde{H}(\rho,\tau,\vartheta) = n\tilde{r}(\rho,\tau,n\vartheta) = nJ(\rho) + \sqrt{2}n^{-3/2}\rho^{1/2}p(\tau)\cos n\vartheta + n\tilde{R}(\rho,\tau,n\vartheta).$$

As n is a positive integer, the function $\tilde{H}(\rho,\tau,\vartheta)$ is 2π-periodic in ϑ.

3. Proof of Lemma 2.3

In this section, we shall give a proof of Lemma 2.3. Throughout this section, we assume the hypotheses of Theorem 1 hold.

It is convenient to introduce the following notation. For $n = 0, 1, \ldots, \infty$ and $r, s \in \mathbb{R}$, we say that a function $f = f(h, t, \theta)$ belongs to the class $C(r)$ if $|f| \le Ch^r$ for $h \gg 1$, uniformly for t and θ, and f belongs to the class $C(r, \epsilon)$ if $|f| \le \varepsilon(h)h^r$ for $h \gg 1$, uniformly for t and θ. Then we have the following observations.

Lemma 3.1. (i) $C(r, \epsilon) \subset C(r)$.

(ii) *If $r_1 < r_2$, then $C(r_1) \subset C(r_2)$ and $C(r_1, \epsilon) \subset C(r_2, \epsilon)$.*

(iii) *If $\alpha, \beta \in \mathbb{R}$ and $f_1 \in C(r_1)$, $f_2 \in C(r_2)$, then $\alpha f_1 + \beta f_2 \in C(\max\{r_1, r_2\})$ and $f_1 \cdot f_2 \in C(r_1 + r_2)$.*

(iv) *If $\alpha, \beta \in \mathbb{R}$ and $f_1 \in C(r_1, \epsilon)$, $f_2 \in C(r_2, \epsilon)$, then $\alpha f_1 + \beta f_2 \in C(\max\{r_1, r_2\}, \epsilon)$ and $f_1 \cdot f_2 \in C(r_1 + r_2, \epsilon)$.*

(v) *If $f_1 \in C(r_1, \epsilon)$, $f_2 \in C(r_2)$, then $f_1 \cdot f_2 \in C(r_1 + r_2, \epsilon)$.*

We say that a C^n-smooth function $f = f(h, t, \theta)$ belongs to the class $C^n_s(r)$ if

$$\frac{\partial^{k+m}}{\partial h^k \partial t^m} f \in C(-k + r), \quad \text{for } k + m \le n,\ m \ge 1,$$

and

$$\frac{\partial^k}{\partial h^k} f \in C(-k + r + s), \quad \text{for } k \le n.$$

We say that a C^n-smooth function $f = f(h, t, \theta)$ belongs to the class $C^n_s(r, \epsilon)$ if

$$\frac{\partial^{k+m}}{\partial h^k \partial t^m} f \in C(-k + r), \quad \text{for } k + m \le n,\ m \ge 1,$$

and

$$\frac{\partial^k}{\partial h^k} f \in C(-k + r + s, \epsilon), \quad \text{for } k \le n.$$

Then it is not hard to show the following

Lemma 3.2. (i) $C^n_s(r, \epsilon) \subset C^n_s(r)$.

(ii) *If $r_1 < r_2$, then $C^n_s(r_1) \subset C^n_s(r_2)$ and $C^n_s(r_1, \epsilon) \subset C^n_s(r_2, \epsilon)$.*

(iii) *If $\alpha, \beta \in \mathbb{R}$ and $f_1 \in C^n_{s_1}(r_1)$, $f_2 \in C^n_{s_2}(r_2)$, then $\alpha f_1 + \beta f_2 \in C^n_{\max\{s_1,s_2\}}(\max\{r_1, r_2\})$ and $f_1 \cdot f_2 \in C^n_{s_1+s_2}(r_1 + r_2)$.*

(iv) *If $\alpha, \beta \in \mathbb{R}$ and $f_1 \in C^n_{s_1}(r_1, \epsilon)$, $f_2 \in C^n_{s_2}(r_2, \epsilon)$, then $\alpha f_1 + \beta f_2 \in C^n_{\max\{s_1,s_2\}}(\max\{r_1, r_2\}, \epsilon)$ and $f_1 \cdot f_2 \in C^n_{s_1+s_2}(r_1 + r_2, \epsilon)$.*

(v) *If $f_1 \in C^n_{s_1}(r_1, \epsilon)$, $f_2 \in C^n_{s_2}(r_2)$, then $f_1 \cdot f_2 \in C^n_{s_1+s_2}(r_1 + r_2, \epsilon)$.*

Lemma 3.3. *Suppose that a smooth function $I = I(h, t, \theta)$ satisfies $I \in C^q_{1/2}(1/2, \epsilon)$ for $q \le 6$. Let*

$$\bar{g} = \bar{g}(h, t, \theta) = [2(n^{-1}h + I)]^{-1/2} g(\sqrt{2(n^{-1}h + I)} \cos\theta) \cos\theta,$$

then $\bar{g} \in C^q_{1/2}(-1/2, \epsilon)$.

Proof. Let $u = u(h, t, \theta) = n^{-1}h + I(h, t, \theta)$, then

$$\begin{aligned}
&\bar{g} = (2u)^{-1/2} g(\sqrt{2u}\cos\theta)\cos\theta, \qquad (3.1)\\
&\frac{\partial u}{\partial h} = n^{-1} + \frac{\partial I}{\partial h}, \quad \frac{\partial^k u}{\partial h^k} = \frac{\partial^k I}{\partial h^k}, \quad k \geq 2,\\
&\frac{\partial^{k+m} u}{\partial h^k \partial t^m} = \frac{\partial^{k+m} I}{\partial h^k \partial t^m}, \quad m \geq 1.
\end{aligned}$$

Since $I \in C^q_{1/2}(1/2, \epsilon)$, we can easily see that $u \in C^q_{1/2}(1/2)$.

For $0 \leq k \leq q$, let

$$Q_k = Q_k(h, t, \theta) = \sum_{j=0}^{k} c_{k,j} (2u)^{-k+\frac{j-1}{2}} g^{(j)}(\sqrt{2u}\cos\theta)\cos^{j+1}\theta, \qquad (3.2)$$

where

$$\begin{aligned}
&c_{k,0} = c_{k-1,0} \cdot (-2k+1), \quad c_{k,k} = 1,\\
&c_{k,j} = c_{k-1,j-1} + c_{k-1,j} \cdot (-2k+1+j), \quad 1 \leq j \leq k-1.
\end{aligned}$$

Then we have

$$Q_0 = \bar{g}, \quad \frac{\partial}{\partial t} Q_k = Q_{k+1} \frac{\partial u}{\partial t}, \quad \frac{\partial}{\partial h} Q_k = Q_{k+1} \frac{\partial u}{\partial h}. \qquad (3.3)$$

Furthermore, it follows from (3.2) and Lemma 2.2 that $Q_k \in C(-k, \epsilon)$. In particular, we have $\bar{g} = Q_0 \in C(0, \epsilon)$.

For $1 \leq k + m \leq q$, by (3.3) and a direct computation, we find

$$\frac{\partial^{k+m}}{\partial h^k \partial t^m} \bar{g} = \sum_{j=1}^{k+m} Q_j P^j_{k,m}(h, t), \qquad (3.4)$$

and $P^j_{k,m} := P^j_{k,m}(h, t)$ is a sum of the following terms

$$\prod_{r=1}^{k} \left(\frac{\partial^r u}{\partial h^r}\right)^{p_r} \cdot \prod_{s=1}^{j-p} \frac{\partial^{q_s} u}{\partial h^{r_s} \partial t^{q_s - r_s}}, \qquad (3.5)$$

where

$$p_r \geq 0, \quad r = 1, \ldots, k, \quad r_s \geq 0, \quad q_s - r_s \geq 1, \quad s = 1, \ldots, j-p,$$

and

$$\sum_{r=1}^{k} p_r = p, \quad \sum_{s=1}^{j-p} r_s + \sum_{r=1}^{k} r p_r = k, \quad \sum_{s=1}^{j-p} q_s + \sum_{r=1}^{k} r p_r = m + k, \quad \sum_{s=1}^{j-p} (q_s - r_s) = m.$$

We notice that $m \geq 1$ if and only if $j - p \geq 1$ and $m = 0$ if and only if $p = j$.

Since $u \in C^q_{1/2}(1/2)$, it follows from (3.5) that

$$P^j_{k,m} \in C(j - k - 1/2), \quad k + m \leq q, m \geq 1, \qquad (3.6)$$

and

$$P^j_{k,0} \in C(j - k), \quad 1 \leq k \leq q. \qquad (3.7)$$

Hence, $Q_j \cdot P^j_{k,m} \in C(-k-1/2,\epsilon), m \geq 1$ and $Q_j \cdot P^j_{k,0} \in C(-k,\epsilon)$. Therefore, we have

$$\frac{\partial^{k+m}}{\partial h^k \partial t^m}\bar{g} \in C(-k-1/2,\epsilon) \subset C(-k-1/2), \quad k+m \leq q, m \geq 1 \tag{3.8}$$

and

$$\frac{\partial^k}{\partial h^k}\bar{g} \in C(-k,\epsilon), \quad 1 \leq k \leq q, \tag{3.9}$$

which, together with the fact that $\bar{g} \in C(0,\epsilon)$, yields $\bar{g} \in C^q_{1/2}(-1/2,\epsilon)$. The proof is complete. □

Lemma 3.4. *Assume that a smooth function $I = I(h,t,\theta)$ satisfies (3.2) and (3.3). Let*

$$\bar{S}(h,t,\theta) = [2(n^{-1}h + I)]^{-1/2} p(t)\cos\theta,$$

then $\bar{S} \in C^q_0(-1/2) \subset C^q_{1/2}(-1/2,\epsilon)$.

Proof. Clearly, $\bar{S} \in C(-1/2)$. Set $u = n^{-1}h + I(h,t,\theta)$, then

$$\bar{S}(h,t,\theta) = (2u)^{-1/2} p(t)\cos\theta. \tag{3.10}$$

For $k \geq 1$, let

$$\bar{Q}_k = (-1)^k (2k-1)!!(2u)^{-k-1/2}. \tag{3.11}$$

Then we have $\bar{Q}_k \in C(-k-1/2)$ and

$$\frac{\partial}{\partial t}\bar{Q}_k = \bar{Q}_{k+1}\frac{\partial u}{\partial t}, \quad \frac{\partial}{\partial h}\bar{Q}_k = \bar{Q}_{k+1}\frac{\partial u}{\partial h}. \tag{3.12}$$

A direct computation shows that

$$\begin{aligned}\frac{\partial^{k+m}}{\partial h^k \partial t^m}\bar{S}(h,t,\theta) &= \sum_{i=0}^{m} C^i_m \frac{\partial^{k+m-i}}{\partial h^k \partial t^{m-i}}(2u)^{-1/2} \cdot p^{(i)}(t)\cos\theta \\ &= \sum_{i=0}^{m} C^i_m \sum_{j=1}^{k+m-i} \bar{Q}_j P^j_{k,m-i}(h,t) \cdot p^{(i)}(t)\cos\theta. \end{aligned} \tag{3.13}$$

By (3.6) and (3.7), we have $P^j_{k,m-i} \in C(j-k-1/2)$ for $i \leq m-1$ and $P^j_{k,0} \in C(j-k)$. So we have $\bar{Q} \cdot P^j_{k,m-i} \in C(-k-1) \subset C(-k-1/2)$ for $i \leq m-1$, and $\bar{Q} \cdot P^j_{k,0} \in C(-k-1/2)$. Therefore, it follows from (3.13) that $\frac{\partial^{k+m}}{\partial h^k \partial t^m}\bar{S} \in C(-k-1/2)$ for $k \geq 1$. Thus, we have $\bar{S} \in C^q_0(1/2)$. The proof is complete. □

Lemma 3.5. *Suppose that a smooth function $I = I(h,t,\theta)$ satisfies*

$$I(h,t,\theta) = -n^{-2}G(\sqrt{2(n^{-1}h+I)}\cos\theta) + n^{-2}\sqrt{2(n^{-1}h+I)}\,p(t)\cos\theta$$

and

$$|I(h,t,\theta)| \leq \varepsilon(h)h.$$

Then $I \in C^7_{1/2}(1/2,\epsilon)$.

Proof. Let $u = u(h, t, \theta) = n^{-1}h + I(h, t, \theta)$, then

$$\frac{\partial u}{\partial h} = n^{-1} + \frac{\partial I}{\partial h}, \quad \frac{\partial u}{\partial t} = \frac{\partial I}{\partial t},$$

and

$$\lim_{h\to\infty} u/h = n^{-1},$$

uniformly for any t and $\theta \in [0, 2\pi]$. Let

$$g_1 = g_1(h, t, \theta) = n^{-2}(2u)^{1/2} p'(t)\cos\theta,$$
$$g_2 = g_2(h, t, \theta) = -n^{-3}(2u)^{-1/2}[g(\sqrt{2u}\cos\theta)\cos\theta - p(t)\cos\theta],$$
$$\Delta = 1 - ng_2(h, t, \theta) = 1 + n^{-2}(2u)^{-1/2}[g(\sqrt{2u}\cos\theta)\cos\theta - p(t)\cos\theta].$$

Then by Lemma 2.2, we have

$$|g_1| \le C \cdot h^{1/2}, \quad |g_2| \le \varepsilon(h), \quad |\Delta| \ge 1 - \varepsilon(h). \tag{3.14}$$

When $k + m = 1$, we have

$$\Delta \cdot \frac{\partial}{\partial t} I(h, t, \theta) = g_1(h, t, \theta), \quad \Delta \cdot \frac{\partial}{\partial h} I(h, t, \theta) = g_2(h, t, \theta). \tag{3.15}$$

Hence, for $h \gg 1$, we have

$$\frac{1}{2}\left|\frac{\partial I}{\partial t}\right| \le |\Delta| \cdot \left|\frac{\partial I}{\partial t}\right| = |g_1| \le C \cdot h^{1/2}, \quad \frac{1}{2}\left|\frac{\partial I}{\partial h}\right| \le |\Delta| \cdot \left|\frac{\partial I}{\partial h}\right| = |g_2| \le \varepsilon(h).$$

That is, $I \in C^1_{1/2}(1/2, \epsilon)$. In general, if $I \in C^q_{1/2}(1/2, \epsilon)$ for $q \le 6$, then by virtue of Lemmas 3.3 and 3.4, we have $g_2, \Delta - 1 \in C^q_{1/2}(-1/2, \epsilon)$. Similarly, we can prove that $g_1 \in C^q_0(1/2)$.

By (3.15) and a direct computation, we find that for $0 \le k + m \le q$,

$$\begin{aligned}\Delta \frac{\partial^{k+m+1}}{\partial h^k \partial t^{m+1}} I &= \frac{\partial^{k+m}}{\partial h^k \partial t^m} g_1 - \sum_{i=0}^{m-1}\sum_{j=0}^{k} C^i_m C^j_k \frac{\partial^{k+m-i-j}}{\partial h^{k-j}\partial t^{m-i}}(\Delta - 1)\frac{\partial^{i+j+1}}{\partial h^j \partial t^{i+1}} I \\ &\quad - \sum_{j=0}^{k-1} C^j_k \frac{\partial^{k-j}}{\partial h^{k-j}}(\Delta - 1)\frac{\partial^{m+j+1}}{\partial h^j \partial t^{m+1}} I,\end{aligned} \tag{3.16}$$

for $m \ge 1$, $k + m \le q$,

$$\begin{aligned}\Delta \frac{\partial^{k+m+1}}{\partial h^{k+1} \partial t^{m}} I &= \frac{\partial^{k+m}}{\partial h^k \partial t^m} g_2 - \sum_{i=1}^{m-1}\sum_{j=0}^{k} C^i_m C^j_k \frac{\partial^{k+m-i-j}}{\partial h^{k-j}\partial t^{m-i}}(\Delta - 1)\frac{\partial^{i+j+1}}{\partial h^{j+1} \partial t^{i}} I \\ &\quad - \sum_{j=0}^{k-1} C^j_k \frac{\partial^{k-j}}{\partial h^{k-j}}(\Delta - 1)\frac{\partial^{m+j+1}}{\partial h^{j+1} \partial t^{m}} I \\ &\quad - \sum_{j=0}^{k} C^j_k \frac{\partial^{k+m-j}}{\partial h^{k-j}\partial t^m}(\Delta - 1)\frac{\partial^{j+1}}{\partial h^{j+1}} I.\end{aligned} \tag{3.17}$$

and for $m = 0$ and $0 \le k \le q$,

$$\Delta \frac{\partial^{k+1}}{\partial h^{k+1}} I = \frac{\partial^k}{\partial h^k} g_2 - \sum_{j=0}^{k-1} C^j_k \frac{\partial^{k-j}}{\partial h^{k-j}} \Delta \cdot \frac{\partial^{j+1}}{\partial h^{j+1}} I. \tag{3.18}$$

Since $\frac{\partial^{k+m-i-j}}{\partial h^{k-j}\partial t^{m-i}}(\Delta-1)\in C(-k+j-1/2)$, $\frac{\partial^{i+j+1}}{\partial h^j\partial t^{i+1}}I\in C(-j+1/2)$ for $0\le i\le m-1$, $0\le j\le k$, and $\frac{\partial^{k-j}}{\partial h^{k-j}}(\Delta-1)\in C(-k+j,\epsilon)$, $\frac{\partial^{m+j+1}}{\partial h^j\partial t^{m+1}}I\in C(-j+1/2)$ for $0\le j\le k-1$, it follows that

$$\begin{aligned}&\frac{\partial^{k+m-i-j}}{\partial h^{k-j}\partial t^{m-i}}(\Delta-1)\frac{\partial^{i+j+1}}{\partial h^j\partial t^{i+1}}I\in C(-k)\subset C(-k+1/2),\\ &\quad\text{for } 0\le i\le m-1,\ 0\le j\le k,\\ &\frac{\partial^{k-j}}{\partial h^{k-j}}(\Delta-1)\frac{\partial^{m+j+1}}{\partial h^j\partial t^{m+1}}I\in C(-k+1/2,\epsilon)\subset C(-k+1/2),\quad \text{for } 0\le j\le k-1,\end{aligned}$$

which, together with (3.16) and the fact that $\frac{\partial^{k+m}}{\partial h^k\partial t^m}g_1\in C(-k+1/2)$, implies that $\Delta\frac{\partial^{k+m+1}}{\partial h^k\partial t^{m+1}}I\in C(-k+1/2)$, and hence $\frac{\partial^{k+m+1}}{\partial h^k\partial t^{m+1}}I\in C(-k+1/2)$ for $0\le k+m\le q$.

By using (3.17) and (3.18) and a similar argument, we can show that $\frac{\partial^{k+m+1}}{\partial h^{k+1}\partial t^m}I\in C(-k-1/2)=C(-(k+1)+1/2)$ for $m\ge 1$, $0\le k+m\le q$, and $\frac{\partial^{k+1}}{\partial h^{k+1}}I\in C(-k,\epsilon)=C(-(k+1)+1,\epsilon)$ for $0\le k\le q$. Therefore, we have $I\in C^{q+1}_{1/2}(1/2,\epsilon)$. Thus, the conclusion follows by induction and the proof is complete. □

Proof to Lemma 2.3. Clearly, we have

$$R_1(h,\theta)=-n^{-2}G(\sqrt{2(n^{-1}h+R_1)}\cos\theta),$$

and

$$R_2(h,t,\theta)=-n^{-2}G(\sqrt{2(n^{-1}h+R_2)}\cos\theta)+n^{-2}[2(n^{-1}h+R_2)]^{1/2}p(t)\cos\theta.$$

Since $|R_1(h,\theta)|, |R_2(h,t,\theta)|\le\varepsilon(h)h$, Lemma 3.5 implies that $R_j\in C^7_{1/2}(1/2,\epsilon)$, $j=1,2$.

Let $R_{12}=R_{12}(h,t,\theta)=R_1(h,\theta)-R_2(h,t,\theta)$, then it follows that

$$\tilde{\Delta}\cdot R_{12}(h,t,\theta)=\tilde{g},\tag{3.19}$$

where

$$\tilde{g}=-n^{-2}[2(n^{-1}h+R_2)]^{1/2}p(t)\cos\theta,$$

and

$$\begin{aligned}&\tilde{\Delta}=1+n^{-2}\int_0^1[2(n^{-1}h+R_\mu)]^{-1/2}g(\sqrt{2(n^{-1}h+R_{mu})}\cos\theta)\cos\theta\,\mathrm{d}\mu,\\ &R_\mu=R_\mu(h,t,\theta)=\mu R_1(h,\theta)+(1-\mu)R_2(h,t,\theta).\end{aligned}$$

Clearly, we have

$$|\tilde{g}|\le Ch^{1/2},\quad |\tilde{\Delta}|\ge 1-\varepsilon(h).\tag{3.20}$$

Moreover, an argument similar to that used in the proof of Lemma 3.3 shows that $\tilde{g}\in C^6_0(1/2)$ and $\tilde{\Delta}-1\in C^6_{1/2}(-1/2,\epsilon)$.

We claim that

$$R_{12}\in C^6_0(1/2).\tag{3.21}$$

Since $R_j\in C^7_{1/2}(1/2,\epsilon)$, $j=1,2$, we have $R_{12}=R_1-R_2\in C^7_{1/2}(1/2,\epsilon)$. Therefore, we have $\frac{\partial^{k+m}}{\partial h^k\partial t^m}R_{12}\in C(-k+1/2)$ for $m\ge 1, k+m\le 6$. So it suffices to show that $\frac{\partial^k}{\partial h^k}R_{12}\in C(-k+1/2)$ for $k\le 6$.

When $k = 0$, by (3.19) and (3.20), we have that for $h \gg 1$,

$$\frac{1}{2}|R_{12}| \leq |\tilde{\Delta}| \cdot |R_{12}| = |\tilde{g}| \leq C \cdot h^{1/2}.$$

That is, $R_{12} \in C(1/2)$. In general, if $\frac{\partial^k}{\partial h^k} R_{12} \in C(-k+1/2)$ for $0 \leq k \leq q < 6$, then for $k \leq q$, by (3.19), we obtain

$$\tilde{\Delta}\frac{\partial^{k+1}}{\partial h^{k+1}} R_{12} = \frac{\partial^{k+1}}{\partial h^{k+1}}\tilde{g} - \sum_{j=0}^{k} C_{k+1}^{j} \frac{\partial^{k+1-j}}{\partial h^{k+1-j}}(\tilde{\Delta} - 1) \cdot \frac{\partial^j}{\partial h^j} R_{12}. \tag{3.22}$$

Since $\frac{\partial^{k+1-j}}{\partial h^{k+1-j}}(\tilde{\Delta} - 1) \in C(-k-1+j, \epsilon)$ and $\frac{\partial^j}{\partial h^j} R_{12} \in C(-j+1/2)$ for $0 \leq j \leq k$, it follows that

$$\frac{\partial^{k+1-j}}{\partial h^{k+1-j}}(\tilde{\Delta} - 1) \cdot \frac{\partial^j}{\partial h^j} R_{12} \in C(-k-1/2, \epsilon) \subset C(-k-1/2), \quad \text{for } 0 \leq j \leq k,$$

which, together with (3.22) and the fact that $\frac{\partial^{k+1}}{\partial h^{k+1}}\tilde{g} \in C(-(k+1)+1/2) = C(-k-1/2)$, yields

$$\tilde{\Delta}\frac{\partial^{k+1}}{\partial h^{k+1}} R_{12} \in C(-k-1/2) = C(-(k+1)+1/2).$$

Therefore, by induction, we conclude that $\frac{\partial^k}{\partial h^k} R_{12} \in C(-k+1/2)$ for $k \leq 6$.

On the other hand, we have

$$\begin{aligned} R(h,t,\theta) &= r(h,t,\theta) - r_0(h,\theta) - \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta \\ &= R_2(h,t,\theta) - R_1(h,\theta) - \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta \\ &= n^{-2}G(\sqrt{2(n^{-1}h+R_1)}\cos\theta) - n^{-2}G(\sqrt{2(n^{-1}h+R_2)}\cos\theta) \\ &\quad + n^{-2}[2(n^{-1}h+R_2)]^{1/2}p(t)\cos\theta - \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta \\ &= n^{-2}\int_0^1 [2(n^{-1}h+R_\mu)]^{-1/2} g(\sqrt{2(n^{-1}h+R_\mu)}\cos\theta)\cos\theta \mathrm{d}\mu \cdot R_{12} \\ &\quad + n^{-2}\int_0^1 [2(n^{-1}h+R_\nu)]^{-1/2} p(t)\cos\theta \mathrm{d}\nu \cdot R_2, \end{aligned} \tag{3.23}$$

where $R_\mu = R_\mu(h,t,\theta) = \mu R_1(h,\theta) + (1-\mu)R_2(h,t,\theta)$ and $R_\nu = R_\nu(h,t,\theta) = \nu R_2(h,t,\theta)$.

Set

$$\hat{g} = n^{-2}\int_0^1 [2(n^{-1}h+R_\mu)]^{-1/2} g(\sqrt{2(n^{-1}h+R_\mu)}\cos\theta)\cos\theta \mathrm{d}\mu,$$

and

$$\hat{S} = n^{-2}\int_0^1 [2(n^{-1}h+R_\nu)]^{-1/2} p(t)\cos\theta \mathrm{d}\nu.$$

Recall that $R_\mu \in C_{1/2}^7(1/2, \epsilon)$ and $R_\nu \in C_{1/2}^7(1/2, \epsilon)$, it follows from Lemmas 3.3 and 3.4 that

$$\hat{g} \in C_{1/2}^6(-1/2, \epsilon), \quad \hat{S} \in C_0^6(-1/2). \tag{3.24}$$

Clearly, (3.21) and (3.24) imply that $\hat{g} \cdot R_{12} \in C_{1/2}^6(0, \epsilon)$ and $\hat{S} \cdot R_2 \in C_{1/2}^6(0, \epsilon)$. Therefore, it follows from (3.23) that

$$R(h,t,\theta) = \hat{g} \cdot R_{12} + \hat{S} \cdot R_2 \in C_{1/2}^6(0, \epsilon),$$

from which we conclude that

$$\left|\frac{\partial^{k+m}}{\partial h^k \partial t^m} R(h,t,\theta)\right| \le Ch^{-k} \le \varepsilon(h) h^{-k+1/2}, \quad \text{for } m \ge 1,\ k+m \le 6$$

and

$$\left|\frac{\partial^{k}}{\partial h^k} R(h,t,\theta)\right| \le \varepsilon(h) h^{-k+1/2}, \quad \text{for } k \le 6.$$

This completes the proof of Lemma 2.3. □

4. Proof of Theorem 1

In this section, we will prove Theorem 1 by using a variant of Moser's small twist theorem.

We first state a new version of Moser's small twist theorem below. Its proof is similar to that of a small twist theorem due to Ortega [18] and, for the reader's convenience, will be given in the Appendix.

Let $A = [a,b] \times S^1$ be a finite cylinder with universal cover $\mathbb{A} = [a,b] \times \mathbb{R}$. The coordinates in A and $\mathbb{A}$ are denoted by $(r, \bar{\theta})$ and (r, θ) respectively, and the circle S^1 is identified with the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. Functions defined on A will be identified with functions defined on $\mathbb{A}$ and satisfying the periodicity condition $F(r, \theta + 2\pi) = F(r, \theta)$ for all $(r, \theta) \in \mathbb{A}$.

Consider the map

$$\bar{f} : A \to \mathbb{R} \times S^1.$$

By an invariant curve of $\bar{f}$, we understand a Jordan curve $\bar{\Gamma} \subset A$ that is homotopic to the circle $\{r = \text{constant}\}$ and satisfies $\bar{f}(\bar{\Gamma}) = \bar{\Gamma}$.

We assume that $\bar{f}$ has the intersection property. By this we mean that every Jordan curve $\bar{\Gamma} \subset A$ that is homotopic to the circle $\{r = \text{constant}\}$ satisfies $\bar{f}(\bar{\Gamma}) \cap \bar{\Gamma} \neq \emptyset$. Besides the intersection property we shall assume that $\bar{f}$ is a continuous mapping that is one-to-one and isotopic to the identity. We sum up all these properties by saying that $\bar{f}$ belongs to the class $M(A)$.

A lift of $\bar{f}$ will be denoted by

$$f : \mathbb{A} \to \mathbb{R} \times \mathbb{R}, \quad f(r,\theta) = (r', \theta')$$

and we shall assume that f can be expressed in the form

$$\begin{cases} \theta' = \theta + 2N\pi + \delta \ell_\delta^1(r,\theta) + \delta \varphi_\delta^1(r,\theta), \\ r' = r + \delta \ell_\delta^2(r,\theta) + \delta \varphi_\delta^2(r,\theta), \end{cases} \tag{4.1}$$

where N is an integer, $\delta \in (0,1)$ is a parameter, and $\ell_\delta^1 \in C^7(A)$ and $\ell_\delta^2, \varphi_\delta^1, \varphi_\delta^2 \in C^5(A)$ are functions satisfying

$$\liminf_{\delta\to 0+} \min_{(r,\theta)\in A} \ell_\delta^1(r,\theta) > 0, \quad \liminf_{\delta\to 0+} \min_{(r,\theta)\in A} \frac{\partial \ell_\delta^1}{\partial r}(r,\theta) > 0, \quad \limsup_{\delta\to 0+} \|\ell_\delta^1\|_{C^7(A)} < +\infty, \tag{4.2}$$

and

$$\limsup_{\delta\to 0+} \|\ell_\delta^2\|_{C^5(A)} < +\infty. \tag{4.3}$$

In addition, we assume that there exists a function $I_\delta \in C^7(A)$ satisfying

$$\liminf_{\delta \to 0+} \min_{(r,\theta)\in A} \frac{\partial I_\delta}{\partial r}(r,\theta) > 0, \quad \limsup_{\delta\to 0+} \|I_\delta\|_{C^7(A)} < +\infty. \tag{4.4}$$

Set

$$\varrho_\delta(r,\theta) := \ell_\delta^1(r,\theta)\frac{\partial I_\delta}{\partial \theta}(r,\theta) + \ell_\delta^2(r,\theta)\frac{\partial I_\delta}{\partial r}(r,\theta), \quad (r,\theta) \in \mathbb{A}, \tag{4.5}$$

we then assume that

$$\lim_{\delta\to 0+} \|\varrho_\delta\|_{C^5(A)} = 0. \tag{4.6}$$

Define the functions

$$\bar{I}_\delta(r) = \max_\theta I_\delta(r,\theta), \quad \underline{I}_\delta(r) = \min_\theta I_\delta(r,\theta), \quad r \in [a,b],$$
$$\bar{I}(r) = \limsup_{\delta\to 0+} \bar{I}_\delta(r), \quad \underline{I}(r) = \liminf_{\delta\to 0+} \underline{I}_\delta(r), \quad r\in[a,b].$$

Theorem 4.1. *Let $\bar{f}$ be given so that (4.1)–(4.3) hold. Assume, in addition, that there exist numbers $\tilde{a}, \tilde{b}, a_1, b_1$, which are independent of δ, and a function I_δ satisfying (4.4)–(4.6) such that*

$$a < \tilde{a} < a_1 < b_1 < \tilde{b} < b, \quad \bar{I}(a) < \underline{I}(\tilde{a}) \le \bar{I}(a_1) < \underline{I}(b_1) \le \bar{I}(\tilde{b}) < \underline{I}(b). \tag{4.7}$$

Then there exist $\varepsilon > 0$ and $\Delta > 0$ such that if $\delta < \Delta$ and $\|\varphi_\delta^1\|_{C^5(A)} + \|\varphi_\delta^2\|_{C^5(A)} < \varepsilon$, the map $\bar{f}$ has an invariant curve $\bar{\Gamma}$. The constants ε and Δ are independent of δ. Furthermore, if we denote by $\mu(\bar{\Gamma},\delta) \in S^1$ the rotation number of the restriction of $\bar{f}$ on $\bar{\Gamma}$, then

$$\lim_{\delta\to 0+} \mu(\bar{\Gamma},\delta) = 0.$$

Remark. It follows from the proof of the theorem that the invariant curve has the form $r = \mu(\bar{\theta})$, where $\mu \in C^3(S^1)$.

Remark. The change of variables $\tilde{\theta} = -\theta, \tilde{r} = r$ shows that the condition (4.2) in the theorem can be replaced by

$$\limsup_{\delta\to 0+} \max_{(r,\theta)\in A} \ell_\delta^1(r,\theta) < 0, \quad \limsup_{\delta\to 0+} \max_{(r,\theta)\in A} \frac{\partial \ell_\delta^1}{\partial r}(r,\theta) < 0, \quad \limsup_{\delta\to 0+} \|\ell_\delta^1\|_{C^7(A)} < +\infty. \tag{4.8}$$

Remark. It is not necessary to assume in the statement of the theorem that $\bar{f}$ is a mapping that is one-to-one and isotopic to the identity because, for small δ, this follows from the remaining conditions. In other words, the assumption $\bar{f} \in M(A)$ can be replaced by the weaker condition: $\bar{f}$ has the intersection property.

Before giving a proof of our main theorem, we first give an expression for the Poincaré map of system (2.23).

In order to calculate the Poincaré map, we introduce a new variable v and a small positive parameter δ by the formula

$$\rho = \delta^{-2} v, \quad v \in [b^{-1}, a^{-1}], \tag{4.9}$$

where $b > a > 0$ are independent of δ and will be specified later.

In the new action and angle variables (v, τ), the system (2.23) can be written in the form

$$\frac{\mathrm{d}v}{\mathrm{d}\vartheta} = -\frac{\partial}{\partial \tau}\hat{H}(v, \tau, \vartheta, \delta), \quad \frac{\mathrm{d}\tau}{\mathrm{d}\vartheta} = \frac{\partial}{\partial v}\hat{H}(v, \tau, \vartheta, \delta), \tag{4.10}$$

where

$$\begin{aligned}\hat{H}(v, \tau, \vartheta, \delta) = \delta^2 \tilde{H}(\delta^{-2} v, \tau, \vartheta) &= \delta^2 n J(\delta^{-2} v) + \sqrt{2}\delta n^{-3/2} v^{1/2} p(\tau) \cos n\vartheta \\ &\quad + \delta^2 n \tilde{R}(\delta^{-2} v, \tau, n\vartheta).\end{aligned}$$

Let

$$\hat{R}(v, \tau, \vartheta, \delta) = \delta^2 n \tilde{R}(\delta^{-2} v, \tau, n\vartheta).$$

By virtue of Lemma 2.4, it is easy to show that

$$\delta^{-1} \cdot \left| \frac{\partial^{k+m}}{\partial v^k \partial \tau^m} \hat{R}(v, \tau, \vartheta, \delta) \right| \le n\varepsilon(\delta^{-2} v) v^{-k+1/2} \to 0 \quad \text{as } \delta \to 0+ \tag{4.11}$$

for $k + m \le 6$.

Since

$$\tau(\delta^{-2} v) = \frac{2\pi}{n} + \delta v^{-1/2} \Gamma(\delta^{-2} v),$$

we may rewrite the system (4.10) explicitly as

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}\vartheta} = -\sqrt{2}\delta n^{-3/2} v^{1/2} p'(\tau) \cos n\vartheta - \dfrac{\partial \hat{R}}{\partial \tau}, \\ \dfrac{\mathrm{d}\tau}{\mathrm{d}\vartheta} = 1 + \dfrac{1}{2\pi}\delta n v^{-1/2} \Gamma(\delta^{-2} v) + \dfrac{\sqrt{2}}{2}\delta n^{-3/2} v^{-1/2} p(\tau) \cos n\vartheta + \dfrac{\partial \hat{R}}{\partial v}. \end{cases} \tag{4.12}$$

In the following, we use the notation $o_5(1)$ and $O_5(1)$. A function $f(v, \tau, \vartheta, \delta)$ is said to be of order $o_5(1)$ if it is C^5 in (v, τ) and

$$\left| \frac{\partial^{k+m}}{\partial v^k \partial \tau^m} f(v, \tau, \vartheta, \delta) \right| \to 0, \quad \text{as } \delta \to 0+,$$

for $k + m \le 5$, uniformly in ϑ. Similarly, a function $f(v, \tau, \vartheta, \delta)$ is said to be of order $O_5(1)$ if it is C^5 in (v, τ) and

$$\left| \frac{\partial^{k+m}}{\partial v^k \partial \tau^m} f(v, \tau, \vartheta, \delta) \right| \le C \quad \text{for } \delta \ll 1,$$

for $k + m \le 5$, uniformly in ϑ.

Denote by $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ the solution of (4.12) with the initial condition

$$(v(0, v_0, \tau_0), \tau(0, v_0, \tau_0)) = (v_0, \tau_0).$$

From (4.11), we know that for $\delta \ll 1$, the solution $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ exists in $[0, 4\pi]$ for any $(v_0, \tau_0) \in [b^{-1}, a^{-1}] \times [0, 2\pi]$. Moreover,

$$0 < \frac{1}{2} b^{-1} \le v(\vartheta, v_0, \tau_0) \le 2a^{-1}, \quad \forall \vartheta \in [0, 4\pi].$$

Assume that the solution $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ has the following expression

$$v(\vartheta, v_0, \tau_0) = v_0 + \delta F_2(\vartheta, v_0, \tau_0), \quad \tau(\vartheta, v_0, \tau_0) = \tau_0 + \vartheta + \delta F_1(\vartheta, v_0, \tau_0). \tag{4.13}$$

Then the Poincaré map of (4.12), denoted by P, has the form

$$P(v_0, \tau_0) = (v_0 + \delta F_2(2\pi, v_0, \tau_0), \tau_0 + 2\pi + \delta F_1(2\pi, v_0, \tau_0)).$$

From the above discussions, we know that if $\delta \ll 1$, this map is well defined in the region $[b^{-1}, a^{-1}] \times [0, 2\pi]$.

If there exists a sequence $\{\delta_m\}_{m=1}^{\infty}$ with $\delta_m \to 0$ as $m \to \infty$, such that for every $\delta = \delta_m$ the map P has an invariant curve which is diffeomorphic to the circle $\{v_0 = \text{constant}\}$, then boundedness of all the solutions of (1.1) follows from the standard arguments (see [3,9,10,18] etc.). In order to prove the existence of such invariant curves for every $\delta = \delta_m$, it suffices to verify that for every $\delta = \delta_m$, the Poincaré map P satisfies all the assumptions of Theorem 4.1.

Since $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ is the solution of (4.12), we have

$$\begin{cases} \dfrac{\mathrm{d}F_1}{\mathrm{d}\vartheta} = \left[\dfrac{1}{2\pi} n\Gamma(\delta^{-2}(v_0 + \delta F_2)) + \dfrac{\sqrt{2}}{2} n^{-3/2} p(\tau)\cos n\vartheta\right](v_0 + \delta F_2)^{-1/2} \\ \qquad\qquad + \delta^{-1}\dfrac{\partial \hat{R}}{\partial v}, \\ \dfrac{\mathrm{d}F_2}{\mathrm{d}\vartheta} = -\sqrt{2} n^{-3/2}(v_0 + \delta F_2)^{1/2} p'(\tau)\cos n\vartheta - \delta^{-1}\dfrac{\partial \hat{R}}{\partial \tau}. \end{cases} \tag{4.14}$$

As in the proof in [3], we can show from (4.11) and (4.14) that

$$\left|\frac{\partial^{k+m}}{\partial v_0^k \partial \tau_0^m} F_1(\vartheta, v_0, \tau_0)\right|, \quad \left|\frac{\partial^{k+m}}{\partial v_0^k \partial \tau_0^m} F_2(\vartheta, v_0, \tau_0)\right| \le C$$

for all $k + m \le 5$, uniformly in ϑ. Hence

$$v(\vartheta, v_0, \tau_0) = v_0 + \delta O_5(1), \quad \tau(\vartheta, v_0, \tau_0) = \tau_0 + \vartheta + \delta O_5(1). \tag{4.15}$$

Notice that $F_1(0, v_0, \tau_0) = F_2(0, v_0, \tau_0) = 0$, it follows from (1.6), (4.11), (4.14) and (4.15) that

$$\begin{aligned} F_1(2\pi, v_0, \tau_0) &= \int_0^{2\pi} \left[\frac{1}{2\pi} n\Gamma(\delta^{-2}(v_0 + \delta F_2)) + \frac{\sqrt{2}}{2} n^{-3/2} p(\tau(\vartheta))\cos n\vartheta\right] \\ &\quad \times (v_0 + \delta F_2)^{-1/2}\mathrm{d}\vartheta + o_5(1) \\ &= n v_0^{-1/2}\Gamma(\delta^{-2} v_0) + \frac{\sqrt{2}}{2} n^{-3/2} v_0^{-1/2}\int_0^{2\pi} p(\tau_0 + \vartheta)\cos n\vartheta \mathrm{d}\vartheta + o_5(1), \\ F_2(2\pi, v_0, \tau_0) &= -\sqrt{2} n^{-3/2}\int_0^{2\pi} (v_0 + \delta F_2)^{1/2} p'(\tau(\vartheta))\cos n\vartheta \mathrm{d}\vartheta + o_5(1) \\ &= -\sqrt{2} n^{-3/2} v_0^{1/2}\int_0^{2\pi} p'(\tau_0 + \vartheta)\cos n\vartheta \mathrm{d}\vartheta + o_5(1). \end{aligned}$$

Set

$$p_n^c := \int_0^{2\pi} p(\vartheta)\cos n\vartheta \mathrm{d}\vartheta, \quad p_n^s := \int_0^{2\pi} p(\vartheta)\sin n\vartheta \mathrm{d}\vartheta.$$

Then, we obtain

$$\begin{aligned}\int_0^{2\pi} p(\tau_0+\vartheta)\cos n\vartheta \mathrm{d}\vartheta &= \int_0^{2\pi} p(\vartheta)\cos n(\vartheta-\tau_0)\mathrm{d}\vartheta \\ &= p_n^c \cos n\tau_0 + p_n^s \sin n\tau_0, \\ \int_0^{2\pi} p'(\tau_0+\vartheta)\cos n\vartheta \mathrm{d}\vartheta &= \int_0^{2\pi} p'(\vartheta)\cos n(\vartheta-\tau_0)\mathrm{d}\vartheta \\ &= n\int_0^{2\pi} p(\vartheta)\sin n(\vartheta-\tau_0)\mathrm{d}\vartheta \\ &= n(p_n^s \cos n\tau_0 - p_n^c \sin n\tau_0).\end{aligned}$$

Now we get an expression of the Poincaré map P as

$$P:\begin{cases}\tau_1 = \tau_0 + 2\pi + \delta\tilde{\ell}_1(v_0,\tau_0,\delta) + \delta\tilde{\phi}_1(v_0,\tau_0,\delta),\\ v_1 = v_0 - \delta\tilde{\ell}_2(v_0,\tau_0,\delta) + \delta\tilde{\phi}_2(v_0,\tau_0,\delta),\end{cases} \tag{4.16}$$

where

$$\begin{cases}\tilde{\ell}_1(v_0,\tau_0,\delta) = n v_0^{-1/2}\left\{\Gamma(\delta^{-2}v_0) + \dfrac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s \sin n\tau_0)\right\},\\ \tilde{\ell}_2(v_0,\tau_0,\delta) = \sqrt{2}n^{-1/2}v_0^{1/2}(p_n^s\cos n\tau_0 - p_n^c\sin n\tau_0).\end{cases} \tag{4.17}$$

and the functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are of order $o_5(1)$.

Since P is the Poincaré map of the Hamiltonian system (4.12), it is symplectic and has the intersection property in the cylinder $[b^{-1}, a^{-1}] \times S^1$. Moreover, the intersection property is preserved under a homeomorphism.

Under the diffeomorphism

$$\tau = \tau, \quad u = v^{-1},$$

the symplectic map P is transformed into the form

$$Q:\begin{cases}\tau_1 = \tau_0 + 2\pi + \delta\ell_\delta^1(u_0,\tau_0) + \delta\phi_1(u_0,\tau_0,\delta),\\ u_1 = u_0 + \delta\ell_\delta^2(u_0,\tau_0) + \delta\phi_2(u_0,\tau_0,\delta),\end{cases} \tag{4.18}$$

where $(u_0, \tau_0) \in [a, b] \times S^1$, and

$$\begin{cases}\ell_\delta^1(u_0,\tau_0) = n u_0^{1/2}\left\{\Gamma(\delta^{-2}u_0^{-1}) + \dfrac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s \sin n\tau_0)\right\},\\ \ell_\delta^2(u_0,\tau_0) = \sqrt{2}n^{-1/2}u_0^{3/2}(p_n^s\cos n\tau_0 - p_n^c\sin n\tau_0),\end{cases} \tag{4.19}$$

and

$$\begin{cases}\phi_1(u_0,\tau_0,\delta) = \tilde{\phi}_1(u_0^{-1},\tau_0,\delta),\\ \phi_2(u_0,\tau_0,\delta) = -u_0^2\tilde{\phi}_2(u_0^{-1},\tau_0,\delta) + \dfrac{\delta u_0^2[\tilde{\ell}_2(u_0^{-1},\tau_0,\delta) + \tilde{\phi}_2(u_0^{-1},\tau_0,\delta)]^2}{1-\delta u_0\tilde{\ell}_2(u_0^{-1},\tau_0,\delta) + \delta u_0\tilde{\phi}_2(u_0^{-1},\tau_0,\delta)}.\end{cases}$$

By (1.6), we know that the functions ϕ_1 and ϕ_2 are of order $o_5(1)$.

By (1.7), we may assume, without loss of generality, that

$$\limsup_{h\to+\infty} \Gamma(h) > \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right|. \tag{4.20}$$

Let $\varpi > 0$ be such that

$$\varpi \le \limsup_{h\to+\infty} \Gamma(h) - \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right|.$$

Since $h\Gamma'(h) \to 0$ as $h \to +\infty$, there exists a number $\bar{h} > 0$ such that

$$|h\Gamma'(h)| \le \min\left\{ \frac{b^{-1}\varpi}{4(a^{-1} - b^{-1})}, \frac{\varpi}{16} \right\}, \tag{4.21}$$

holds for all $h \ge \bar{h}$.

By (4.20), we can choose a sequence $\{h_m\}_{m=1}^{\infty}$ with $\bar{h} \le h_m \to +\infty$ as $m \to +\infty$ such that

$$h_{m+1} > \frac{b}{a} h_m \tag{4.22}$$

and

$$\Gamma(h_m) \ge \frac{3}{4}\varpi + \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right|. \tag{4.23}$$

Take $\delta_m = (bh_m)^{-1/2}$. Then we have $\delta_m \to 0$ as $m \to +\infty$. It follows from (4.22) that

$$\delta_m^{-2}(a^{-1} - b^{-1}) = h_m \left(\frac{b}{a} - 1 \right) \le h_{m+1} - h_m,$$

and hence

$$[\delta_m^{-2} b^{-1}, \delta_m^{-2} a^{-1}] \subset [h_m, h_{m+1}].$$

We claim that for any $h \in [\delta_m^{-2} b^{-1}, \delta_m^{-2} a^{-1}]$,

$$\Gamma(h) > \frac{1}{4}\varpi + \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right|. \tag{4.24}$$

In fact, since $h_m = \delta_m^{-2} b^{-1}$, it follows from (4.23) that (4.24) holds for $h \ge h_m$ with $|h - h_m|$ sufficiently small. Suppose that there is an $h_m^* \in [\delta_m^{-2} b^{-1}, \delta_m^{-2} a^{-1}]$ such that

$$\Gamma(h_m^*) = \frac{1}{4}\varpi + \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right|.$$

Then it follows from (4.21) and (4.23) that

$$\begin{aligned} \frac{1}{2}\varpi &\le |\Gamma(h_m^*) - \Gamma(h_m)| \\ &= |\Gamma'(h_m + \mu(h_m^* - h_m))|(h_m^* - h_m) \quad (\mu \in [0, 1]) \\ &= \frac{|(h_m + \mu(h_m^* - h_m))\Gamma'(h_m + \mu(h_m^* - h_m))|}{h_m + \mu(h_m^* - h_m)}(h_m^* - h_m) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{b^{-1}\varpi}{4(a^{-1}-b^{-1})h_m}(h_m^* - h_m)\\
&\le \frac{b^{-1}\varpi}{4(a^{-1}-b^{-1})\delta_m^{-2}b^{-1}}\delta_m^{-2}(a^{-1}-b^{-1}) = \frac{\varpi}{4},
\end{aligned}
$$

which is a contradiction. Therefore, (4.24) holds for all $h \in [\delta_m^{-2}b^{-1}, \delta_m^{-2}a^{-1}]$.

Since $\limsup_{h\to+\infty} |\Gamma(h)| < +\infty$, we can assume, without loss of generality, that

$$
\sup_{h\ge\bar{h}} |\Gamma(h)| \le M. \tag{4.25}
$$

For simplicity, we set $\ell_m^i(u_0,\tau_0) = \ell_{\delta_m}^i(u_0,\tau_0),\ i = 1, 2$. Since

$$
\begin{aligned}
\frac{\partial}{\partial u_0}\ell_m^1(u_0,\tau_0) = \frac{1}{2}nu_0^{-1/2}\Bigg\{&\Gamma(\delta_m^{-2}u_0^{-1}) - 2\delta_m^{-2}u_0^{-1}\Gamma'(\delta_m^{-2}u_0^{-1})\\
&+\frac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0)\Bigg\},
\end{aligned}
$$

and

$$
|p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0| \le \sqrt{(p_n^c)^2 + (p_n^s)^2} = \left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|,
$$

it follows from (4.19), (4.21) and (4.24) that for $u_0 \in [a, b]$,

$$
\ell_m^1(u_0,\tau_0) \ge \frac{n\varpi}{4}u_0^{1/2} \ge \frac{n\varpi}{4}a^{1/2} > 0,
$$

and

$$
\frac{\partial}{\partial u_0}\ell_m^1(u_0,\tau_0) \ge \frac{n\varpi}{16}u_0^{-1/2} \ge \frac{n\varpi}{16}b^{-1/2} > 0.
$$

Furthermore, by (4.21) and (4.25), we also have

$$
\ell_m^1(u_0,\tau_0) \le nb^{1/2}\left[M + \frac{\sqrt{2}}{2}n^{-5/2}\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|\right],
$$

and

$$
\frac{\partial}{\partial u_0}\ell_m^1(u_0,\tau_0) \le \frac{1}{2}na^{-1/2}\left[M + \frac{\varpi}{8} + \frac{\sqrt{2}}{2}n^{-5/2}\left|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|\right].
$$

Notice that for $k = 1, 2, \ldots, 7$, $\limsup_{h\to+\infty} |h^k\Gamma^{(k)}(h)| < +\infty$, a similar argument can be used to show that

$$
\limsup_{m\to+\infty} \|\ell_m^1\|_{C^7(A)} < +\infty, \quad \limsup_{m\to+\infty} \|\ell_m^2\|_{C^5(A)} < +\infty,
$$

here and in what follows, we always denote by A the set $[a, b] \times S^1$.

Let

$$I_m(u_0,\tau_0) = -u_0^{-1/2}\left[\Gamma(\delta_m^{-2}u_0^{-1}) + \frac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0)\right]$$
$$-\int_a^{u_0}\delta_m^{-2}u^{-5/2}\Gamma'(\delta_m^{-2}u^{-1})\mathrm{d}u, \quad u_0\in[a,b].$$

Then it follows from (4.21) and (4.24) that

$$\frac{\partial}{\partial u_0}I_m(u_0,\tau_0) = \frac{1}{2}u_0^{-3/2}\left[\Gamma(\delta_m^{-2}u_0^{-1}) + \frac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0)\right]$$
$$\geq \frac{\varpi}{8}u_0^{-3/2} \geq \frac{\varpi}{8}b^{-3/2} > 0.$$

Clearly, by (4.21), (4.24) and (4.25), we can also obtain

$$\frac{\varpi}{16}a^{-3/2}(b-a) \geq I_m(u_0,\tau_0) \geq -a^{-1/2}\left[M + \frac{\sqrt{2}}{2}n^{-5/2}\left|\int_0^{2\pi}p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|\right]$$
$$-\frac{\varpi}{16}a^{-3/2}(b-a)$$

and

$$\frac{\partial}{\partial u_0}I_m(u_0,\tau_0) \leq \frac{1}{2}a^{-3/2}\left[M + \frac{\sqrt{2}}{2}n^{-5/2}\left|\int_0^{2\pi}p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|\right].$$

In a similar way, we can show that

$$\limsup_{m\to+\infty}\|I_m\|_{C^7(A)} < +\infty.$$

Moreover, we have

$$\begin{aligned}
\varrho_m(u_0,\tau_0) :&= \varrho_{\delta_m}(u_0,\tau_0)\\
&= \ell_m^1(u_0,\tau_0)\frac{\partial}{\partial\tau_0}I_m(u_0,\tau_0) + \ell_m^2(u_0,\tau_0)\frac{\partial}{\partial u_0}I_m(u_0,\tau_0)\\
&= nu_0^{1/2}\left\{\Gamma(\delta_m^{-2}u_0^{-1}) + \frac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0)\right\}\\
&\quad\cdot\frac{\sqrt{2}}{2}n^{-3/2}u_0^{-1/2}(p_n^c\sin n\tau_0 - p_n^s\cos n\tau_0)\\
&\quad+\sqrt{2}n^{-1/2}u_0^{3/2}(p_n^s\cos n\tau_0 - p_n^c\sin n\tau_0)\\
&\quad\cdot\frac{1}{2}u_0^{-3/2}\left\{\Gamma(\delta_m^{-2}u_0^{-1}) + \frac{\sqrt{2}}{2}n^{-5/2}(p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0)\right\}\\
&= 0.
\end{aligned}$$

On the other hand, for any $x, y \in [a,b]$, we have

$$\bar{I}_m(x) = -x^{-1/2}\left\{\Gamma(\delta_m^{-2}x^{-1}) - \frac{\sqrt{2}}{2}n^{-5/2}\left|\int_0^{2\pi}p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\right|\right\}$$

$$- \int_a^x \delta_m^{-2} u^{-5/2} \Gamma'(\delta_m^{-2} u^{-1}) \mathrm{d}u,$$

and

$$\underline{I}_m(y) = -y^{-1/2} \left\{ \Gamma(\delta_m^{-2} y^{-1}) + \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right| \right\} - \int_a^y \delta_m^{-2} u^{-5/2} \Gamma'(\delta_m^{-2} u^{-1}) \mathrm{d}u.$$

Therefore, by virtue of (1.6) and the dominated convergence theorem, we obtain

$$\bar{I}(x) = \limsup_{m\to+\infty} \bar{I}_m(x) \le -x^{-1/2} \left\{ \liminf_{m\to+\infty} \Gamma(\delta_m^{-2} x^{-1}) - \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right| \right\},$$

and

$$\underline{I}(y) = \liminf_{m\to+\infty} \underline{I}_m(y) \ge -y^{-1/2} \left\{ \limsup_{m\to+\infty} \Gamma(\delta_m^{-2} y^{-1}) + \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right| \right\}.$$

If

$$y/x \ge \frac{256 M^2}{\varpi^2}, \tag{4.26}$$

then it follows from (4.24)–(4.26) that

$$\begin{aligned}
\underline{I}(y) - \bar{I}(x) &\ge x^{-1/2} \left\{ \liminf_{m\to+\infty} \Gamma(\delta_m^{-2} x^{-1}) - \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right| \right\} \\
&\quad - y^{-1/2} \left\{ \limsup_{m\to+\infty} \Gamma(\delta_m^{-2} y^{-1}) + \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right| \right\} \\
&= (x^{-1/2} + y^{-1/2}) \left\{ \liminf_{m\to+\infty} \Gamma(\delta_m^{-2} x^{-1}) - \frac{\sqrt{2}}{2} n^{-5/2} \left| \int_0^{2\pi} p(t) \mathrm{e}^{-\mathrm{i}nt} \mathrm{d}t \right| \right. \\
&\quad \left. - \frac{1}{1 + (y/x)^{1/2}} \left[\liminf_{m\to+\infty} \Gamma(\delta_m^{-2} x^{-1}) + \limsup_{m\to+\infty} \Gamma(\delta_m^{-2} y^{-1}) \right] \right\} \\
&\ge (x^{-1/2} + y^{-1/2}) \left[\frac{1}{4}\varpi - \frac{1}{8}\varpi \right] \\
&= \frac{\varpi}{8}(x^{-1/2} + y^{-1/2}) \ge \frac{\varpi}{4} b^{-1/2} > 0,
\end{aligned}$$

and hence

$$\bar{I}(x) < \underline{I}(y). \tag{4.27}$$

Now we choose the constants a and b as

$$b = \frac{4096 M^2 (M+1)}{\varpi^3} > 1, \quad a = b^{-1}.$$

Set

$$\tilde{b} = \frac{16(M+1)}{\varpi}, \quad \tilde{a} = \tilde{b}^{-1}, \quad b_1 = \frac{16M}{\varpi}, \quad a_1 = b_1^{-1}.$$

Then we have

$$a < \tilde{a} < a_1 < b_1 < \tilde{b} < b,$$

and, by (4.26) and (4.27), we also have

$$\bar{I}(a) < \underline{I}(\tilde{a}) \le \bar{I}(a_1) < \underline{I}(b_1) \le \bar{I}(\tilde{b}) < \underline{I}(b).$$

Therefore, for the map Q, all the conditions of Theorem 4.1 are fulfilled. Hence for every positive integer m large enough, the map Q, and so the map P, has an invariant closed curve which is diffeomorphic to $\{u_0 = \text{constant}\}$. This completes the proof of Theorem 1.

5. Proofs of Theorems 2 and 3

In this section, we will give proofs of Theorems 2 and 3. We start with a few technical lemmas.

Lemma 5.1. *Let*

$$\Gamma_k(h) := [h^{-1/2}\Gamma(h)]^{(k)}, \quad \text{for } 0 \le k \le 7.$$

Then

$$h^k \Gamma^{(k)}(h) = h^{k+1/2}\Gamma_k(h) - 2^{-k}\prod_{j=1}^{k}(-2j+1)\Gamma(h) - \sum_{j=1}^{k-1}\tilde{c}_{k,j}h^j\Gamma^{(j)}(h), \tag{5.1}$$

where

$$\tilde{c}_{k,0} = \tilde{c}_{k-1,0}\cdot\left(-\frac{2k-1}{2}\right) = \prod_{j=1}^{k}\left(-\frac{2j-1}{2}\right), \quad \tilde{c}_{k,k} = 1,$$

$$\tilde{c}_{k,j} = \tilde{c}_{k-1,j-1} + \tilde{c}_{k-1,j}\cdot\left(-\frac{2k-1}{2} + j\right), \quad 1 \le j \le k-1.$$

Proof. It is easy to see that

$$\Gamma_k(h) = \sum_{j=0}^{k}\tilde{c}_{k,j}h^{-\frac{2k+1}{2}+j}\Gamma^{(j)}(h).$$

Hence

$$\Gamma^{(k)}(h) = h^{1/2}\Gamma_k(h) - \sum_{j=0}^{k-1}\tilde{c}_{k,j}h^{-k+j}\Gamma^{(j)}(h),$$

from which the conclusion follows. □

Lemma 5.2. *Let*

$$q_h = q(h,\theta) = n + n^{-1}(2h)^{-1/2}g(\sqrt{2h}\cos\theta)\cos\theta,$$

$$q_h^{(k)} = \frac{\partial^k}{\partial h^k}q(h,\theta).$$

Then

$$q_h^{(k)} = n^{-1} \sum_{j=0}^{k} c_{k,j} (2h)^{-k+\frac{j-1}{2}} g^{(j)}(\sqrt{2h}\cos\theta)\cos^{j+1}\theta, \tag{5.2}$$

for $1 \le k \le 8$, *where*

$$c_{0,0} = 1, \quad c_{k,0} = c_{k-1,0} \cdot (-2k+1) = \sum_{j=1}^{k} (-2j+1), \quad c_{k,k} = 1,$$
$$c_{k,j} = c_{k-1,j-1} + c_{k-1,j} \cdot (-2k+1+j), \quad 1 \le j \le k-1.$$

If, in addition, (1.8) *holds, then*

$$|q_h^{(k)}| \le \varepsilon(h) h^{-k-1/4}, \quad 1 \le k \le 8. \tag{5.3}$$

Proof. The proof is easy and is omitted. □

Lemma 5.3. *If* (1.8) *holds, then*

$$\Gamma_k(h) = -n^{-k-2} \int_0^{2\pi} q_{r_0}^{(k)} \mathrm{d}\theta + \varepsilon(h) h^{-k-1/2}, \quad 1 \le k \le 7. \tag{5.4}$$

where $r_0 = r_0(h, \theta)$.

Proof. By (2.9), we see that

$$\Gamma_0(h) = h^{-1/2}\Gamma(h) = \tau(h) - \frac{2\pi}{n} = \int_0^{2\pi} \frac{1}{q_{r_0}} \mathrm{d}\theta - \frac{2\pi}{n}.$$

Notice that

$$\frac{\partial}{\partial h} r_0(h,\theta) = \frac{1}{\frac{\partial}{\partial r} h_0(r_0, \theta)} = \frac{1}{q_{r_0}},$$

by using a direct computation, we can easily check that

$$\Gamma_k(h) = -\int_0^{2\pi} \frac{q_{r_0}^{(k)} [q_{r_0}]^{k-1} + Q_k(h,\theta)}{[q_{r_0}]^{2k+1}} \mathrm{d}\theta, \quad 1 \le k \le 7, \tag{5.5}$$

where

$$Q_k(h,\theta) = \sum_{s=2}^{k} \left(\sum_{\{i_j\} \in P_s} C_{ks}\{i_j\} \prod_{j=1}^{s} q_{r_0}^{(i_j)} \right) q_{r_0}^{k-s}, \quad 1 \le k \le 7,$$

where $P_s = \{(i_1, i_2, \ldots, i_s);\ 1 \le i_j \le k-1,\ \sum_{j=1}^{s} i_j = k\}$ and $C_{ks}\{i_j\}$ are constants.

It follows from (1.8) and (5.3) that

$$|q_{r_0}^{(k)}| \le \varepsilon(h) h^{-k-1/4}, \quad 1 \le k \le 7, \tag{5.6}$$

and hence

$$|Q_k(h,\theta)| \le \varepsilon(h) h^{-k-1/2}, \quad 1 \le k \le 7. \tag{5.7}$$

Set

$$g_{r_0} = g(\sqrt{2r_0}\cos\theta)\cos\theta.$$

Notice that

$$|q_{r_0} - n| = n^{-1}|(2r_0)^{-1/2} g(\sqrt{2r_0}\cos\theta)\cos\theta| \le \varepsilon(h)h^{-1/4},$$

it follows from (5.5)–(5.7) that

$$\begin{aligned}
\Gamma_k(h) &= -\int_0^{2\pi} \frac{q_{r_0}^{(k)}}{[q_{r_0}]^{k+2}} \mathrm{d}\theta + \varepsilon(h)h^{-k-1/2} \\
&= -n^{-k-2}\int_0^{2\pi} \frac{q_{r_0}^{(k)}}{[1 + n^{-2}(2r_0)^{-1/2} g_{r_0}]^{k+2}} \mathrm{d}\theta \\
&= -n^{-k-2}\int_0^{2\pi} q_{r_0}^{(k)}[1 - n^{-2}(2r_0)^{-1/2} g_{r_0}]^{k+2} \mathrm{d}\theta + \varepsilon(h)h^{-k-1/2} \\
&= -n^{-k-2}\int_0^{2\pi} q_{r_0}^{(k)} \mathrm{d}\theta + \varepsilon(h)h^{-k-1/2}.
\end{aligned}$$

The proof is complete. □

Now, we are in the position to prove Theorem 2.

Proof of Theorem 2. Notice that $R_1(h,\theta) = r_0(h,\theta) - n^{-1}h$ satisfies

$$R_1(h,\theta) = -n^{-2} G(\sqrt{2(n^{-1}h + R_1)}\cos\theta), \quad |R_1(h,\theta)| \le \varepsilon(h)h,$$

it follows that

$$\bar{\Delta} \cdot R_1(h,\theta) = -n^{-2} G(\sqrt{2n^{-1}h}\cos\theta), \tag{5.8}$$

where

$$\bar{\Delta} = 1 + n^{-2}[2(n^{-1}h + \mu R_1)]^{-1/2} g(\sqrt{2(n^{-1}h + \mu R_1)}\cos\theta)\cos\theta, \quad \mu \in [0,1].$$

Since $|R_1(h,\theta)| \le \varepsilon(h)h$, we have

$$\bar{\Delta} = 1 + \varepsilon(h). \tag{5.9}$$

From (5.8) and (5.9) and the rule of L'Hospital, it follows that

$$\begin{aligned}
\lim_{h\to+\infty} \frac{R_1(h,\theta)}{h^{3/4}} &= \lim_{h\to+\infty} \bar{\Delta}\frac{R_1(h,\theta)}{h^{3/4}} \\
&= -n^{-2}\lim_{h\to+\infty} \frac{G(\sqrt{2n^{-1}h}\cos\theta)}{h^{3/4}} \\
&= -n^{-2}\lim_{h\to+\infty} \frac{4n^{-1} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta}{3\sqrt{2n^{-1}h}\cdot h^{-1/4}} \\
&= -\frac{2\sqrt{2}}{3} n^{-5/2}\lim_{h\to+\infty} h^{-1/4} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta \\
&= 0.
\end{aligned}$$

Hence,

$$|R_1(h,\theta)| \le \varepsilon(h)h^{3/4}. \tag{5.10}$$

For $1 \le k \le 7$, we have

$$q_{r_0}^{(k)} = q_{n^{-1}h}^{(k)} + \int_0^1 q_{n^{-1}h+\alpha R_1}^{(k+1)} \mathrm{d}\alpha \cdot R_1.$$

Therefore, (5.3) and (5.10) imply that

$$\left| \int_0^1 q_{n^{-1}h+\alpha R_1}^{(k+1)} \mathrm{d}\alpha \cdot R_1 \right| \le \varepsilon(h) h^{-(k+1)-1/4} \cdot \varepsilon(h) h^{3/4} \le \varepsilon(h) h^{-k-1/2},$$

and hence, from (5.4), it follows that

$$\Gamma_k(h) = -n^{-k-2} \int_0^{2\pi} q_{n^{-1}h}^{(k)} \mathrm{d}\theta + \varepsilon(h) h^{-k-1/2}, \quad 1 \le k \le 7. \tag{5.11}$$

Therefore, by (5.2) and (1.10), we have

$$\begin{aligned} h^{k+1/2}\Gamma_k(h) &= -n^{-k-2} \int_0^{2\pi} h^{k+1/2} q_{n^{-1}h}^{(k)} \mathrm{d}\theta + \varepsilon(h) \\ &= -2^{-k-1/2} n^{-5/2} \sum_{j=0}^{k} c_{k,j} \int_0^{2\pi} (2n^{-1}h)^{j/2} g^{(j)}(\sqrt{2n^{-1}h}\cos\theta) \cos^{j+1}\theta \mathrm{d}\theta \\ &\quad + \varepsilon(h) \\ &= -2^{-k-1/2} n^{-5/2} c_{k,0} \int_0^{2\pi} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta \mathrm{d}\theta + \varepsilon(h) \\ &= -2^{-k-1/2} n^{-5/2} \prod_{j=1}^{k}(-2j+1) \int_0^{2\pi} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta \mathrm{d}\theta + \varepsilon(h). \end{aligned} \tag{5.12}$$

On the other hand, by using a similar argument, we can show that

$$\Gamma(h) = -2^{-1/2} n^{-5/2} \int_0^{2\pi} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta \mathrm{d}\theta + \varepsilon(h). \tag{5.13}$$

By (5.1), (5.12) and (5.13), we have

$$|h^k \Gamma^{(k)}(h)| \le \varepsilon(h), \quad 1 \le k \le 7.$$

That is, (1.6) holds. Clearly, (1.5) also holds and the conclusion of Theorem 2 follows from (1.7) and (5.13). The proof is complete. □

Proof of Theorem 3. Firstly, we notice that if (1.12) and (1.13) hold, then (1.8), and hence (5.3) holds for $1 \le k \le 7$. Therefore, (5.4) and (5.13) also hold and we can prove that

$$\begin{aligned} h^{k+1/2}\Gamma_k(h) &= -2^{-k-1/2} n^{-5/2} \prod_{j=1}^{k}(-2j+1) \int_0^{2\pi} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta \mathrm{d}\theta \\ &\quad - n^{-k-3} \sum_{j=1}^{k} c_{k,j} \int_0^{2\pi} \left[\frac{h}{2r_0}\right]^{k+1/2} (2r_0)^{j/2} g^{(j)}(\sqrt{2r_0}\cos\theta)\cos^{j+1}\theta \mathrm{d}\theta \\ &\quad + \varepsilon(h), \end{aligned}$$

which, together with (1.12), (5.1) and (5.13), yields (1.6). Now the conclusion of Theorem 3 follows from Theorem 1. □

Appendix

In this appendix, we sketch the proof of Theorem 4.1 in three steps. For more details, we refer the reader to [18].

1. *The New Coordinates*. For each $\theta \in \mathbb{R}$ and $h \in [\bar{I}_\delta(a), \underline{I}_\delta(b)]$, we denote by $R_\delta = R_\delta(h,\theta)$ the unique solution of $I_\delta(R_\delta(h,\theta),\theta) = h$. Then the implicit function theorem and (4.4) imply that R_δ is well defined and of class C^7. Moreover, $a \le R_\delta(h,\theta) \le b$, R_δ is 2π-periodic in θ and satisfies $R_\delta(I_\delta(r,\theta),\theta) = r$ for all $(r,\theta) \in \mathbb{A}$ with $I_\delta(r,\theta) \in [\bar{I}_\delta(a), \underline{I}_\delta(b)]$. By using an induction argument, we can show that

$$\limsup_{\delta\to 0+} \|R_\delta\|_{C^7([\bar{I}_\delta(a),\underline{I}_\delta(b)]\times[0,2\pi])} < +\infty. \tag{A.1}$$

Define

$$T_\delta : [\bar{I}_\delta(a), \underline{I}_\delta(b)] \to \mathbb{R}, \quad T_\delta(h) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\ell_\delta^1(R_\delta(h,\theta),\theta)},$$

and

$$\omega_\delta(h) = \frac{2\pi}{T_\delta(h)}, \quad \text{for } h \in [\bar{I}_\delta(a), \underline{I}_\delta(b)].$$

Then T_δ and ω_δ are of class C^7, and by using (4.2) and (4.4), we can show that

$$\liminf_{\delta\to 0+} \min_{h\in[\bar{I}_\delta(a),\underline{I}_\delta(b)]} \omega_\delta(h) > 0, \quad \liminf_{\delta\to 0+} \min_{h\in[\bar{I}_\delta(a),\underline{I}_\delta(b)]} \omega_\delta'(h) > 0, \tag{A.2}$$

and

$$\limsup_{\delta\to 0+} \|\omega_\delta\|_{C^7[\bar{I}_\delta(a),\underline{I}_\delta(b)]} < +\infty. \tag{A.3}$$

Next we consider the region $\tilde{\mathbb{A}} = \{(r,\theta) : \theta \in \mathbb{R}, \tilde{a} \le r \le \tilde{b}\}$ and define the function

$$K_\delta : \tilde{\mathbb{A}} \to \mathbb{R}, \quad K_\delta(r,\theta) = \int_0^\theta \frac{\mathrm{d}s}{\ell_\delta^1(R_\delta(I_\delta(r,\theta),s),s)}.$$

It follows from (4.7) that it is well defined. Moreover, it is of class C^7 and satisfies

$$K_\delta(r,\theta+2\pi) = K_\delta(r,\theta) + T_\delta(I_\delta(r,\theta)), \quad \text{for all } (r,\theta) \in \tilde{\mathbb{A}}.$$

Moreover, it is easily seen that the derivatives of K_δ are given by

$$\frac{\partial K_\delta}{\partial\theta}(r,\theta) = \frac{1}{\ell_\delta^1(r,\theta)} + \frac{\partial I_\delta}{\partial\theta}(r,\theta)S_\delta(r,\theta), \quad \frac{\partial K_\delta}{\partial r}(r,\theta) = \frac{\partial I_\delta}{\partial\theta}(r,\theta)S_\delta(r,\theta), \tag{A.4}$$

where $S_\delta(r,\theta)$ is of class C^6 and given by

$$S_\delta(r,\theta) = -\int_0^\theta \frac{1}{\ell_\delta^1(R_\delta,s)^2}\frac{\partial\ell_\delta^1}{\partial r}(R_\delta,s)\frac{\partial R_\delta}{\partial h}(I_\delta,s)\mathrm{d}s.$$

From (4.2) and (4.4) and (B.1), we can easily obtain

$$\limsup_{\delta\to 0+} \|S_\delta\|_{C^6([\tilde{a},\tilde{b}]\times[0,2\pi])} < +\infty, \quad \limsup_{\delta\to 0+} \|K_\delta\|_{C^7([\tilde{a},\tilde{b}]\times[0,2\pi])} < +\infty. \tag{A.5}$$

Finally, we define

$$\tau_\delta : \tilde{\mathbb{A}} \to \mathbb{R}, \quad \tau_\delta(r,\theta) = \omega_\delta(I_\delta(r,\theta))K_\delta(r,\theta).$$

This function is of class C^7 and satisfies

$$\tau_\delta(r,\theta+2\pi) = \tau_\delta(r,\theta) + 2\pi, \quad \text{for all } (r,\theta)\in\tilde{\mathbb{A}}, \tag{A.6}$$

and

$$\limsup_{\delta\to 0+} \|\tau_\delta\|_{C^7([\tilde{a},\tilde{b}]\times[0,2\pi])} < +\infty. \tag{A.7}$$

By using (4.5) and a direct computation, it can be easily checked that

$$\ell_\delta^1 \frac{\partial \tau_\delta}{\partial\theta} + \ell_\delta^2 \frac{\partial\tau_\delta}{\partial r} = \omega_\delta \circ I_\delta + [(\omega'_\delta \circ I_\delta)K_\delta + (\omega_\delta\circ I_\delta)S_\delta]\varrho_\delta. \tag{A.8}$$

If $\varrho_\delta \equiv 0$, then all the functions previously defined have an interpretation in terms of the differential equation

$$\begin{cases}\theta' = \ell_\delta^1(r,\theta),\\ r' = \ell_\delta^2(r,\theta).\end{cases}$$

For each h the equation $r = R_\delta(h,\theta)$ describes an orbit with period $T_\delta(h)$ and frequency $\omega_\delta(h)$. Given (r,θ), the quantity $K_\delta(r,\theta)$ is the time employed by an orbit to go from the horizontal axis $\theta = 0$ to the point (r,θ).

We can now define the mapping

$$\Psi_\delta : \tilde{\mathbb{A}} \to \mathbb{R}^2, \quad (r,\theta) \to (I_\delta(r,\theta), \tau_\delta(r,\theta)). \tag{A.9}$$

The periodicity of I_δ and (B.5) imply that Ψ_δ satisfies

$$\Psi_\delta(r,\theta+2\pi) = \Psi_\delta(r,\theta) + (0,2\pi), \quad \text{for all } (r,\theta)\in\tilde{\mathbb{A}}. \tag{A.10}$$

Therefore, Ψ_δ is the lift of a mapping $\bar{\Psi}_\delta : \tilde{A} \to \mathbb{R}\times S^1$.

From (4.5), we obtain $(I_\delta)_\theta = -(\ell_\delta^2/\ell_\delta^1)(I_\delta)_r + \varrho_\delta/\ell_\delta^1$, and combining this with (A.6), we get

$$\begin{aligned}\det \Psi'_\delta &= (I_\delta)_r(\tau_\delta)_\theta - (I_\delta)_\theta(\tau_\delta)_r\\ &= (\ell_\delta^1(\tau_\delta)_\theta + \ell_\delta^2(\tau_\delta)_r)\frac{(I_\delta)_r}{\ell_\delta^1} - \frac{\varrho_\delta}{\ell_\delta^1}(\tau_\delta)_r\\ &= \frac{\omega_\delta\circ I_\delta}{\ell_\delta^1}(I_\delta)_r + \frac{\varrho_\delta}{\ell_\delta^1}\{[(\omega'_\delta\circ I_\delta)K_\delta + (\omega_\delta\circ I_\delta)S_\delta](I_\delta)_r - (\tau_\delta)_r\},\end{aligned}$$

which, together with (4.2), (4.4), (4.6), (A.2), (A.3), (A.5) and (A.7), implies that

$$\liminf_{\delta\to 0+} \min_{(r,\theta)\in[\tilde{a},\tilde{b}]\times[0,2\pi]} \det\Psi'_\delta > 0, \quad \limsup_{\delta\to 0+} \|\det\Psi'_\delta\|_{C^5([\tilde{a},\tilde{b}]\times[0,2\pi])} < +\infty. \tag{A.11}$$

By using (A.5), (A.7) and (A.11) and an argument similar to that used in [18], we can show that Ψ_δ is a change of variables. More precisely, we have the following

Proposition A.1. *Let Ψ_δ be defined by (A.9). Then*

(i) *Ψ_δ is a C^5-diffeomorphism from $\tilde{\mathbb{A}}$ onto $\Psi_\delta(\tilde{\mathbb{A}})$;*

(ii) *the following inclusion holds:*

$$\mathbb{A}^* := [\bar{I}_\delta(a_1), \underline{I}_\delta(b_1)] \times \mathbb{R} \subset \Psi_\delta([a_1, b_1] \times \mathbb{R});$$

(iii) *Ψ_δ and Ψ_δ^{-1} can be expressed in the form*

$$\Psi_\delta : \begin{cases} \tau = \theta + \psi_1(r, \theta), \\ I = r + \psi_2(r, \theta), \end{cases} \qquad \Psi_\delta^{-1} : \begin{cases} \theta = \tau + \psi^1(I, \tau), \\ r = I + \psi^2(I, \tau), \end{cases}$$

with $\limsup_{\delta\to 0+} \|\psi_i\|_{C^5(\tilde{A})} < +\infty$ *and* $\limsup_{\delta\to 0+} \|\psi^i\|_{C^5(A^*)} < +\infty$.

2. *The New Mapping*. Define

$$\mathbb{A}_0 = \tilde{\mathbb{A}}, \quad \mathbb{A}_1 = \{(r, \theta) : \theta \in \mathbb{R}, a_1 \le r \le b_1\},$$
$$\mathbb{A}_2 = \mathbb{A}^* = \{(I, \tau) : \tau \in \mathbb{R}, \bar{I}_\delta(a_1) \le I \le \underline{I}_\delta(b_1)\}.$$

Then

$$\mathbb{A}_1 \subset \mathbb{A}_0, \quad \mathbb{A}_2 \subset \Psi_\delta(\mathbb{A}_1). \tag{A.12}$$

From now on, we set

$$L_1 = \limsup_{\delta\to 0+} \|\ell_\delta^1\|_{C(A)}, \qquad L_2 = \limsup_{\delta\to 0+} \|\ell_\delta^2\|_{C(A)},$$

and assume, without loss of generality, that

$$\|\varphi_\delta^1\|_{C^5(A)} + \|\varphi_\delta^2\|_{C^5(A)} < 1. \tag{A.13}$$

Choose $\Delta_1 > 0$ with

$$\Delta_1 \le \min\left\{\frac{a_1 - \tilde{a}}{2 + L_2}, \frac{\tilde{b} - b_1}{2 + L_2}\right\},$$

such that for $0 < \delta < \Delta_1$,

$$\|\ell_\delta^1\|_{C(A)} < L_1 + 1, \quad \|\ell_\delta^2\|_{C(A)} < L_2 + 1.$$

Notice that L_1, L_2 and Δ_1 are independent of δ. A computation based on (4.1) shows that if $0 < \delta < \Delta_1$, then

$$f(\mathbb{A}_1) \subset \mathbb{A}_0. \tag{A.14}$$

We can now define

$$g : \mathbb{A}_2 \to \mathbb{R} \times \mathbb{R}, \quad g = \Psi_\delta \circ f \circ \Psi_\delta^{-1}.$$

Then it is easily seen that g is the lift of a mapping $\bar{g} : A_2 \to \mathbb{R} \times S^1$. Moreover, we have the following observation:

Proposition A.2. *In the above setting, $\bar{g}$ has the intersection property in A_2. Moreover, if $\bar{\Sigma}$ is an invariant curve of $\bar{g}$, then $\bar{\Gamma} = \bar{\Psi}_\delta^{-1}(\bar{\Sigma})$ is an invariant curve of $\bar{f}$.*

By virtue of Proposition A.2, it suffices to show that $\bar{g}$ has an invariant curve in A_2. To this end, we express the mapping $\bar{g}$ in terms of the new variables (I, τ).

Proposition A.3. *Assume that (A.13) holds and $\delta < \Delta_1$. Let $g(I,\tau) = (I', \tau')$ be the mapping defined above. Then*

$$\begin{cases} \tau' = \tau + 2N\pi + \delta\omega_\delta(I) + \delta S_\delta^1(I,\tau), \\ I' = I + \delta S_\delta^2(I,\tau), \end{cases}$$

where S_δ^1, S_δ^2 belong to $C^5(A_2)$ and satisfy

$$\|S_\delta^1\|_{C^5(A_2)} + \|S_\delta^2\|_{C^5(A_2)} \le k[\|\varphi_\delta^1\|_{C^5(A)} + \|\varphi_\delta^2\|_{C^5(A)} + \|\varrho_\delta\|_{C^5(A)} + \delta].$$

Here k is a constant and is independent of δ and $\varphi_\delta^1, \varphi_\delta^2$.

By (4.6) and (A.13) and Proposition A.3, we can further restrict the size of Δ_1 in such a way that Δ_1 is still independent of δ and if $0 < \delta < \Delta_1$, then $\bar{g} \in M(A_2)$.

To prove Proposition A.3, we need a preliminary result. Given a function $F : \tilde{\mathbb{A}} \to \mathbb{R}$ we define

$$F^* : \mathbb{A}_1 \to \mathbb{R}, \quad F^*(r,\theta) = F(r', \theta'),$$

that is, $F^* = F \circ f$. If $\delta < \Delta_1$, then F^* is well defined.

Lemma A.4. *Assume that $F \in C^7(\tilde{A})$, $\|F\|_{C^7(\tilde{A})} \le H$ and $\delta < \Delta_1$. Then, for each $(r,\theta) \in \mathbb{A}_1$,*

$$F^*(r,\theta) = F(r,\theta) + \delta\dot{F}(r,\theta) + \delta R(r,\theta), \tag{A.15}$$

with

$$\dot{F}(r,\theta) = \frac{\partial F}{\partial\theta}(r,\theta)\ell_\delta^1(r,\theta) + \frac{\partial F}{\partial r}(r,\theta)\ell_\delta^2(r,\theta)$$

and

$$\|R\|_{C^5(A_1)} \le K(\|\varphi_\delta^1\|_{C^5(A)} + \|\varphi_\delta^2\|_{C^5(A)} + \delta),$$

where K depends on H, L_1, L_2. In particular, K is independent of δ.

Proof. We prove by induction that for each $j = 0, 1, 2, 3, 4, 5$ and $F \in C^{j+2}(\tilde{A})$ the identity (A.15) holds with

$$\|R\|_{C^j(A_1)} \le K_j(\|\varphi_\delta^1\|_{C^j(A)} + \|\varphi_\delta^2\|_{C^j(A)} + \delta), \tag{A.16}$$

where K_j depends on H, L_1, L_2.

Assume $j = 0$. By (4.1) and (A.13) and the mean value theorem, we obtain

$$\left| F^*(r,\theta) - F(r,\theta) - \frac{\partial F}{\partial\theta}(r,\theta)(\theta' - \theta - 2N\pi) - \frac{\partial F}{\partial r}(r,\theta)(r' - r) \right| \\ \le H[|\theta' - \theta - 2N\pi| + |r' - r|]^2 \le H(L_1 + L_2 + 4)^2\delta^2.$$

Combining this estimate with (4.1), we obtain

$$|F^* - F - \delta\dot{F}| \le \delta \left| \frac{\partial F}{\partial\theta}\varphi_\delta^1 \right| + \delta \left| \frac{\partial F}{\partial r}\varphi_\delta^2 \right| + H(L_1 + L_2 + 4)^2\delta^2.$$

Let $K_0 = H(L_1 + L_2 + 4)^2$, then the estimate (A.16) follows for $j = 0$.

We now prove that (A.16) is also valid for $j \geq 1$ when it holds for $0, 1, \ldots, j-1$. First we apply the induction hypothesis to $\partial F/\partial\theta$ and $\partial F/\partial r$ to obtain

$$(F_\theta)^* = F_\theta + \delta \dot{F}_\theta + \delta R_1, \quad (F_r)^* = F_r + \delta \dot{F}_r + \delta R_2$$

with

$$\|R_i\|_{C^{j-1}(A_1)} \leq K^*(\|\varphi_\delta^1\|_{C^{j-1}(A)} + \|\varphi_\delta^2\|_{C^{j-1}(A)} + \delta), \quad \text{for } i = 1, 2,$$

where K^* depends on H, L_1, L_2. Now,

$$\begin{aligned}
\frac{\partial F^*}{\partial\theta} &= \left(\frac{\partial F}{\partial\theta}\right)^* \frac{\partial\theta'}{\partial\theta} + \left(\frac{\partial F}{\partial r}\right)^* \frac{\partial r'}{\partial\theta} \\
&= \left(\frac{\partial F}{\partial\theta}\right)^* \left(1 + \delta\frac{\partial\ell_\delta^1}{\partial\theta} + \delta\frac{\partial\varphi_\delta^1}{\partial\theta}\right) + \left(\frac{\partial F}{\partial r}\right)^* \left(\delta\frac{\partial\ell_\delta^2}{\partial\theta} + \delta\frac{\partial\varphi_\delta^2}{\partial\theta}\right) \\
&= F_\theta + \delta\dot{F}_\theta + \delta\left\{F_\theta\frac{\partial\ell_\delta^1}{\partial\theta} + F_r\frac{\partial\ell_\delta^2}{\partial\theta}\right\} + \delta R_3 \\
&= F_\theta + \delta\frac{\partial}{\partial\theta}\{\dot{F}\} + \delta R_3,
\end{aligned}$$

with

$$\begin{aligned}
R_3 = {} & R_1\left(1 + \delta\frac{\partial\ell_\delta^1}{\partial\theta} + \delta\frac{\partial\varphi_\delta^1}{\partial\theta}\right) + F_\theta\frac{\partial\varphi_\delta^1}{\partial\theta} + \delta\dot{F}_\theta\left(\frac{\partial\ell_\delta^1}{\partial\theta} + \frac{\partial\varphi_\delta^1}{\partial\theta}\right) \\
& + F_r\frac{\partial\varphi_\delta^2}{\partial\theta} + \delta(\dot{F}_r + R_2)\left(\frac{\partial\ell_\delta^2}{\partial\theta} + \frac{\partial\varphi_\delta^2}{\partial\theta}\right).
\end{aligned}$$

The remainder R_3 satisfies an estimate of the form

$$\|R_3\|_{C^{j-1}(A_1)} \leq K_j(\|\varphi_\delta^1\|_{C^j(A)} + \|\varphi_\delta^2\|_{C^j(A)} + \delta),$$

where K_j depends on H, L_1, L_2. (Notice that we have used (A.13).)

In a similar way, we obtain

$$\frac{\partial F^*}{\partial r} = F_r + \delta\frac{\partial}{\partial r}\{\dot{F}\} + \delta R_4,$$

with R_4 satisfying the same estimate as R_3.

This completes the proof because the remainder in (A.15) satisfies

$$\frac{\partial R}{\partial\theta} = R_3, \quad \frac{\partial R}{\partial r} = R_4. \quad \square$$

Proof of Proposition A.3. Since

$$\limsup_{\delta\to 0+} \|\tau_\delta\|_{C^7([\tilde{a},\tilde{b}]\times[0,2\pi])} < +\infty, \quad \limsup_{\delta\to 0+} \|I_\delta\|_{C^7(A)} < +\infty,$$

for small δ, we can apply the previous lemma to the functions $\alpha_\delta(r,\theta) = \tau_\delta(r,\theta) - \theta$ and $I_\delta(r,\theta)$ and obtain, in $\mathbb{A}_1$,

$$\begin{cases} \tau_\delta^* = \tau_\delta + 2N\pi + \delta\dot{\tau}_\delta + \delta R_\delta^1, \\ I_\delta^* = I_\delta + \delta\dot{I}_\delta + \delta R_\delta^2, \end{cases}$$

where R^1_δ, R^2_δ satisfy

$$\|R^i_\delta\|_{C^5(A_1)} \le K_i(\|\varphi^1_\delta\|_{C^5(A)} + \|\varphi^2_\delta\|_{C^5(A)} + \delta), \tag{A.17}$$

for $i = 1, 2$, with K_i independent of δ.

From (4.5) and (A.8), $\dot{\tau}_\delta = \omega_\delta \circ I_\delta + [(\omega'_\delta \circ I_\delta)K_\delta + (\omega_\delta \circ I_\delta)S_\delta]\varrho_\delta$ and $\dot{I}_\delta = \varrho_\delta$. Thus

$$\begin{cases} \tau^*_\delta(r,\theta) = \tau_\delta(r,\theta) + 2N\pi + \delta\omega_\delta(I_\delta(r,\theta)) + \delta\tilde{S}^1_\delta(r,\theta), \\ I^*_\delta(r,\theta) = I_\delta(r,\theta) + \delta\tilde{S}^2_\delta(r,\theta), \end{cases} \tag{A.18}$$

where

$$\begin{cases} \tilde{S}^1_\delta(r,\theta) = R^1_\delta(r,\theta) + [\omega'_\delta(I_\delta(r,\theta))K_\delta(r,\theta) + \omega_\delta(I_\delta(r,\theta))S_\delta(r,\theta)]\varrho_\delta(r,\theta), \\ \tilde{S}^2_\delta(r,\theta) = R^2_\delta(r,\theta) + \varrho_\delta(r,\theta). \end{cases}$$

By (4.4), (A.3), (A.5) and (A.17), we obtain

$$\|\tilde{S}^i_\delta\|_{C^5(A_1)} \le \tilde{K}_i(\|\varphi^1_\delta\|_{C^5(A)} + \|\varphi^2_\delta\|_{C^5(A)} + \|\varrho_\delta\|_{C^5(A)} + \delta), \tag{A.19}$$

for $i = 1, 2$, with $\tilde{K}_i$ independent of δ.

The mapping $\Psi_\delta \circ f : \mathbb{A}_1 \to \mathbb{R} \times \mathbb{R}$ is given by $\Psi_\delta \circ f(r,\theta) = (I^*_\delta(r,\theta), \tau^*_\delta(r,\theta))$ and, denoting by (I,τ) the independent variables in $\mathbb{A}_2$, we have

$$(I',\tau') = g(I,\tau) = (I^*_\delta(\Psi^{-1}_\delta(I,\tau)), \tau^*_\delta(\Psi^{-1}_\delta(I,\tau))).$$

This identity, together with (A.18), allows us to obtain the expansion of the proposition with $S^i_\delta = \tilde{S}^i_\delta \circ \Psi^{-1}_\delta$. It is easy to see that

$$\|S^i_\delta\|_{C^5(A_2)} \le C\|\tilde{S}^i_\delta\|_{C^5(A_1)}\|\Psi^{-1}_\delta\|^5_{C^5(A_2)},$$

where C is a constant independent of δ. An application of Proposition A.1 ends the proof. □

3. *The Proof of* Theorem 4.1. To prove Theorem 4.1, we give a version of the small twist theorem, whose proof is similar to that of Theorem 3.6 in [18] and is omitted.

Proposition A.5. *Let $\bar{f} \in M(A)$ be a mapping with a lift f that can be written in the form*

$$\begin{cases} \theta' = \theta + 2N\pi + \delta\alpha_\delta(r) + \delta\phi^1_\delta(r,\theta), \\ r' = r + \delta\phi^2_\delta(r,\theta), \end{cases}$$

where $\delta \in (0,1)$ is a parameter and

$$\alpha_\delta \in C^5[a,b], \quad \limsup_{\delta\to 0+} \|\alpha_\delta\|_{C^5[a,b]} < +\infty, \quad \liminf_{\delta\to 0+} \min_{r\in[a,b]} \frac{\mathrm{d}\alpha_\delta}{\mathrm{d}r}(r) > 0,$$

$$\phi^1_\delta \in C^5[a,b], \quad \phi^2_\delta \in C^4[a,b].$$

Then there exist constants $\varepsilon > 0$ and $\Delta > 0$ such that if $\delta < \Delta$ and

$$\|\phi^1_\delta\|_{C^5(A)} + \|\phi^2_\delta\|_{C^4(A)} < \varepsilon,$$

the map $\bar{f}$ has an invariant curve.

Theorem 4.1 is now a consequence of Proposition A.5. We apply the latter to the mapping described in Proposition A.3 to find an invariant curve of $g = \Psi_\delta \circ f \circ \Psi^{-1}_\delta$ in A_2. According to [4], this curve has rotation number α_0 and can be expressed in the form $I = \psi_\delta(\tau)$, where

ψ_δ is a 2π-periodic function of class C^3 such that ψ'_δ is small. So $\Psi_\delta^{-1}(\psi_\delta(\tau), \tau)$ is an invariant curve of $\bar{f}$, which is defined implicitly by the equation

$$I_\delta(r, \theta) = \psi_\delta(\tau_\delta(r, \theta)).$$

It also has rotation number α_0, and can be explicitly described as $r = \mu(\theta)$ by the implicit function theorem and (4.4), where μ independent of δ. This proves the first remark after Theorem 4.1. $\square$

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