

The pricing of options for securities markets with delayed response

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Abstract

The analogue of Black–Scholes formula for vanilla call option price in conditions of (B, S) -securities market with delayed response is derived. A special case of continuous-time version of GARCH is considered. The results are compared with the results of Black and Scholes.

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1. Introduction

In the early 1970s, Black and Scholes [3] made a major breakthrough by deriving pricing formulas for vanilla options written on the stock. Their model and its extensions assume that the probability distribution of the underlying cash flow at any given future time is lognormal. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see [16]), and the assumption of constant volatility σ in financial model (such as the original Black–Scholes model) is incompatible with derivatives prices observed in the market.

The above issues have been addressed and studied in several ways, such as

- (i) volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$ (see [31]);
- (ii) volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$ (see [16]); the dynamics of the stock price satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t) dt + \sigma(t, S(t)) S(t) dW_1(t),$$

where $W_1(t)$ is a standard Wiener process;

- (iii) the time variation of the volatility involves an additional source of randomness represented by $W_2(t)$ and is given by

$$d\sigma(t) = a(t, \sigma(t)) dt + b(t, \sigma(t)) dW_2(t),$$

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where $W_2(t)$ and $W_1(t)$ (the initial Wiener process that governs the price process) may be correlated (see [5,17]);

- (iv) the volatility depends on a random parameter x such as $\sigma(t) \equiv \sigma(x(t))$, where $x(t)$ is some random process (see [13,27–29]).

In the approach (i), the volatility coefficient is independent of the current level of the underlying stochastic process $S(t)$. This is a deterministic volatility model, and the special case where σ is a constant reduces to the well-known Black–Scholes model that suggests changes in stock prices are lognormally distributed. But the empirical test by Bollerslev [4] seems to indicate otherwise. One explanation for this problem of a lognormal model is the possibility that the variance of $\log(S(t)/S(t-1))$ changes randomly. This motivated the work of Chesney and Scott [7], where the prices are analyzed for European options using the modified Black–Scholes model of foreign currency options and a random variance model. In their works the results of Hull and White [17], Scott [24] and Wiggins [30] were used in order to incorporate randomly changing variance rates.

In the approach (ii), several ways have been developed to derive the corresponding Black–Scholes formula: one can obtain the formula by using stochastic calculus and, in particular, the Ito’s formula (see [23], for example). In the book by Cox and Rubinstein [8], an alternative approach was developed: the Black–Scholes formula is interpreted as the continuous-time limit of a binomial random model. A generalized volatility coefficient of the form $\sigma(t, S(t))$ is said to be *level-dependent*. Because volatility and asset price are perfectly correlated, we have only one source of randomness given by $W_1(t)$. A time and level-dependent volatility coefficient makes the arithmetic more challenging and usually precludes the existence of a closed-form solution. However, the *arbitrage argument* based on portfolio replication and a completeness of the market remain unchanged.

The situation becomes different if the volatility is influenced by a second “nontradable” source of randomness. This is addressed in the approach (iii) and (iv) and one usually obtains a *stochastic volatility model*, which is general enough to include the deterministic model as a special case. The concept of stochastic volatility was introduced by Hull and White [17], and subsequent development includes the work of Wiggins [30], Johnson and Shanno [18], Scott [24], Stein and Stein [26] and Heston [14]. We also refer to Frey [11] for an excellent survey on level-dependent and stochastic volatility models. We should mention that the approach (iv) is taken by, for example, Griego and Swishchuk [13].

There is yet another approach connected with stochastic volatility, namely, uncertain volatility scenario (see [5]). This approach is based on the uncertain volatility model developed in Avelanda et al. [2], where a concrete volatility surface is selected among a candidate set of volatility surfaces. This approach addresses the sensitivity question by computing an upper bound for the value of the portfolio under arbitrary candidate volatility, and this is achieved by choosing the local volatility $\sigma(t, S(t))$ among two extremal values σ_{\min} and σ_{\max} such that the value of the portfolio is maximized locally.

Assumption made implicitly by Black and Scholes [3] is that the historical performance of the (B, S) -securities markets can be ignored. In particular, the so-called efficient market hypothesis implies that all information available is already reflected in the present price of the stock and the past stock performance gives no information that can aid in predicting future performance. However, some statistical studies of stock prices (see [25,1]) indicate the dependence on past returns. For example, Kind et al. [20] obtained a diffusion approximation result for processes satisfying some equations with past-dependent coefficients, and they applied this result to a model of option pricing, in which the underlying asset price volatility depends on the past evolution to obtain a generalized (asymptotic) Black–Scholes formula. Hobson and Rogers [15] suggested a new class of nonconstant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. The volatility is nonconstant and can be regarded as an endogenous factor in the sense that it is defined in terms of the *past behavior* of the stock price. This is done in such a way that the price and volatility form a multi-dimensional Markov process.

Chang and Yoree [6] studied the pricing of an European contingent claim for the (B, S) -securities markets with a hereditary price structure in the sense that the rate of change of the unit price of the bond account and rate of change of the stock account S depend not only on the current unit price but also on their historical prices. The price dynamics for the bank account and that of the stock account are described by a linear functional differential equation and a linear stochastic functional differential equation, respectively. They show that the rational price for an European contingent claim is independent of the mean growth rate of the stock.

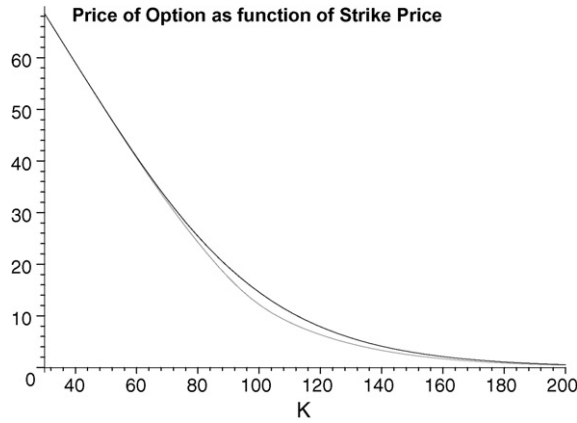


Fig. 1. The upper curve is the original Black–Scholes price and the lower curve is the option price given by the formula (5.5). Here $S(0) = 100$, $r = 0.05$, $\sigma(0) = 0.316$, $T = 1$, $V = 0.127$, $\alpha = 0.0626$, $\gamma = 0.0428$, $\tau = 0.002$.

Other alternatives to the Black–Scholes model include models where a company’s equity is assumed to be an option of its assets (see [12]); models of (B, S, X) -securities market with Markov or semi-Markov stochastic volatility $\sigma(x(t))$ (see [13,27]); models of fractional (B, S) -securities markets with Hurst index $H > 1/2$, $H \in (0, 1)$ or combined with the assumption of Markov or semi-Markov volatility (see [10]).

Clearly related to our work is the work by Mohammed et al. [22] devoted to the derivation of a delayed Black–Scholes formula for the (B, S) -securities market using PDE approach. In their paper, the stock price satisfies the following equation:

$$dS(t) = \mu S(t - a)S(t) dt + \sigma(S(t - b))S(t) dW(t),$$

where a and b are positive constants and σ is a continuous function, and the price of the option at time t has the form $F(t, S(t))$.

The subject of our work is the study of stochastic delay differential equations (SDDE), which arise in the pricing of options for security markets with delayed response. We show that a continuous-time equivalent of GARCH arises as a stochastic volatility model with delayed dependence on the stock value. We derive an analogue of Ito’s lemma for a general type of SDDEs and we obtain an integro-differential equation for a function of option price with boundary conditions specified according to the type of option to be priced. This equation is solved using a simplifying assumption and the graph of the closed-form solution is shown in Fig. 1. An implied volatility plot is generated to demonstrate the difference between Black–Scholes model and our model (see Fig. 2). Hobson and Rogers [15] also observed in their past-dependent model that the resulting implied volatility is U-shaped as a function of strike price. However, they dealt

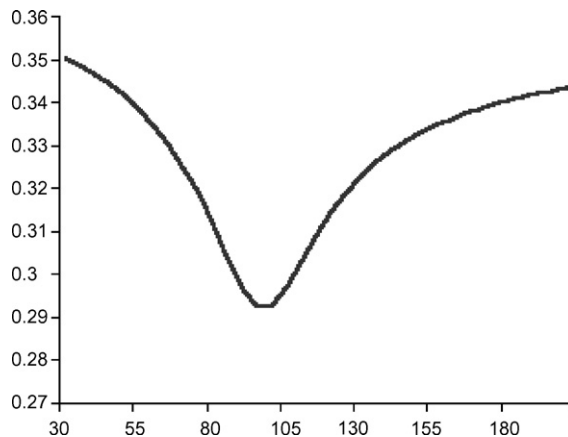


Fig. 2. Implied volatility of the call option price computed by (5.5) vs. strike price. The set of parameters is the same as for Fig. 1.

with only a special case where the model can be reduced to a system of SDEs. Unfortunately, not every past-dependent model can be reduced to a system of SDEs, and a more sophisticated approach, as developed in this paper, is needed.

More precisely, we consider the model of $S(t)$ with volatility σ depending on t and the path $S_t = \{S(t + \theta), \theta \in [-\tau, 0]\}$. We call it a *level-and-past-dependent volatility* $\sigma \equiv \sigma(t, S_t)$, contrary to the level-dependent volatility (that is clearly a special case of the former one when the time delay parameter $\tau = 0$). Our model of stochastic volatility exhibits past-dependence: the behavior of a stock price right after a given time t not only depends on the situation at t , but also on the whole past (history) of the process $S(t)$ up to time t . This draws some similarities with fractional Brownian motion models due to a long-range dependence property. Our work is also based on the GARCH(1,1) model (see [4]) and the celebrated work of Duan [9] where he showed that it is possible to use the GARCH model as the basis for an internally consistent option pricing model. We should mention that in the work of Kind et al. [20], a past-dependent model was defined by diffusion approximation. In their model, the volatility depends on the quadratic variation of the process, while our model deals with more general dependence of the volatility on the history of the process over a finite interval.

In the future work, we wish to derive the continuous-time GARCH model for stochastic volatility with delayed response which incorporates conditional expectation of log-returns, and we expect to develop also a method of estimation of the time delay parameter (as well as all the other parameters). Some preliminary results of this paper have been reported in conference proceedings (see [19]).

2. Stochastic delay differential equations

For any path $x : [-\tau, \infty) \rightarrow \mathbb{R}^d$ at each $t \geq 0$ define the segment $x_t : [-\tau, 0] \rightarrow \mathbb{R}^d$ by

$$x_t(s) := x(t + s), \quad \text{a.s., } t \geq 0, s \in [-\tau, 0].$$

Denote by $C := C([-\tau, 0], \mathbb{R}^d)$ the Banach space of all continuous paths $\eta : [-\tau, 0] \rightarrow \mathbb{R}^d$ equipped with the supremum norm

$$\|\eta\|_C := \sup_{s \in [-\tau, 0]} |\eta(s)|, \quad \eta \in C.$$

Consider the following *stochastic delay differential equation* (see [21])

$$\begin{cases} dx(t) = H(t, x_t) dt + G(t, x_t) dW(t), & t \geq 0 \\ x_0 = \phi \in C \end{cases} \quad (2.1)$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying *the usual conditions*; that is, the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and each \mathcal{F}_t , $t \geq 0$, contains all P -null sets in \mathcal{F} . $W(t)$ represents the m -dimensional Brownian motion.

The SDDE (2.1) has a *drift coefficient* function $H : [0, T] \times C \rightarrow \mathbb{R}^d$ and a *diffusion coefficient* function $G : [0, T] \times C \rightarrow \mathbb{R}^{d \times m}$ satisfying the following.

Hypotheses 2.1.

- (i) H and G are Lipschitz on bounded sets of C uniformly with respect to the first variable, i.e. for each integer $n \geq 1$, there exists a constant $L_n > 0$ (independent of $t \in [0, T]$) such that

$$|H(t, \eta_1) - H(t, \eta_2)| + \|G(t, \eta_1) - G(t, \eta_2)\| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\|_C \leq n, \|\eta_2\|_C \leq n$.

- (ii) There is a constant $K > 0$ such that

$$|H(t, \eta)| + \|G(t, \eta)\| \leq K(1 + \|\eta\|_C)$$

for all $t \in [0, T]$ and $\eta \in C$.

A *solution* of (2.1) is a measurable, sample-continuous process $x : [-\tau, T] \times \Omega \rightarrow \mathbb{R}^d$ such that $x|_{[0, T]}$ is $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted and x satisfies (2.1) almost surely.

In Mohammed [21] it was shown that if Hypotheses 2.1 holds then for each $\phi \in C$ the SDDE (2.1) has a unique solution $x^\phi : [-\tau, \infty) \times \Omega \rightarrow R^d$ with $x^\phi|_{[-\tau, 0]} = \phi \in C$ and $[0, T] \ni t \rightarrow x_t^\phi \in C$ being sample-continuous.

3. General formulation

The stock price value is assumed to satisfy the following SDDE:

$$dS(t) = rS(t) dt + \sigma(t, S_t)S(t) dW(t) \tag{3.1}$$

with continuous deterministic initial data $S_0 = \varphi \in C := C([-\tau, 0], R)$, here σ represents a volatility which is a continuous function of time and the elements of C .

As mentioned in last section, the existence and uniqueness of solution of (3.1) are guaranteed if the coefficients in (3.1) satisfy the following local Lipschitz and growth conditions:

$$\begin{aligned} \forall n \geq 1 \quad \exists L_n > 0 \quad \forall t \in [0, T] \quad \forall \eta_1, \eta_2 \in C, \|\eta_1\|_C \leq n, \|\eta_2\|_C \leq n : |\sigma(t, \eta_1)\eta_1(t) - \sigma(t, \eta_2)\eta_2(t)| \\ \leq L_n \|\eta_1 - \eta_2\|_C \end{aligned} \tag{3.2}$$

and

$$\exists K > 0 \quad \forall t \in [0, T], \eta \in C : |\sigma(t, \eta)\eta(t)| \leq K(1 + \|\eta\|_C). \tag{3.3}$$

Note that the stock price values are positive with probability 1 if the initial data is positive, that is, $\varphi(\theta) > 0$ for all $\theta \in [-\tau, 0]$.

We are primarily interested in an option price value, which is assumed to depend on the current and previous stock price values in the following way:

$$F(t, S_t) = \int_{-\tau}^0 e^{-r\theta} H(S(t + \theta), S(t), t) d\theta, \tag{3.4}$$

where $H \in C^{0,2,1}(R \times R \times R_+)$. Such a representation is chosen since it includes sufficiently general functionals for which an analogue of Ito’s lemma can be derived.

Lemma 3.1. Suppose a functional $F : R_+ \times C \rightarrow R$ has the following form

$$F(t, S_t) = \int_{-\tau}^0 h(\theta)H(S_t(\theta), S_t(0), t) d\theta,$$

for $H \in C^{0,2,1}(R \times R \times R_+)$ and $h \in C^1([-\tau, 0], R)$. Then

$$F(t, S_t) = F(0, \varphi) + \int_0^t \mathcal{A}F(s, S_s) ds + \int_0^t \sigma(s, S_s)S(s)\mathcal{B}F(s, S_s) dW(s), \tag{3.5}$$

where for $(t, x) \in R_+ \times C$,

$$\begin{aligned} \mathcal{A}F(t, x) &= h(0)H(x(0), x(0), t) - h(-\tau)H(x(-\tau), x(0), t) - \int_{-\tau}^0 h'(\theta)H(x(\theta), x(0), t) d\theta \\ &\quad + \int_{-\tau}^0 h(\theta)LH(x(\theta), x(0), t) d\theta, \\ \mathcal{B}F(t, x) &= \int_{-\tau}^0 h(\theta)H_2'(x(\theta), x(0), t) d\theta, \end{aligned}$$

and

$$LH(x(\theta), x(0), t) = rx(0)H_2'(x(\theta), x(0), t) + \frac{\sigma^2(t, x)x^2(0)}{2}H_{22}''(x(\theta), x(0), t) + H_3'(x(\theta), x(0), t),$$

where H_i' , $i = 1, 2, 3$, represents the derivative of $H(x(\theta), x(0), t)$ with respect to i th argument.

Proof. We defer the detailed proof to the Appendix A. □

In what follows, we assume that a riskless portfolio consisting of a position in the option and a position in the underlying stock is set up. In the absence of arbitrage opportunities, the return from the portfolio must be risk-free with the interest rate r . The portfolio Π has to be riskless during the time interval $[t, t + dt]$ and must instantaneously earn the same rate of return as other short-term risk-free securities. It follows that $d\Pi(t) = r\Pi(t) dt$ and this will be used in the proof of the following theorem.

Theorem 3.1. *Suppose the functional F is given by (3.4) with $S(t)$ satisfying (3.1) and $H \in C^{0,2,1}(R \times R \times R_+)$. Then, $H(S(t + \theta), S(t), t)$ satisfies the following equation*

$$0 = H|_{\theta=0} - e^{-r\theta} H|_{\theta=-\tau} + \int_{-\tau}^0 e^{-r\theta} \left(H'_3 + rS(t)H'_2 + \frac{1}{2}\sigma^2(t, S_t)S^2(t)H''_{22} \right) d\theta \quad (3.6)$$

for all $t \in [0, T]$.

Proof. To construct an equation for F , we need to consider a portfolio which consists of -1 derivative and $\mathcal{B}F(t, S_t)$ shares. Then, the portfolio value $\Pi(t)$ is equal to

$$\Pi(t) = -F(t, S_t) + \mathcal{B}F(t, S_t)S(t),$$

and the associated infinitesimal change in the time interval $[t, t + dt]$ is

$$d\Pi = -dF + \mathcal{B}F dS.$$

We should point out here that in the last statement we suppose that $(\mathcal{B}F)$ is held constant during the time-step dt , and hence term $d(\mathcal{B}F)$ is equal to zero. If this were not the case then $d\Pi$ would contain term $d(\mathcal{B}F)$.

Using (3.5) and (3.1), we obtain

$$d\Pi = -\mathcal{A}F dt - \sigma S \mathcal{B}F dW + \mathcal{B}F(rS dt + \sigma S dW).$$

Consideration of risk-free during the time dt then leads to

$$d\Pi = r\Pi dt,$$

that is,

$$-\mathcal{A}F(t, S_t) + rS(t)\mathcal{B}F(t, S_t) = r(-F(t, S_t) + \mathcal{B}F(t, S_t)S(t)),$$

or

$$\mathcal{A}F(t, S_t) = rF(t, S_t).$$

Therefore, the equation for $H(S(t + \theta), S(t), t)$ becomes

$$0 = H|_{\theta=0} - e^{-r\theta} H|_{\theta=-\tau} + \int_{-\tau}^0 e^{-r\theta} \left(H'_3 + rS(t)H'_2 + \frac{1}{2}\sigma^2(t, S_t)S^2(t)H''_{22} \right) d\theta.$$

This completes the proof. □

Consider a *European call option* with the final payoff $\max(S - K, 0)$ at the maturity time T Hull [16]. Then problem (3.6) has the boundary condition at the time T , either

$$F(T, S_T) = \max(S(T) - K, 0) \quad (3.7)$$

or, induced by the functional nature of F ,

$$F(T, S_T) = \frac{1}{\tau} \int_{-\tau}^0 \max(e^{-r\theta} S_T(\theta) - K, 0) d\theta, \quad (3.8)$$

where $1/\tau$ is a normalizing factor such that $F(T, S_T) \rightarrow \max(S(T) - K, 0)$ as $\tau \rightarrow 0$.

4. A simplified problem

We now consider the simplified problem (3.6), assuming that $H(S(t + \theta), S(t), t)$ is a sum of two functions, one of which depends on the current value of stock price $S(t)$ and another depends on the previous values of stock price $\{S(t + \theta), \theta \in [-\tau, 0)\}$. That is, our option price (3.4) takes the form:

$$F(t, S_t) = h_1(S(t), t) + \int_{-\tau}^0 e^{-r\theta} h_2(S(t + \theta), t) d\theta, \tag{4.1}$$

where $h_1(S(t), t)$ is a classical Black–Scholes call option price (see [3]) with the variance assumed equal to a long-run variance rate V (it is known that the stock price variance rate exhibits the so-called *mean reversion*, see [16])

$$h_1(S(t), t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{4.2}$$

where $N(x) = (1/\sqrt{2\pi}) \int_{-\infty}^x e^{-x^2/2} dx$ and d_1 and d_2 are defined as

$$d_1 = \frac{\ln(S(t)/K) + (r + V/2)(T - t)}{\sqrt{V(T - t)}},$$

$$d_2 = d_1 - \sqrt{V(T - t)}.$$

Note that the functional (4.1) seems to be a natural choice since we are interested in the difference between the original Black–Scholes option price and the one implied by the market with delayed response.

Theorem 4.1. *Assume the functional F is given by (4.1) with h_1 given by (4.2). Then,*

$$F(t, S_t) = h_1(S(t), t) + \frac{1}{2} \int_t^T e^{r(t-\xi)} [\sigma^2(\xi, S_t) - V] S^2(t) \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi. \tag{4.3}$$

Proof. Substituting (4.1) into (3.6) yields the following equation for h_2 .

$$h_2(S(t), t) - e^{r\tau} h_2(S(t - \tau), t) + \int_{-\tau}^0 e^{-r\theta} \frac{\partial h_2}{\partial t} d\theta = \frac{1}{2} (V - \sigma^2(t, S_t)) S^2(t) \frac{\partial^2 h_1}{\partial S^2}.$$

Also from (4.1) we derive that

$$\begin{aligned} \frac{dF}{dt} &= \frac{dh_1}{dt} + \frac{d}{dt} \left[\int_{t-\tau}^t e^{-r(s-t)} h_2(S(s), t) ds \right] = \frac{dh_1}{dt} + h_2(S(t), t) - e^{r\tau} h_2(S(t - \tau), t) \\ &\quad + \int_{t-\tau}^t r e^{-r(s-t)} h_2(S(s), t) ds + \int_{t-\tau}^t e^{-r(s-t)} \frac{\partial h_2}{\partial t} ds. \end{aligned}$$

Combining this expression with the one considered above we get the following equation for F :

$$\frac{\partial F}{\partial t} + rS(t) \frac{\partial h_1}{\partial S} + \frac{1}{2} \sigma^2(t, S_t) S^2(t) \frac{\partial^2 h_1}{\partial S^2} = rF. \tag{4.4}$$

We remark that the above equation is very similar to the well-known Black–Scholes PDE. Observing that h_1 satisfies the following PDE:

$$\frac{\partial h_1}{\partial t} + rS(t) \frac{\partial h_1}{\partial S} + \frac{1}{2} VS^2(t) \frac{\partial^2 h_1}{\partial S^2} = rh_1,$$

we have a new PDE for $f(t, S_t) := F(t, S_t) - h_1(S(t), t)$ as follows

$$\frac{\partial f}{\partial t} = rf + \frac{1}{2} [V - \sigma^2(t, S_t)] S^2(t) \frac{\partial^2 h_1}{\partial S^2}(S(t), t).$$

We can easily solve the above equation by using the variation-of-constants formula to obtain (4.3). □

5. Continuous-time version of GARCH model

Assume the $\sigma^2(t)$ satisfies the following equation

$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \ln^2 \left(\frac{S(t)}{S(t-\tau)} \right) - (\alpha + \gamma)\sigma^2(t), \quad (5.1)$$

where V is a long-run average variance rate, and α and γ are positive constants such that $\alpha + \gamma < 1$. Here, $S(t)$ is a solution of the SDDE (3.1) with positive initial data $\varphi \in C$.

Consider a grid $-\tau = t_{-l} < t_{-l+1} < \dots < t_0 = 0 < t_1 < \dots < t_N = T$ with the time step size Δ_l of the form

$$\Delta_l = \frac{\tau}{l},$$

where $l \geq 2$. Then a discrete time analogue of (5.1) is

$$\sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2 \left(\frac{S_{n-1}}{S_{n-1-l}} \right) + (1 - \alpha - \gamma)\sigma_{n-1}^2,$$

where $\sigma_n^2 = \sigma^2(t_n)$ and $S_n = S(t_n)$. Note that the process described by this difference equation is very similar to the GARCH(1,1) model proposed by Bollerslev in 1986 (with returns assumed to have mean zero), which seems to provide good explanation of stock price data,

$$\sigma_n^2 = \gamma V + \alpha \ln^2 \left(\frac{S_{n-1}}{S_{n-2}} \right) + (1 - \alpha - \gamma)\sigma_{n-1}^2. \quad (5.2)$$

Now, using a variation of constants formula for (5.1) we obtain

$$\sigma^2(t) = \frac{\gamma V}{\alpha + \gamma} + \left(\sigma^2(t_0) - \frac{\gamma V}{\alpha + \gamma} \right) e^{-(\alpha + \gamma)(t - t_0)} + \frac{\alpha}{\tau} \int_{t_0}^t e^{(\alpha + \gamma)(\xi - t)} \ln^2 \left(\frac{S(\xi)}{S(\xi - \tau)} \right) d\xi \quad (5.3)$$

for $t_0 \geq 0$. It is then natural that we consider the following expression for variance:

$$\bar{\sigma}^2(t, S_t) = \sigma^2(t_0) e^{-(\alpha + \gamma)(t - t_0)} + \left[\gamma V + \frac{\alpha}{\tau} \ln^2 \left(\frac{S(t)}{S(t - \tau)} \right) \right] \frac{1 - e^{-(\alpha + \gamma)(t - t_0)}}{\alpha + \gamma}, \quad (5.4)$$

since functions $\bar{\sigma}^2$ and σ^2 are close to each other in the following sense:

$$\sigma^2(t) = \bar{\sigma}^2(t) + O(|t - t_0|), \quad \text{as } t \rightarrow t_0.$$

Expression (5.4) for the volatility allows us to obtain a closed-form for a call option price involving delayed market response.

Theorem 5.1. *Assume that the stock price satisfies SDDE (3.1) with the initial data $\varphi \in C$ and assume its volatility is given by (5.4). Then, if the European call option price with the strike price K and maturity T at the time t can be written in the form (4.1), it is given by*

$$F(t, S_t) = h_1(S(t), t) + (\Sigma(S_t) - V)\mathcal{I}(r, t, S(t)) + (\sigma^2(t) - \Sigma(S_t))\mathcal{I}(r + \alpha + \gamma, t, S(t)), \quad (5.5)$$

where $h_1(S(t), t)$ is given by (4.2) and

$$\Sigma(S_t) = \frac{\alpha}{\tau(\alpha + \gamma)} \ln^2 \left(\frac{S(t)}{S(t - \tau)} \right) + \frac{\gamma V}{\alpha + \gamma},$$

$$\mathcal{I}(p, t, S(t)) = \frac{1}{2} S^2(t) \int_t^T e^{p(t - \xi)} \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi, \quad \text{for } p \geq 0.$$

Proof. Substituting the expression (5.4) for σ^2 in (4.3), we obtain

$$F(t, S_t) = h_1(S(t), t) + \frac{1}{2} S^2(t) \int_t^T e^{r(t-\xi)} [\sigma^2(t_0) e^{-(\alpha+\gamma)(\xi-t_0)} + \Sigma(S_t)(1 - e^{-(\alpha+\gamma)(\xi-t_0)}) - V] \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi,$$

which can be rewritten as

$$F(t, S_t) = h_1(S(t), t) + (\Sigma(S_t) - V) \frac{1}{2} S^2(t) \int_t^T e^{r(t-\xi)} \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi + (\sigma^2(t_0) - \Sigma(S_t)) e^{-(\alpha+\gamma)(t-t_0)} \frac{1}{2} S^2(t) \int_t^T e^{(r+\alpha+\gamma)(t-\xi)} \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi,$$

which, in the case $t = t_0$, is

$$F(t, S_t) = h_1(S(t), t) + (\Sigma(S_t) - V) \frac{1}{2} S^2(t) \int_t^T e^{r(t-\xi)} \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi + (\sigma^2(t) - \Sigma(S_t)) \frac{1}{2} S^2(t) \int_t^T e^{(r+\alpha+\gamma)(t-\xi)} \frac{\partial^2 h_1}{\partial S^2}(S(t), \xi) d\xi. \quad \square$$

6. Conclusion

In this article we derive an analogue (5.5) of Black–Scholes formula for a call option value in the market with stock price satisfying a *stochastic delay differential equation* (3.1). The motivation for considering such a market with delayed response comes from the continuous-time analogue of GARCH(1,1) model (see [4,9]). The implied volatility plot is generated to show the difference between obtained closed-form solution and the Black–Scholes formula (see Fig. 2).

Many related problems remain open. In particular, we would like to know how to find an option price value by solving Eq. (3.6) in general form without our simplifying assumption. Moreover, it is desirable to estimate the time delay τ involved and to consider the case where (3.1) experiences random Poisson jumps.

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Appendix A

Here we give a proof of Lemma 3.1. Fix $t > 0$ and denote $C \ni x = S_t$ with $S(t)$ satisfying (3.1). Then for a sufficiently small s

$$[F(t + s, x_s) - F(t, x)] = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \int_{-\tau}^0 h(\theta - s)[H(x(\theta), x(s), t + s) - H(x(\theta), x(s), t)] d\theta,$$

$$I_2 = \int_{-\tau}^0 (h(\theta - s) - h(\theta))H(x(\theta), x(s), t) d\theta,$$

$$I_3 = \int_{-\tau}^0 h(\theta)[H(x(\theta), x(s), t) - H(x(\theta), x(0), t)] d\theta,$$

$$I_4 = \int_0^s h(\theta - s)H(x(\theta), x(s), t + s) d\theta,$$

$$I_5 = - \int_{-\tau}^{-\tau+s} h(\theta - s)H(x(\theta), x(s), t + s) d\theta.$$

Then, by letting $s \rightarrow 0$,

$$I_1 \rightarrow \int_{-\tau}^0 h(\theta)H'_3(x(\theta), x(0), t) d\theta dt,$$

$$I_2 \rightarrow - \int_{-\tau}^0 h'(\theta)H(x(\theta), x(0), t) d\theta dt,$$

$$I_3 \rightarrow \int_{-\tau}^0 h(\theta)TH(x(\theta), x(0), t) d\theta dt + \int_{-\tau}^0 h(\theta)\sigma(t, x)x(0)H'_2(x(\theta), x(0), t) d\theta dW(t),$$

$$I_4 \rightarrow h(0)H(x(0), x(0), t) dt,$$

$$I_5 \rightarrow -h(-\tau)H(x(-\tau), x(0), t) dt,$$

where

$$TH(x(\theta), x(0), t) = rx(0)H'_2(x(\theta), x(0), t) + \frac{\sigma^2(t, x)x^2(0)}{2}H''_{22}(x(\theta), x(0), t).$$

The limit for I_3 is obtained by using the Ito's lemma. Then expression (3.5) follows.

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