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Existence, Uniqueness and Asymptotic Stability of Traveling Wavefronts in A Non-Local Delayed Diffusion Equation

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In this paper, we study the existence, uniqueness, and global asymptotic stability of traveling wave fronts in a non-local reaction–diffusion model for a single species population with two age classes and a fixed maturation period living in a spatially unbounded environment. Under realistic assumptions on the birth function, we construct various pairs of super and sub solutions and utilize the comparison and squeezing technique to prove that the equation has exactly one non-decreasing traveling wavefront (up to a translation) which is monotonically increasing and globally asymptotic stable with phase shift.

KEY WORDS: Non-local reaction-diffusion equation; traveling wave front; existence; uniqueness; asymptotic stability; comparison principle.

AMS (1991) SUBJECT CLASSIFICATION: 34K30; 35B40; 35R10; 58D25.

1. INTRODUCTION

We consider the following system

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - dw + \int_{-\infty}^{\infty} b(w(t-r,y))f(x-y)dy. \tag{1.1}$$

This model describes the evolution of the adult population of a single species population with two age classes and moving around in a unbounded

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1-dimensional spatial domain. In this context, $D>0$ and $d>0$ denote the diffusion rate and death rate of the adult population, respectively, $r\geq 0$ is the maturation time for the species, b is related to the birth function, and the function $f\in C^{\infty}(\mathbb{R},\mathbb{R})$ satisfies

$$f(x)\geq 0, \qquad \int_{-\infty}^{\infty} f(y)dy=1 \quad \text{and} \quad \int_{-\infty}^{\infty} |y|f(y)dy<+\infty.$$

The main feature of the above model is the non-local interaction represented by the convolution (in space) term, and this is biologically realistic because the matured population at the current time t and spatial location x was born at time $t-r$ or earlier and might be at different spatial location. Therefore, the kernel function f describes the diffusion pattern of the immature population during the maturation process. We refer to Ref. [8] for more details and some specific forms of f, obtained from integration along characteristic equations of a structured population model and following the pioneering work of Smith and Thieme [6] (see also ref. [3] for a survey of the short history and the current status of the study of reaction diffusion equations with non-local delayed interactions). For related topics and some theoretical aspects of delayed reaction-diffusion equations, we refer the readers to [4,9].

In this paper, we always assume that the birth function $b\in C^1(\mathbb{R},\mathbb{R})$ and there exists a constant $K>0$ such that

$$b(0)=dK-b(K)=0.$$

Therefore (1.1) has at least two spatially homogeneous equilibria

$$w_1=0, \qquad w_2=K.$$

We are interested in the existence and other qualitative properties of a traveling wavefront $w(t,x)=U(x+ct)$ of (1.1), with U saturating at w_1 and w_2. Let $\xi=x+ct$, we then have the following equation for the profile U:

$$cU'(\xi)=DU''(\xi)-dU(\xi)+\int_{-\infty}^{\infty} b(U(\xi-cr-y))f(y)dy \tag{1.2}$$

subject to the boundary conditions

$$U(-\infty)=w_1, \qquad U(\infty)=w_2. \tag{1.3}$$

The so-called non-local monostable case (where there is no other zero of $du=b(u)$) on $[w_1,w_2]$ was addressed by So et al. [8] and Faria et al. [2]. Here, we consider the so-called non-local bistable case. Namely, let

$$u^+:=\sup\{u\in[0,K);\, du=b(u)\}, \qquad u^-:=\inf\{u\in(0,K];\, du=b(u)\}.$$

We assume the following conditions are satisfied:

(H1) $b'(\eta) \geq 0, \quad \text{for } \eta \in [0, K];$
(H2) $d > \max\{b'(0), b'(K)\};$
(H3) $u^* := u^+ = u^-$ and $b'(u^*) > d$.

A specific function which has been widely used in the mathematical biology literature given by $b(w) = pw^2e^{-\alpha w}$ with $p > 0$ and $\alpha > 0$ does satisfy the above conditions for a wide range of parameters p, α.

Denote by $[0, K]_C$ the set $\{\varphi \in C([-r, 0] \times \mathbb{R}, \mathbb{R}); \quad 0 \leq \varphi(s, x) \leq K, \; s \in [-r, 0], x \in \mathbb{R}\}$. We can now formulate our main result as follows:

Theorem 1.1. *Assume that* (H1)–(H3) *hold. Then* (1.1) *has exactly one traveling wavefront* $U(x + ct)$ *with* $0 \leq U \leq K$ *and* $|c| \leq C$ *for some positive constant* C*, which is independent of* r*. The unique traveling wavefront* $U(x + ct)$ *is strictly increasing with respect to* $\xi = x + ct$ *and globally asymptotically stable with phase shift in the sense that there exists* $\gamma > 0$ *such that for any* $\varphi \in [0, K]_C$ *with*

$$\liminf_{x \to +\infty} \min_{s \in [-r,0]} \varphi(s, x) > u^*, \qquad \limsup_{x \to -\infty} \max_{s \in [-r,0]} \varphi(s, x) < u^*$$

the solution $w(t, x, \varphi)$ *of* (1.1)*, with* $w(s, x, \phi) = \phi(s, x)$ *for* $s \in [-r, 0]$ *and* $x \in R$*, satisfies*

$$|w(t, x, \varphi) - U(x + ct + \xi_0)| \leq Me^{-\gamma t}, \quad t \geq 0, \;\; x \in \mathbb{R}$$

for some $M = M(\varphi) > 0$ *and* $\xi_0 = \xi_0(\varphi) \in \mathbb{R}$.

We should remark that time delay does implicitly play a significant role in the dynamical behaviors of system (1.1), as the function b involves the size of this time lag. In terms of modeling the dynamics of adult population, the function b should be the birth rate times e^{-rd_i} when the death rate of the immature population is assumed to be a constant $d_i > 0$, and this factor must be used to account for the survival probability of a new born during its maturation phase. Theorem 1.1 seems to remain true when we replace the linear death rate dw by a strictly increasing function (and of course some relevant technical conditions), but it is not so clear if the arguments developed in the main body of this paper can be generalized to a more general class of scalar delay differential equations with non-local interaction, mainly due to the technical details of the construction of pairs of super and sub solutions.

The rest of this paper is organized as follows. In Section 2, we establish a comparison result and then prove the uniqueness of a traveling wavefront. The asymptotic stability of the unique traveling wavefront is obtained in Section 3 and the existence is given in Section 4. We prove the asymptotic stability of the traveling wavefront by using the comparison and squeezing technique, this trick was used previously in [1,7]. Some results needed in the existence proof can be obtained by using arguments very much similar to those in [1], and these results are summarized in the appendix, with sketched proofs that we believe are needed to fill out some gaps in the arguments of Chen [1].

2. UNIQUENESS OF TRAVELING WAVEFRONTS

We start with the following initial value problem

$$\begin{aligned}\frac{\partial w(t,x)}{\partial t} &= D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x)\\ &\quad + \int_{-\infty}^{\infty} b(w(t-r,y))f(x-y)dy, \quad t>0, \ x\in\mathbb{R},\\ w(s,x) &= \phi(s,x), \quad s\in[-r,0], \ x\in\mathbb{R}.\end{aligned} \tag{2.1}$$

Let $X=\mathrm{BUC}(\mathbb{R},\mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|_X$, and let $X^+=\{\phi\in X:\phi(x)\geq 0, \forall x\in\mathbb{R}\}$. It is easy to see that X^+ is a closed cone of X and X is a Banach lattice under the partial ordering induced by X^+.

The heat equation

$$\begin{aligned}\frac{\partial w(t,x)}{\partial t} &= D\frac{\partial^2 w(t,x)}{\partial x^2}, \quad t>0, \ x\in\mathbb{R},\\ w(0,x) &= \phi(x), \quad x\in\mathbb{R}\end{aligned} \tag{2.2}$$

has the solution

$$\begin{aligned}&T(t)\phi(x)\\ &= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4Dt}\right)\phi(y)dy, \quad t>0, \ x\in\mathbb{R}, \ \phi\in X\end{aligned} \tag{2.3}$$

and $T(t): X\to X$ is an analytic semigroup on X with $T(t)X^+\subset X^+$ for all $t\geq 0$.

Let $C=C([-r,0],X)$ be the Banach space of continuous functions from $[-r,0]$ into X with the supremum norm $\|\cdot\|$ and let $C^+=\{\phi\in C: \phi(s)\in X^+, \forall s\in[-r,0]\}$. Then C^+ is a closed cone of C. As usual, we identify an element $\phi\in C$ as a function from $[-r,0]\times\mathbb{R}$ into $\mathbb{R}$ defined by $\phi(s,x)=\phi(s)(x)$. For any continuous function $y:[-r,b)\to X$, where $b>0$, we define $y_t\in C, t\in[0,b)$, by $y_t(s)=y(t+s), s\in[-r,0]$. Then $t\mapsto y_t$ is a continuous function from $[0,b)$ to C.

Under assumptions (H1) and (H2), we can choose a positive constant $\delta_0 > 0$ such that

$$du < b(u), \quad \text{for } u \in [-\delta_0, 0) \tag{2.4}$$

and

$$du > b(u), \quad \text{for } u \in (K, K + \delta_0]. \tag{2.5}$$

We also assume

$$b'(u) \geq 0, \quad \text{for } u \in [-\delta_0, K + \delta_0]. \tag{2.6}$$

By (H1), this can be achieved by modifying (if necessary) the definition of b outside the closed interval $[-\delta_0, K+\delta_0]$ to a new C^1-smooth function and apply our results below to the new function b.

For any $\phi \in [-\delta_0, K + \delta_0]_C = \{\phi \in C; \ \phi(s,x) \in [-\delta_0, K + \delta_0], \ s \in [-r, 0], x \in \mathbb{R}\}$, define

$$F(\phi)(x) = -d\phi(0,x) + \int_{-\infty}^{\infty} b(\phi(-r,y)) f(x-y)dy, \quad x \in \mathbb{R}.$$

Then $F(\phi) \in X$ and $F\colon [-\delta_0, K+\delta_0]_C \to X$ is globally Lipschitz continuous.

Definition 2.1. A continuous function $\upsilon\colon [-r, b) \to X, b > 0$, is called a supersolution (subsolution) of (2.1) on $[0, b)$ if

$$\upsilon(t) \geq (\leq) T(t-t_0)\upsilon(t_0) + \int_{t_0}^{t} T(t-s)F(\upsilon_s)ds \tag{2.7}$$

for all $b > t > t_0 \geq 0$. If υ is both a supersolution and a subsolution on $[0, b)$, then it is said to be a mild solution of (2.1).

Remark 2.2. Assume that there is a bounded and continuous υ: $[-r, b) \times \mathbb{R} \to \mathbb{R}$, with $b > 0$ and such that υ is C^2 in $x \in \mathbb{R}$, C^1 in $t \in (0, b)$, and

$$\begin{aligned}\frac{\partial \upsilon(t,x)}{\partial t} \geq (\leq) D \frac{\partial^2 \upsilon(t,x)}{\partial x^2} - d\upsilon(t,x) \\ + \int_{-\infty}^{\infty} b(\upsilon(t-r,y)) f(x-y)dy, \quad t \in (0,b), \ \ x \in \mathbb{R}.\end{aligned}$$

Then, by the fact that $T(t)X^+ \subset X^+$, it follows that (2.7) holds, and hence $\upsilon(t,x)$ is a supersolution (subsolution) of (2.1) on $[0, b)$.

Define

$$\Theta(J,t)=\frac{1}{\sqrt{4\pi Dt}}\exp\left(-dt-\frac{(J+1)^2}{4Dt}\right),\quad J\geq 0,\quad t>0.$$

Clearly, $\Theta\in C([0,\infty)\times(0,\infty),\mathbb{R})$.

We first establish the following existence and comparison result.

Lemma 2.3. *For any $\phi\in[-\delta_0,K+\delta_0]_C$,* (2.1) *has a unique mild solution $w(t,x,\phi)$ on $[0,+\infty)$ with $-\delta_0\leq w(t,x)\leq K+\delta_0$ for $(t,x)\in[-r,+\infty)\times\mathbb{R}$ and $w(t,x,\phi)$ is a classical solution to* (1.1) *for $(t,x)\in(r,+\infty)\times\mathbb{R}$. Furthermore, for any pair of supersolution $w^+(t,x)$ and subsolution $w^-(t,x)$ of (1.1) on $[0,+\infty)$ with $-\delta_0\leq w^+(t,x), w^-(t,x)\leq K+\delta_0$ for $t\in[-r,+\infty)$ and $x\in\mathbb{R}$, and $w^+(s,x)\geq w^-(s,x)$ for $s\in[-r,0]$ and $x\in\mathbb{R}$, there holds $w^+(t,x)\geq w^-(t,x)$ for $t\geq 0, x\in\mathbb{R}$, and*

$$w^+(t,x)-w^-(t,x)\geq\Theta(|x-z|,t-t_0)\int_z^{z+1}[w^+(t_0,y)-w^-(t_0,y)]dy \quad (2.8)$$

for every $z\in\mathbb{R}$ and $t>t_0\geq 0$.

Proof. Under the abstract setting [5], a mild solution of (2.1) is a solution to its associated integral equation

$$\begin{aligned} w(t) &= T(t)\phi(0)+\textstyle\int_0^t T(t-s)F(w_s)ds, \quad t>0,\\ w_0 &= \phi\in[-\delta_0,K+\delta_0]_C. \end{aligned} \quad (2.9)$$

By the choice of δ_0 in (2.4) and (2.5), it is easy to see that $\upsilon^+(t)=K+\delta_0$ and $\upsilon^-(t)=-\delta_0$ are an ordered pair of super and subsolutions of (2.1) on $[0,\infty)$. Since $b\in C^1(\mathbb{R},\mathbb{R})$, it can be easily checked that $F:[-\delta_0,K+\delta_0]_C\to X$ is globally Lipschitz continuous. We further claim that F is quasi-monotonic on $[-\delta_0,K+\delta_0]_C$ in the sense that

$$\lim_{h\to 0+}\frac{1}{h}\text{dist}\left(\phi(0)-\psi(0)+h[F(\phi)-F(\psi)];\,X^+\right)=0 \quad (2.10)$$

for all $\phi,\psi\in[-\delta_0,K+\delta_0]_C$ with $\phi\geq\psi$. In fact, it follows from (2.6) that

$$\begin{aligned} F(\phi)(x)-F(\psi)(x) &= -d[\phi(0,x)-\psi(0,x)]\\ &\quad+\textstyle\int_{-\infty}^{\infty}[b(\phi(-r,y)-\psi(-r,y)]f(x-y)dy \\ &\geq -d[\phi(0,x)-\psi(0,x)] \end{aligned} \quad (2.11)$$

and hence, for any $h>0$ with $hd<1$,

$$\phi(0)-\psi(0)+h[F(\phi)-F(\psi)]\geq(1-hd)(\phi(0)-\psi(0))\geq 0$$

from which (2.10) follows. Therefore, the existence and uniqueness of $w(t,x,\phi)$ follows from Corollary 5 in [5] with $S(t,s)=T(t,s)=T(t-s)$

for $t \geq s \geq 0$ and $B(t,\phi) = F(\phi)$. Moreover, a semigroup theory argument given in the proof of Theorem 1 in [5] yields that $w(t,x,\phi)$ is a classical solution for $t > r$.

Since $w^+, w^- \in [-\delta_0, K+\delta_0]_C$ and $w^+ \geq w^-$, it follows from Corollary 5 in [5] that

$$-\delta_0 \leq w(t,x,w^-) \leq w(t,x,w^+) \leq K+\delta_0, \qquad t \geq 0, \quad x \in \mathbb{R}. \quad (2.12)$$

By applying Corollary 5 in [5] with [$v^+(t,x) = K+\delta_0$ and $v^-(t,x) = w^-(t,x)$], [$v^+(t,x) = w^+(t,x)$ and $v^-(t,x) = -\delta_0$], respectively, we get

$$w^-(t,x) \leq w(t,x,w^-) \leq K+\delta_0, \quad t \geq 0, \ x \in \mathbb{R} \quad (2.13)$$

and

$$-\delta_0 \leq w(t,x,w^+) \leq w^+(t,x), \quad t \geq 0, \ x \in \mathbb{R}. \quad (2.14)$$

Combining (2.12)–(2.14), we have $w^-(t,x) \leq w^+(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$.

It remains to prove the last inequality in the lemma. Let $v(t,x) = w^+(t,x) - w^-(t,x)$, $t \in [-r,\infty), x \in \mathbb{R}$. Then we have that $v(t,x) \geq 0$, $t \in [-r,\infty), x \in \mathbb{R}$ and $w_t^+, w_t^- \in [-\delta_0, K+\delta_0]_C$ with $w_t^+ \geq w_t^-$ in C for all $t \geq 0$. For any given $t_0 \geq 0$, by Definition 2.1 (2.11) and the fact that $T(t)X^+ \subset X^+$ for $t \geq 0$ that, for all $t \geq t_0$,

$$\begin{aligned} v(t) &\geq T(t-t_0)v(t_0) + \textstyle\int_{t_0}^t T(t-s)[F(w_s^+) - F(w_s^-)]ds \\ &\geq T(t-t_0)v(t_0) - d\textstyle\int_{t_0}^t T(t-s)v(s)ds. \end{aligned} \quad (2.15)$$

Let

$$z(t) = e^{-d(t-t_0)}T(t-t_0)v(t_0), \quad t \geq t_0.$$

Then $z(t)$ satisfies

$$z(t) = T(t-t_0)z(t_0) - d\int_{t_0}^t T(t-s)z(s)ds, \quad t \geq t_0. \quad (2.16)$$

By Proposition 3 in [5] with $v^-(t) = z(t)$, $v^+(t) = +\infty$, $S(t,\phi) = B^-(t,\phi) = -d\phi(0)$, we get $v(t) \geq z(t)$ for all $t \geq t_0$. Thus it follows that

$$w^+(t) - w^-(t) \geq e^{-d(t-t_0)}T(t-t_0)(w^+(t_0) - w^-(t_0)), \quad t \geq t_0. \quad (2.17)$$

Combining (2.3), (2.17) and the definition of $\Theta \in C([0,\infty)\times(0,\infty),\mathbb{R})$, we have that for all $t > t_0 \geq 0$ and $x \in \mathbb{R}$,

$$w^+(t,x) - w^-(t,x) \geq \Theta(|x-z|, t-t_0)\int_z^{z+1}[w^+(t_0,y) - w^-(t_0,y)]dy.$$

The proof is complete. □

Remark 2.4. By virtue of Lemma 2.3, it follows that if $w^+(t,x)$ and $w^-(t,x)$ are the pair of supersolution and subsolution of (1.1) given in Lemma 2.3 and $w^+(0,x) \not\equiv w^-(0,x)$, then for any $t>0$,

$$w^+(t,x) - w^-(t,x) \geq \max_{z\in\mathbb{R}} \Theta(|x-z|,t) \int_z^{z+1} [w^+(0,y) - w^-(0,y)]dy > 0.$$

In particular, if $v(t,x,\phi)$ is a solution of (1.1) with the initial data $\phi \in [-\delta_0, K+\delta_0]_C$ and $\phi(0,x)$ ($\not\equiv$ const.) is a non-decreasing function on $\mathbb{R}$, then for any fixed $t>0$, $v(t,x)$ is strictly increasing in $x\in\mathbb{R}$.

Lemma 2.5. *Assume that (H1) and (H2) hold. Let $U(x+ct)$ be a non-decreasing traveling wavefront of (1.1). Then*

$$0 < U'(\xi) \leq \frac{b(K)}{2\sqrt{Dd}}, \quad \text{for all } \xi\in\mathbb{R} \tag{2.18}$$

and

$$\lim_{|\xi|\to\infty} U'(\xi) = 0. \tag{2.19}$$

Proof. By Lemma 2.3, we have that for $\xi = x+ct$ and every $h>0$,

$$U(\xi+h) - U(\xi) \geq \max_{z\in\mathbb{R}} \Theta(|x-z|,t) \int_z^{z+1} [U(y+h) - U(y)]dy > 0$$

from which, it follows that

$$\begin{aligned} U'(\xi) &\geq \max_{z\in\mathbb{R}} \Theta(|x-z|,t) \\ \int_z^{z+1} U'(y)dy &= \max_{z\in\mathbb{R}} \Theta(|x-z|,t)[U(z+1) - U(z)] > 0. \end{aligned} \tag{2.20}$$

Let

$$\lambda_1 = \frac{c-\sqrt{c^2+4Dd}}{2D} < 0, \qquad \lambda_2 = \frac{c+\sqrt{c^2+4Dd}}{2D} > 0.$$

Then, it follows from (1.2) that

$$U(\xi) = \frac{1}{D(\lambda_2-\lambda_1)} \left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)} H(U)(s)ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)} H(U)(s)ds \right]$$

and hence,

$$U'(\xi) = \frac{1}{D(\lambda_2 - \lambda_1)} \times \left[\lambda_1 \int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} H(U)(s)ds + \lambda_2 \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)} H(U)(s)ds\right], \tag{2.21}$$

where

$$H(U)(s) = \int_{-\infty}^{\infty} b(U(s - cr - y)) f(y)dy. \tag{2.22}$$

Since $\lambda_2 - \lambda_1 \ge 2\sqrt{\frac{d}{D}}$, it follows from (2.21) and (2.22) that

$$U'(\xi) \le \frac{\lambda_2 b(K)}{2\sqrt{Dd}} \int_{\xi}^{\infty} e^{\lambda_2(\xi - s)} ds = \frac{b(K)}{2\sqrt{Dd}},$$

which together with (2.20) yields (2.18). Finally, (2.19) follows from (1.3), (2.21), (2.22), and the dominant convergence theorem. This completes the proof. □

Lemma 2.6. *Assume that* (H1) *and* (H2) *hold. Let* $U(x+ct)$ *be a non-decreasing traveling wavefront of* (1.1)*. Then there exist three positive numbers* β_0 *(which is independent of* U*),* σ_0 *and* $\bar{\delta}$ *such that for any* $\delta \in (0, \bar{\delta}]$ *and every* $\hat{\xi} \in \mathbb{R}$*, the function* w^+ *and* w^- *defined by*

$$w^{\pm}(t, x) := U(x + ct + \hat{\xi} \pm \sigma_0 \delta (e^{\beta_0 r} - e^{-\beta_0 t})) \pm \delta e^{-\beta_0 t}$$

are a supersolution and a subsolution of (1.1) *on* $[0, +\infty)$*, respectively.*

Proof. By (H2), we can choose a $\beta_0 > 0$ and an $\epsilon^* > 0$ such that

$$d > \beta_0 + e^{\beta_0 r}(\max\{b'(0), b'(K)\} + \epsilon^*). \tag{2.23}$$

By (2.6), there exists a $\delta^* > 0$ such that

$$0 \le b'(\eta) \le b'(0) + \epsilon^*, \quad \text{for all } \eta \in [-\delta^*, \delta^*], \tag{2.24}$$

$$0 \le b'(\eta) \le b'(K) + \epsilon^*, \quad \text{for all } \eta \in [K - \delta^*, K + \delta^*]. \tag{2.25}$$

Let $c_0 = |c|r + (e^{\beta_0 r} - 1)$. Since $\lim_{\xi \to +\infty} U(\xi) = K$ and $\lim_{\xi \to -\infty} U(\xi) = 0$, there exists a constant $M_0 = M_0(U, \beta_0, \epsilon^*, \delta^*) > 0$ such that

$$U(\xi) \le \delta^*, \quad \text{for all } \xi \le -M_0/2 + c_0, \tag{2.26}$$

$$U(\xi) \ge K - \delta^*, \quad \text{for all } \xi \ge M_0/2 - c_0, \tag{2.27}$$

and

$$d > \beta_0 + e^{\beta_0 r}(\max\{b'(0), b'(K)\} + \epsilon^*) \\ + e^{\beta_0 r} b'_{\max}\left[\int_{\frac{1}{2}M_0}^{\infty} + \int_{-\infty}^{-\frac{1}{2}M_0} f(y)dy\right]. \tag{2.28}$$

By virtue of Lemma 2.5, we have $m_0 := m_0(U, \beta_0, \epsilon^*, \delta^*) = \min\{U'(\xi); |\xi| \leq M_0\} > 0$. Define

$$\sigma_0 := \frac{1}{\beta_0 m_0}[(e^{\beta_0 r} b'_{\max} - d) + \beta_0] > 0 \tag{2.29}$$

and

$$\bar{\delta} = \min\left\{\frac{1}{\sigma_0}, \delta^* e^{-\beta_0 r}\right\}.$$

We only prove $w^+(t,x)$ is a supersolution of (1.1). The proof for $w^-(t,x)$ is analogous and is omitted. By a translation, we can assume that $\hat{\xi} = 0$. For any given $\delta \in (0, \bar{\delta}]$, let $\xi(t) = x + ct + \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 t})$. Then for any $t \geq 0$, we have

$$\begin{aligned}
S(w^+)(t,x) :=\ & \frac{\partial w^+(t,x)}{\partial t} - D\frac{\partial^2 w^+(t,x)}{\partial t} + dw^+(t,x) \\
& - \int_{-\infty}^{\infty} b(w^+(t-r, x-y)) f(y)dy \\
=\ & U'(\xi(t))(c + \sigma\delta\beta_0 e^{-\beta_0 t}) - \beta_0\delta e^{-\beta_0 t} \\
& - DU''(\xi(t)) + dU(\xi(t)) + d\delta e^{-\beta_0 t} \\
& - \int_{-\infty}^{\infty} b\{U[\xi(t) - cr - y + \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(t-r)}) - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 t})] \\
& + \delta e^{-\beta_0(t-r)}\} \cdot f(y)dy \\
=\ & \sigma_0\delta\beta_0 U'(\xi(t))e^{-\beta_0 t} - \beta_0\delta e^{-\beta_0 t} + d\delta e^{-\beta_0 t} \\
& - \int_{-\infty}^{\infty} b\{U[\xi(t) - cr - y + \sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t}] + \delta e^{-\beta_0(t-r)}\} \cdot f(y)dy \\
& + \int_{-\infty}^{\infty} b\{U[\xi(t) - cr - y]\} \cdot f(y)dy \\
=\ & [\sigma_0\delta\beta_0 U'(\xi(t)) - \beta_0\delta + d_m\delta]e^{-\beta_0 t} \\
& - \int_{-\infty}^{\infty} b'(\tilde{\eta})\{U[\xi(t) - cr - y + \sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t}] + \delta e^{-\beta_0(t-r)}
\end{aligned}$$

$$
\begin{aligned}
&-U[\xi(t)-cr-y]\}\cdot f(y)dy\\
&= [\sigma_0\delta\beta_0 U'(\xi(t))-\beta_0\delta+d\delta]e^{-\beta_0 t}\\
&\quad -\int_{-\infty}^{\infty} b'(\tilde{\eta})[U'(\tilde{\xi})\sigma_0\delta(1-e^{\beta_0 r})e^{-\beta_0 t}+\delta e^{-\beta_0(t-r)}]\cdot f(y)dy\\
&= \{\sigma_0\beta_0 U'(\xi(t))-\beta_0+d\\
&\quad +\int_{-\infty}^{\infty} b'(\tilde{\eta})[U'(\tilde{\xi})\sigma_0(e^{\beta_0 r}-1)-e^{\beta_0 r}]\cdot f(y)dy\}\delta e^{-\beta_0 t},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\xi} &= \xi(t)-cr-y+\theta\sigma_0(1-e^{\beta_0 r})e^{-\beta_0 t},\\
\tilde{\eta} &= \theta U[\xi(t)-cr-y+\sigma_0\delta(1-e^{\beta_0 r})e^{-\beta_0 t}]\\
&\quad +\theta\delta e^{-\beta_0(t-r)}+(1-\theta)U[\xi(t)-cr-y].
\end{aligned}
$$

Clearly, $0\le\tilde{\eta}\le K+\delta e^{\beta_0 r}\le K+\delta^*$. Therefore, $b'(\tilde{\eta})\ge 0$, and hence,

$$
S(w^+)(t,x)\ge\left\{\sigma_0\beta_0 U'(\xi(t))-\beta_0+d-e^{\beta_0 r}\int_{-\infty}^{\infty} b'(\tilde{\eta})f(y)dy\right\}\delta e^{-\beta_0 t}. \tag{2.30}
$$

We distinguish among three cases.

Case (i): $|\xi(t)|\le M_0$. In this case, by (2.29) and (2.30), we have

$$
\begin{aligned}
S(w^+)(t,x) &\ge \{\sigma_0\beta_0 U'(\xi(t))-\beta_0+d-e^{\beta_0 r}\int_{-\infty}^{\infty} b'(\tilde{\eta})f(y)dy\}\delta e^{-\beta_0 t}\\
&\ge \{\sigma_0\beta_0 m_0-\beta_0+d-e^{\beta_0 r}b'_{\max}\}\delta e^{-\beta_0 t}\\
&\ge 0.
\end{aligned}
$$

Case (ii): $\xi(t)\ge M_0$. For $y\in[-\frac{1}{2}\xi(t),\frac{1}{2}\xi(t)]$, we have

$$
\frac{1}{2}M_0\le\frac{1}{2}\xi(t)\le\xi(t)-y\le\frac{3}{2}\xi(t).
$$

By the choice of $\bar{\delta}$, for any $\delta\in(0,\bar{\delta}]$, we have $\sigma_0\delta\le 1$, and hence,

$$
\begin{aligned}
&\xi(t)-y-cr+\sigma_0\delta(1-e^{\beta_0 r})e^{-\beta_0 t}\\
&\quad\ge\frac{1}{2}M_0-cr+\sigma_0\delta(1-e^{\beta_0 r})\ge\frac{1}{2}M_0-c_0,
\end{aligned}
$$

and

$$
\xi(t)-y-cr\ge\frac{1}{2}M_0-cr\ge\frac{1}{2}M_0-c_0.
$$

Therefore, it follows from (2.25) and (2.27) that

$$
K+\delta^*\ge K+\delta e^{\beta_0 r}\ge\tilde{\eta}\ge K-\delta^*
$$

and

$$b'(\tilde{\eta}) \le b'(K) + \epsilon^*.$$

Hence, by (2.28) and (2.30), we have

$$\begin{aligned} S(w^+)(t,x) &\ge \{\sigma_0\beta_0 U'(\xi(t)) - \beta_0 + d - e^{\beta_0 r}\int_{-\infty}^{\infty} b'(\tilde{\eta}) f(y)dy\}\delta e^{-\beta_0 t} \\ &\ge \{-\beta_0 + d - e^{\beta_0 r}\int_{-\frac{1}{2}\xi(t)}^{\frac{1}{2}\xi(t)} b'(\tilde{\eta}) f(y)dy \\ &\quad - e^{\beta_0 r}\int_{\frac{1}{2}\xi(t)}^{\infty} b'(\tilde{\eta}) f(y)dy - e^{\beta_0 r}\int_{-\infty}^{-\frac{1}{2}\xi(t)} b'(\tilde{\eta}) f(y)dy\}\delta e^{-\beta_0 t} \\ &\ge \{-\beta_0 + d - e^{\beta_0 r}(b'(K) + \epsilon^*) \\ &\quad - e^{\beta_0 r} b'_{\max}[\int_{\frac{1}{2}M_0}^{\infty} + \int_{-\infty}^{-\frac{1}{2}M_0} f(y)dy]\}\delta e^{-\beta_0 t} \\ &\ge 0. \end{aligned}$$

Case (iii): $\xi(t) \le -M_0$. The proof for this case similar to that for Case (ii) and thus is omitted.

The proof of Lemma 2.4 is complete. □

Theorem 2.7. *Assume that* (H1) *and* (H2) *hold. Also assume that* (1.1) *has a non-decreasing traveling wavefront* $U(x+ct)$*, then for any traveling wavefront* $\tilde{U}(x+\tilde{c}t)$ *with* $0 \le \tilde{U} \le K$*, we have* $\tilde{c} = c$ *and* $\tilde{U}(\cdot) = U(\cdot + \xi_0)$ *for some* $\xi_0 \in \mathbb{R}$.

Proof. Since $\tilde{U}(\xi)$ and $U(\xi)$ have the same limits as $\xi \to \pm\infty$, there exist $\tilde{\xi} \in \mathbb{R}$ and a sufficiently large number $h > 0$ such that for every $s \in [-r, 0]$ and $x \in \mathbb{R}$,

$$U(x + cs + \tilde{\xi}) - \bar{\delta} < \tilde{U}(x + \tilde{c}s) < U(x + cs + \tilde{\xi} + h) + \bar{\delta}$$

and hence,

$$\begin{aligned} &U(x + cs + \tilde{\xi} - \sigma_0\bar{\delta}(e^{\beta_0 r} - e^{-\beta_0 s})) - \bar{\delta}e^{-\beta_0 s} \\ &\quad < \tilde{U}(x + \tilde{c}s) < U(x + cs + \tilde{\xi} + h + \sigma_0\bar{\delta}(e^{\beta_0 r} - e^{-\beta_0 s})) + \bar{\delta}e^{-\beta_0 s}, \end{aligned}$$

where β_0, σ_0 and $\bar{\delta}$ are given in Lemma 2.6. By the comparison, we obtain that for all $t \ge 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} &U(x + ct + \tilde{\xi} - \sigma_0\bar{\delta}(e^{\beta_0 r} - e^{-\beta_0 t})) - \bar{\delta}e^{-\beta_0 t} \\ &\quad < \tilde{U}(x + \tilde{c}t) < U(x + ct + \tilde{\xi} + h + \sigma_0\bar{\delta}(e^{\beta_0 r} - e^{-\beta_0 t})) + \bar{\delta}e^{-\beta_0 t}. \end{aligned}$$

Keeping $\xi = x + ct$ fixed and letting $t \to \infty$, we then obtain from the first inequality that $c \le \tilde{c}$ and from the second inequality that $c \ge \tilde{c}$, so that $\tilde{c} = c$. In addition,

$$U(\xi + \tilde{\xi} - \sigma_0 \bar{\delta} e^{\beta_0 r}) \le \tilde{U}(\xi) \le U(\xi + \tilde{\xi} + h + \sigma_0 \bar{\delta} e^{\beta_0 r}) \quad \text{for } \xi \in \mathbb{R}. \quad (2.31)$$

Define

$$\xi^* := \inf\{\xi; \tilde{U}(\cdot) \le U(\cdot + \xi)\}, \qquad \xi_* := \sup\{\xi; \tilde{U}(\cdot) \ge U(\cdot + \xi)\}.$$

Then from (2.31), both ξ^* and ξ_* are well defined. Since $U(\cdot + \xi_*) \le \tilde{U}(\cdot) \le U(\cdot + \xi^*)$, we have $\xi_* \le \xi^*$.

To complete the proof, it suffices to show that $\xi_* = \xi^*$. By way of contradiction, assume that $\xi_* < \xi^*$ and $\tilde{U}(\cdot) \not\equiv U(\cdot + \xi^*)$. Since $\lim_{|\xi| \to \infty} U'(\xi) = 0$, there exists a large positive constant $\hat{M} = \hat{M}(U) > 0$ such that

$$2\sigma_0 e^{\beta_0 r} U'(\xi) \le 1, \quad \text{if } |\xi| \ge \hat{M}. \quad (2.32)$$

Note that $\tilde{U}(\cdot) \le U(\cdot + \xi^*)$ and $\tilde{U}(\cdot) \not\equiv U(\cdot + \xi^*)$, by Lemma 2.3 and Remark 2.4, it follows that $\tilde{U}(\cdot) < U(\cdot + \xi^*)$ on $\mathbb{R}$. Consequently, by the continuity of U and $\tilde{U}$, there exists a small constant $\hat{h} \in (0, \bar{\delta}]$ with $\hat{h} \le \frac{1}{2\sigma_0} e^{-\beta_0 r}$, such that

$$\tilde{U}(\xi) < U(\xi + \xi^* - 2\sigma_0 e^{\beta_0 r} \hat{h}), \quad \text{if } \xi \in [-\hat{M} - 1 - \xi^*, \hat{M} + 1 - \xi^*]. \quad (2.33)$$

When $|\xi + \xi^*| \ge \hat{M} + 1$, we have

$$\begin{aligned} U(\xi + \xi^* - 2\sigma_0 e^{\beta_0 r} \hat{h}) - \tilde{U}(\xi) &> U(\xi + \xi^* - 2\sigma_0 e^{\beta_0 r} \hat{h}) - U(\xi + \xi^*) \\ &= -2\sigma_0 e^{\beta_0 r} \hat{h} U'(\xi + \xi^* - 2\theta \sigma_0 e^{\beta_0 r} \hat{h}) \ge -\hat{h}, \quad (2.34) \end{aligned}$$

which, together with (2.33), implies that for any $s \in [-r, 0]$ and $x \in \mathbb{R}$,

$$U(x + cs + \xi^* - 2\sigma_0 e^{\beta_0 r} \hat{h} + \sigma_0 \hat{h}(e^{\beta_0 r} - e^{-\beta_0 s})) + \hat{h} e^{-\beta_0 s} \ge \tilde{U}(x + cs).$$

Therefore, the comparison implies that for any $t \ge 0$ and $x \in \mathbb{R}$,

$$U(x + ct + \xi^* - 2\sigma_0 e^{\beta_0 r} \hat{h} + \sigma_0 \hat{h}(e^{\beta_0 r} - e^{-\beta_0 t})) + \hat{h} e^{-\beta_0 t} \ge \tilde{U}(x + ct). (2.35)$$

In (2.35), keeping $\xi = x + ct$ fixed and letting $t \to \infty$, we obtain $U(\xi + \xi^* - \sigma_0 e^{\beta_0 r} \hat{h}) \ge \tilde{U}(\xi)$ for all $\xi \in \mathbb{R}$. This contradicts the definition of ξ^*. Hence, $\xi_* = \xi^*$ and this completes the proof. □

3. ASYMPTOTIC STABILITY OF TRAVELING WAVEFRONTS

Let $\zeta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a fixed function with the following properties:

$$\zeta(s) = 0, \quad \text{if } s \leq -2; \qquad \zeta(s) = 1, \quad \text{if } s \geq 2;$$

$$0 < \zeta'(s) < 1; \qquad |\zeta''(s)| \leq 1, \quad \text{if } s \in (-2, 2).$$

Then we have the following result.

Lemma 3.1. *Assume that* (H1) *and* (H2) *hold. Then there exists a positive number* $\bar{\delta}_0 < \min\{\frac{u^-}{2}, \frac{K-u^+}{2}, \delta_0/2\}$ *with the following property: for any* $\delta \in (0, \bar{\delta}_0]$, *there exist two positive numbers* $\epsilon = \epsilon(\delta)$ *and* $C = C(\delta)$ *such that for every* $\xi^\pm \in \mathbb{R}$, *the functions* $\upsilon^+(t,x)$ *and* $\upsilon^-(t,x)$ *defined by*

$$\upsilon^+(t,x) := (K+\delta) - [K - (u^- - 2\delta)e^{-\epsilon t}]\zeta(-\epsilon(x - \xi^+ + Ct)),$$
$$\upsilon^-(t,x) := -\delta + [K - (K - u^+ - 2\delta)e^{-\epsilon t}]\zeta(\epsilon(x - \xi^- - Ct))$$

are a supersolution and a subsolution of (1.1) *on* $[0, +\infty)$, *respectively.*

Proof. By a translation, we can assume that $\xi^\pm = 0$. By (H2), we can find two constants $\varrho \in [1/2, 1)$ and $\iota > 0$ satisfying

$$\varrho d > \max\{b'(0), b'(K)\} + \iota \tag{3.1}$$

and then, by (2.6), we can choose a positive constant $\bar{\delta}_0 < \min\{\frac{u^-}{2}, \frac{K-u^+}{2}, \frac{\delta_0}{2}\}$ such that

$$\left(\frac{1}{\varrho} - \varrho\right)\bar{\delta}_0 < K \tag{3.2}$$

$$0 \leq b'(\eta) < b'(0) + \iota \quad \text{for } \eta \in [-2\bar{\delta}_0, 2\bar{\delta}_0] \tag{3.3}$$

and

$$0 \leq b'(\eta) < b'(K) + \iota \quad \text{for } \eta \in [K - 2\bar{\delta}_0, K + 2\bar{\delta}_0]. \tag{3.4}$$

In what follows, we always assume that $\delta \in (0, \bar{\delta}_0]$. We note that the constants ϱ, ι and $\bar{\delta}_0$ are independent of δ.

Since

$$du < \varepsilon b(u) \quad \text{for } u \in [-\delta_0, 0) \cup (u^+, K),$$

we have

$$M_1 = M_1(\delta) := \min\{b(u) - du;\ u \in [-\delta, -\delta/2]\} > 0,$$

$$M_2 = M_2(\delta) := \min\{b(u) - du;\ u \in [u^+ + \delta/2, K - \delta]\} > 0.$$

Therefore, we can choose two positive constants $\epsilon^* = \epsilon^*(\delta) > 0$ and $M_0 = M_0(\delta) > 0$, with ϵ^* sufficiently small and M_0 sufficiently large, such that

$$K\epsilon^* < 2(1-\varrho)\delta \tag{3.5}$$

and

$$-\min\{M_1, M_2\} + Kb'_{\max}\epsilon^* + 2Kb'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} f(y)dy\right] < 0. \tag{3.6}$$

Take $\kappa = \kappa(\delta) \in (0,1)$ sufficiently small such that

$$0 \le \zeta(s) < \epsilon^*/2, \quad \text{if } s < -2+\kappa, \tag{3.7}$$

$$1 \ge \zeta(s) > 1-\epsilon^*/2, \quad \text{if } s > 2-\kappa. \tag{3.8}$$

Take $\varpi = \varpi(\delta) > 0$ small enough so that

$$(1-\varpi)(2-\kappa/2) > 2-\kappa. \tag{3.9}$$

By (3.1) and (3.6), we can take $\epsilon = \epsilon(\delta) > 0$ small enough such that

$$(K-u^+)e^{\epsilon r} < K, \qquad u^- e^{\epsilon r} < K, \quad \epsilon M_0 \le \varpi(2-\kappa), \tag{3.10}$$

$$\epsilon K + D\epsilon^2 K - \delta[\varrho d - (\max\{b'(0), b'(K)\} + \iota)] < 0 \tag{3.11}$$

and

$$\begin{aligned}\epsilon K + D\epsilon^2 K + rKb'_{\max}\epsilon e^{\epsilon r} \;-\; &\min\{M_1, M_2\} + Kb'_{\max}\epsilon^* \\ &+2Kb'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} f(y)dy\right] < 0.\end{aligned} \tag{3.12}$$

Finally, we set

$$\tilde{M} := \min\{\zeta'(s);\ -2+\kappa/2 \le s \le 2-\kappa/2\} > 0.$$

Then take $C = C(\delta) > 0$ large enough so that

$$\begin{aligned}&-C\epsilon u^+\tilde{M} + \epsilon K + D\epsilon^2 K + rKb'_{\max}\epsilon e^{\epsilon r} \\ &\quad + \max\{|du - b(u)|; u \in [-\delta, K+\delta]\} + 2Kb'_{\max} < 0.\end{aligned} \tag{3.13}$$

Clearly, for any $t \geq -r$ and $x \in \mathbb{R}$, we have

$$\delta \leq \upsilon^+(t,x) \leq K+\delta, \qquad -\delta \leq \upsilon^-(t,x) \leq K-\delta.$$

Set $\xi = x - Ct$. Then for $t \geq 0$, we have

$$\begin{aligned} S(\upsilon^-)(t,x) := &\frac{\partial \upsilon^-(t,x)}{\partial t} - D\frac{\partial^2 \upsilon^-(t,x)}{\partial x^2} + d\upsilon^-(t,x) \\ &- \int_{-\infty}^{\infty} b(\upsilon^-(t-r, x-y)) f(y) dy \\ = &-C\epsilon[K-(K-u^+-2\delta)e^{-\epsilon t}]\zeta'(\epsilon\xi) + \epsilon(K-u^+-2\delta)e^{-\epsilon t}\zeta(\epsilon\xi) \\ &- D\epsilon^2[K-(K-u^+-2\delta)^{-\epsilon t}]\zeta''(\epsilon\xi) + d\upsilon^-(t,x) \\ &- \int_{-\infty}^{\infty} b(\upsilon^-(t-r, x-y)) f(y) dy \\ \leq &-C\epsilon u^+\zeta'(\epsilon\xi) + \epsilon K + D\epsilon^2 K + d_m\upsilon^-(t,x) \\ &- \int_{-\infty}^{\infty} b(\upsilon^-(t-r, x-y)) f(y) dy, \end{aligned} \tag{3.14}$$

For $t \geq -r$, we have

$$\begin{aligned} \frac{\partial \upsilon^-(t,x)}{\partial t} &= -C\epsilon[K-(K-u^+-2\delta)^{-\epsilon t}]\zeta'(\epsilon\xi) + \epsilon(K-u^+-2\delta)^{-\epsilon t}\zeta(\epsilon\xi) \\ &\leq \epsilon(K-u^+-2\delta)e^{\epsilon r} \leq \epsilon K e^{\epsilon r} \end{aligned}$$

and hence, for $t \geq 0$,

$$\begin{aligned} b(\upsilon^-(t-r,x)) - b(\upsilon^-(t,x)) &= b'(\hat{\eta})[\upsilon^-(t-r,x) - \upsilon^-(t,x)] \\ &= -rb'(\hat{\eta})\frac{\partial \upsilon^-(t^*,x)}{\partial t} \\ &\geq -rb'(\hat{\eta})\epsilon K e^{\epsilon r} \geq -rb'_{\max}\epsilon K e^{\epsilon r}, \end{aligned} \tag{3.15}$$

where $t^* \in [t-r, t]$ and $\hat{\eta} = \theta\upsilon^-(t,x) + (1-\theta)\upsilon^-(t-r,x) \in [-\delta, K-\delta]$.

On the other hand, for $t \geq 0$, we have

$$\begin{aligned} &|\upsilon^-(t-r, x-y) - \upsilon^-(t-r,x)| \\ &\quad = |K-(K-u^+-2\delta)^{-\epsilon(t-r)}| \cdot |\zeta(\epsilon(\xi-y+Cr)) - \zeta(\epsilon(\xi+Cr))| \\ &\quad \leq K|\zeta(\epsilon(\xi-y+Cr)) - \zeta(\epsilon(\xi+Cr))|. \end{aligned} \tag{3.16}$$

Therefore, it follows that

$$
\begin{aligned}
S(\upsilon^-)(t,x) &\le -C\epsilon u^+\zeta'(\epsilon\xi)+\epsilon K+D\epsilon^2K+d\upsilon^-(t,x)-b(\upsilon^-(t-r,x))\\
&\quad -\int_{-\infty}^{\infty} b'(\tilde\eta)[\upsilon^-(t-r,x-y)-\upsilon^-(t-r,x)]f(y)dy\\
&\le -C\epsilon u^+\zeta'(\epsilon\xi)+\epsilon K+D\epsilon^2K\\
&\quad +d\upsilon^-(t,x)-b(\upsilon^-(t,x))-[b(\upsilon^-(t-r,x))-b(\upsilon^-(t,x))]\\
&\quad +Kb'_{\max}\int_{-\infty}^{\infty}|\zeta(\epsilon(\xi-y+Cr))-\zeta(\epsilon(\xi+Cr))|f(y)dy\\
&\le -C\epsilon u^+\zeta'(\epsilon(\xi))+\epsilon K+D\epsilon^2K+rKb'_{\max}\epsilon e^{\epsilon r}\\
&\quad +d\upsilon^-(t,x)-b(\upsilon^-(t,x))\\
&\quad +Kb'_{\max}\int_{-\infty}^{\infty}|\zeta(\epsilon(\xi-y+Cr))-\zeta(\epsilon(\xi+Cr))|f(y)dy,
\end{aligned}
\tag{3.17}
$$

where $\tilde\eta=\theta\upsilon^-(t-r,x)+(1-\theta)\upsilon^-(t-r,x-y)\in[-\delta,\,K-\delta]$.

We distinguish among three cases:

Case (i): $\epsilon\xi\le -2+\kappa/2$.

In this case, $\epsilon\xi\le-2+\kappa$, $0\le\zeta(\epsilon\xi)\le\epsilon^*/2$, and hence

$$
\begin{aligned}
-\delta\le\upsilon^-(t,x) &\le -\delta+[K-(K-u^+-2\delta)e^{-\epsilon t}]\epsilon^*/2\\
&\le -\delta+K\epsilon^*/2\\
&< -\delta+(1-\varrho)\delta\\
&= -\varrho\delta\le-\frac{1}{2}\delta,
\end{aligned}
$$

for all $t\ge0$ and $x\in\mathbb{R}$. Set $E(t,x)=\{y\in\mathbb{R};\,\upsilon^-(t-r,x-y)\le0\}$. It then follows from (3.14) that

$$
\begin{aligned}
S(\upsilon^-)(t,x) &\le \epsilon K+D\epsilon^2K+d\upsilon^-(t,x)-\int_{E(t,x)} b'(\check\eta)\upsilon^-(t-r,x-y)f(y)dy\\
&\le \epsilon K+D\epsilon^2K+d\upsilon^-(t,x)-\int_{E(t,x)} b'(\check\eta)\upsilon^-(t-r,x-y)f(y)dy\\
&\le K+D\epsilon^2K-d\varrho\delta+\delta\int_{-\infty}^{\infty} b'(\check\eta)f(y)dy,
\end{aligned}
$$

where $\check\eta=\theta\upsilon^-(t-r,x-y)\in[-\delta,0]\subset[-2\bar\delta_0,2\bar\delta_0]$.

Hence, by (3.11), we have

$$
\begin{aligned}
S(\upsilon)(t,x) &\le \epsilon K+D\epsilon^2K-d\varrho\delta+\delta(b'(0)+\iota)\\
&= \epsilon K+D\epsilon^2K-\delta[\varrho d-(b'(0)+\iota)]\\
&< 0.
\end{aligned}
$$

Case (ii): $\epsilon\xi \geq 2-\kappa/2$.

In this case, $\epsilon\xi \geq 2-\kappa$, $1-\epsilon^*/2 \leq \zeta(\epsilon\xi) \leq 1$, and hence

$$\begin{aligned} K-\delta \geq \upsilon^-(t,x) &\geq -\delta + [K-(K-u^+-2\delta)e^{-\epsilon t}](1-\epsilon^*/2) \\ &\geq -\delta + (u^+ + 2\delta)(1-\epsilon^*/2) \\ &= u^+ + \delta - (u^+ + 2\delta)\epsilon^*/2 \\ &\geq u^+ + \delta - K\epsilon^*/2 \\ &\geq u^+ + \delta - (1-\varrho)\delta \\ &= u^+ + \varrho\delta \geq u^+ + \delta/2 \end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}$. It then follows that

$$d\upsilon^-(t,x) - b(\upsilon^-(t,x)) \leq -\min\{b(u)-du;\ u \in [u^+ + \delta/2, K-\delta]\} = -M_2.$$

By the Choice of ϵ and ϖ, we see that

$$\varpi(x-Ct) \geq \frac{\varpi(2-\kappa/2)}{\epsilon} \geq \frac{\varpi(2-\kappa)}{\epsilon} \geq M_0$$

and for any $y \in [-\varpi\xi, \varpi\xi]$,

$$\begin{aligned} &\epsilon(\xi - y + Cr) \geq \epsilon(1-\varpi)\xi + \epsilon Cr \geq (1-\varpi)(2-\kappa/2) > 2-\kappa, \\ &\epsilon(\xi + Cr) \geq 2-\kappa/2 + \epsilon Cr > 2-\kappa. \end{aligned}$$

Hence, it follows from (3.8) that

$$\int_{-\varpi\xi}^{\varpi\xi} |\zeta(\epsilon(\xi-y+Cr)) - \zeta(\epsilon(\xi+Cr))| f(y)dy \leq \epsilon^* \int_{-\varpi\xi}^{\varpi\xi} f(y)dy \leq \epsilon^*.$$

Therefore, by (3.17) and (3.12), we have

$$\begin{aligned} S(\upsilon^-)(t,x) &\leq \epsilon K + D\epsilon^2 K + rKb'_{\max}\epsilon e^{\epsilon r} - M_2 \\ &\quad + Kb'_{\max} \int_{-\varpi\xi}^{\varpi\xi} |\zeta(\epsilon(\xi-y+Cr)) - \zeta(\epsilon(\xi+Cr))| f(y)dy \\ &\quad + 2Kb'_{\max}\left[\int_{\varpi\xi}^{\infty} + \int_{-\infty}^{-\varpi\xi} f(y)dy\right] \\ &\leq \epsilon K + D\epsilon^2 K + rKb'_{\max}\epsilon e^{\epsilon r} - M_2 + Kb'_{\max}\epsilon^* \\ &\quad + 2Kb'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} f(y)dy\right] \\ &< 0. \end{aligned}$$

Case (iii): $-2+\kappa/2\leq\epsilon\xi\leq 2-\kappa/2$.
In this case, by (3.13), we also have

$$\begin{aligned}S(\upsilon^-)(t,x) &\leq -C\epsilon u^+\tilde{M}+\epsilon K+D\epsilon^2 K+rKb'_{\max}\epsilon e^{\epsilon r}\\ &\qquad +\max\{|du-b(u)|;\,u\in[-\delta,\,K+\delta]\}+2Kb'_{\max}\\ &< 0.\end{aligned}$$

Combining cases (i)–(iii), we have

$$\begin{aligned}&\frac{\partial\upsilon^-(t,x)}{\partial t}-D\frac{\partial^2\upsilon^-(t,x)}{\partial x^2}+d\upsilon^-(t,x)\\ &\quad -\int_{-\infty}^{\infty}b(\upsilon^-(t-r,x-y))f(y)dy\leq 0,\quad t\geq 0,\ \ x\in\mathbb{R}.\end{aligned}$$

Thus $\upsilon^-(t,x)$ is a subsolution of (1.1) on $[0,+\infty)$. In a similar way, we can prove that $\upsilon^+(t,x)$ is a supersolution of (1.1) on $[0,+\infty)$. This completes the proof. □

Remark 3.2. Clearly, the functions υ^+ and υ^- have the following properties:

$$\begin{aligned}&\upsilon^+(s,x)=K+\delta,\quad \text{if } s\in[-r,0],\quad \text{and } x\geq\xi^+-Cs+2\epsilon^{-1},\\ &\upsilon^+(s,x)\geq u^--\delta,\quad \text{for all } s\in[-r,0],\quad \text{and } x\in\mathbb{R},\\ &\upsilon^+(t,x)=\delta+(u^--2\delta)e^{-\epsilon t},\quad \text{for all } t\geq -r \text{ and } x\leq\xi^+-Ct-2\epsilon^{-1},\\ &\upsilon^-(s,x)=-\delta,\quad \text{if } s\in[-r,0],\ \text{and } x\leq\xi^-+Cs-2\epsilon^{-1},\\ &\upsilon^-(s,x)\leq u^++\delta\quad \text{for all } s\in[-r,0],\ \text{and } x\in\mathbb{R},\\ &\upsilon^-(t,x)=K-\delta-(K-u^+-2\delta)e^{-\epsilon t},\quad \text{for all } t\geq -r \text{ and } x\geq\xi^-+Ct+2\epsilon^{-1}.\end{aligned}$$

Let $U(x+ct)$ be a non-decreasing traveling wavefront of (1.1). We define the following two functions:

$$w^{\pm}(x,t,\eta,\delta)=U(x+ct+\eta\pm\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0 t}))\pm\delta e^{-\beta_0 t},$$

where σ_0 and β_0 are as in Lemma 2.4. By the proof of Lemma 2.4, we can choose $\beta_0>0$ as small as we wish.

Lemma 3.3. *Assume that* (H1) *and* (H2) *hold. Let $U(x+ct)$ be a non-decreasing traveling wavefront of* (1.1)*, and $\varphi\in[0,K]_C$ be such that*

$$\liminf_{x\to\infty}\min_{s\in[-r,0]}\varphi(s,x)>u^+,\qquad \limsup_{x\to-\infty}\max_{s\in[-r,0]}\varphi(s,x)<u^-.$$

Then, for any $\delta>0$, there exist $T=T(\varphi,\delta)>0,\xi=\xi(\varphi,\delta)\in\mathbb{R}$, and $h=h(\varphi,\delta)>0$ such that

$$w_0^-(x,cT+\xi,\delta)(s)\leq w_T(x,\varphi)(s)\leq w_0^+(x,cT+\xi+h,\delta)(s),\quad s\in[-r,0],\ \ x\in\mathbb{R}.$$

Proof. By Lemma 2.3, $w(t,x,\varphi)$ exists globally on $[0,\infty)\times\mathbb{R}$ and $0\le w(t,x,\varphi)\le K$ for all $t\ge 0$ and $x\in\mathbb{R}$. For any $\delta>0$, we can choose a positive constant $\delta_1=\delta_1(\delta,\varphi)<\min\{\delta,\bar{\delta}_0\}$ such that

$$\liminf_{x\to\infty}\min_{s\in[-r,0]}\varphi(s,x)>u^+ +\delta_1$$

and

$$\limsup_{x\to-\infty}\max_{s\in[-r,0]}\varphi(s,x)<u^- -\delta_1.$$

Hence, there exists a constant $M_3=M_3(\delta_1,\varphi)>0$ such that

$$\varphi(s,x)\le u^- -\delta_1,\quad \text{for all } s\in[-r,0],\ x\le M_3, \tag{3.18}$$

$$\varphi(s,x)\ge u^+ +\delta_1,\quad \text{for all } s\in[-r,0],\ x\ge M_3. \tag{3.19}$$

Let $\epsilon=\epsilon(\delta_1)$ and $C=C(\delta_1)$ be defined in Lemma 3.1 with δ replaced by δ_1. Define $\xi^+=-M_3-Cr-2\epsilon^{-1}$ and $\xi^-=M_3+Cr+2\epsilon^{-1}$, and let $\upsilon^{\pm}(t,x)$ be defined in Lemma 3.1. By (3.18), (3.19) and Remark 3.2, it follows that for all $s\in[-r,0]$,

$$\varphi(s,x)\le u^- -\delta_1\le \upsilon^+(s,x),\quad \text{for } x\le -M_3,$$

$$\varphi(s,x)\le K<K+\delta_1\le \upsilon^+(s,x),\quad \text{for } x\ge \xi^+ +Cr+2\epsilon^{-1}=-M_3$$

and

$$\varphi(s,x)\ge u^+ +\delta_1\ge \upsilon^-(s,x),\quad \text{for } x\ge M_3,$$

$$\varphi(s,x)\ge 0>-\delta_1\ge \upsilon^-(s,x),\quad \text{for } x\le \xi^- -Cr-2\epsilon^{-1}=M_3.$$

Therefore, we have

$$\upsilon^-(s,x)\le\varphi(s,x)\le\upsilon^+(s,x),\quad s\in[-r,0],\ \ x\in\mathbb{R}. \tag{3.20}$$

By Lemma 3.1 and the comparison, it follows that

$$\upsilon^-(t,x)\le w(t,x,\varphi)\le \upsilon^+(t,x),\quad t\ge 0,\ \ x\in\mathbb{R}. \tag{3.21}$$

Since $\delta_1<\delta$, we can choose a sufficiently large constant $T>r$ such that, for all $t\ge T-r$,

$$\delta_1+(u^- -2\delta_1)e^{-\epsilon t}<\delta,\qquad \text{and}\quad K-\delta_1-(K-u^+ -2\delta)e^{-\epsilon t}>K-\delta$$

and hence, again by Remark 3.2,

$$w(t,x,\varphi) \leq v^+(t,x) < \delta, \quad \text{for} \quad x \leq \xi^+ - Ct - 2\epsilon^{-1} \tag{3.22}$$

and

$$w(t,x,\varphi) \geq v^-(t,x) > K - \delta, \quad \text{for } x \geq \xi^- + Ct + 2\epsilon^{-1}. \tag{3.23}$$

Let $x^- = \xi^+ - CT - 2\epsilon^{-1}$ and $x^+ = \xi^- + CT + 2\epsilon^{-1}$. By (3.22) and (3.23), it follows that, for all $t \in [T-r, T]$,

$$w(t,x,\varphi) < \delta \quad \text{for} \quad x \leq x^-, \qquad w(t,x,\varphi) > K - \delta, \quad \text{for} \quad x \geq x^+. \tag{3.24}$$

Take a large constant $M_4 > 0$ so that

$$U(x) \leq \delta \quad \text{for } x \leq -M_4, \qquad U(x) \geq K - \delta, \quad \text{for } x \geq M_4.$$

Let $\xi = -M_4 - x^+ - |c|(T+r)$, then for $x \geq x^+$ and $s \in [-r, 0]$,

$$U(x + cs + cT + \xi) - \delta \leq K - \delta \leq w_T(x,\varphi)(s)$$

and for $x \leq x^+$ and $s \in [-r, 0]$,

$$\begin{aligned} U(x + cs + cT + \xi) - \delta &\leq U(x^+ + |c|(T+r) + \xi) - \delta \\ &= U(-M_4) - \delta \leq 0 \leq w_T(x,\varphi)(s). \end{aligned}$$

Therefore, we have

$$U(x + cs + cT + \xi) - \delta \leq w_T(x,\varphi)(s), \quad \text{for} \quad s \in [-r, 0], x \in \mathbb{R}. \tag{3.25}$$

Let $h = M_4 - x^- + |c|(T+r) - \xi = 2(M_3 + M_4) + 2(C + |c|)(T+r) + 8\epsilon^{-1} > 0$. Then for $x \leq x^-$ and $s \in [-r, 0]$,

$$U(x + cs + cT + \xi + h) + \delta \geq \delta \geq w_T(x,\varphi)(s)$$

and for $x \geq x^-$ and $s \in [-r, 0]$,

$$\begin{aligned} U(x + cs + cT + \xi + h) + \delta &\geq U(x^- - |c|(T+r) + \xi + h) + \delta \\ &= U(M_4) + \delta \geq K \geq w_T(x,\varphi)(s). \end{aligned}$$

Therefore, we have

$$U(x + cs + cT + \xi + h) + \delta \geq w_T(x,\varphi)(s), \quad \text{for} \quad s \in [-r, 0], \quad x \in \mathbb{R}. \tag{3.26}$$

Thus, it follows that

$$\begin{aligned} &U(x + cs + cT + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta^s})) - \delta e^{-\beta_0 s} \\ &\quad \leq w_T(x,\varphi)(s) \\ &\quad \leq U(x + cs + cT + \xi + h + \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 s})) + \delta e^{-\beta_0 s}, \\ &s \in [-r, 0], \quad x \in \mathbb{R}. \end{aligned}$$

This completes the proof. □

Lemma 3.4. *Assume that* (H1) *and* (H2) *hold. Let* $U(x+ct)$ *be a non-decreasing traveling wavefront of* (1.1)*. Then there exists a positive number* ε^* *such that if* $w(t,x)$ *is a solution of* (1.1) *on* $[0,+\infty)$ *with* $0\le w(t,x)\le K$ *for* $t\in[0,+\infty)$ *and* $x\in\mathbb{R}$*, and for some* $\xi\in\mathbb{R}, h>0, \delta>0$ *and* $T\ge 0$*, there holds*

$$w_0^-(x, cT+\xi, \delta)(s) \le w_T(x)(s) \le w_0^+(x, cT+\xi+h, \delta)(s), \quad s\in[-r,0], \quad x\in\mathbb{R}$$

then for any $t\ge T+r+1$*, there exist* $\hat{\xi}(t), \hat{\delta}(t)$ *and* $\hat{h}(t)$ *such that*

$$\begin{aligned} &w_0^-(x, ct+\hat{\xi}(t), \hat{\delta}(t))(s) \\ &\quad \le w_t(x)(s) \le w_0^+(x, ct+\hat{\xi}(t)+\hat{h}(t), \hat{\delta}(t))(s), \quad s\in[-r,0], \quad x\in\mathbb{R} \end{aligned}$$

with $\hat{\xi}(t), \hat{\delta}(t)$ *and* $\hat{h}(t)$ *satisfying*

$$\hat{\xi}(t)\in[\xi-\sigma_0(2\delta+\varepsilon^*\min\{1,h\})e^{\beta_0 r}, \xi+h-\sigma_0(2\delta+\varepsilon^*\min\{1,h\})e^{\beta_0 r}];$$

$$\hat{\delta}(t) = (\delta+\varepsilon^*\min\{1,h\})e^{-\beta_0[t-T-r-1]};$$

$$\begin{aligned} \hat{h}(t) &= h-2\sigma_0\varepsilon^*\min\{1,h\}+\sigma_0(3\delta+\varepsilon^*\min\{1,h\})e^{\beta_0 r} \\ &= h-\sigma_0[\varepsilon^*\min\{1,h\}(2-e^{\beta_0 r})-3\delta e^{\beta_0 r}]>0. \end{aligned}$$

Proof. By Lemma 2.6, $w^+(t,x,cT+\xi,\delta)$ and $w^-(x,t,cT+\xi,\delta)$ are a supersolution and a subsolution of (1.1), respectively. Clearly, $\upsilon(t,x) = w(T+t,x)$, $t\ge 0$, is also a solution of (1.1) with $\upsilon_0(x)(s) = w_T(x)(s)$, $s\in[-r,0], x\in\mathbb{R}$. Then the comparison implies that

$$w^-(t,x,cT+\xi,\delta) \le w(T+t,x) \le w^+(t,x,cT+\xi+h,\delta), \quad t\ge 0, \quad x\in\mathbb{R}.$$

That is,

$$\begin{aligned} &U[x+c(T+t)+\xi-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0 t})]-\delta e^{-\beta_0 t} \\ &\quad \le w(T+t,x) \\ &\quad \le U[x+c(T+t)+\xi+h+\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0 t})]+\delta e^{-\beta_0 t}, \\ &t\ge 0, \quad x\in\mathbb{R}. \end{aligned} \tag{3.27}$$

Let $z=-\xi-cT$. Then it follows from Lemma 2.3 that for any $J\ge 0, x\in\mathbb{R}$ with $|x-z|\le J$, and all $t>0$,

$$\begin{aligned} &w(T+t,x)-w^-(t,x,cT+\xi,\delta) \\ &\quad \ge \Theta(J,t)\int_z^{z+1}[w(T,y)-w^-(0,y,cT+\xi,\delta)]dy \\ &\quad = \Theta(J,t)\int_z^{z+1}[w(T,y)-U(y+cT+\xi-\sigma_0\delta(e^{\beta_0 r}-1))+\delta]dy \\ &\quad \ge \Theta(J,t)\int_z^{z+1}[w(T,y)-U(y+cT+\xi)+\delta]dy. \end{aligned} \tag{3.28}$$

Since $\lim_{|\eta|\to+\infty} U'(\eta)=0$, we can fix a positive number $M_5>0$ such that

$$U'(\eta)\le \frac{1}{2\sigma_0}, \quad \text{for all } |\eta|\ge M_5. \tag{3.29}$$

Let $J=M_5+|c|(1+r)+3, \bar{h}=\min\{1,h\}$, and

$$\varepsilon_1=\frac{1}{2}\min\{U'(\eta);\ |\eta|\le 2\}>0.$$

Then

$$\int_z^{z+1}[U(y+cT+\xi+\bar{h})-U(y+cT+\xi)]dy$$
$$=\int_0^1[U(y+\bar{h})-U(y)]dy\ge 2\varepsilon_1\bar{h}$$

and hence, one of the following must hold:

(i) $\int_z^{z+1}[w(T,y)-U(y+cT+\xi)]dy\ge\varepsilon_1\bar{h}$,
(ii) $\int_z^{z+1}[U(y+cT+\xi+\bar{h})-w(T,y)]dy\ge\varepsilon_1\bar{h}$.

In what follows, we consider only Case (i). Case (ii) is similar and thus the proof is omitted.

For any $s\in[-r,0]$, $|x-z|\le J$, letting $t=1+r+s\ge 1$ in (3.28), we have

$$\begin{aligned} & w(T+1+r+s,x)\\ &\quad \ge U[x+c(T+1+r+s)+\xi-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]-\delta e^{-\beta_0(1+r+s)}\\ &\qquad +\Theta(J,1+r+s)\textstyle\int_z^{z+1}[w(T,y)-U(y+cT+\xi)+\delta]dy\\ &\quad \ge U[x-z+c(1+r+s)-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]\\ &\qquad -\delta e^{-\beta_0(1+r+s)}+\Theta_0(J)\varepsilon_1\bar{h}, \end{aligned} \tag{3.30}$$

where $\Theta_0(J)=\min_{s\in[-r,0]}\Theta(J,1+r+s)$. Let $J_1=J+|c|(1+r)+3$, and choose a positive constant $\varepsilon^*>0$ such that

$$\varepsilon^*\le\min\left\{\min_{|\eta|\le J_1}\frac{\Theta_0(J)\varepsilon_1}{2\sigma_0 U'(\eta)},\ \frac{1}{3\sigma_0}\right\}, \tag{3.31}$$

then

$$\begin{aligned} & U[x-z+c(1+r+s)+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]\\ & -U[x-z+c(1+r+s)-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]\\ & =U'(\eta_1)2\sigma_0\varepsilon^*\bar{h}\le\Theta_0(J)\varepsilon_1\bar{h}, \end{aligned} \tag{3.32}$$

where $\eta_1=x-z+c(1+r+s)-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})+\theta\cdot 2\sigma_0\varepsilon^*\bar{h}$, and in the last inequality, we have used (3.31) and the estimate

$$|\eta_1|\le|x-z|+|c|(1+r)+\sigma_0\delta e^{\beta_0 r}+2\sigma_0\varepsilon^*\le J_1.$$

Hence, (3.30) and (3.32) imply that

$$w(T+1+r+s,x) \geq U[x+c(T+1+r+s)+\xi +2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]-\delta e^{-\beta_0(1+r+s)}. \tag{3.33}$$

For $s\in[-r,0], |x-z|\geq J$, it follows that

$$\begin{aligned} &U[x+c(T+1+r+s)+\xi-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ &\quad -U[x+c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ &\quad = -U'(\eta_2)2\sigma_0\varepsilon^*\bar{h}\geq-\varepsilon^*\bar{h}, \end{aligned}$$

where $\eta_2=x-z+c(1+r+s)-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})+\theta\cdot 2\sigma_0\varepsilon^*\bar{h}$, and in the last inequality, we have used (3.29) and the estimate

$$|\eta_2|\geq|x-z|-\{|c|(1+r)+\sigma_0\delta e^{\beta_0 r}+2\sigma_0\varepsilon^*\}\geq M_5.$$

Therefore, for all $s\in[-r,0], |x-z|\geq J$,

$$\begin{aligned} &U[x+c(T+1+r+s)+\xi-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ &\quad \geq U[x+c(T+1+r+s)+\xi \\ &\qquad +2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]-\varepsilon^*\bar{h}, \end{aligned}$$

which together with (3.27) yields

$$\begin{aligned} w(T+1+r+s,x) \geq\ & U[x+c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h} \\ & -\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ & -\delta e^{-\beta_0(1+r+s)}-\varepsilon^*\bar{h} \end{aligned} \tag{3.34}$$

for all $s\in[-r,0], |x-z|\geq J$.

Combining (3.33) and (3.34), we find that for all $s\in[-r,0], \quad x\in\mathbb{R}$,

$$\begin{aligned} w(T+1+r+s,x) \geq\ & U[x+c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h} \\ & -\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]-\delta e^{-\beta_0(1+r+s)}-\varepsilon^*\bar{h}, \end{aligned}$$

and hence,

$$\begin{aligned} w_{T+1+r}(x)(s) \geq\ & U[x+cs+c(T+1+r)+\xi \\ & +2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)}) \\ & -\sigma_0(\delta+\varepsilon^*\bar{h})(e^{\beta_0 r}-e^{-\beta_0 s})]-(\delta+\varepsilon^*\bar{h})e^{-\beta_0 s} \\ \geq\ & U[x+cs+c(T+1+r)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta e^{\beta_0 r} \\ & -\sigma_0(\delta+\varepsilon^*\bar{h})(e^{\beta_0 r}-e^{-\beta_0 s})]-(\delta+\varepsilon^*\bar{h})e^{-\beta_0 s} \\ =\ & w_0^-(x,\eta,\delta+\varepsilon^*\bar{h})(s), \quad s\in[-r,0], \quad x\in\mathbb{R}, \end{aligned} \tag{3.35}$$

where $\eta=c(T+1+r)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta e^{\beta_0 r}$.

Therefore, by the comparison, we have

$$w_{T+1+r+t'}(x)(s) \geq w^-_{t'}(x,\eta,\delta+\varepsilon^*\bar{h})(s), \quad \text{for all } t' \geq 0, s \in [-r,0], x \in \mathbb{R}. \tag{3.36}$$

Hence, for $t \geq T+1+r$, setting $t' = t-(T+1+r)$ in (3.36), we get

$$\begin{aligned} w_t(x)(s) &\geq w^-_{t-(T+1+r)}(x,\eta,\delta+\varepsilon^*\bar{h})(s) \\ &= w^-(t-(T+1+r)+s, x, \eta, \delta+\varepsilon^*\bar{h}) \\ &= U[x+cs+ct-c(T+1+r)+\eta-\sigma_0(\delta+\varepsilon^*\bar{h}) \\ &\quad (e^{\beta_0 r} - e^{-\beta_0 s}\cdot e^{-\beta_0[t-(T+1+r)]})] \\ &\quad -(\delta+\varepsilon^*\bar{h})e^{-\beta_0[t-(T+1+r)]}\cdot e^{-\beta_0 s} \\ &\geq U[x+cs+ct-c(T+1+r)+\eta \\ &\qquad -\sigma_0(\delta+\varepsilon^*\bar{h})e^{\beta_0 r} - \sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})] - \hat{\delta}(t)e^{-\beta_0 s}, \end{aligned} \tag{3.37}$$

where

$$\hat{\delta}(t) = (\delta+\varepsilon^*\bar{h})e^{-\beta_0[t-(T+1+r)]}. \tag{3.38}$$

It then follows that for $t \geq T+1+r$,

$$w_t(x)(s) \geq w^-_0(x, ct+\hat{\xi}(t), \hat{\delta}(t))(s), \quad s \in [-r,0], x \in \mathbb{R}, \tag{3.39}$$

where

$$\begin{aligned} \hat{\xi}(t) &= -c(T+1+r)+\eta-\sigma_0(\delta+\varepsilon^*\bar{h})e^{\beta_0 r} \\ &= \xi + 2\sigma_0\varepsilon^*\bar{h} - \sigma_0\delta e^{\beta_0 r} - \sigma_0(\delta+\varepsilon^*\bar{h})e^{\beta_0 r} \\ &= \xi + 2\sigma_0\varepsilon^*\bar{h} - \sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}. \end{aligned} \tag{3.40}$$

Clearly, by (3.31), it is easy to see that

$$\hat{\xi}(t) \in [\xi - \sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}, \xi + h - \sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}]. \tag{3.41}$$

On the other hand, for $t \geq T$, by (3.27), we have

$$w(t,x) \leq U[x+ct+\xi+h+\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(t-T)})] + \delta e^{-\beta_0(t-T)},$$

which implies, for all $t \geq T+1+r$, that

$$\begin{aligned} w_t(x)(s) &= w(t+s,x) \\ &\leq U[x+c(t+s)+\xi+h+\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(t+s-T)})] + \delta e^{-\beta_0(t+s-T)} \\ &\leq U[x+cs+ct+\xi+h+\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(t+s-T)}) \\ &\quad +\sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})] + \hat{\delta}(t)e^{-\beta_0 s} \\ &\leq U[x+cs+ct+\xi+h+\sigma_0\delta e^{\beta_0 r}+\sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})] + \hat{\delta}(t)e^{-\beta_0 s} \\ &= U[x+cs+ct+(\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}) + (h-2\sigma_0\varepsilon^*\bar{h} \\ &\quad +\sigma_0(3\delta+\varepsilon^*\bar{h})e^{\beta_0 r}) + \sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})] + \hat{\delta}(t)e^{-\beta_0 s} \\ &= U[x+cs+ct+\hat{\xi}(t)+(h-2\sigma_0\varepsilon^*\bar{h}+\sigma_0(3\delta+\varepsilon^*\bar{h})e^{\beta_0 r}) \\ &\quad +\sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})] + \hat{\delta}(t)e^{-\beta_0 s}, \quad s \in [-r,0], \ x \in \mathbb{R}. \end{aligned}$$

Therefore, for $t \geq T+1+r$, we find

$$w_t(x)(s) \leq w_0^+(x, ct+\hat{\xi}(t)+\hat{h}(t), \hat{\delta}(t))(s), \quad s \in [-r, 0], \ x \in \mathbb{R}, \quad (3.42)$$

where

$$\begin{aligned} \hat{h}(t) &= h - 2\sigma_0 \varepsilon^* \bar{h} + \sigma_0(3\delta + \varepsilon^* \bar{h}) e^{\beta_0 r} \\ &= h - \sigma_0[2\varepsilon^* \bar{h} - \varepsilon^* \bar{h} e^{\beta_0 r} - 3\delta e^{\beta_0 r}] \\ &= h - \sigma_0[\varepsilon^* \bar{h}(2 - e^{\beta_0 r}) - 3\delta e^{\beta_0 r}] > 0 \end{aligned} \quad (3.43)$$

and in (3.43), we have used (3.31) and the estimate

$$h - 2\sigma_0 \varepsilon^* \bar{h} > h - 3\sigma_0 \varepsilon^* \bar{h} \geq h - \bar{h} \geq 0.$$

Now the conclusions of the lemma follow from (3.38), (3.39) and (3.41)–(3.43).

Theorem 3.5. *Assume that* (H1) *and* (H2) *hold. Also assume that* (1.1) *has a non-decreasing traveling wavefront* $U(x+ct)$. *Then* $U(x+ct)$ *is globally asymptotically stable with phase shift in the sense that there exists* $\gamma > 0$ *such that for any* $\varphi \in [0, K]_C$ *with*

$$\liminf_{x \to +\infty} \min_{s \in [-r,0]} \varphi(s, x) > u^+, \qquad \limsup_{x \to -\infty} \max_{s \in [-r,0]} \varphi(s, x) < u^-$$

the solution $w(t, x, \varphi)$ *of* (1.1) *satisfies*

$$|w(t, x, \varphi) - U(x+ct+\xi_0)| \leq Me^{-\gamma t}, \qquad t \geq 0, \ \ x \in \mathbb{R}$$

for some $M = M(\varphi) > 0$ *and* $\xi_0 = \xi_0(\varphi) \in \mathbb{R}$.

Proof. Let $\beta_0, \sigma_0, \bar{\delta}$ be as in Lemma 2.4 with β_0 chosen such that $e^{\beta_0 r} < 2$, and let ε^* be as in Lemma 3.3 with ε^* chosen such that $\sigma_0 \varepsilon^*(2 - e^{\beta_0 r}) < 1$. We further choose a $0 < \delta^* < \min\{\frac{\delta_0}{2}, \bar{\delta}, \frac{1}{\sigma_0}\}$ such that

$$1 > k^* := \sigma_0[\varepsilon^*(2 - e^{\beta_0 r}) - 3\delta^* e^{\beta_0 r}] > 0$$

and then fix a $t^* \geq r+1$ such that

$$e^{-\beta_0(t^*-r-1)}(1 + \varepsilon^*/\delta^*) < 1 - k^*.$$

We first prove the following two claims.

Claim 1. *There exist* $T^* = T^*(\varphi) > 0, \xi^* = \xi^*(\varphi) \in \mathbb{R}$ *such that*

$$\begin{aligned} w_0^-(x, cT^*+\xi^*, \delta^*)(s) &\leq w_{T^*}(x, \varphi)(s) \leq w_0^+(x, cT^*+\xi^*+1, \delta^*)(s), \\ s &\in [-r, 0], \ \ x \in \mathbb{R}. \end{aligned} \quad (3.44)$$

Indeed, by Lemma 3.3, there exist $T = T(\varphi) > 0, \xi = \xi(\varphi) \in \mathbb{R}$ and $h = h(\varphi) > 0$ such that

$$w_0^-(x, cT + \xi, \delta^*)(s) \le w_T(x, \varphi)(s) \le w_0^+(x, cT + \xi + h, \delta^*)(s), \quad s \in [-r, 0], \; x \in \mathbb{R}. \tag{3.45}$$

If $h \le 1$, then Claim 1 follows from the monotonicity of $U(\cdot)$. In what follows, we assume that $h > 1$, and let

$$N = \max\{m; m \text{ is a non}-\text{negative integer and } mk^* < h\}.$$

Since $0 < k^* < 1$ and $h > 1$, we have $N \ge 1$, $Nk^* < h \le (N+1)k^*$, and hence, $0 < h - Nk^* \le (N+1)k^* - Nk^* = k^* < 1$. Clearly, $\bar{h} = \min\{1, h\} = 1$. By (3.45), the choice of k^* and t^*, and Lemma 3.3, we have

$$\begin{aligned} & w_0^-(x, c(T+t^*) + \hat{\xi}(T+t^*), \hat{\delta}(T+t^*))(s) \\ & \quad \le w_{T+t^*}(x, \varphi)(s) \\ & \quad \le w_0^+(x, c(T+t^*) + \hat{\xi}(T+t^*) + \hat{h}(T+t^*), \hat{\delta}(T+t^*))(s), \\ & s \in [-r, 0], \quad x \in \mathbb{R}, \end{aligned} \tag{3.46}$$

where

$$\hat{\xi}(T+t^*) \in [\xi - \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}, \xi + h - \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}],$$

$$\hat{\delta}(T+t^*) = (\delta^* + \varepsilon^*)e^{-\beta_0[t^* - r - 1]} < \delta^*(1 - k^*),$$

$$0 \le \hat{h}(T+t^*) \le h - \sigma_0[\varepsilon^*(2 - e^{\beta_0 r}) - 3\delta^* e^{\beta_0 r}] = h - k^*.$$

Repeating the same process N times, we then have that (3.46), with $T + t^*$ replaced by $T + Nt^*$, holds for some $\hat{\xi} \in \mathbb{R}, 0 < \hat{\delta} \le \delta^*(1-k^*)^N$, and $0 \le \hat{h} \le h - Nk^* < 1$. Let $T^* = T + Nt^*, \xi^* = \hat{\xi}$. Again by the monotonicity of $U(\cdot)$, (3.44) then follows.

Claim 2. *Let* $p = 1 + \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}, T_m = T^* + mt^*, \delta_m^* = (1-k^*)^m \delta^*$ *and* $h_m = (1-k^*)^m < 1, m \ge 0$. *Then there exists a sequence* $\{\hat{\xi}_m\}_{m=0}^{\infty}$ *with* $\hat{\xi}_0 = \xi^*$ *such that*

$$|\hat{\xi}_{m+1} - \hat{\xi}_m| \le p h_m, \quad m \ge 0 \tag{3.47}$$

and

$$w_0^-(x, cT_m + \hat{\xi}_m, \delta_m)(s) \le w_{T_m}(x, \varphi)(s) \le w_0^+(x, cT_m + \hat{\xi}_m + h_m, \delta_m^*)(s), \quad s \in [-r, 0], \quad x \in \mathbb{R}, \quad m \ge 0. \tag{3.48}$$

In fact, Claim 1 implies that (3.48) holds for $m=0$. Now suppose that (3.48) holds for some $m=\ell\geq 0$. By Lemma 3.4, with $T=T_\ell, \xi=\hat{\xi}_\ell, h=h_\ell, \delta=\delta^*_\ell$ and $t=T_\ell+t^*=T_{\ell+1}\geq T_\ell+r+1$, we then have

$$\begin{aligned} w_0^-(x, cT_{\ell+1}+\hat{\xi}, \hat{\delta})(s) &\leq w_{T_{\ell+1}}(x,\varphi)(s) \\ &\leq w_0^+(x, cT_{\ell+1}+\hat{\xi}+\hat{h}, \hat{\delta})(s), \quad s\in[-r,0], \quad x\in\mathbb{R}, \end{aligned}$$

where

$$\hat{\xi}\in[\hat{\xi}_\ell-\sigma_0(2\delta^*_\ell+\varepsilon^* h_\ell)e^{\beta_0 r}, \hat{\xi}_\ell+h_\ell-\sigma_0(2\delta^*_\ell+\varepsilon^* h_\ell)e^{\beta_0 r}],$$

$$\begin{aligned} \hat{\delta} &= (\delta^*_\ell+\varepsilon^* h_\ell)e^{-\beta_0[T_{\ell+1}-T_\ell-r-1]} \\ &= (1-k^*)^\ell(\delta^*+\varepsilon^*)e^{-\beta_0[t^*-r-1]} \\ &\leq (1-k^*)^\ell\cdot\delta^*(1-k^*) \\ &= (1-k^*)^{\ell+1}\cdot\delta^* = \delta^*_{\ell+1}, \end{aligned}$$

$$\begin{aligned} \hat{h} &= h_\ell-\sigma_0[\varepsilon^* h_\ell(2-e^{\beta_0 r})-3\delta^*_\ell e^{\beta_0 r}] \\ &= (1-k^*)^\ell\{1-\sigma_0[\varepsilon^*(2-e^{\beta_0 r})-3\delta^* e^{\beta_0 r}]\} \\ &= (1-k^*)^{\ell+1} = h_{\ell+1}. \end{aligned}$$

We choose $\hat{\xi}_{\ell+1}=\hat{\xi}$. Then

$$\begin{aligned} |\hat{\xi}_{\ell+1}-\hat{\xi}_\ell| &\leq h_\ell+\sigma_0(2\delta^*_\ell+\varepsilon^* h_\ell)e^{\beta_0 r} \\ &= [1+\sigma_0(2\delta^*+\varepsilon^*)e^{\beta_0 r}]h_\ell \\ &= ph_\ell. \end{aligned}$$

It follows that (3.47) holds for $m=\ell$, and (3.48) holds for $m=\ell+1$. By induction, (3.47) and (3.48) holds for all $m\geq 0$.

For every $m\geq 0$, by (3.48) and the comparison, it follows that for all $t\geq T_m,\ x\in\mathbb{R}$,

$$\begin{aligned} &U(x+ct+\hat{\xi}_m-\sigma_0\delta^*_m(e^{\beta_0 r}-e^{-\beta_0(t-T_m)}))-\delta^*_m e^{-\beta_0(t-T_m)} \\ &\quad\leq w(t,x,\varphi) \\ &\quad\leq U(x+ct+\hat{\xi}_m+h_m+\sigma_0\delta^*_m(e^{\beta_0 r}-e^{-\beta_0(t-T_m)}))+\delta^*_m e^{-\beta_0(t-T_m)}. \end{aligned} \tag{3.49}$$

For any $t\geq T^*$, let $m=\left[\frac{t-T^*}{t^*}\right]\geq 0$ be the largest integer not greater than $\frac{t-T^*}{t^*}$, and define $\delta(t)=\delta^*_m, \xi(t)=\hat{\xi}_m-\sigma_0\delta^*_m e^{\beta_0 r}$, and $h(t)=h_m+2\sigma_0\delta^*_m e^{\beta_0 r}$, then we have $T_m=T^*+mt^*\leq t<T^*+(m+1)t^*=T_{m+1}$. By (3.49), it follows that for all $t\geq T^*, x\in\mathbb{R}$,

$$\begin{aligned} U(x+ct+\xi(t))-\delta(t) &\leq w(t,x,\varphi) \\ &\leq U(x+ct+\xi(t)+h(t))+\delta(t). \end{aligned} \tag{3.50}$$

Set $\gamma := -\frac{1}{t^*}\ln(1-k^*) > 0$ and $q = \exp\{-(1+T^*/t^*)\ln(1-k^*)\}$. Since $0 \le m \le \frac{t-T^*}{t^*} < m+1$, we have

$$(1-k^*)^m < (1-k^*)^{\frac{t-T^*}{t^*}-1} = \exp\left\{\left(\frac{t-T^*}{t^*}-1\right)\ln(1-k^*)\right\} = qe^{-\gamma t}.$$

Therefore, for any $t \ge T^*$, we have

$$\delta(t) = \delta_m^* = (1-k^*)^m \delta^* \le \delta^* q e^{-\gamma t}, \tag{3.51}$$

$$\begin{aligned} h(t) &= h_m + 2\sigma_0 \delta_m^* e^{\beta_0 r} \\ &= (1+2\sigma_0\delta^* e^{\beta_0 r})(1-k^*)^m \le (1+2\sigma_0\delta^* e^{\beta_0 r}) q e^{-kt} \end{aligned} \tag{3.52}$$

and for any $t' \ge t \ge T^*$, by (3.47), we have

$$\begin{aligned} |\xi(t')-\xi(t)| &= |\hat{\xi}_n - \sigma_0\delta_m^* e^{\beta_0 r} - (\hat{\xi}_m - \sigma_0\delta_m^* e^{\beta_0 r})| \\ &\le |\hat{\xi}_n - \hat{\xi}_m| + \sigma_0|\delta_n^* - \delta_m^*| e^{\beta_0 r} \\ &\le \textstyle\sum_{\ell=m}^{n-1} p \times h_\ell + \sigma_0 \delta_m^* e^{\beta_0 r} \\ &= \left[\frac{p}{\delta^*}\textstyle\sum_{\ell=0}^{n-m-1}(1-k^*)^\ell + \sigma_0 e^{\beta_0 r}\right]\delta_m^* \\ &\le \left(\frac{p}{k^*\delta^*} + \sigma_0 e^{\beta_0 r}\right)\delta(t) \\ &\le \left(\frac{p}{k^*} + \sigma_0\delta^* e^{\beta_0 r}\right) q e^{-\gamma t}, \end{aligned} \tag{3.53}$$

where $n = \left[\frac{t'-T^*}{t^*}\right] \ge m = \left[\frac{t-T^*}{t^*}\right]$. Therefore, it follows from (3.53) that $\xi_0 := \lim_{t\to\infty}\xi(t)$ exists, and for $t \ge T^*$, we have

$$|\xi_0 - \xi(t)| \le \left(\frac{p}{k^*} + \sigma_0\delta^* e^{\beta_0 r}\right) q e^{-\gamma t}. \tag{3.54}$$

Set

$$M := \max\left\{\frac{b(K)}{2\sqrt{Dd}}\left[\frac{p}{k^*} + 3\sigma_0\delta^* e^{\beta_0 r} + 1\right] q + \delta^* q,\, 2K e^{kT^*}\right\}.$$

Since $0 < U'(\xi) \le \frac{b(K)}{2\sqrt{Dd}}$, it follows from (3.50)–(3.54) that for all $t \ge T^*$,

$$\begin{aligned} |w(t,x,\varphi) - U(x+ct+\xi_0)| &\le \tfrac{b(K)}{2\sqrt{Dd}}[|\xi(t)-\xi_0| + h(t)] + \delta(t) \\ &\le \left\{\tfrac{b(K)}{2\sqrt{Dd}}\left[\tfrac{p}{k^*} + 3\sigma_0\delta^* e^{\beta_0 r} + 1\right] q + \delta^* q\right\} e^{-\gamma t}, \end{aligned}$$

which together with the fact that $|w(t,x,\varphi) - U(x+ct+\xi_0)| \le 2K,\ \ t \in [0, T^*]$ yields $|w(t,x,\varphi) - U(x+ct+\xi_0)| \le Me^{-\gamma t}$ for all $t \ge 0$. The proof is complete. □

4. EXISTENCE OF TRAVELING WAVEFRONTS

First, we consider the following reaction–diffusion equation without delay:

$$\begin{aligned}\frac{\partial w(t,x)}{\partial t} &= D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x)\\ &\quad + \int_{-\infty}^{\infty} b(w(t,x-y))f(y-cr)dy, \quad t>0, \quad x\in\mathbb{R}. \end{aligned} \tag{4.1}$$

where $c\in\mathbb{R}$ is a parameter.

Lemma 4.1. *Assume that* (H1)–(H3) *hold. Then there exist two small constants $\delta^*>0$, $\epsilon_0>0$ and a large constant $C_0>0$, which are independent of c and r, such that*

(I) *the functions $\upsilon_1^+(t,x)$ and $\upsilon_1^-(t,x)$ defined by*

$$\upsilon_1^+(t,x) = K + \delta^* - K\zeta(-\epsilon_0(x+C_0t))$$

and

$$\upsilon_1^-(t,x) = -\delta^* + K\zeta(\epsilon_0(x-C_0t))$$

are a supersolution of (4.1) *for $c\geq 0$, and a subsolution of* (4.1) *for $c\leq 0$;*

(II) *the functions $\upsilon_2^+(t,x)$ and $\upsilon_2^-(t,x)$ defined by*

$$\upsilon_2^+(t,x) = K + \delta^* - K\zeta(-\epsilon_c(x+C_ct))$$

and

$$\upsilon_1^-(t,x) = -\delta^* + K\zeta(\epsilon_c(x-C_ct))$$

are a supersolution and a subsolution of (4.1) *for all $c\in\mathbb{R}$, respectively, here $\epsilon_c = \frac{\epsilon_0}{1+|c|r}$ and $C_c = (1+|c|r)C_0$.*

Proof. By (H2), we can find two constants $\varrho\in[1/2,1)$ and $\iota>0$ satisfying

$$\varrho d > \max\{b'(0), b'(K)\} + \iota \tag{4.2}$$

and then, by (2.6), take a positive constant $\delta^* < \min\{\frac{u^-}{2}, \frac{K-u^+}{2}, \frac{\delta_0}{2}\}$ such that

$$\left(\frac{1}{\varrho} - \varrho\right)\delta^* < K, \tag{4.3}$$

$$0 \le b'(\eta) < b'(0) + \iota, \quad \text{for } \eta \in [-2\delta^*, 2\delta^*] \tag{4.4}$$

and

$$0 \le b'(\eta) < b'(K) + \iota, \quad \text{for } \eta \in [K - 2\delta^*, K + 2\delta^*]. \tag{4.5}$$

Since

$$M_1 := \min\{b(u) - du; u \in [K - \frac{3}{2}\delta^*, K - \delta^*]\} > 0,$$

$$M_2 := \min\{du - b(u); u \in [\delta^*, \frac{3}{2}\delta^*]\} > 0,$$

we can choose two positive constants $\epsilon^* > 0$ and $M_0 > 0$, with ϵ^* sufficiently small and M_0 sufficiently large, such that

$$K\epsilon^* < 2(1 - \varrho)\delta^* \tag{4.6}$$

and

$$-\min\{M_1, M_2\} + Kb'_{\max}\epsilon^* + 2Kb'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} f(y)dy\right] < 0. \tag{4.7}$$

Take $\kappa \in (0, 1)$ sufficiently small such that

$$0 \le \zeta(s) < \epsilon^*/2, \quad \text{if } s < -2 + \kappa, \tag{4.8}$$

$$1 \ge \zeta(s) > 1 - \epsilon^*/2, \quad \text{if } s > 2 - \kappa. \tag{4.9}$$

Take $\varpi > 0$ small enough so that

$$(1 - \varpi)(2 - \kappa/2) > 2 - \kappa. \tag{4.10}$$

Then we take $\epsilon_0 > 0$ small enough such that

$$\epsilon_0 M_0 \le \varpi(2 - \kappa), \tag{4.11}$$

$$\epsilon_0 < (1 - \varpi)(2 - \kappa/2) - 2 + \kappa, \tag{4.12}$$

$$D\epsilon_0^2 K - \delta^*[\varrho d - (\max\{b'(0), b'(K)\} + \iota)] < 0 \tag{4.13}$$

and

$$D\epsilon_0^2 K - \min\{M_1, M_2\} + Kb'_{\max}\epsilon^* + 2Kb'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} f(y)dy\right] < 0. \tag{4.14}$$

Finally, we set

$$\tilde{M} := \min\{\zeta'(s);\ \ -2+\kappa/2 \le s \le 2-\kappa/2\} > 0.$$

Then take

$$C_0 = \frac{1}{\epsilon_0 K \tilde{M}}[D\epsilon_0^2 K + \max\{|du - b(u)|;\\ u \in [-\delta^*, K+\delta^*]\} + 2Kb'_{\max}] > 0. \tag{4.15}$$

We note that δ^*, ϵ_0 and C_0 are independent of c and r.

Clearly, for any $t \ge 0$ and $x \in \mathbb{R}$, we have

$$\delta^* \le \upsilon_i^+(t,x) \le K+\delta^*, \qquad -\delta^* \le \upsilon_i^-(t,x) \le K-\delta^*, \quad i=1,2.$$

Denote $\epsilon_1 = \epsilon_0, \epsilon_2 = \epsilon_c = \frac{\epsilon_0}{1+|c|r} \le \epsilon_0, C_1 = C_0, C_2 = C_c = (1+|c|r)C_0$. Set $\xi_i = x - C_i t$. Then for $i=1,2$ and $t \ge 0$, we have

$$\begin{aligned} S(\upsilon_i^-)(t,x) :=\ & \frac{\partial \upsilon_i^-(t,x)}{\partial t} - D\frac{\partial^2 \upsilon_i^-(t,x)}{\partial x^2} + d\upsilon_i^-(t,x) \\ & - \int_{-\infty}^{\infty} b(\upsilon_i^-(t,x-y)) f(y-cr)dy \\ =\ & -C_i\epsilon_i K\zeta'(\epsilon_i(\xi_i)) - D\epsilon_i^2 K\zeta''(\epsilon_i(\xi_i)) \\ & + d\upsilon_i^-(t,x) - \int_{-\infty}^{\infty} b(\upsilon_i^-(t,x-y)) f(y-cr)dy \\ =\ & -C_i\epsilon_i K\zeta'(\epsilon_i(\xi_i)) - D\epsilon_i^2 K\zeta''(\epsilon_i(\xi_i)) + d\upsilon_i^-(t,x) \\ & - \int_{-\infty}^{\infty} b(\upsilon_i^-(t,x-cr-y)) f(y)dy \\ =\ & -C_i\epsilon_i K\zeta'(\epsilon_i(\xi_i)) - D\epsilon_i^2 K\zeta''(\epsilon_i(\xi_i)) + d\upsilon_i^-(t,x) \\ & - b(\upsilon_i^-(t,x)) - \int_{-\infty}^{\infty} b'(\eta)[\upsilon_i^-(t,x-cr-y) - \upsilon_i^-(t,x)] f(y)dy \\ \le\ & -C_i\epsilon_i K\zeta'(\epsilon_i(\xi_i)) - D\epsilon_i^2 K\zeta''(\epsilon_i(\xi_i)) + d\upsilon_i^-(t,x) \\ & - b(\upsilon_i^-(t,x)) - Kb'_{\max}\int_{-\infty}^{\infty} |\zeta(\epsilon_i(\xi_i - cr - y)) - \zeta(\epsilon_i(\xi_i))| f(y)dy, \end{aligned} \tag{4.16}$$

where $\eta = \theta\upsilon_i^-(t,x) + (1-\theta)\upsilon_i^-(t,x-cr-y)$, and in the last inequality, we have used the estimate $|\upsilon_i^-(t,x-cr-y) - \upsilon_i^-(t,x)| \le K|\zeta(\epsilon_i(\xi_i - cr - y)) - \zeta(\epsilon_i(\xi_i))|$.

We distinguish among three cases:

Case (i): $\epsilon_i \xi_i \leq -2+\kappa/2$.

In this case, $\epsilon_i \xi_i \leq -2+\kappa$, $0 \leq \zeta(\epsilon_i \xi_i) \leq \epsilon^*/2$, and hence

$$-\delta^* \leq \upsilon_i^-(t,x) \leq -\delta^* + K\epsilon^*/2 < -\delta^* + (1-\varrho)\delta^* = -\varrho\delta^* \leq -\frac{1}{2}\delta^*.$$

Set $E_i(t,x) = \{y \in \mathbb{R};\ \upsilon_i^-(t, x-cr-y) \leq 0\}$. It then follows from (4.4), (4.13) and (4.16) that

$$\begin{aligned} S(\upsilon_i^-)(t,x) &\leq D\epsilon_i^2 K + d\upsilon_i^-(t,x) - \int_{E_i(t,x)} b'(\check{\eta})\upsilon_i^-(t,x-cr-y)f(y)dy \\ &\leq D\epsilon_i^2 K + d\upsilon_i^-(t,x) - \int_{E_i(t,x)} b'(\check{\eta})\upsilon_i^-(t,x-cr-y)f(y)dy \\ &\leq D\epsilon_i^2 K - d\varrho\delta^* + \delta^* \int_{-\infty}^{\infty} b'(\check{\eta})f(y)dy \\ &= D\epsilon_0^2 K - \delta^*[\varrho d - (b'(0)+\iota)] \\ &< 0. \end{aligned}$$

where $\check{\eta} = \theta\upsilon_i^-(t,x-cr-y) \in [-\delta^*, 0] \subset [-2\delta^*, 2\delta^*]$.

Case (ii): $\epsilon_i \xi_i \geq 2-\kappa/2$.

In this case, $\epsilon_i \xi_i \geq 2-\kappa$, $1-\epsilon^*/2 \leq \zeta(\epsilon_i \xi_i) \leq 1$, and hence

$$K-\delta^* \geq \upsilon_i^-(t,x) \geq -\delta^* + K(1-\epsilon^*/2) \geq K-\delta^* - (1-\varrho)\delta^* \geq K - \frac{3}{2}\delta^*.$$

It then follows that

$$d\upsilon_i^-(t,x) - b(\upsilon_i^-(t,x)) \leq -\min\left\{b(u)-du;\ \ u \in [K-\frac{3}{2}\delta^*, K-\delta^*]\right\} = -M_2.$$

By the Choice of ϵ_0 and ϖ, we see that

$$\varpi\xi_i \geq \frac{\varpi(2-\kappa/2)}{\epsilon_i} \geq \frac{\varpi(2-\kappa)}{\epsilon_0} \geq M_0.$$

Let $y \in [-\varpi\xi_i, \varpi\xi_i]$. Then for $c \leq 0$ and $i=1$, by (4.10), we find

$$\epsilon_1(\xi_1 - cr - y) \geq \epsilon_1(1-\varpi)\xi_1 - \epsilon_1 cr \geq (1-\varpi)(2-\kappa/2) > 2-\kappa$$

for $c \in \mathbb{R}$ and $i=2$, by (4.12), we find

$$\epsilon_2(\xi_2 - cr - y) \geq \epsilon_2(1-\varpi)\xi_2 - \epsilon_2 cr \geq (1-\varpi)(2-\kappa/2) - \epsilon_0 > 2-\kappa.$$

Noting that $\epsilon_i \xi_i \geq 2-\kappa/2 > 2-\kappa$, by (4.9), we get

$$\int_{-\varpi\xi_i}^{\varpi\xi_i} |\zeta((\epsilon_i(\xi_i - cr - y)) - \zeta(\epsilon_i(\xi_i))| f(y)dy \leq \epsilon^* \int_{-\varpi\xi_i}^{\varpi\xi_i} f(y)dy \leq \epsilon^*.$$

Therefore, it follows from (4.14) and (4.16) that

$$\begin{aligned} S(\upsilon_i^-)(t,x) &\le D\epsilon_i^2 K - M_2 \\ &\quad + Kb'_{\max}\int_{-\varpi\xi_i}^{\varpi\xi_i} |\zeta(\epsilon_i(\xi_i - cr - y)) - \zeta(\epsilon_i(\xi_i))| f(y)dy \\ &\quad + 2Kb'_{\max}\left[\int_{\varpi\xi_i}^{\infty} + \int_{-\infty}^{-\varpi\xi_i} f(y)dy\right] \\ &\le D\epsilon_0^2 K - M_2 + Kb'_{\max}\epsilon^* + 2Kb'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} f(y)dy\right] \\ &< 0. \end{aligned}$$

Case (iii): $-2+\kappa/2 \le \epsilon_i \xi_i \le 2-\kappa/2$.

In this case, by (4.15) and (4.16), we also have

$$\begin{aligned} S(\upsilon_i^-)(t,x) &\le -C_i\epsilon_i K\tilde{M} + D\epsilon_i^2 K \\ &\quad + \max\{|du - b(u)|;\, u\in[-\delta^*, K+\delta^*]\} + 2Kb'_{\max} \\ &\le -C_0\epsilon_0 K\tilde{M} + D\epsilon_0^2 K \\ &\quad + \max\{|du - b(u)|;\, u\in[-\delta^*, K+\delta^*]\} + 2Kb'_{\max} \\ &= 0. \end{aligned}$$

Combining Cases (i)–(iii), we have

$$\begin{aligned} &\frac{\partial \upsilon_i^-(t,x)}{\partial t} - D\frac{\partial^2 \upsilon_i^-(t,x)}{\partial x^2} + d\upsilon_i^-(t,x) \\ &\quad - \int_{-\infty}^{\infty} b(\upsilon_i^-(t,x-y)) f(y)dy \le 0, \quad t\ge 0, \ x\in\mathbb{R}. \end{aligned}$$

Thus for $c \le 0$, $\upsilon_1^-(t,x)$ is a subsolution of (4.1) on $[0,+\infty)$ and for all $c\in\mathbb{R}$, $\upsilon_2^-(t,x)$ is a subsolution of (4.1) on $[0,+\infty)$. In a similar way, we can prove that for $c\ge 0$, $\upsilon_1^+(t,x)$ is a supersolution of (4.1) on $[0,+\infty)$ and for all $c\in\mathbb{R}$, $\upsilon_2^+(t,x)$ is a supersolution of (4.1) on $[0,+\infty)$. This completes the proof. □

Lemma 4.2. *Assume that (H1)–(H3) hold. Then for every $c\in\mathbb{R}$, (4.1) admits a unique strictly monotonic traveling wavefront with speed $C(c)$ which is a continuous function of c.*

Proof. The proof for the existence of non-decreasing traveling wavefronts is similar to that of Theorem 5.1 in [1], and will be given in the appendix. It follows from Lemma 2.5 that the obtained traveling wavefront is strictly monotonic. The uniqueness follows directly from Theorem 2.7.

For any $c\in\mathbb{R}$, we denote by $U_c(x+C(c)t)$ the corresponding traveling wavefront with the wave speed $C(c)$. Without loss of generality, we assume that $U_c(0)=0$ for each $c\in\mathbb{R}$. Then we have

$$C(c)U_c'(\xi) = DU_c''(\xi) - dU_c(\xi) + \int_{-\infty}^{\infty} b(U_c(\xi-y)) f(y-cr)dy, \quad (4.17)$$

where $\xi = x + C(c)t$. Hence

$$U_c(\xi) = \frac{1}{D(\lambda_2(C(c)) - \lambda_1(C(c)))}\left[\int_{-\infty}^{\xi} e^{\lambda_1(C(c))(\xi-s)} H_c(U_c)(s)ds + \int_{\xi}^{\infty} e^{\lambda_2(C(c))(\xi-s)} H_c(U_c)(s)ds\right], \tag{4.18}$$

where

$$\lambda_1(C(c)) = \frac{C(c) - \sqrt{C^2(c)+4Dd}}{2D} < 0, \qquad \lambda_2(C(c)) = \frac{C(c) + \sqrt{C^2(c)+4Dd}}{2D} > 0$$

and

$$H_c(U_c)(s) = \int_{-\infty}^{\infty} b(U_c(s-y)) f(y-cr)dy. \tag{4.19}$$

Since $0 \le U_c(\xi) \le K$, and

$$\lambda_2(C(c)) - \lambda_1(C(c)) = \frac{\sqrt{C^2(c)+4Dd}}{D} \ge 2\sqrt{\frac{d}{D}},$$

it is easy to show that

$$|U_c'(\xi)| \le \frac{b(K)}{2\sqrt{Dd}} \tag{4.20}$$

for every $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$.

Suppose $c_n \to c$, but $C(c_n)$ does not converge to $C(c)$, then there exists a subsequence $c_{n_k} \to c$ so that $C(c_{n_k}) \to b \neq C(c)$. By the Arzela–Ascoli theorem, we can choose a subsequence of $\{c_{n_k}\}$, also denoting it by $\{c_{n_k}\}$, such that $U_{c_{n_k}}(\cdot)$ converges to a continuous function $\tilde{U}(\cdot)$ in $\mathbb{R}$. Let $H^* = \sup\{|c_n|\}$. Since $U_{c_{n_k}}(\cdot)$ is non-decreasing, $U_{c_{n_k}}(0) = 0$, and by (A.3) and (A.4) in Appendix, $U_{c_{n_k}}(\cdot)$ also satisfies

$$U_{c_{n_k}}(x) \le u^* - \delta_*, \quad \text{if} \quad x \le -M^* - L^*H^*r \le -M^* - L^*|c_{n_k}|r,$$

$$U_{c_{n_k}}(x) \ge u^* + \delta_*, \quad \text{if} \quad x \ge M^* + L^*H^*r \ge M^* + L^*|c_{n_k}|r,$$

it follows that $\tilde{U}(\cdot)$ is non-decreasing, $0 \le \tilde{U}(\cdot) \le K$ and,

$$\limsup_{x\to-\infty} \tilde{U}(x) \le u^* - \delta_* \tag{4.21}$$

and

$$\liminf_{x\to\infty} \tilde{U}(x) \ge u^* + \delta_*. \tag{4.22}$$

In Eq. (4.18) with c being replaced by c_{n_k}, we let $k \to \infty$ and apply the dominant convergence theorem to get

$$\tilde{U}(\xi) = \frac{1}{D(\lambda_2(b)) - \lambda_1(b))} \left[\int_{-\infty}^{\xi} e^{\lambda_1(b)(\xi - s)} H_c(\tilde{U})(s) ds + \int_{\xi}^{\infty} e^{\lambda_2(b)(\xi - s)} H_c(\tilde{U})(s) ds \right] \tag{4.23}$$

and hence $\tilde{U}(x+bt)$ is a solution of (4.1). By virtue of Theorem 3.5, we conclude that there exists a constant $\gamma > 0$ such that

$$|\tilde{U}(x+bt) - U_c(x + C(c)t + \xi_0)| \leq M_* e^{-\gamma t}, \quad t \geq 0 \tag{4.24}$$

for some $\xi_0 \in \mathbb{R}$ and $M_* > 0$.

If $b > C(c)$, keeping $x+bt$ fixed and letting $t \to \infty$ in (4.24)(and noting that $U_c(-\infty) = 0$), we get $\tilde{U}(\cdot) \equiv 0$, which contradicts to (4.22). If $b < C(c)$, a similar argument yields $\tilde{U}(\cdot) \equiv K$, which contradicts to (4.21). Therefore, we have $b = C(c)$, which is also a contradiction. This completes the proof. □

Theorem 4.3. *Assume that* (H1)–(H3) *hold. Then* (1.1) *admits a strictly monotonic traveling wavefront* $U(x+c^*t)$ *with* $|c^*| \leq C_0$, *where* C_0 *is given in Lemma 4.1.*

Proof. It is easy to see that if there exists a $c^* \in \mathbb{R}$ such that $C(c^*) = c^*$, and $U(x+c^*t)$ is a monotonic traveling wavefront of (4.1), then $U(x+c^*t)$ is also a traveling wavefront of (1.1). Therefore, it suffices to show that the curves $y = c$ and $y = C(c)$ have at least one common point in the (c, y) plane.

For $c \leq 0$, let $v_1^-(t, x)$ be the subsolution of (4.1) given in Lemma 4.1. Then there exists a large constant $X > 1$ such that $U(\cdot) \geq v_1^-(0, \cdot - X)$. Therefore, by the comparison, it follows that $U(x + C(c)t) \geq v_1^-(t, x - X)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Thus there exists a large $\bar{X}$, which is independent of t, such that $x + C(c)t \geq x - C_0 t - \bar{X}$ for all $t \geq 0$. Letting $t \to \infty$, we then obtain that $C(c) \geq -C_0$ for $c \leq 0$. Similarly, we can show that $C(c) \leq C_0$ for $c \geq 0$. Finally, we note that the common point c^* of two curves satisfies $|c^*| \leq C_0$. This completes the proof. □

APPENDIX

In this section, we sketch of the proof for the existence of traveling wavefront of (4.1).

Let $w(t,x)$ be the solution to the following initial value problem

$$\begin{aligned}\frac{\partial w(t,x)}{\partial t} &= D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x)\\ &\quad + \int_{-\infty}^{\infty} b(w(t,x-y))f(y-cr)dy, \quad t>0, \ x\in\mathbb{R},\\ w(0,x) &= K\zeta(x), \quad x\in\mathbb{R},\end{aligned} \tag{A.1}$$

where $c\in R$ is a parameter and $\zeta(\cdot)$ refers to the function ζ given in Section 3.

By Lemma 2.1, it can be shown that

$$w_x(t,x)>0, \quad \text{for all } (t,x)\in(0,\infty)\times\mathbb{R}$$

and since the constant δ^* in Lemma 4.1 can be chosen as small as we wish, by Lemma 4.1 and the comparison, it is not hard to show that

$$\lim_{x\to\infty} w(t,x)=K, \quad \text{and} \quad \lim_{x\to-\infty} w(t,x)=0, \quad \text{for all} \quad t>0.$$

Therefore, there exists a unique function $z_c(\alpha,t)$ defined on $(0,K)\times[0,\infty)$ such that

$$w(t,z_c(\alpha,t))=\alpha, \quad \alpha\in(0,K), \ \ t\in[0,\infty).$$

Following the idea in [1], we can show that, for some sequence $\{t_j\}_{j=1}^{\infty}$, the sequence $\{w(t_j,\cdot+z_c(u^*,t_j))\}_{j=1}^{\infty}$ has a limit $U_c(\cdot)$, which is the profile of a traveling wave front.

The proof for the existence of traveling wavefront of (4.1) is based on Propositions A.1 and A.6. In what follows, we only give proofs of Propositions A.1 and A.6, the rest of the proof is similar to that of Theorem 4.1 in [1] (see also Theorem 5.1), and thus is omitted.

The following Proposition A.1 is a directly consequence of Lemma A.5.

Proposition A.1. *Assume that (H1)–(H3) hold. Then there exist a small positive constant δ_*, two large positive constants M^* and $L^*\geq 1$, which are independent of c and r, such that*

$$z_c(u^*+\delta_*,t)-z_c(u^*-\delta_*,t)\leq M^*+L^*|c|r, \quad \text{for all } t\geq 0. \tag{A.2}$$

Remark. Clearly, it follows from (A.2) that, as a limit $w(t_j,\cdot+z_c(u^*,t_j))$ as $j\to\infty$, the function $U_c(\cdot)$ satisfies $U(0)=u^*$ and

$$U_c(x)\leq u^*-\delta_*, \quad \text{if} \quad x\leq -M^*-L^*|c|r \tag{A.3}$$

and

$$U_c(x)\geq u^*+\delta_*, \quad \text{if} \quad x\geq M^*+L^*|c|r. \tag{A.4}$$

Using the assertion of Proposition A.1, we can also derive the following.

(a) For every $\delta \in (0, \delta_*/2]$, there exists $m(\delta) > 0$ such that

$$z_c(K-\delta, t) - z_c(\delta, t) \leq m(\delta), \quad \text{for all } t > 0. \tag{A.5}$$

(b) For every $M > 0$, there exists a constant $\hat{\eta}(M) > 0$ such that

$$w_x(t, x + z_c(u^*, t)) \geq \hat{\eta}(M), \quad \text{for all } t \geq 1 \text{ and } x \in [-M, M]. \tag{A.6}$$

Lemma A.2. *Let w_{11} and w_{12} be the solutions to the following linear evolution problems*

$$\begin{aligned} \frac{\partial w_{11}(t,x)}{\partial t} &= D\frac{\partial^2 w_{11}(t,x)}{\partial x^2} - dw_{11}(t,x) + b'(u^*)\int_{-\infty}^{\infty} w_{11}(t, x-y) f(y-cr)dy, \\ w_{11}(0,x) &= H(x) \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} \frac{\partial w_{12}(t,x)}{\partial t} &= D\frac{\partial^2 w_{12}(t,x)}{\partial x^2} - dw_{12}(t,x) + b'(u^*)\int_{-\infty}^{\infty} w_{12}(t, x-y) f(y-cr)dy, \\ w_{12}(0,x) &= -1 + 2H(x) \end{aligned} \tag{A.8}$$

then there exist constants $\tau_0 > 0$, $\hat{\epsilon}_0 > 0$, which are independent of c and r, and $x_c \in \mathbb{R}$ with $|x_c| \leq 4(1+|c|r)\hat{\epsilon}_0^{-1}$ such that

$$w_{11}(\tau_0, x_c + 0) \geq 3, \quad w_{12}(\tau_0, x_c - 0) \leq -3$$

here and in the sequel, $H(x)$ is the Heaviside function equal to 1 when $x > 0$, 1/2 when $x = 0$ and 0 when $x < 0$, and $w_{ij}(\tau_0, x_c \pm 0) = \lim_{y \to x_c \pm} w_{ij}(\tau_0, y)$.

Proof. By (H3), we can show that the flows of the equation

$$\frac{\partial w(t,x)}{\partial t} = D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x) + b'(u^*)\int_{-\infty}^{\infty} w(t, x-y) f(y-cr)dy$$

also satisfies the comparison principle.

Let $\lambda^* = -d + b'(u^*)$. Then, by (H3), $\lambda^* > 0$ and $e^{\lambda^* t}$ is a solution of (A.5). Since $0 \leq w_{11}(0,x) = H(x) \leq 1$, by comparison, we have

$$0 \leq w_{11}(t,x) \leq e^{\lambda^* t}, \quad \text{for all } t \geq 0.$$

In addition, since for each $h > 0$, $w_{11}(0, \cdot + h) \geq w_{11}(0, \cdot)$, we have $w_{11}(t, \cdot + h) \geq w_{11}(t, \cdot)$ for all $t \geq 0$. That is, w_{11} is non-decreasing in $x \in \mathbb{R}$. Thus, for any $x \in \mathbb{R}$, $w_{11}(t, x \pm 0) = \lim_{y \to x\pm} w_{11}(t, y)$ exist.

Take a small positive constant $\hat{\epsilon}_0 > 0$ such that

$$\rho(\hat{\epsilon}_0) := \frac{1}{\lambda^*}[D\hat{\epsilon}_0^2 + \hat{\epsilon}_0 b'(u^*)\int_{-\infty}^{\infty}|y|f(y)dy + \hat{\epsilon}_0 b'(u^*)] < \frac{1}{81}.$$

Let $\hat{\epsilon}_c = \frac{\hat{\epsilon}}{1+|c|r}$ and $w(t,x) = \rho(\hat{\epsilon}_0)e^{2\lambda^* t} + \zeta(\hat{\epsilon}_c x)e^{\lambda^* t}$. Then

$$\begin{aligned}
S(w)(t,x) &= \frac{\partial w}{\partial t} - D\frac{\partial^2 w}{\partial x^2} + dw - b'(u^*)\int_{-\infty}^{\infty} w(t,x-y)f(y-cr)dy \\
&= (2\lambda^* + d)\rho(\hat{\epsilon}_0)e^{2\lambda^* t} + (\lambda^* + d)\zeta(\hat{\epsilon}_c x)e^{\lambda^* t} - D\hat{\epsilon}_c^2\zeta''(\hat{\epsilon}_c x)e^{\lambda^* t} \\
&\quad -b'(u^*)\int_{-\infty}^{\infty}(\rho(\hat{\epsilon}_0)e^{2\lambda^* t} + \zeta(\hat{\epsilon}_c(x-y))e^{\lambda^* t})f(y-cr)dy \\
&\geq [2\lambda^* + d - b'(u^*)]\rho(\hat{\epsilon}_0)e^{2\lambda^* t} + [(\lambda^* + d)\zeta(\hat{\epsilon}_c x) \\
&\quad -D\hat{\epsilon}_c^2 - b'(u^*)\int_{-\infty}^{\infty}\zeta(\hat{\epsilon}_c(x-y))f(y-cr)dy]e^{\lambda^* t} \\
&\geq \lambda^*\rho(\hat{\epsilon}_0)e^{2\lambda^* t} \\
&\quad +[-D\hat{\epsilon}_c^2 + b'(u^*)\int_{-\infty}^{\infty}[\zeta(\hat{\epsilon}_c x) - \zeta(\hat{\epsilon}_c(x-y)]f(y-cr)dy]e^{\lambda^* t} \\
&= \lambda^*\rho(\hat{\epsilon}_0)e^{2\lambda^* t} + [-D\hat{\epsilon}_c^2 + \hat{\epsilon}_c b'(u^*)\int_{-\infty}^{\infty}\zeta'(\eta)yf(y-cr)dy]e^{\lambda^* t} \\
&\geq \lambda^*\rho(\hat{\epsilon}_0)e^{2\lambda^* t} - [D\hat{\epsilon}_c^2 + \hat{\epsilon}_c b'(u^*)\int_{-\infty}^{\infty}|y|f(y-cr)dy]e^{\lambda^* t} \\
&\geq \{\lambda^*\rho(\hat{\epsilon}_0)e^{\lambda^* t} - [D\hat{\epsilon}_c^2 + \hat{\epsilon}_c b'(u^*)\int_{-\infty}^{\infty}|y|f(y)dy + \hat{\epsilon}_c b'(u^*)|c|r]\}e^{\lambda^* t} \\
&\geq \{\lambda^*\rho(\hat{\epsilon}_0) - [D\hat{\epsilon}_0^2 + \hat{\epsilon}_0 b'(u^*)\int_{-\infty}^{\infty}|y|f(y)dy + \hat{\epsilon}_0 b'(u^*)]\}e^{\lambda^* t} \\
&= 0.
\end{aligned}$$

Hence, $w(t,x) = \rho(\hat{\epsilon}_0)e^{2\lambda^* t} + \zeta(\hat{\epsilon}_c x)e^{\lambda^* t}$ is a supersolution of (A.7). Since $w_{11}(0,x) = H(x) \leq \zeta(\hat{\epsilon}_c x + 2) \leq w(0, x + 2\hat{\epsilon}_c^{-1})$, $x \in \mathbb{R}$, the comparison yields $w_{11}(t,x) \leq w(t, x + 2\hat{\epsilon}_c^{-1})$ in $[0,\infty) \times \mathbb{R}$. That is,

$$w_{11}(t,x) \leq \rho(\hat{\epsilon}_0)e^{2\lambda^* t} + \zeta(\hat{\epsilon}_c(x + 2\hat{\epsilon}_c^{-1}))e^{\lambda^* t}.$$

Consequently,

$$0 \leq w_{11}(t,x) \leq \rho(\hat{\epsilon}_0)e^{2\lambda^* t}, \quad \text{if} \quad x \leq -4\hat{\epsilon}_c^{-1}.$$

By using a similar argument, we can shows that $w(t,x) = (1 - \zeta(-\hat{\epsilon}_c x))e^{\lambda^* t} - \rho(\hat{\epsilon}_0)e^{2\lambda^* t}$ is a subsolution of (A.7). Since $w_{11}(0,x) = H(x) \geq 1 - \zeta(-\hat{\epsilon}_c x + 2) \geq w(0, x - 2\hat{\epsilon}_c^{-1})$. By the comparison, we have

$$w_{11}(t,x) \geq w(t, x - 2\hat{\epsilon}_c^{-1}) = (1 - \zeta(-\hat{\epsilon}_c x + 2))e^{\lambda^* t} - \rho(\hat{\epsilon}_0)e^{2\lambda^* t},$$

which implies that

$$e^{\lambda^* t} \geq w_{11}(t,x) \geq e^{\lambda^* t} - \rho(\hat{\epsilon}_0)e^{2\lambda^* t}, \quad \text{if} \quad x \geq 4\hat{\epsilon}_c^{-1}.$$

Now set $\tau_0 = \frac{1}{\lambda^*}\ln 9 > 0$, so that $e^{\lambda^*\tau_0} = 9$. Therefore, by the choice of $\hat{\epsilon}_0$, we have

$$0 \leq w_{11}(\tau_0, x) \leq 81\rho(\hat{\epsilon}_0) \leq 1, \quad \text{if} \quad x \leq -4(1+|c|r)\hat{\epsilon}_0^{-1},$$

$$8 \leq 9 - 81\rho(\hat{\epsilon}_0) \leq w_{11}(\tau_0, x) \leq 9, \quad \text{if} \quad x \geq 4(1+|c|r)\hat{\epsilon}_0^{-1}.$$

By the monotonicity of $w_{11}(\tau_0, \cdot)$, there exists $x_c \in \mathbb{R}$ with $|x_c| \leq 4(1 + |c|r)\hat{\epsilon}_0^{-1}$ such that

$$w_{11}(\tau_0, x_c + 0) \geq 3, \qquad w_{11}(\tau_0, x_c - 0) \leq 3.$$

Using the identity $w_{12}(t,x) = -e^{\lambda^* t} + 2w_{11}(t,x)$ on $[0,\infty) \times \mathbb{R}$, we also have

$$w_{12}(\tau_0, x_c - 0) = -e^{\lambda^*\tau_0} + 2w_{11}(\tau_0, x_c - 0) \leq -3.$$

This completes the proof. □

Lemma A.3. *There exists a small positive constant $\delta_* > 0$, which is independent of c and r, such that the solutions w_{21} and w_{22} to*

$$\begin{aligned}\frac{\partial w_{21}(t,x)}{\partial t} &= D\frac{\partial^2 w_{21}(t,x)}{\partial x^2} - dw_{21}(t,x) \\ &\quad + \int_{-\infty}^{\infty} b(w_{21}(t,x-y))f(y-cr)dy, \\ w_{21}(0,x) &= u^* + \delta_* H(x),\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}\frac{\partial w_{22}(t,x)}{\partial t} &= D\frac{\partial^2 w_{22}(t,x)}{\partial x^2} - dw_{22}(t,x) \\ &\quad + \int_{-\infty}^{\infty} b(w_{22}(t,x-y))f(y-cr)dy, \\ w_{22}(0,x) &= u^* + \delta_*[-1+2H(x)]\end{aligned} \tag{A.10}$$

satisfy

$$w_{21}(\tau_0, x_c + 0) \geq u^* + 2\delta_*, \qquad w_{22}(\tau_0, x_c - 0) \leq u^* - 2\delta_*.$$

Proof. Let $\lambda > \lambda^*$ be fixed and $w(t,x) = u^* + \delta_* w_{12}(t,x) + \delta_*^2 e^{\lambda t}$. Since $0 \le w_{11}(t,x) \le e^{\lambda^* t}$, we have

$$-e^{\lambda^* t} \le w_{12}(t,x) = -e^{\lambda^* t} + 2w_{11}(t,x) \le e^{\lambda^* t}$$

and hence,

$$u^* - \delta_* e^{\lambda^* t} + \delta_*^2 e^{\lambda t} \le w(t,x) \le u^* + \delta_* e^{\lambda^* t} + \delta_*^2 e^{\lambda t}$$

for all $t \ge 0$ and $x \in \mathbb{R}$. Therefore, for $t \in [0, \tau_0]$, $|w(t,x) - u^*| \le B_*$ for some constant B_*, and hence there exists a small constant $\delta_* > 0$, which is independent of c and r, such that

$$\delta_* e^{\lambda \tau_0} \le 1$$

and

$$\begin{aligned}
S(w)(t,x) &= \frac{\partial w}{\partial t} - D\frac{\partial^2 w}{\partial x^2} + dw - \int_{-\infty}^{\infty} b(w(t,x-y)) f(y-cr)dy \\
&= \delta_*[\frac{\partial w_{12}}{\partial t} - D\frac{\partial^2 w_{12}}{\partial x^2} + dw_{12}] + \delta_*^2(\lambda + d)e^{\lambda t} \\
&\quad + du^* - \int_{-\infty}^{\infty} b(w(t,x-y)) f(y-cr)dy \\
&= \delta_* b'(u^*) \int_{-\infty}^{\infty} w_{12}(t,x-y) f(y-cr)dy + \delta_*^2(\lambda + d)e^{\lambda t} \\
&\quad - b'(u^*) \int_{-\infty}^{\infty} [w(t,x-y) - u^*] f(y-cr)dy + o(\delta_*^2) \\
&= -\delta_*^2 b'(u^*) e^{\lambda t} + \delta_*^2(\lambda + d)e^{\lambda t} + o(\delta_*^2) \\
&= \delta_*^2[\lambda + d - b'(u^*)]e^{\lambda t} + o(\delta_*^2) > 0 \\
&\ge \delta_*^2[\lambda - \lambda^*] + o(\delta_*^2) > 0.
\end{aligned}$$

Therefore, $w(t,x) = u^* + \delta_* w_{12}(t,x) + \delta_*^2 e^{\lambda t}$ is a supersolution of (A.10) on $[0, \tau_0]$. Moreover, we have

$$\begin{aligned}
w_{22}(0,x) &= u^* + \delta_*[-1 + 2H(x)] \\
&= u^* + \delta_* w_{12}(0,x) \le w(0,x).
\end{aligned}$$

Therefore, the comparison implies that $w_{22}(t,x) \le w(t,x), \forall t \in [0, \tau_0]$ and $x \in \mathbb{R}$. In particular, we have

$$\begin{aligned}
w_{22}(\tau_0, x_c - 0) &\le w(\tau_0, x_c - 0) \\
&= u^* + \delta_* w_{12}(\tau_0, x_c - 0) + \delta_*^2 e^{\lambda \tau_0} \\
&\le u^* - 3\delta_* + \delta_* = u^* - 2\delta_*.
\end{aligned}$$

In a similar way, we can show that $w(t,x)=u^*+\delta_* w_{11}(t,x)-\delta_*^2 e^{\lambda t}$ is subsolution of (A.9) on $[0,\tau_0]$. Since $w_{21}(0,x)=u^*+\delta_* H(x)=u^*+\delta_* w_{11}(0,x)\geq w(0,x)$, by the comparison, we have

$$\begin{aligned} w_{21}(\tau_0, x_c+0) &\geq w(\tau_0, x_c+0) \\ &= u^*+\delta_* w_{11}(\tau_0, x_c+0)-\delta_*^2 e^{\lambda\tau_0} \\ &\geq u^*+3\delta_*-\delta_*=u^*+2\delta_*. \end{aligned}$$

The proof is complete.

Lemma A.4. *There exist a large positive constant h_0, which is independent of c and r, such that for $h_c=(1+|c|r)h_0$, the solutions w_{31} and w_{32} to*

$$\begin{aligned} \frac{\partial w_{31}(t,x)}{\partial t} &= D\frac{\partial^2 w_{31}(t,x)}{\partial x^2}-dw_{31}(t,x) \\ &\quad +\int_{-\infty}^{\infty} b(w_{31}(t,x-y))f(y-cr)dy, \qquad \text{(A.11)} \\ w_{31}(0,x) &= u^*+\delta_* H(x)-u^* H(-x-h_c) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial w_{32}(t,x)}{\partial t} &= D\frac{\partial^2 w_{32}(t,x)}{\partial x^2}-dw_{32}(t,x) \\ &\quad +\int_{-\infty}^{\infty} b(w_{32}(t,x-y))f(y-cr)dy, \qquad \text{(A.12)} \\ w_{32}(0,x) &= u^*+\delta_*[-1+2H(x)]+[K-u^*-\delta_*]H(x-h_c) \end{aligned}$$

satisfy

$$w_{31}(\tau_0, x_c+0)\geq u^*+\delta_*, \qquad w_{32}(\tau_0, x_c-0)\leq u^*-\delta_*.$$

Proof. Let $\lambda=-d+b'_{\max}$. Then $\lambda>0$. Let

$$w(t,x)=w_{21}(t,x)+\psi(t,x),$$

$$\psi(t,x)=-\hat{\varepsilon}_0^{1/2}e^{2\lambda t}-u^* e^{\lambda t}\zeta(-\hat{\varepsilon}_c(x-x_c)-2),$$

where $\hat{\varepsilon}_0$ is a small positive constant and $\hat{\varepsilon}_c=\frac{\hat{\varepsilon}_0}{1+|c|r}$ satisfying

$$\hat{\varepsilon}_0^{1/2}e^{2\lambda\tau_0}\leq\delta_*$$

and

$$\lambda\geq[D\hat{\varepsilon}_0^{3/2}+\hat{\varepsilon}_0^{1/2}(\lambda+d)\int_{-\infty}^{\infty}|y|f(y)dy+\hat{\varepsilon}_0^{1/2}(\lambda+d)]u^*.$$

Denote $\xi_c = -\hat{\varepsilon}_c(x-x_c)-2$. Then for $t\in[0,\tau_0]$, we have

$$
\begin{aligned}
S(w)(t,x) &= \frac{\partial w}{\partial t} - D\frac{\partial^2 w}{\partial x^2} + dw - \int_{-\infty}^{\infty} b(w(t,x-y))f(y-cr)dy \\
&= \frac{\partial w_{21}}{\partial t} - D\frac{\partial^2 w_{21}}{\partial x^2} + dw_{21} + \frac{\partial \psi}{\partial t} - D\frac{\partial^2 \psi}{\partial x^2} + d\psi \\
&\quad - \int_{-\infty}^{\infty} b(w(t,x-y))f(y-cr)dy \\
&= \frac{\partial \psi}{\partial t} - D\frac{\partial^2 \psi}{\partial x^2} + d\psi \\
&\quad + \int_{-\infty}^{\infty} [b(w_{21}(t,x-y)) - b(w(t,x-y))]f(y-cr)dy \\
&\leq -(2\lambda+d)\hat{\varepsilon}_0^{1/2}e^{2\lambda t} - [(\lambda+d)\zeta(\xi_c) - D\hat{\varepsilon}_c^2\zeta''(\xi_c)]u^*e^{\lambda t} \\
&\quad - b'_{\max}\int_{-\infty}^{\infty} \psi(t,x-y)f(y-cr)dy \\
&\leq -(2\lambda+d-b'_{\max})\hat{\varepsilon}_0^{1/2}e^{2\lambda t} - [(\lambda+d)\zeta(\xi_c) - D\hat{\varepsilon}_c^2]u^*e^{\lambda t} \\
&\quad + b'_{\max}u^*e^{\lambda t}\int_{-\infty}^{\infty} \zeta(\xi_c+\hat{\varepsilon}_c y)f(y-cr)dy \\
&\leq -\lambda\hat{\varepsilon}_0^{1/2}e^{2\lambda t} - [(\lambda+d)\zeta(\xi_c) - D\hat{\varepsilon}_c^2 \\
&\quad - b'_{\max}\int_{-\infty}^{\infty} \zeta(\xi_c+\hat{\varepsilon}_c y)f(y-cr)dy]u^*e^{\lambda t} \\
&= -\lambda\hat{\varepsilon}_0^{1/2}e^{2\lambda t} - [(\lambda+d)\zeta(\xi_c) - D\hat{\varepsilon}_c^2 \\
&\quad -(\lambda+d)\int_{-\infty}^{\infty} \zeta(\xi_c+\hat{\varepsilon}_c y)f(y-cr)dy]u^*e^{\lambda t} \leq -\lambda\hat{\varepsilon}_0^{1/2}e^{2\lambda t} \\
&\quad + [D\hat{\varepsilon}_c^2 - (\lambda+d)\int_{-\infty}^{\infty} [\zeta(\xi_c) - \zeta(\xi_c+\hat{\varepsilon}_c y)]f(y-cr)dy]u^*e^{\lambda t} \\
&\leq -\lambda\hat{\varepsilon}_0^{1/2}e^{2\lambda t} + [D\hat{\varepsilon}_c^2 - (\lambda+d)\int_{-\infty}^{\infty} \zeta'(\eta)[-\hat{\varepsilon}_c y]f(y-cr)dy]u^*e^{\lambda t} \\
&\leq -\lambda\hat{\varepsilon}_0^{1/2}e^{2\lambda t} + [D\hat{\varepsilon}_c^2 + \hat{\varepsilon}_c(\lambda+d)\int_{-\infty}^{\infty} |y|f(y-cr)dy]u^*e^{\lambda t} \\
&\leq -\{\lambda\hat{\varepsilon}_0^{1/2}e^{\lambda t} - [D\hat{\varepsilon}_c^2 + \hat{\varepsilon}_c(\lambda+d)\int_{-\infty}^{\infty} |y|f(y)dy + \hat{\varepsilon}_c(\lambda+d)|c|r]u^*\}e^{\lambda t} \\
&\leq -\{\lambda\hat{\varepsilon}_0^{1/2} - [D\hat{\varepsilon}_0^2 + \hat{\varepsilon}_0(\lambda+d)\int_{-\infty}^{\infty} |y|f(y)dy + \hat{\varepsilon}_0(\lambda+d)]u^*\}e^{\lambda t} \\
&\leq 0.
\end{aligned}
$$

Therefore, $w(t,x)$ is a subsolution of (A.11) on $[0,\tau_0]\times\mathbb{R}$. Clearly, if we take $h_0=4(\hat{\varepsilon}_0^{-1}+\hat{\epsilon}_0^{-1})$, then $h_c=(1+|c|r)h_0\geq 4\hat{\varepsilon}_c^{-1}+|x_c|$ and hence,

$$\begin{aligned} w(0,x) &= w_{21}(0,x)+\psi(0,x)\\ &= w_{21}(0,x)-\hat{\varepsilon}^{1/2}-u^*\zeta(-\hat{\varepsilon}_c(x-x_c)-2)\\ &\leq w_{21}(0,x)-u^*\zeta(-\hat{\varepsilon}_c(x-x_c)-2)\\ &\leq w_{31}(0,x)\end{aligned}$$

so that by the comparison, we have

$$\begin{aligned} w_{31}(\tau_0,x_c+0) &\geq w(\tau_0,x_c+0)=w_{21}(\tau_0,x_c+0)-\hat{\varepsilon}^{1/2}e^{2\lambda\tau_0}\\ &\geq u^*+2\delta_*-\delta_*=u^*+\delta_*.\end{aligned}$$

Let

$$w(t,x)=w_{22}(t,x)+\psi(t,x),$$

$$\psi(t,x)=\hat{\varepsilon}_0^{1/2}e^{2\lambda t}+[K-u^*-\delta_*]e^{\lambda t}\zeta(\hat{\varepsilon}_c(x-x_c)-2).$$

Then by using a similar argument, we can show that $w(t,x)$ is a supersolution of (A.12) on $[0,\tau_0]\times\mathbb{R}$, and

$$\begin{aligned} w(0,x) &= w_{22}(0,x)+\psi(0,x)\\ &= w_{22}(0,x)+\hat{\varepsilon}^{1/2}+[K-u^*-\delta_*]\zeta(\hat{\varepsilon}_c(x-x_c)-2)\\ &\geq w_{22}(0,x)+[K-u^*\delta_*]\zeta(\hat{\varepsilon}_c(x-x_c)-2)\\ &\geq w_{32}(0,x),\end{aligned}$$

so that by the comparison, we have

$$\begin{aligned} w_{32}(\tau_0,x_c-0) &\leq w(\tau_0,x_c-0)=w_{22}(\tau_0,x_c-0)+\hat{\varepsilon}_0^{1/2}e^{2\lambda\tau_0}\\ &\leq u^*-2\delta_*+\delta_*=u^*-\delta_*.\end{aligned}$$

The proof is complete. □

Lemma A.5. *Let $w(t,x)$ be the solution of* (A.1)*, then for every $t>0$, there exist $\xi_-(t)$ and $\xi_+(t)$ such that*

$$w(t,\xi_-(t))=u^*-\delta_*,\qquad w(t,\xi_+(t))=u^*+\delta_*$$

and

$$\xi_+(t)-\xi_-(t)\leq\max\{4+2(1+|c|r)(4\epsilon_0^{-1}+C_0\tau_0),2h_c\},$$

where $h_c=(1+|c|r)h_0$, ϵ_0 and C_0 are given in Lemma 4.1.

Proof. Let $\upsilon_2^{\pm}(t,x)$ be as in Lemma 4.1. We note that

$$\upsilon_2^-(0, x-2-2\epsilon_c^{-1}) \le w(0,x) = K\zeta(x) \le \upsilon_2^+(0, x+2+2\epsilon_c^{-1})$$

and hence, by the comparison, we have

$$\upsilon_2^-(t, x-2-2\epsilon_c^{-1}) \le w(t,x) \le \upsilon_2^+(t, x+2+2\epsilon_c^{-1}), \quad \text{for all} \quad t \ge 0.$$

Therefore, for every $t \ge 0$, $\xi_+(t)$ and $\xi_-(t)$ exist and satisfy

$$\xi_+(t) - \xi_-(t) \le 4 + 2(1+|c|r)(4\epsilon_0^{-1} + C_0 t).$$

In particular, for all $t \in [0, \tau_0]$, we have

$$\xi_+(t) - \xi_-(t) \le 4 + 2(1+|c|r)(4\epsilon_0^{-1} + C_0 \tau_0).$$

To finish the proof, we need only prove the following: for every $t_1 \ge 0$, $\xi_+(t_1+\tau_0) - \xi_-(t_1+\tau_0) \le \max\{\xi_+(t_1) - \xi_-(t_1), 2h_c\}$.

By translation, we can assume that $w(t_1, 0) = u^*$, so that $\xi_-(t) < 0 < \xi_+(t)$. By symmetry, we need only consider the case $\xi_+(t) \ge |\xi_-(t)|$.

Set $h_+ = \max\{\xi_+(t_1), h_c\}$. Then, $w(t_1, \cdot + h_+) \ge w_{31}(0, \cdot)$ in $\mathbb{R}$, so that, by the comparison,

$$w(t_1+\tau_0, x_c + h_+) \ge w_{31}(\tau_0, x_c + 0) \ge u^* + \delta_*,$$

which implies that $\xi_+(t_1+\tau_0) \le x_c + h_+$.

Set $h_- = \max\{\xi_+(t_1) - \xi_-(t_1), h_c\}$. Then $w(t_1, \cdot + \xi_+(t_1) - h_-) \le w_{31}(0, \cdot)$, and hence, by the comparison,

$$w(t_1+\tau_0, x_c + \xi_+(t_1) - h_-) \le w_{32}(\tau_0, x_c - 0) \le u^* - \delta_*.$$

Therefore, $\xi_-(t_1+\tau_0) \ge x_c + \xi_+(t_1) - h_-$.

Combining the two estimates for $\xi_+(t_1+\tau_0)$ and $\xi_-(t_1+\tau_0)$, we have

$$\xi_+(t_1+\tau_0) - \xi_-(t_1+\tau_0) \le h_+ - \xi_+(t_1) + h_- \le \max\{\xi_+(t_1) - \xi_-(t_1), 2h_c\}.$$

This completes the proof. □

Proposition A.6. *Let $w(t,x)$ be the solution of* (A.1)*, then there exist three positive constants β, σ_1 and $\delta_1 > 0$ such that for every $\delta \in (0, \delta_1]$, every $T > 0$ and every $\xi \in \mathbb{R}$, the functions $w^+(t,x)$ and $w^-(t,x)$ defined by*

$$w^{\pm}(t,x) = w(x + \xi \pm \sigma_1 \delta(e^{\beta r} - e^{-\beta t}), t+T) \pm \delta e^{-\beta t}$$

are, respectively, a supersolution and a subsolution of (A.1) *in* $(0, +\infty) \times \mathbb{R}$.

Proof. Proposition A.6 can be proved in a similar way as used in the proof of Lemma 2.6 and the details will be omitted. □

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