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Persistence of wavefronts in delayed nonlocal reaction–diffusion equations

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Abstract

We develop a perturbation argument based on existing results on asymptotic autonomous systems and the Fredholm alternative theory that yields the persistence of traveling wavefronts for reaction–diffusion equations with nonlocal and delayed nonlinearities, when the time lag is relatively small. This persistence result holds when the nonlinearity of the corresponding ordinary reaction–diffusion system is either monostable or bistable. We then illustrate this general result using five different models from population biology, epidemiology and bio-reactors.

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1. Introduction

Since the pioneering work of Fisher [19] and Kolmogorov, Petrowsky and Piscounov [31], traveling wave solutions for reaction–diffusion equations have been extensively investigated, and this investigation has also inspired rapid development in nonlinear analysis and nonlinear dynamical systems, see [8,17,39,55] and the vast references therein.

The simplest scalar case is the following parabolic equation

$$u_t = u_{xx} + f(u) \tag{1.1}$$

with the nonlinearity f satisfying the following conditions

$$f(0) = f(1) = 0, \qquad f'(0) \neq 0 > f'(1).$$

Two prototypes of the nonlinearity have been considered: the monostable case (or the Fisher nonlinearity) where $f(u) > 0$ for $0 < u < 1$, $f'(0) > 0$; and the bistable case (or the Huxley nonlinearity) where there is $a \in (0,1)$ such that $f(u) < 0$ for $u \in (0,a)$ and $f(u) > 0$ for $u \in (a,1)$, $f'(0) < 0$. The existence of traveling waves of Eq. (1.1) can be studied using phase plane analysis: in the monostable case, Eq. (1.1) has a family of traveling waves $u = U(x - ct)$ for all wave speed $c \geqslant c^*(f) =$ minimal speed, whereas in the bistable case, there exists a unique traveling wave solution $u = U(x - ct)$ for some constant c, see for example, [52] and [18].

The study of traveling waves for many systems of reaction–diffusion equations arising from biological and physical applications becomes more complicated due to the lack of general techniques for phase space analysis, and some other approaches such as monotone iteration schemes [4,5], homotopy arguments [55] and perturbation analysis (for large speed waves) [1,39,48] have been developed.

It has been recognized that some of the well-known existence results of traveling waves must be extended to delay reaction–diffusion equations since time lags enter the dynamical models in a very natural way due to the slow signal and biochemical processes in many biological and physical systems, but such an extension becomes highly nontrivial. A fundamental difficulty arises since the equations describing the waves are no-longer systems of ordinary differential equations, but rather functional differential equations. Nevertheless, there has been substantial progress. Notably, Schaaf [47] studied a scalar reaction–diffusion equation with a single discrete delay in both the monostable case and bistable case by using ideas from phase space analysis, and he obtained the existence of traveling wavefront(s) under quasimonotonicity condition of the delayed nonlinearity. This quasimonotonicity also allows [59] to obtain the existence of traveling wavefronts for a very general delayed reaction–diffusion system, via a monotone iteration scheme coupled with the standard upper-lower solutions technique.

The difficulty in obtaining the existence of traveling waves for systems involving both spatial diffusion and temporal time lags also arises from the recent observation (see [51]) that this interaction of time lags and spatial diffusion leads to the so-called delay induced nonlocality and the resulted models taking into account biological realities take the form of reaction–diffusion equations with nonlocal and delayed nonlinearities: individuals have not been at the same point in space at previous times. For example, So, Wu and Zou [51] adopted Smith–Thieme's approach—reduction from a structured population model—to obtain a functional differential equation for the matured population in a biological system with two age classes [49] to the case of spatial diffusion in continuous and unbounded space, and they obtained the following reaction–diffusion equation with time delay and nonlocal effects

$$\frac{\partial w}{\partial t}=D\frac{\partial^2 w}{\partial x^2}-dw+\varepsilon\int_{-\infty}^{\infty}b\big(w(y,t-\tau)\big)f_\alpha(x-y)\,dy,\quad x\in(-\infty,\infty),\tag{1.2}$$

where $D,\tau,d>0$ and $0\leqslant\varepsilon\leqslant 1$ are given constants, and the kernel function is given by $f_\alpha(x)=\frac{1}{\sqrt{4\pi\alpha}}e^{-x^2/4\alpha}$ with $\alpha=\tau D_I>0$, D_I being the diffusion rate for the immature population, and $b(\cdot)$ the nonlinear birth rate. One can easily show that the equation describing the waves is now not only a functional differential equation, but also with mixed arguments (both advanced and retarded arguments). We should mention that delayed nonlocal reaction–diffusion models similar to (1.2) can also be obtained from predator–prey interaction and from the spread of infectious diseases when the carriers of the diseases involve age structure and maturation time, See Gourley and Kuang [21], and the two survey articles [22,24].

Due to the nonlocality and time lags, even for the model (1.2) the classical partition of nonlinearity and nonlinear dynamics into the monostable and bistable cases is no longer valid and we can expect more complex patterns of traveling waves. For example, for (1.2) with a monotone birth function b, we can obtain analogue existence results in both monostable and bistable cases (see So, Wu and Zou [51], and Ma and Wu [37]). However, when the function b is no-longer monotone (as it should be in order to reflect the crowding effect), we should expect oscillatory waves (either periodic waves around a positive equilibrium, or a nonmonotone traveling wave from the trivial solution to the positive equilibrium, see [23]).

Such an oscillatory pattern is associated with large delay. It is therefore very natural to ask if "small delay is harmless" in the sense that traveling waves for Eq. (1.1) persist when the time τ is incorporated in such an equation (note that Eq. (1.2) reduces to Eq. (1.1) when the maturation τ approaches zero). Such a persistence issue or harmless small delay has been addressed in other aspects of qualitative behaviors of delay differential equations. Despite simple biological intuition, rigorous mathematical justification seems to be quite difficult and unresolved even for the global stability of a unique positive equilibrium, see [20,46,54] and the book [32] by Kuang.

To describe precisely our goal in this paper about the persistence of traveling wavefront(s) of an ordinary reaction–diffusion equation subject to the nonlocal interaction induced by a small time lag, we consider the following system

$$\frac{\partial u(x,t)}{\partial t}=D\Delta u(x,t)+F\left(u(x,t),\int_{-\tau}^{0}\int_{\Omega}d\mu_\tau(\theta,y)g\big(u(x+y,t+\theta)\big)\right),\tag{1.3}$$

where $x\in\mathbf{R}^m$, $t\geqslant 0$, $u(x,t)\in\mathbf{R}^n$, $D=\operatorname{diag}(d_1,\dots,d_n)$ with positive constants d_i, $i=1,\dots,n$, $\Delta=\sum_{i=1}^m\frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator, τ is a positive constant, μ_τ is a bounded variation function on $[-\tau,0]\times\Omega\subseteq[-\tau,0]\times\mathbf{R}^m$ with values in $\mathbf{R}^{n\times n}$ and normalized so that $\int_{-\tau}^0\int_\Omega d\mu_\tau(\theta,y)=1$, and this measure may be dependent on τ, $F:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ and $g:\mathbf{R}^n\to\mathbf{R}^n$ are C^2-smooth functions. Our result shows that Eq. (1.3) has traveling wavefronts as long as the reduced version of an ordinary reaction–diffusion system, by setting $\tau=0$,

$$\frac{\partial u(x,t)}{\partial t}=D\Delta u(x,t)+F\big(u(x,t),g\big(u(x,t)\big)\big),$$

has ones: this is true in both monostable and bistable cases, and the quasimonotonicity condition on F and g is no longer required. This general result can then find applications in a number of models arising from ecology, biology and epidemiology, providing a framework and a systematic solution in a very general setting.

We illustrate these applications by five models. The first one is about Eq. (1.2). Direct applications of our general results yield that in the monostable case where $b(m) = pme^{-am}$, $\varepsilon p > d$, for any $c \geqslant c_0^* = 2\sqrt{D(\varepsilon p - d)}$, Eq. (1.2) has a family of traveling wavefronts $u = U(x - c_\tau^* ct/c_0^*)$ when the maturation time τ is small, where the minimal speed c_τ^* is determined by the characteristic equation associated with Eq. (1.2) at the equilibrium $u = 0$; in the bistable case where $b(m) = pm^2e^{-am}$, $\varepsilon p > dae$, Eq. (1.2) has a traveling wavefront $u = U(x - ct)$ for some wave speed c.

The second illustrative example is the following nonlocal Fisher model in the form of an integro-differential reaction–diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u + u\left[1 + \alpha u - (1+\alpha) g * u\right], \quad x = (x_1, x_2, \ldots, x_m) \in R^m,\ t, u \in R, \tag{1.4}$$

where $\alpha > 0$ and $g * u$ represents a temporal-spatial convolution, given by

$$g * u = \int_{\mathbf{R}^m} \frac{e^{-\|y\|^2/4\tau}}{(4\pi\tau)^{m/2}} u(x - y, t - \tau)\, dy.$$

This model was first proposed by Britton [9,10] and was recently studied by Wang, Li and Ruan [56] in the scalar case by using a nonstandard ordering. By applying our general results, we find that for any given $c \geqslant 2$, there exists a constant $\delta(c) > 0$ so that for $\tau \in [0, \delta)$, Eq. (1.4) has a traveling wavefront $u = U(\nu \cdot x - ct)$ which satisfying

$$\lim_{s \to -\infty} U(s) = 1, \qquad \lim_{s \to \infty} U(s) = 0,$$

where ν is a unit vector in $\mathbf{R}^m$. This result improves the main conclusion in [56] that shows the existence of traveling wavefronts for $c \geqslant 2\sqrt{1+\alpha}$ for small delay τ.

We also consider a system associated with the spatial spread of rabies among red foxes in Europe. Incorporating incubation time into a well-known model, we obtain the following delay reaction–diffusion model

$$\begin{cases} \dfrac{\partial S}{\partial t} = -KIS, \\ \dfrac{\partial I}{\partial t} = D\dfrac{\partial^2 I}{\partial x^2} + e^{-d\tau} KI(x, t-\tau) S(x, t-\tau) - \mu I, \end{cases} \tag{1.5}$$

where S and I are the susceptible and infective population densities respectively, τ is the incubation period of rabies and the constant d is the death rate of susceptible foxes, the parameter K is the transmission coefficient and μ is the death rate of the infective. Our result shows that Eq. (1.5) has traveling wavefronts with the minimal speed being a decreasing function of τ when the delay τ is small.

The same idea and technique applies to the following bio-reactor model:

$$\begin{cases} S_t = -vS_x - f(S)u, \\ u_t = u_{xx} - vu_x + e^{-d\tau} \int_{-\infty}^{\infty} f\big(S(y - v\tau, t-\tau)\big)u(y - v\tau, t-\tau) f_\alpha(x-y)\,dy - ku \end{cases} \tag{1.6}$$

where $f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-x^2/4\alpha}$ with $\alpha = \tau D > 0$, $S(x,t)$ and $u(x,t)$ are the concentrations of nutrient and microbial populations at position x and time t, respectively. Parameter $v \geqslant 0$ is the flowing velocity, $k > 0$ is the cell death rate (or dilution rate) and d is the dilution rate of the nutrient. In [58,60] the authors assumed that $k = d$. The constant $\tau \geqslant 0$ denotes the time delay involved in the conversion of nutrient to viable biomass. Our result shows again that if the non-delay model has a wavefront, so does the delayed model, but with a new minimal speed which is a function (decreasing) of τ.

The general results can also be extended to cover second-order hyperbolic–parabolic systems with time delay. We demonstrate this using the following hyperbolic model arising from the evolution of a single species by considering the time delay in the spatial movement of individuals in the nonlocal model (1.2):

$$\begin{aligned} &\frac{\partial}{\partial t} m(x,t) + r_1 \frac{\partial^2}{\partial t^2} m(x,t) \\ &\quad = D\frac{\partial^2}{\partial x^2} m(x,t) - dm(x,t) + \varepsilon \int_{-\infty}^{\infty} f_\alpha(x-y) b\big(m(y,t-\tau)\big)\,dy \\ &\qquad + r_1 \frac{\partial}{\partial t}\left[\varepsilon \int_{-\infty}^{\infty} f_\alpha(x-y) b\big(m(y,t-\tau)\big)\,dy\right], \end{aligned} \tag{1.7}$$

where the new parameter $r_1 > 0$ is related to the delay of the spatial movement. This is a nonlocal hyperbolic model for which the maximum principle and the comparison theorem seem to be unavailable, but the existence of traveling waves can be established using our general theory in both monostable and bistable cases.

While there has been significant progress towards the existence of traveling wavefronts in reaction–diffusion equations with nonlocal and delayed nonlinearities, very few general results that can be applied to a wide range of models important for applications have been established, with basically two exceptions: the celebrated work [35,53] for the case where the nonlinearity satisfies the quasimonotonicity condition, and the recent work [16] (based on a perturbation argument developed in [1,39,48]) that ensures the existence of traveling waves with large wave speeds in the neighborhood of a heteroclinic connecting orbit of a corresponding ordinary delay differential equation obtained through spatial averaging. Our results do not require the quasi-monotonicity condition on the nonlinearity, and ensure the existence of wavefronts with speeds close to the minimal wave speed. Both seem to be significant, in particular, for applications since most application problems involve nonmonotone nonlinearity, and wavefronts with the speeds close to the minimal speed are more biologically relevant because the solutions of the corresponding Cauchy initial value problem normally converge to these particular waves, see [39].

The remaining part of this paper is organized as follows. In Section 2, we present the existence result of traveling wavefronts in the monostable case and in Section 3 we establish the corresponding results in the bistable case. In Sections 4 and 5, we give the proofs of our main results. The last section is devoted to the illustration of the main results by their applications to the aforementioned models arising in ecology, biology and epidemiology.

2. Persistence of traveling wavefronts: monostable case

We consider the following system of reaction–diffusion equations with nonlocal and delayed nonlinearity

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\left(u(x,t), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y)g\big(u(x+y,t+\theta)\big)\right), \tag{2.1}$$

where $x \in \mathbf{R}^m$, $t \geqslant 0$, $u(x,t) \in \mathbf{R}^n$, $D = \mathrm{diag}(d_1, \ldots, d_n)$ with positive constants d_i, $i = 1, \ldots, n$, $\Delta = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator, τ is a positive constant, μ_τ is a bounded variation measure with values on $\mathbf{R}^{n\times n}$, and this measure may be dependent on τ, $F : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^n$ are C^2-smooth functions.

For later use, we introduce some notations. For a vector $x \in \mathbf{R}^n$, we set $\|x\| = \|x\|_{\mathbf{R}^n}$ and for a $n \times n$ matrix A, $\|A\| = \|A\|_{\mathbf{R}^n}$ denotes the norm of A as a linear operator. Let $C = C(\mathbf{R}, \mathbf{R}^n)$ be the Banach space of continuous and bounded functions from $\mathbf{R}$ to $\mathbf{R}^n$ equipped with the standard norm $\|\phi\| = \sup\{\|\phi(t)\|_{\mathbf{R}^n},\ t \in \mathbf{R}\}$. Denote $C^1 = C^1(\mathbf{R}, \mathbf{R}^n) = \{\phi \in C\colon\ \phi' \in C\}$, $C^2 = \{\phi \in C\colon\ \phi'' \in C\}$, $C_0 = \{\phi \in C\colon\ \lim_{t\to\pm\infty} \phi = \mathbf{0}\}$ and $C_0^1 = \{\phi \in C_0\colon\ \phi' \in C_0\}$.

We let $F_u(u,v)$, $F_v(u,v)$ denote the partial derivatives of F with respect to the variables $u \in \mathbf{R}^n$ and $v \in \mathbf{R}^n$, and let $g_u(u)$ be the derivative of g with respect to the variable $u \in \mathbf{R}^n$. Let $f(u) = F(u, g(u))$. Then

$$f'(u) = F_u(u,u) + F_v\big(u, g(u)\big)g'(u).$$

We assume that μ_τ satisfies the normalization condition

$$\int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) = 1, \tag{2.2}$$

and the limiting condition

$$\lim_{\tau\to 0}\int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y)g\big(u(x+y,t+\theta)\big) = g\big(u(x,t)\big)$$

for any fixed x and t. Thus when $\tau \to 0$, Eq. (2.1) reduces to a standard reaction–diffusion equation

$$\frac{\partial u}{\partial t} = D\Delta u + F\big(u, g(u)\big) = D\Delta u + f(u), \tag{2.3}$$

that includes the well-known Fisher [19] and KPP [31] model as a special case.

We assume that Eq. (2.1) has only two equilibria $E_1 = \mathbf{0}$ and $E_2 = \mathbf{K} \neq \mathbf{0}$, where $\mathbf{K}$ is a constant vector in $\mathbf{R}^n$ which is independent of τ. This assumption is reasonable because when Eq. (2.1) has two equilibria $E_1 = \mathbf{K}_1(\tau) \neq E_2 = \mathbf{K}_2(\tau)$, we can always use the following linear scaling transform

$$\begin{cases} \tilde{u}_i = \dfrac{u_i - (E_1)_i}{(E_2)_i - (E_1)_i} & \text{if } (E_2)_i \neq (E_1)_i, \\ \tilde{u}_i = u_i - (E_1)_i & \text{if } (E_2)_i = (E_1)_i \end{cases}$$

to have a new system for $\tilde{u}$ with two new equilibria $E_1 = \mathbf{0}$ and E_2 that are independent of τ.

Let $u(x,t) = U_0(s)$ with $s = \nu \cdot x - ct$ be a traveling wave, where ν is a unit vector in $\mathbf{R}^m$ and c is the wave speed. Then we have from (2.3) the following profile equation

$$-cU_0' = DU_0'' + f(U_0). \tag{2.4}$$

For Eq. (2.4), $U_0 = 0$ is a solution. The behaviors of solutions near the equilibrium $U_0 = 0$ are determined by the linearization of (2.4) around the point $U_0 = 0$:

$$-cU' = DU'' + f'(\mathbf{0})U,$$

and the corresponding characteristic equation

$$\Lambda_0^{\mathbf{0}}(\lambda) := \det\left[D\lambda^2 + c\lambda I + f'(\mathbf{0})\right] = 0,$$

where I is the identity matrix. Similarly, we obtain the characteristic equation at E_2 as follows

$$\Lambda_0^{\mathbf{K}}(\lambda) := \det\left[D\lambda^2 + c\lambda I + f'(\mathbf{K})\right] = 0.$$

The following conditions are usually required for the existence of traveling waves connecting the two equilibria E_1 and E_2 for the ordinary reaction–diffusion equation (2.3):

(H_1) The equilibria E_1 and E_2 are hyperbolic in the sense that $\Lambda_0^{\mathbf{0}}(iv) \neq 0$, $\Lambda_0^{\mathbf{K}}(iv) \neq 0$ for any real number v.

(H_2) There exists a constant $c_0^* > 0$ so that for every $c \geqslant c_0^*$, equation $\Lambda_0^{\mathbf{0}}(\lambda) = 0$ has a negative *real* root. Moreover, $E_1 = \mathbf{0}$ is a stable node of Eq. (2.4) in the sense that the real part of every complex zero of equation $\Lambda_0^{\mathbf{0}}(\lambda) = 0$ is negative.

(H_3) For any $c \geqslant c_0^*$, Eq. (2.4) has a solution U_0 satisfying

$$\lim_{s\to-\infty} U_0(s) = E_2 = \mathbf{K}, \qquad \lim_{s\to\infty} U_0(s) = E_1 = \mathbf{0}. \tag{2.5}$$

In addition, we shall require

(H_4) $\mu_\tau(\theta, y)$ satisfies

$$\int_{-\tau}^{0} \int_{\Omega} d\mu_\tau(\theta, y)(\nu \cdot y) = o(1) \quad \text{as } \tau \to 0$$

and

$$\int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y)\theta = o(1) \quad \text{as } \tau \to 0.$$

Remark 2.1. c_0^* is called the minimal speed of the traveling wavefronts to Eq. (2.3). It is usually defined as

$$c_0^* = \inf\{c > 0 \mid \Lambda_0^{\mathbf{0}}(\lambda) = 0 \text{ has a negative real zero}\}.$$

Remark 2.2. In some applications, η may be a function of bounded variation from $(-\infty, 0] \times \Omega$ to the space $\mathbf{R}^{n\times n}$, satisfying the normalization condition

$$\int_{-\infty}^{0}\int_{\Omega} d\mu_\tau(\theta, y) = 1$$

instead of (2.2), see [21,42] and [45]. In this case, our main results below remain valid as long as

$$\lim_{\tau\to 0}\int_{-\infty}^{0}\int_{\Omega} d\mu_\tau(\theta, y) g\big(u(x+y, t+\theta)\big) = g\big(u(x,t)\big).$$

We want to seek traveling wavefronts for Eq. (2.1), that is, we suppose that $u(x,t) = U(s)$, $s = \nu\cdot x - ct$, and obtain a wave profile equation from Eq. (2.1):

$$-cU' = DU'' + F\left(U(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) g\big(U(s+\nu\cdot y - c\theta)\big)\right), \quad s \in \mathbf{R}. \tag{2.6}$$

As usual, around $E_1 = \mathbf{0}$, we linearize Eq. (2.6) and define the following function

$$\Lambda_\tau^{\mathbf{0}}(\lambda) := \det\left[\lambda^2 D + \lambda c I + B_1 + B_2 \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) e^{\lambda(\nu\cdot y - c\theta)}\right],$$

where λ is a complex variable and the two matrices B_1 and B_2 are

$$B_1 = F_u\big(\mathbf{0}, g(\mathbf{0})\big), \qquad B_2 = F_v\big(\mathbf{0}, g(\mathbf{0})\big)g'(\mathbf{0}).$$

Define the minimal speed c_τ^* as

$$c_\tau^* = \inf\{c > 0 \mid \Lambda_\tau^{\mathbf{0}}(\lambda) = 0 \text{ has a negative real zero}\}.$$

Here we assume that

$$\int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y)e^{\lambda(\nu\cdot y-c\theta)}$$

is a continuous function of τ so that c_τ^* depends continuously on τ and

$$\lim_{\tau\to 0} c_\tau^* = c_0^*.$$

Our main result can be stated as follows:

Theorem 2.1. *For any given $c \geqslant c_0^*$, there exists a constant $\delta = \delta(c) > 0$ so that for $\tau \in (0, \delta)$, Eq.* (2.1) *has a traveling wavefront $u = U(\nu \cdot x - c_\tau^* ct/c_0^*)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = E_2 = \mathbf{K}, \qquad \lim_{s\to\infty} U(s) = E_1 = \mathbf{0}.$$

In recent existence studies of traveling wavefronts, most authors assume that the delayed nonlinearities $F_v(u, v)$ and $g_u(u)$ are positive and these assumptions result in monotone traveling wavefronts for system (2.1), see [11,12,33–36,41,47,57,59]. Our theorem addresses the long standing issue on the existence of traveling wavefronts in the nonmonotone and nonlocal cases, as shown in the next two examples.

A natural corollary of Theorem 2.1 for the following scalar equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + F\big(u(x,t), u(x,t-\tau)\big), \quad x \in \mathbf{R},\ t \geqslant 0 \tag{2.7}$$

can be derived, where F satisfies

$$F(0,0) = F(1,1) = 0 \quad \text{and} \quad F(s,t) > 0 \quad \text{for } 0 < s,\ t < 1.$$

Suppose that $\alpha = F_u(0,0)$, $\beta = F_v(0,0)$, $\alpha+\beta > 0$ and $F_u(1,1) + F_v(1,1) < 0$. The characteristic functions $\Lambda_\tau^0(\lambda)$ and $\Lambda_0^0(\lambda)$ become

$$\Lambda_\tau^0(\lambda) = \lambda^2 + \lambda c + \alpha + \beta e^{\lambda c\tau}$$

and

$$\Lambda_0^0(\lambda) = \lambda^2 + \lambda c + \alpha + \beta.$$

Set

$$c_\tau^* = \inf\big\{c > 0 \bigm| \Lambda_\tau^0(\lambda) = 0 \text{ has a real negative zero}\big\}.$$

By the implicit function theorem, it is readily seen that

$$\left.\frac{dc_\tau^*}{d\tau}\right|_{\tau=0} < 0$$

which means that the minimal speed is a decreasing function of τ for τ in some right neighborhood of $\tau = 0$.

As a corollary of Theorem 2.1, we have the following

Corollary 2.1. *For any given $c \geqslant c_0^* = 2\sqrt{\alpha+\beta}$, there exists a constant $\delta = \delta(c) > 0$ so that for $\tau \in [0, \delta)$, Eq.* (2.7) *has a traveling wavefront $u = U(x - c_\tau^* ct/c_0^*)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = 1, \qquad \lim_{s\to\infty} U(s) = 0.$$

We remark that we do not impose any monotonicity condition on the function of $F(u,\cdot)$: $\mathbf{R} \to \mathbf{R}$ for any fixed $u \in \mathbf{R}$.

We can also deal with a nonlocal hyperbolic equation with delay

$$\frac{\partial u(x,t)}{\partial t} + r_1 \frac{\partial^2 u(x,t)}{\partial t^2} = D\Delta u(x,t) + F\left(u(x,t), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) g\big(u(x+y, t+\theta)\big)\right), \quad (2.8)$$

where $x \in \mathbf{R}^m$, $u \in \mathbf{R}^n$, $r_1 > 0$, $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ with positive constants d_i. At the solution $u \equiv \mathbf{0}$, the characteristic function is defined as

$$\Lambda_{\tau,r_1}^{\mathbf{0}}(\lambda) := \det\left[\lambda^2\big(D - c^2 r_1 I\big) + \lambda c I + B_1 + B_2 \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) e^{\lambda(\nu\cdot y - c\theta)}\right].$$

As above, define the minimal speed c_{τ,r_1}^* by

$$c_{\tau,r_1}^* = \inf\big\{c > 0 \bigm| \Lambda_{\tau,r_1}^{\mathbf{0}}(\lambda) = 0 \text{ has a negative real zero}\big\},$$

and the minimal speed $c_{0,0}^*$ is defined by

$$c_{0,0}^* = \inf\big\{c > 0 \bigm| \det\big[\lambda^2 D + \lambda c I + f'(\mathbf{0})\big] = 0 \text{ has a negative real zero}\big\}.$$

Then under the conditions (H_1)–(H_4), we have the following

Theorem 2.2. *For any given $c \geqslant c_{0,0}^*$, there exist constants $\delta_1 = \delta_1(c)$ and $\delta_2 = \delta_2(c) > 0$ so that for $\tau \in [0, \delta_1)$ and $r_1 \in [0, \delta_2)$, Eq.* (2.8) *has a traveling wavefront $u = U(\nu \cdot x - c_{\tau,r_1}^* ct/c_{0,0}^*)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = E_2 = \mathbf{K}, \qquad \lim_{s\to\infty} U(s) = E_1 = \mathbf{0}.$$

3. Persistence of traveling wavefronts: bistable case

A natural question is whether or not we can obtain similar results about the persistence of traveling wavefront in the bistable case. The answer is affirmative as shown in this section.

Still consider the equation

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\left(u(x,t), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big(u(x+y,t+\theta)\big)\right) \tag{3.1}$$

and its reduced version when $\tau = 0$

$$\frac{\partial u}{\partial t} = D\Delta u + F\big(u, g(u)\big) = D\Delta u + f(u), \tag{3.2}$$

where $x \in \mathbf{R}^m$, $t \geqslant 0$, $u(x,t) \in \mathbf{R}^n$. Here all the parameters D, $\Delta = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}$, τ, μ_τ are defined the same as in Section 2, but the nonlinearities F and g have different behaviors. In the bistable case, $f(u) = F(u, g(u)) = 0$ has three equilibria:

$$E_1 = \mathbf{0}, \qquad E_2 \text{ and } E_3 = \mathbf{K}$$

where the vector $E_2 \in \mathbf{R}^n$ is between 0 and $\mathbf{K}$ with respect to a certain ordering in $\mathbf{R}^n$.

As before, by setting $u(x,t) = U(\nu \cdot x - ct)$ and $u(x,t) = U_0(\nu \cdot x - c_0 t)$, traveling wave profiles to Eqs. (3.1) and (3.2) are given respectively by

$$-cU'(s) = DU''(s) + F\left(U(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big(U(s+\nu\cdot y - c\theta)\big)\right), \quad s \in \mathbf{R}, \tag{3.3}$$

and

$$-c_0 U_0' = DU_0'' + f(U_0), \quad s \in \mathbf{R}. \tag{3.4}$$

Equation (3.4) has neither time delay nor spatial averaging effect. We will assume that there exists a traveling wavefront (U_0, c_0) for Eq. (3.4) connecting E_1 and E_3. For the presentation of our results, we introduce a linear operator $Q : C^2 \to C$, accompanied by the linearization of Eq. (3.4), as

$$Q(u)(s) = -c_0 u'(s) - Du''(s) - f'\big(U_0(s)\big)u(s), \quad s \in \mathbf{R}.$$

The adjoint operator of Q is given by

$$Q^*(u)(s) = c_0 u'(s) - Du''(s) - f'(U_0)^* u(s), \quad s \in \mathbf{R},$$

where $f'(U_0)^*$ is the transpose of the matrix $f'(U_0)$.

Our basic assumptions, about Eq. (3.4), are as follows:

(G_1) When $\tau = 0$, Eq. (3.4) has a wavefront U_0 with the wave speed c_0 satisfying

$$\lim_{s\to-\infty} U_0(s) = E_3 = \mathbf{K}, \qquad \lim_{s\to\infty} U_0(s) = E_1 = \mathbf{0}.$$

(G_2) When $\tau = 0$, the two equilibria $E_1 = \mathbf{0}$ and $E_3 = \mathbf{K}$ with the wavespeed $c = c_0$ are hyperbolic in the sense that $\Lambda_0^{\mathbf{0}}(iv) \neq 0$, $\Lambda_0^{\mathbf{K}}(iv) \neq 0$ for any real number v when $c = c_0$. Furthermore, there exists a unique (up to scalar multiple) bounded element $p = U_0'$ such that $Q(p) = 0$, and correspondingly a unique (up to scalar multiple) nontrivial bounded element p^* so that $Q^*(p) = 0$ and

$$\int_{-\infty}^{\infty} p(s) \cdot p^*(s)\, ds \neq 0.$$

Based on a combination of the Fredholm theory and the fixed-point theorem, we can obtain the following result by a local continuation of the parameter τ:

Theorem 3.1. *Under the conditions* (G_1), (G_2) *and* (H_4)*, there exists a* $\delta > 0$ *so that for any* $\tau \in [0, \delta)$*, Eq.* (3.1) *has a traveling wavefront* $u = U(\nu \cdot x - ct)$ *satisfying*

$$\lim_{s \to -\infty} U(s) = E_3 = \mathbf{K}, \qquad \lim_{s \to \infty} U(s) = E_1 = \mathbf{0}$$

for some $c \in \mathbf{R}$.

As an example, we consider the scalar equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + F\left(u(x,t), \int_{-\tau}^{0} \int_{\Omega} d\mu_\tau(\theta, y) g\big(u(x+y, t+\theta)\big)\right), \tag{3.5}$$

where $u(x,t) \in \mathbf{R}$ and $f(u) = F(u, g(u)) = u(u-a)(1-u)$, $0 < a < 1$. When $\tau = 0$, from the classical paper of Fife and McLeod [18], we know that equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) \tag{3.6}$$

has a unique traveling wavefront $u = U_0(x - c_0 t)$, where U_0 is a strictly decreasing function. By Lemma 3.1 in [15], we know that for the nonlinear function $f(u) = u(u-a)(1-u)$, condition (G_2) is satisfied. Therefore, under condition (H_4), we have the following

Corollary 3.1. *There exists a* $\delta > 0$ *so that for any* $\tau \in [0, \delta)$*, scalar Eq.* (3.5) *has a traveling wavefront* $u = U(x - ct)$ *satisfying*

$$\lim_{s \to -\infty} U(s) = 1, \qquad \lim_{s \to \infty} U(s) = 0$$

for some $c \in \mathbf{R}$.

We can also consider the nonlocal hyperbolic equation (2.8) in the bistable case when the nonlinear function $f(u) = F(u, g(u)) = 0$ has three equilibria $E_1 = \mathbf{0}$, E_2 and $E_3 = \mathbf{K}$. The wave profile equation becomes

$$-cU' = \left(D - r_1 c^2\right)U'' + F\left(U(s), \int_{-\tau}^{0} \int_{\Omega} d\mu_\tau(\theta, y) g\left(U(s + \nu \cdot y - rc\theta)\right)\right).$$

Under the conditions (G_1), (G_2) and (H_4), we have

Theorem 3.2. *There exist constants δ_1 and $\delta_2 > 0$ so that for any $\tau \in [0, \delta_1)$ and $r_1 \in [0, \delta_2)$, Eq.* (3.1) *has a traveling wavefront $u = U(\nu \cdot x - ct)$ satisfying*

$$\lim_{s \to -\infty} U(s) = E_3 = \mathbf{K}, \qquad \lim_{s \to \infty} U(s) = E_1 = \mathbf{0}$$

for some $c \in \mathbf{R}$.

4. Proofs: the monostable case

In this section, we give a proof of Theorem 2.1. Our main approach is based on a combination of some perturbation analysis, the Fredholm theory and the Banach fixed point theorem, by showing that a traveling wavefront to Eq. (2.1) can be approximated by the corresponding wavefront $U_0(z)$ of (2.4) when τ is small.

For any given $c \geqslant c_0^*$, we want to show that Eq. (2.1) has a traveling wavefront $u = U(s)$, $s = \nu \cdot x - rct$, where $r = c_\tau^*/c_0^*$ which is dependent on τ and tends to 1 as $\tau \to 0$. That is, we want to find a solution U to the following equation

$$-rcU' = DU'' + F\left(U(s), \int_{-\tau}^{0} \int_{\Omega} d\mu_\tau(\theta, y) g\left(U(s + \nu \cdot y - rc\theta)\right)\right). \tag{4.1}$$

We suppose that U can be approximated by U_0, and hence we write $U = U_0 + W$. It is then easy to derive an equation for W:

$$\begin{aligned} -rcW' = DW'' &+ F\left(U_0 + W, \int_{-\tau}^{0} \int_{\Omega} d\mu_\tau(\theta, y) g\left([U_0 + W](s + \nu \cdot y - rc\theta)\right)\right) \\ &- F\left(U_0(s), g\left(U_0(s)\right)\right) + rcU_0' - cU_0' \end{aligned} \tag{4.2}$$

subject to

$$W(\pm\infty) = \mathbf{0}.$$

Expanding the term

$$F\left(U_0(s)+W(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big([U_0+W](s+\nu\cdot y-rc\theta)\big)\right)$$

yields

$$\begin{aligned} &F\left(U_0(s)+W(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big([U_0+W](s+\nu\cdot y-rc\theta)\big)\right) \\ &\quad = F\left(U_0(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big(U_0(s+\nu\cdot y-rc\theta)\big)\right) \\ &\qquad + A^\tau(s)W + B^\tau(s)C^\tau(s,W) + R_1(\tau,s,W), \end{aligned} \tag{4.3}$$

with

$$A^\tau(s) = F_u\left(U_0(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big(U_0(s+\nu\cdot y-rc\theta)\big)\right), \quad s\in\mathbf{R},$$

$$B^\tau(s) = F_v\left(U_0(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big(U_0(s+\nu\cdot y-rc\theta)\big)\right),$$

and

$$C^\tau(s,W) = \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g_u\big(U_0(s+\nu\cdot y-rc\theta)\big) W(s+\nu\cdot y-rc\theta), \tag{4.4}$$

where $R_1(\tau,s,W)$ is the remainder of the higher orders, that is,

$$\begin{aligned} R_1(\tau,s,W) = {} & F\left(U_0(s)+W(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big([U_0+W](s+\nu\cdot y-rc\theta)\big)\right) \\ & - F\left(U_0(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta,y) g\big(U_0(s+\nu\cdot y-rc\theta)\big)\right) \\ & - A^\tau(s)W - B^\tau(s)C^\tau(s,W). \end{aligned} \tag{4.5}$$

Therefore, by (4.3) and the fact that

$$f'(U_0) = F_u\big(U_0, g(U_0)\big) + F_v\big(U_0, g(U_0)\big) g_u(U_0),$$

Eq. (4.1) becomes

$$-crW' = DW'' + f'(U_0)W + R_1(\tau, s, W) + R_2(\tau, s, W) + R_3(\tau, s, W), \tag{4.6}$$

where

$$R_2(\tau, s, W) = F\left(U_0(s), \int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) g\big(U_0(s + \nu \cdot y - rc\theta)\big)\right) - f(U_0) + rcU_0' - cU_0',$$

$$R_3(\tau, s, W) = A^\tau(s)W - A^0(s)W(s) + B^\tau(s)C^\tau(s, W) - B^0(s)g_u\big(U_0(s)\big)W(s) \tag{4.7}$$

with

$$A^0(s) = F_u\big(U_0(s), g\big(U_0(s)\big)\big), \qquad B^0(s) = F_v\big(U_0(s), g\big(U_0(s)\big)\big)g_u\big(U_0(s)\big), \quad s \in \mathbf{R}.$$

Next we transform Eq. (4.6) into an integral equation. We re-write Eq. (4.6) as

$$d_i \frac{d^2 W_i}{ds^2} + cr\frac{dW_i}{ds} - W_i = -W_i - \big[f'(U_0)W\big]_i - \big[R_1(\tau, s, W)\big]_i - \big[R_2(\tau, s, W)\big]_i - \big[R_3(\tau, s, W)\big]_i, \quad 1 \leqslant i \leqslant n, \tag{4.8}$$

where the index i denotes the ith component for the corresponding functions. Equation

$$d_i \frac{d^2 W_i}{ds^2} + cr\frac{dW_i}{ds} - W_i = 0$$

has a characteristic equation

$$d_i\lambda^2 + cr\lambda - 1 = 0$$

with two real roots

$$\alpha_i^\tau = \frac{-cr - \sqrt{c^2r^2 + 4d_i}}{2d_i} < 0, \qquad \beta_i^\tau = \frac{-cr + \sqrt{c^2r^2 + 4d_i}}{2d_i} > 0$$

satisfying

$$\lim_{\tau\to 0} \alpha_i^\tau = \alpha_i^0 := \frac{-c - \sqrt{c^2 + 4d_i}}{2d_i}, \qquad \lim_{\tau\to 0} \beta_i^\tau = \beta_i^0 := \frac{-c + \sqrt{c^2 + 4d_i}}{2d_i}. \tag{4.9}$$

We thus conclude that (4.8) is equivalent to the following integral equation

$$W_i(s) = \frac{1}{d_i(\beta_i^\tau - \alpha_i^\tau)} \int_{-\infty}^{s} e^{\alpha_i^\tau(s-t)} \left\{ W_i + \left[f'(U_0)W\right]_i + \sum_{j=1}^{3} \left[R_j(\tau, s, W)\right]_i \right\} dt$$
$$+ \frac{1}{d_i(\beta_i^\tau - \alpha_i^\tau)} \int_{s}^{\infty} e^{\beta_i^\tau(s-t)} \left\{ W_i + \left[f'(U_0)W\right]_i + \sum_{j=1}^{3} \left[R_j(\tau, s, W)\right]_i \right\} dt, \tag{4.10}$$

for $i = 1, 2, \ldots, n$. Recall that $d_i(\beta_i^\tau - \alpha_i^\tau) = \sqrt{c^2r^2 + 4d_i}$. Equation (4.10) can then be written as

$$W_i(s) - \frac{1}{\sqrt{c^2 + 4d_i}} \int_{-\infty}^{s} e^{\alpha_i^0(s-t)} \left\{ W_i + \left[f'(U_0)W\right]_i \right\} dt$$
$$- \frac{1}{\sqrt{c^2 + 4d_i}} \int_{s}^{\infty} e^{\beta_i^0(s-t)} \left\{ W_i + \left[f'(U_0)W\right]_i \right\} dt$$
$$= \int_{-\infty}^{s} \left\{ \frac{e^{\alpha_i^\tau(s-t)}}{\sqrt{c^2r^2 + 4d_i}} - \frac{e^{\alpha_i^0(s-t)}}{\sqrt{c^2 + 4d_i}} \right\} \left\{ W_i + \left[f'(U_0)W\right]_i \right\} dt$$
$$+ \int_{s}^{+\infty} \left\{ \frac{e^{\beta_i^\tau(s-t)}}{\sqrt{c^2r^2 + 4d_i}} - \frac{e^{\beta_i^0(s-t)}}{\sqrt{c^2 + 4d_i}} \right\} \left\{ W_i + \left[f'(U_0)W\right]_i \right\} dt$$
$$+ \frac{1}{\sqrt{c^2r^2 + 4d_i}} \int_{-\infty}^{s} e^{\alpha_i^\tau(s-t)} \sum_{j=1}^{3} \left[R_j(\tau, s, W)\right]_i dt$$
$$+ \frac{1}{\sqrt{c^2r^2 + 4d_i}} \int_{s}^{\infty} e^{\beta_i^\tau(s-t)} \sum_{j=1}^{3} \left[R_j(\tau, s, W)\right]_i dt \tag{4.11}$$

with the boundary conditions

$$W(\pm\infty) = \mathbf{0}.$$

To express (4.11) in a simple form, we introduce a number of matrices:

$$E_1^\tau(s) = \operatorname{diag}\left(\frac{e^{\alpha_1^\tau s}}{\sqrt{c^2r^2 + 4d_1}}, \frac{e^{\alpha_2^\tau s}}{\sqrt{c^2r^2 + 4d_2}}, \ldots, \frac{e^{\alpha_n^\tau s}}{\sqrt{c^2r^2 + 4d_n}} \right), \quad s \in \mathbf{R},$$
$$E_2^\tau(s) = \operatorname{diag}\left(\frac{e^{\beta_1^\tau s}}{\sqrt{c^2r^2 + 4d_1}}, \frac{e^{\beta_2^\tau s}}{\sqrt{c^2r^2 + 4d_2}}, \ldots, \frac{e^{\beta_n^\tau s}}{\sqrt{c^2r^2 + 4d_n}} \right), \quad s \in \mathbf{R},$$
$$E_1^0(s) = \operatorname{diag}\left(\frac{e^{\alpha_2^0 s}}{\sqrt{c^2 + 4d_1}}, \frac{e^{\alpha_2^0 s}}{\sqrt{c^2 + 4d_2}}, \ldots, \frac{e^{\alpha_n^0 s}}{\sqrt{c^2 + 4d_n}} \right), \quad s \in \mathbf{R},$$

$$E_2^0(s) = \operatorname{diag}\left(\frac{e^{\beta_2^0 s}}{\sqrt{c^2+4d_1}}, \frac{e^{\beta_2^0 s}}{\sqrt{c^2+4d_2}}, \dots, \frac{e^{\beta_n^0 s}}{\sqrt{c^2+4d_n}}\right), \quad s \in \mathbf{R},$$
$$E_3^\tau(s) = E_1^\tau(s) - E_1^0(s), \quad s \in \mathbf{R},$$

and

$$E_4^\tau(s) = E_2^\tau(s) - E_2^0(s), \quad s \in \mathbf{R}.$$

Then Eq. (4.11) can be finally written as

$$\begin{aligned} W - \int_{-\infty}^{s} E_1^0(s-t)\bigl\{W(t) + f'(U_0)W(t)\bigr\}\,dt \\ - \int_{s}^{\infty} E_2^0(s-t)\bigl\{W(t) + f'(U_0)W(t)\bigr\}\,dt \\ = H(\tau, s, W), \quad W(\pm\infty) = \mathbf{0}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} H(\tau, s, W) = {} & \int_{-\infty}^{s} E_3^\tau(s-t)\bigl\{W(t) + f'(U_0)W(t)\bigr\}\,dt \\ & + \int_{s}^{+\infty} E_4^\tau(s-t)\bigl\{W(t) + f'(U_0)W(t)\bigr\}\,dt \\ & + \int_{-\infty}^{s} E_1^\tau(s-t)\sum_{j=1}^{3} R_j(\tau, t, W)\,dt \\ & + \int_{s}^{\infty} E_1^\tau(s-t)\sum_{j=1}^{3} R_j(\tau, t, W)\,dt. \end{aligned}$$

We will study the existence of the solution W to (4.11) or (4.12). Define a linear operator $L: C_0 \to C_0$ from the left side of (4.12) by

$$\begin{aligned} L(W)(s) = W(s) - \int_{-\infty}^{s} E_1^0(s-t)\bigl\{W(t) + f'(U_0)W(t)\bigr\}\,dt \\ - \int_{s}^{\infty} E_2^0(s-t)\bigl\{W(t) + f'(U_0)W(t)\bigr\}\,dt. \end{aligned} \tag{4.13}$$

It is obvious that $L(W) \in C_0$ if $W \in C_0$. Now we establish some estimations for the integration on the right side of (4.11).

Lemma 4.1. *When $\tau \to 0$, we have*

$$\left| \int_{-\infty}^{s} \left\{ \frac{e^{\alpha_i^\tau (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\alpha_i^0 (s-t)}}{\sqrt{c^2 + 4d_i}} \right\} \left\{ W_i + \left[f'(U_0) W \right]_i \right\} dt \right| = o(1) \| W \|$$

and

$$\left| \int_{s}^{+\infty} \left\{ \frac{e^{\beta_i^\tau (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\beta_i^0 (s-t)}}{\sqrt{c^2 + 4d_i}} \right\} \left\{ W_i + \left[f'(U_0) W \right]_i \right\} dt \right| = o(1) \| W \|.$$

Proof. Since $\| f'(U_0) \|$ is bounded and independent of τ, we only need to prove that

$$\left| \int_{-\infty}^{s} \left\{ \frac{e^{\alpha_i^\tau (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\alpha_i^0 (s-t)}}{\sqrt{c^2 + 4d_i}} \right\} dt \right| = o(1) \tag{4.14}$$

and

$$\left| \int_{s}^{+\infty} \left\{ \frac{e^{\beta_i^\tau (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\beta_i^0 (s-t)}}{\sqrt{c^2 + 4d_i}} \right\} dt \right| = o(1) \tag{4.15}$$

as $\tau \to 0$. We will show that (4.14) is true and leave the proof of (4.15) to interested readers. Using $\alpha_i^0 < \alpha_i^\tau < 0$, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{s} \left\{ \frac{e^{\alpha_i^\tau (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\alpha_i^0 (s-t)}}{\sqrt{c^2 + 4d_i}} \right\} dt \right| \\
&\quad \leqslant \int_{-\infty}^{s} \left| \frac{e^{\alpha_i^\tau (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\alpha_i^0 (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} \right| dt + \int_{-\infty}^{s} \left| \frac{e^{\alpha_i^0 (s-t)}}{\sqrt{c^2 r^2 + 4d_i}} - \frac{e^{\alpha_i^0 (s-t)}}{\sqrt{c^2 + 4d_i}} \right| dt \\
&\quad = \frac{1}{\sqrt{c^2 r^2 + 4d_i}} \left(-\frac{1}{\alpha_i^\tau} + \frac{1}{\alpha_i^0} \right) + \left| \frac{1}{\sqrt{c^2 r^2 + 4d_i}} - \frac{1}{\sqrt{c^2 + 4d_i}} \right| \left(-\frac{1}{\alpha_i^0} \right) \\
&\quad = o(1)
\end{aligned}$$

as $\tau \to 0$. The proof of this lemma is complete. □

Lemma 4.2. *For each $\delta > 0$, there is a $\sigma > 0$ such that*

$$\left| \left[R_1(s, \tau, \phi) \right]_i - \left[R_1(s, \tau, \varphi) \right]_i \right| \leqslant \delta \| \phi - \varphi \|_{C_0} \tag{4.16}$$

and

$$\int_{-\infty}^{s} e^{\alpha_i^\tau (s-t)} \left|\left[R_1(t,\tau,\phi)\right]_i - \left[R_1(t,\tau,\varphi)\right]_i\right| dt + \int_{s}^{\infty} e^{\beta_i^\tau (s-t)} \left|\left[R_1(t,\tau,\phi)\right]_i - \left[R_1(t,\tau,\varphi)\right]_i\right| dt \leqslant \delta \|\phi-\varphi\|_{C_0} \tag{4.17}$$

uniformly for all $\phi, \varphi \in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof. It is obvious that the remainder $[R_1(s,\tau,\phi)]_i$ satisfies

$$\left|\left[R_1(s,\tau,\phi)\right]_i\right| = O\left(\|\phi\|^2\right) \quad \text{as } \|\phi\| \to 0. \tag{4.18}$$

Therefore, as

$$\frac{1}{\beta_i^\tau} - \frac{1}{\alpha_i^\tau} = \sqrt{c^2 r^2 + 4d_i},$$

(4.16) and (4.17) follow from (4.18). □

Lemma 4.3. *As $\tau \to 0$, we have*

$$\left| \int_{-\infty}^{s} e^{\alpha_i^\tau (s-t)} \left[R_2(t,\tau,W)\right]_i ds + \int_{s}^{\infty} e^{\beta_i^\tau (s-t)} \left[R_2(t,\tau,W)\right]_i dt \right| = o(1).$$

Proof. It is obvious from the expression of $R_2(s,\tau)$ that $R_2(s,\tau)$ approach 0 as $\tau \to 0$. □

Lemma 4.4. *There exists an $M_0 > 0$ such that for all $W \in C_0$, we have*

$$\left\| \int_{-\infty}^{s} E_1^\tau(s-t) R_3(t,\tau,W)\, dt + \int_{s}^{\infty} E_2^\tau(s-t) R_3(t,\tau,W)\, dt \right\| = o(1)\|W\|_{C_0}. \tag{4.19}$$

Furthermore, for any two functions ϕ_1 and ϕ_2 in C_0, we have

$$\left\| \int_{-\infty}^{s} E_1^\tau(s-t)\big(R_3(t,\tau,\phi_1) - R_3(t,\tau,\phi_1)\big)\, dt + \int_{s}^{\infty} E_2^\tau(s-t)\big(R_3(t,\tau,\phi_1) - R_3(t,\tau,\phi_1)\big)\, dt \right\| = O(\sqrt{\tau})\|\phi_1 - \phi_2\|_{C_0}. \tag{4.20}$$

Proof. We re-write R_3 in (4.7) as

$$\begin{aligned} R_3(s,\tau,W) = {} & \big(A^\tau(s) - A^0(s)\big)W(s) \\ & + \big(B^\tau(s) - B^0(s)\big)C^\tau(s,W) \\ & + B^0(s)W(s)\big(C^\tau(s,\mathbf{1}) - g_u\big(U_0(s)\big)\big) \\ & + B^0(s)\big(C^\tau(s,W) - C^\tau(s,\mathbf{1})W(s)\big), \end{aligned} \tag{4.21}$$

where

$$C^\tau(s,\mathbf{1}) = \int_{-\tau}^{0}\int_{\Omega} d_\tau(\theta,y) g_u\big(U_0(s+\nu\cdot y - rc\theta)\big), \quad s \in \mathbf{R}. \tag{4.22}$$

Since F is C^2-smooth and $U_0 \in C^1$, by (H_4) we have as $\tau \to 0$ that

$$A^\tau(s) - A^0(s) = F_u\big(U_0(s), C^\tau(s,\mathbf{1})\big) - F_u\big(U_0(s), g\big(U_0(s)\big)\big) = o(1),$$
$$B^\tau(s) - B^0(s) = o(1),$$

and

$$C^\tau(s,\mathbf{1}) - g_u\big(U_0(s)\big) = o(1).$$

Therefore, for the integrations of the first three terms on the right-hand side of (4.21), we have the following estimates:

$$\left\| \int_{-\infty}^{s} E_1^\tau(s-t)\big(A^\tau(t) - A^0(t)\big)W(t)\,dt \right\| = o(1)\|W\|, \tag{4.23}$$

and

$$\left\| \int_{s}^{\infty} E_2^\tau(s-t)\big(A^\tau(t) - A^0(t)\big)W(t)\,dt \right\| = o(1)\|W\|, \tag{4.24}$$

$$\left\| \int_{-\infty}^{s} E_1^\tau(s-t)\big(B^\tau(t) - B^0(t)\big)\big(C^\tau(t,\mathbf{1})W(t+\nu\cdot y - rc\theta)\big)\,dt \right\| = o(1)\|W\|, \tag{4.25}$$

and

$$\left\| \int_{s}^{\infty} E_2^\tau(s-t)\big(B^\tau(t) - B^0(t)\big)\big(C^\tau(t,\mathbf{1})W(t+\nu\cdot y - rc\theta)\big)\,dt \right\| = o(1)\|W\|, \tag{4.26}$$

$$\left\| \int_{-\infty}^{s} E_1^\tau(s-t)B^0(t)W(t)\big(C^\tau(t,\mathbf{1}) - g_u\big(U_0(t)\big)\big)\,dt \right\| = o(1)\|W\|, \tag{4.27}$$

$$\left\| \int_{s}^{\infty} E_2^\tau(s-t)B^0(t)W(t)\big(C^\tau(t,\mathbf{1}) - g_u\big(U_0(t)\big)\big)\,dt \right\| = o(1)\|W\|. \tag{4.28}$$

Now we estimate the convolution integration of the last term in (4.21). We assume first that $W \in C_0^1$. When $W \in C_0^1$, we have

$$W(t+\nu\cdot y - rc\theta) - W(t) = \int_0^{\nu\cdot y - rc\theta} W'(t+v)\,dv.$$

Using (4.4) and (4.22), and exchanging the order of integration and integration by parts, we obtain

$$\begin{aligned}
&\left\| \int_{-\infty}^{s} E_1^\tau(s-t)B^0(t)\big(C^\tau(t,W) - C^\tau(t,\mathbf{1})W(t)\big)\,dt \right\| \\
&\quad = \left\| \int_{-\infty}^{s} E_1^\tau(s-t)B^0(t)C^\tau(t,\mathbf{1})\big(W(t+\nu\cdot y - rc\theta) - W(t)\big)\,dt \right\| \\
&\quad = \left\| \int_{-\infty}^{s} E_1^\tau(s-t)B^0(t)C^\tau(t,\mathbf{1}) \int_0^{\nu\cdot y - rc\theta} W'(t+v)\,dv\,dt \right\| \\
&\quad = \left\| \int_0^{\nu\cdot y - rc\theta} dv \int_{-\infty}^{s} E_1^\tau(s-t)B^0(t)C^\tau(t,\mathbf{1})W'(t+v)\,dt \right\| \\
&\quad \leqslant \left\| \int_0^{\nu\cdot y - rc\theta} E_1^\tau(0)B^0(s)C^\tau(s,\mathbf{1})W(s+v)\,dv \right\| \\
&\qquad + \left\| \int_0^{\nu\cdot y - rc\theta} dv \int_{-\infty}^{s} W(t+v)\frac{d(E_1^\tau(s-t)B^0(t)C^\tau(t,\mathbf{1}))}{dt}\,dt \right\| \\
&\quad = o(1)\|W\|.
\end{aligned} \tag{4.29}$$

To obtain the above estimate, we have made use of the condition (H_4). Similarly, we can prove that

$$\left\| \int_{s}^{\infty} E_2^\tau(s-t)B^0(t)C^\tau(t,\mathbf{1})\big(W(t+\nu\cdot y - rc\theta) - W(t)\big)\,dt \right\| = o(1)\|W\|. \tag{4.30}$$

Since C_0^1 is dense in C_0, we know that (4.29) and (4.30) hold for any $W \in C_0$. Combining (4.23)–(4.30), we obtain (4.19). Thus (4.20) holds due to the fact that $R_3(s,\tau,W)$ is a linear functional of W and the proof is complete. □

We are ready to reformulate Theorem 2.1 and present a proof.

Theorem 4.1. *For any given $c \geqslant c_0^*$, there exist a constant $\delta = \delta(c) > 0$ so that for any $\tau \in (0,\delta)$, Eq.* (2.1) *has a traveling wavefront $u = U(\nu \cdot x - c_\tau^* ct/c_0^*)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = E_2 = \mathbf{K}, \qquad \lim_{s\to\infty} U(s) = E_1 = \mathbf{0}.$$

Proof. Define an operator $T : \Psi \in C^2 \to C$ from the homogeneous part of (4.6) as follows:

$$T\Psi(s) = -c\Psi'(s) - D\Psi''(s) - f'\big(U_0(s)\big)\Psi(s). \tag{4.31}$$

The formal adjoint equation of $T\Psi = 0$ is given by

$$c\Phi'(s) - D\Phi''(s) - f'\big(U_0(s)\big)\Phi(s) = 0, \quad s \in \mathbf{R}. \tag{4.32}$$

We now divide our proof into following steps:

Step 1. We claim that if $\Phi \in C$ is a solution of (4.32) and Φ is C^2-smooth, then $\Phi = 0$. Moreover, we have $R(T) = C$ where $R(T)$ is the range space of T.

Indeed, when $s \to \infty$, $U_0(s) \to \mathbf{0}$ and $f'(U_0(s)) \to f'(\mathbf{0})$. Then Eq. (4.32) asymptotically tends to equation with a constant coefficients

$$c\Phi'(s) - D\Phi''(s) - f'(\mathbf{0})\Phi(s) = 0. \tag{4.33}$$

Since for (4.33) the solution $\mathbf{0}$ is a unstable node and all eigenvalues to (4.33) have positive real parts, we conclude that any bounded solution to (4.33) must be the zero solution. So as $s \to \infty$, any solution to (4.32) other than the zero solution must grow exponentially for large s. Then the only solution satisfying $\Phi(\pm\infty) = 0$ is the zero solution. By the Fredholm theory (see Lemma 4.2 in [43]) we have that $R(T) = C$.

Step 2. Let $\Theta \in C_0$ be given. If Ψ is a bounded solution to equation $T\Psi = \Theta$, then we have $\lim_{s\to\pm\infty} \Psi(s) = 0$.

Actually when $s \to \infty$, equation

$$-c\Psi'(s) - D\Psi''(s) - f'\big(U_0(s)\big)\Psi(s) = \Theta \tag{4.34}$$

asymptotically tends to

$$-c\Psi'(s) - D\Psi''(s) - f'(\mathbf{0})\Psi(s) = \mathbf{0}. \tag{4.35}$$

Note for (4.35), the ω-limit set of every bounded solution is just the critical point $\Psi = \mathbf{0}$. Using the result from [38], every bounded solution of (4.34) also satisfies

$$\lim_{s\to\infty} \Psi(s) = \mathbf{0}.$$

When $s \to -\infty$, Eq. (4.34) asymptotically tends to

$$-c\Psi'(s) - D\Psi''(s) - f'(\mathbf{K})\Psi(s) = \mathbf{0}. \tag{4.36}$$

Since $\mathbf{K}$ is hyperbolic in the sense that the characteristic equation to (4.36) has no roots with zero real parts, every bounded solution to (4.36) must satisfy

$$\lim_{s\to-\infty} \Psi(s) = \mathbf{0}.$$

Reverting the time variable from s to $-s$ and using the result in [38] again, we know that any bounded solution to (4.34) satisfies $\lim_{s\to-\infty} \Psi(s) = 0$. The claim of this step is then verified.

Step 3. For a linear operator $L : C_0 \to C_0$ defined in (4.13). We want to prove that $R(L) = C_0$, that is, for each $Z(\cdot) \in C_0$, we have a $W(\cdot) \in C_0$ so that

$$\begin{aligned}
W_i(s) &- \frac{1}{\beta_i^0 - \alpha_i^0} \int_{-\infty}^{s} e^{\alpha_i^0(s-t)} \big\{W_i + \big[f'(U_0)W\big]_i\big\}\,dt \\
&- \frac{1}{\beta_i^0 - \alpha_i^0} \int_{s}^{\infty} e^{\beta_i^0(s-t)} \big\{W_i + \big[f'(U_0)W\big]_i\big\}\,dt \\
&= Z_i(s), \quad i = 1, 2, \ldots, n.
\end{aligned}$$

To see this, we assume that $\xi = W - Z$ and obtain a new equation for ξ as follows

$$\begin{aligned}
\xi_i(s) = {} & \frac{1}{\beta_i^0 - \alpha_i^0} \Bigg(\int_{-\infty}^{s} e^{\alpha_i^0(s-t)} \big(\xi_i(t) + \big[f'\big(U_0(t)\big)\xi(t)\big]_i\big)\,dt \\
& + \int_{s}^{\infty} e^{\beta_i^0(s-t)} \big(\xi_i(t) + \big[f'\big(U_0(t)\big)\xi(t)\big]_i\big)\,dt \Bigg) \\
& + \frac{1}{\beta_i^0 - \alpha_i^0} \Bigg(\int_{-\infty}^{s} e^{\alpha_i^0(s-t)} \big(Z_i(t) + \big[f'\big(U_0(t)\big)Z(t)\big]_i\big)\,dt \\
& + \int_{s}^{\infty} e^{\lambda_2(s-t)} \big(Z_i(t) + \big[f'\big(U_0(t)\big)Z(t)\big]_i\big)\,ds \Bigg), \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Differentiating both sides twice yields

$$-c\xi'(z) - D\xi''(z) - f'\big(U_0(z)\big)\xi(z) = \big(I + f'\big(U_0(z)\big)\big)Z(z), \quad z \in \mathbf{R}. \tag{4.37}$$

Using the results that $R(T) = C$ in Step 1 and $Z \in C_0$, we obtain by results in Step 2 that there exists a solution ξ satisfying (4.37) and $\xi(\pm\infty) = 0$. Returning to the variable W, we have $W = \xi + Z \in C_0$.

Step 4. Let $N(L)$ be the null space of the operator L. Define the quotient space $N^{\perp}(L)$ as $N^{\perp}(L) = C_0/N(L)$. It is clear that $N^{\perp}(L)$ is a Banach space. If we let $S = L|_{N^{\perp}(L)}$ be the restriction of L to $N^{\perp}(L)$, then $S: N^{\perp}(L) \to C_0$ is one-to-one and onto. By the well-known Banach inverse operator theorem, we have that $S^{-1}: C_0 \to C_0/N(L)$ is a linear bound operator.

Step 5. When L is restricted to $N^{\perp}(L)$, Eq. (4.11) can be written as

$$S(W) = H(\tau, s, W). \tag{4.38}$$

Since the norm $\|S^{-1}\|$ is independent of τ but dependent on c, it follows from Lemmas 4.1–4.4 that there exist $\sigma = \sigma(c) > 0$, $\delta = \delta(c) > 0$ and $\rho(c)$ with $0 < \rho < 1$ such that for all $\tau \in (0, \delta]$ and $W, \varphi, \psi \in B(\sigma) \subset N^{\perp}(L)$,

$$\left\|S^{-1}H(\tau, s, W)\right\| \leqslant \frac{1}{3}\bigl(\|W\| + \sigma\bigr)$$

and

$$\left\|S^{-1}H(\tau, s, \varphi) - S^{-1}H(\tau, s, \psi)\right\| \leqslant \rho\|\varphi - \psi\|.$$

Hence, $S^{-1}H$ is a uniform contractive mapping for $W \in N^{\perp}(L) \cap B(\sigma)$. By using the classical fixed point theorem, it follows that for $\tau \in [0, \delta]$, (4.38) has a unique solution $W \in C_0/N(L)$. Returning to the original variable, $W + U_0$ is a heteroclinic connection between the two equilibria $\mathbf{K}$ and $\mathbf{0}$. This completes the proof. □

Remark 4.1. The uniform contraction mapping principle [13, pp. 25, 26] implies that the fixed point W is continuous on (s, τ) and C^1-smooth on s.

Remark 4.2. In some applications, some of the diffusion coefficients are zero and thus we have a hybrid system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = D\Delta w(x,t) + F\left(u(x,t), \displaystyle\int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) g_1\bigl(w(x+y, t+\theta)\bigr)\right), \\ \dfrac{\partial v(x,t)}{\partial t} = G\left(w(x,t), \displaystyle\int_{-\tau}^{0}\int_{\Omega} d\mu_\tau(\theta, y) g_2\bigl(w(x+y, t+\theta)\bigr)\right) \end{cases} \tag{4.39}$$

with $u \in R^{m_1}$, $v \in R^{m_2}$, and $w = (u, v)^T$. We should mention that under the same assumptions (H_1)–(H_4) on the nonlinearities F, G, g_1, g_2, our result remains true for the hybrid system (4.39). Actually, in the above proof if for some index i, the coefficient d_i is zero, then (4.8) can be rewritten as

$$\begin{aligned} \frac{dW_i}{ds} + W_i = W_i - \frac{1}{cr}\bigl[f'(U_0)W\bigr]_i - \frac{1}{cr}\bigl[R_1(\tau, s, W)\bigr]_i \\ - \frac{1}{cr}\bigl[R_2(\tau, s, W)\bigr]_i - \frac{1}{cr}\bigl[R_3(\tau, s, W)\bigr]_i. \end{aligned} \tag{4.40}$$

This equation can be transformed into an integral equation as follows:

$$\begin{cases} W_i - \int\limits_{-\infty}^{s} e^{-(s-t)} \left\{ W_i - \frac{1}{c}\big[f'(U_0)W(t)\big]_i \right\} dt \\ \quad = \int\limits_{-\infty}^{s} e^{-(s-t)} \left\{ \frac{1}{c}\big[f'(U_0)W(t)\big]_i - \frac{1}{cr}\big[f'(U_0)W\big]_i \right\} dt \\ \qquad + \int\limits_{-\infty}^{s} e^{-(s-t)} \left\{ -\frac{1}{cr}\big[R_1(\tau,t,W)\big]_i - \frac{1}{cr}\big[R_2(\tau,t,W)\big]_i - \frac{1}{cr}\big[R_3(\tau,t,W)\big]_i \right\} dt. \end{cases}$$

All other arguments remain virtually unchanged and thus Theorem 2.1 holds true for the hybrid system (4.39).

It remains to prove the existence of traveling wavefronts in the hyperbolic system (2.8). Despite the hyperbolic nature, we can obtain a similar profile equation as shown below. This similarity guarantees that the above arguments work again.

Proof of Theorem 2.2. As in (4.1), a traveling wave of Eq. (2.8) satisfies

$$-rcU' = \big(D - r_1c^2\big)U'' + F\left(U(s), \int\limits_{-\tau}^{0}\int\limits_{\Omega} d\mu_\tau(\theta,y) g\big(U(s+\nu\cdot y - rc\theta)\big)\right).$$

Using the same arguments as in proof of Theorem 4.1, we can show that when r_1 and τ are small, the traveling wave can be approximated by U_0 in (2.4) and thus we obtain the existence by minor modification the proof of Theorem 4.1. We leave the details to interesting readers. □

Remark 4.3. Unfortunately, we cannot have an explicit formula for $\delta(c)$. This quantity is dependent on the norm of the operator S^{-1}.

5. Proofs: the bistable case

We now present a proof of Theorem 3.1, the proof of Theorem 3.2 is similar and is thus omitted.

We start with Eqs. (3.3) and (3.4). As before, suppose that $U = U_0 + W$ and $c = c_0 + b$, we then have an equation for (W, b):

$$\begin{aligned} -(c_0+b)W' = DW'' &+ F\left(U_0 + W, \int\limits_{-\tau}^{0}\int\limits_{\Omega} d\mu_\tau(\theta,y) g\big([U_0+W](s+\nu\cdot y - c\theta)\big)\right) \\ &- F\big(U_0(s), g\big(U_0(s)\big)\big) + bU_0', \end{aligned} \tag{5.1}$$

subject to $W(\pm\infty) = \mathbf{0}$, and our goal is to prove the existence of a solution (W, b) to (5.1).

Using Eq. (3.4) and expanding the second term in the right side of (5.1), we have

$$\begin{aligned} -(c_0+b)W' = DW'' + f'(U_0)W + bU_0' + R_1(\tau, s, W, b) \\ + R_2(\tau, s, W, b) + R_3(\tau, s, W, b), \end{aligned} \tag{5.2}$$

where $R_1(\tau, s, W, b)$ and $R_3(\tau, s, W, b)$ are the same as in (4.5) and (4.7) with $r = 1$ and $c = c_0 + b$, and the functional $R_2(\tau, s, W, b)$ is given by

$$R_2(\tau, s, W, b) = F\left(U_0(s), \int_{-\tau}^{0}\int_{\Omega} d\eta(\theta)\, d\mu_\tau(y)\, g\big(U_0(s+\nu\cdot y - c\theta)\big)\right) - f(U_0).$$

Now we transform Eq. (5.2) into an integral equation. We re-write Eq. (5.2) as

$$\begin{aligned} d_i\frac{d^2W_i}{ds^2} + (c_0+b)\frac{dW_i}{ds} - W_i = -W_i - \big[f'(U_0)W\big]_i - \big[bU_0'\big]_i - \big[R_1(\tau, s, W, b)\big]_i \\ - \big[R_2(\tau, s, W, b)\big]_i - \big[R_3(\tau, s, W, b)\big]_i. \end{aligned} \tag{5.3}$$

Since the equation

$$d_i\frac{d^2W_i}{ds^2} + (c_0+b)\frac{dW_i}{ds} - W_i = 0$$

has a characteristic equation

$$d_i\lambda^2 + (c_0+b)\lambda - 1 = 0$$

with two real roots:

$$\alpha_i^b = \frac{-(c_0+b) - \sqrt{(c_0+b)^2 + 4d_i}}{2d_i} < 0, \qquad \beta_i^b = \frac{-(c_0+b) + \sqrt{(c_0+b)^2 + 4d_i}}{2d_i} > 0$$

satisfying

$$\lim_{b\to 0}\alpha_i^b = \alpha_i^0 = \frac{-c_0 - \sqrt{c_0^2 + 4d_i}}{2d_i}, \qquad \lim_{\tau\to 0}\beta_i^b = \beta_i^0 = \frac{-c_0 + \sqrt{c_0^2 + 4d_i}}{2d_i}, \tag{5.4}$$

we conclude that (5.3) is equivalent to the following integral equation:

$$\begin{aligned} W_i(s) = \frac{1}{d_i(\beta_i^b - \alpha_i^b)}\int_{-\infty}^{s} e^{\alpha_i^b(s-t)}\left(W_i + \big[f'(U_0)W + bU_0'\big]_i + \sum_{j=1}^{3}\big[R_j(\tau, s, W, b)\big]_i\right)dt \\ + \frac{1}{d_i(\beta_i^b - \alpha_i^b)}\int_{s}^{\infty} e^{\beta_i^b(s-t)}\left(W_i + \big[f'(U_0)W + bU_0'\big]_i + \sum_{j=1}^{3}\big[R_j(\tau, s, W, b)\big]_i\right)dt, \end{aligned}$$

for $i = 1, 2, \ldots, n$. Recall that $d_i(\beta_i^b - \alpha_i^b) = \sqrt{(c_0+b)^2 + 4d_i}$. The above equation can also be written as

$$
\begin{aligned}
W_i(s) &- \frac{1}{\sqrt{c_0^2+4d_i}} \int_{-\infty}^{s} e^{\alpha_i^0(s-t)} \bigl\{ W_i + \bigl[f'(U_0)W \bigr]_i + \bigl[bU_0' \bigr]_i \bigr\}\, dt \\
&- \frac{1}{\sqrt{c_0^2+4d_i}} \int_{s}^{\infty} e^{\beta_i^0(s-t)} \bigl\{ W_i + \bigl[f'(U_0)W \bigr]_i + \bigl[bU_0' \bigr]_i \bigr\}\, dt \\
= &\int_{-\infty}^{s} \biggl\{ \frac{e^{\alpha_i^b(s-t)}}{\sqrt{(c_0+b)^2+4d_i}} - \frac{e^{\alpha_i^0(s-t)}}{\sqrt{c_0^2+4d_i}} \biggr\} \bigl\{ W_i + \bigl[f'(U_0)W \bigr]_i + \bigl[bU_0' \bigr]_i \bigr\}\, dt \\
&+ \int_{s}^{+\infty} \biggl\{ \frac{e^{\beta_i^b(s-t)}}{\sqrt{(c_0+b)^2+4d_i}} - \frac{e^{\beta_i^0(s-t)}}{\sqrt{c_0^2+4d_i}} \biggr\} \bigl\{ W_i + \bigl[f'(U_0)W \bigr]_i + \bigl[bU_0' \bigr]_i \bigr\}\, dt \\
&+ \frac{1}{\sqrt{(c_0+b)^2+4d_i}} \int_{-\infty}^{s} e^{\alpha_i^b(s-t)} \sum_{j=1}^{3} \bigl[R_j(\tau,s,W,b) \bigr]_i\, dt \\
&+ \frac{1}{\sqrt{(c_0+b)^2+4d_i}} \int_{s}^{\infty} e^{\beta_i^b(s-t)} \sum_{j=1}^{3} \bigl[R_j(\tau,s,W,b) \bigr]_i\, dt. \qquad (5.5)
\end{aligned}
$$

Set

$$
E_1^b(s) = \operatorname{diag}\biggl(\frac{e^{\alpha_1^b s}}{\sqrt{(c_0+b)^2+4d_1}}, \frac{e^{\alpha_2^b s}}{\sqrt{(c_0+b)^2+4d_2}}, \ldots, \frac{e^{\alpha_n^b s}}{\sqrt{(c_0+b)^2+4d_n}} \biggr), \quad s \in \mathbf{R},
$$

and

$$
E_2^b(s) = \operatorname{diag}\biggl(\frac{e^{\beta_1^b s}}{\sqrt{(c_0+b)^2+4d_1}}, \frac{e^{\beta_2^b s}}{\sqrt{(c_0+b)^2+4d_2}}, \ldots, \frac{e^{\beta_n^b s}}{\sqrt{(c_0+b)^2+4d_n}} \biggr), \quad s \in \mathbf{R},
$$

$$
E_1^0(s) = \operatorname{diag}\biggl(\frac{e^{\alpha_2^0 s}}{\sqrt{c_0^2+4d_1}}, \frac{e^{\alpha_2^0 s}}{\sqrt{c_0^2+4d_2}}, \ldots, \frac{e^{\alpha_n^0 s}}{\sqrt{c_0^2+4d_n}} \biggr), \quad s \in \mathbf{R},
$$

$$
E_2^0(s) = \operatorname{diag}\biggl(\frac{e^{\beta_2^0 s}}{\sqrt{c_0^2+4d_1}}, \frac{e^{\beta_2^0 s}}{\sqrt{c_0^2+4d_2}}, \ldots, \frac{e^{\beta_n^0 s}}{\sqrt{c_0^2+4d_n}} \biggr), \quad s \in \mathbf{R},
$$

$$
E_3^b(s) = E_1^b(s) - E_1^0(s), \quad s \in \mathbf{R},
$$

and

$$
E_4^b(s) = E_2^b(s) - E_2^0(s), \quad s \in \mathbf{R}.
$$

Then Eq. (5.5) can be written as

$$\begin{aligned} W &- \int_{-\infty}^{s} E_1^0(s-t)\{W(t)+f'(U_0)W(t)+bU_0'\}\,dt \\ &- \int_{s}^{\infty} E_2^0(s-t)\{W(t)+f'(U_0)W(t)+bU_0'\}\,dt \\ &= H(\tau,s,W,b), \quad W(\pm\infty)=\mathbf{0}, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} H(\tau,s,W,b) &= \int_{-\infty}^{s} E_3^b(s-t)\{W(t)+f'(U_0)W(t)+bU_0'\}\,dt \\ &+ \int_{s}^{+\infty} E_4^b(s-t)\{W(t)+f'(U_0)W(t)+bU_0'\}\,dt \\ &+ \int_{-\infty}^{s} E_1^b(s-t)\sum_{j=1}^{3} R_j(\tau,t,W,b)\,dt \\ &+ \int_{s}^{\infty} E_1^b(s-t)\sum_{j=1}^{3} R_j(\tau,t,W,b)\,dt. \end{aligned}$$

Now we need to study the existence of a solution (W,b) to (5.6). To this end, we define a linear operator $L:(W,b)\in C_0\times\mathbf{R}\to C_0$ from the left side of (5.6) by

$$\begin{aligned} L(W,b)(s) = W(s) &- \int_{-\infty}^{s} E_1^0(s-t)\{W(t)+f'(U_0)W(t)+bU_0'\}\,dt \\ &- \int_{s}^{\infty} E_2^0(s-t)\{W(t)+f'(U_0)W(t)+bU_0'\}\,dt. \end{aligned} \tag{5.7}$$

It follows from a straightforward verification that $L((W,b))\in C_0$ if $W\in C_0$. We need some estimations for the right side of (5.6).

Lemma 5.1. *We have*

$$\left|\int_{-\infty}^{s}\left\{\frac{e^{\alpha_i^b(s-t)}}{\sqrt{(c_0+b)^2+4d_i}}-\frac{e^{\alpha_i^0(s-t)}}{\sqrt{c_0^2+4d_i}}\right\}\left\{W_i+\left[f'(U_0)W\right]_i\right\}dt\right| = O\left(b\|W\|\right)$$

and

$$\left|\int_s^{+\infty}\left\{\frac{e^{\beta_i^\tau(s-t)}}{\sqrt{(c_0+b)^2+4d_i}}-\frac{e^{\beta_i^0(s-t)}}{\sqrt{c_0^2+4d_i}}\right\}\left\{W_i+\left[f'(U_0)W\right]_i\right\}dt\right|=O\big(b\|W\|\big).$$

Lemma 5.2. *For each* $\delta>0$*, there is a* $\sigma>0$ *such that*

$$\left|\left[R_1(s,\tau,\phi,b_1)\right]_i-\left[R_1(s,\tau,\varphi,b_2)\right]_i\right|\leqslant\delta\left\|(\phi,b_1)-(\varphi,b_2)\right\|$$

and

$$\begin{aligned}&\int_{-\infty}^{s}e^{\alpha_i^b(s-t)}\left|\left[R_1(t,\tau,\phi,b_1)\right]_i-\left[R_1(t,\tau,\varphi,b_2)\right]_i\right|dt\\&\qquad+\int_s^{\infty}e^{\beta_i^b(s-t)}\left|\left[R_1(t,\tau,\phi,b_1)\right]_i-\left[R_1(t,\tau,\varphi,b_2)\right]_i\right|dt\\&\quad\leqslant\delta\left\|(\phi,b_1)-(\varphi,b_2)\right\|\end{aligned}$$

for all $(\phi,b),(\varphi,b)\in B(\sigma)$*, where* $B(\sigma)$ *is the ball in* $C_0\times\mathbf{R}$ *with radius* σ *and center at the origin and*

$$\left\|(\phi,b_1)-(\varphi,b_2)\right\|=\max\left\{\|\phi-\varphi\|_{C_0},|b_2-b_1|\right\}.$$

Lemma 5.3. *As* $\tau\to0$*, we have*

$$\left|\int_{-\infty}^{s}e^{\alpha_i^b(s-t)}\left[R_2(t,\tau,W,b)\right]_i\,ds+\int_s^{\infty}e^{\beta_i^b(s-t)}\left[R_2(t,\tau,W,b)\right]_i\,dt\right|=o(1)+o\big(|b|\big).$$

Lemma 5.4. *As* $\tau\to0$*, there exists an* $M_0>0$ *such that for all* $(W,b)\in B(\sigma)\cap(C_0\times\mathbf{R})$*, we have*

$$\left\|\int_{-\infty}^{s}E_1^b(s-t)R_3(t,\tau,W,b)\,dt+\int_s^{\infty}E_2^b(s-t)R_3(t,\tau,W,b)\,dt\right\|=\big(o(1)+o\big(|b|\big)\big)\|W\|_{C_0}.$$

The above lemmas can be proved in a similar fashion to those analogues in Section 4. We can now give a

Proof of Theorem 3.1. Define an operator $T:(\Psi,b)\in C^2\times\mathbf{R}\to C$ from the homogeneous part of (4.6) as follows:

$$T\Psi(s)=-c_0\Psi'(s)-D\Psi''(s)-f'\big(U_0(s)\big)\Psi(s)-bU_0'.\tag{5.8}$$

The formal adjoint equation of $T\Psi = 0$ is given by

$$c_0\Phi'(s) - D\Phi''(s) - f'\big(U_0(s)\big)\Phi(s) - bU_0' = \mathbf{0}, \quad s \in \mathbf{R}. \tag{5.9}$$

First of all, it is easy to know, by condition (G_2) and the Fredholm theory in [43], that $R(T) = C$ and $\operatorname{Ker}(T) = \operatorname{span}\{U_0'\} \times \{0\}$. Furthermore, for given $\Theta \in C$, if Ψ is a bounded solution of equation

$$-c_0\Psi'(s) - D\Psi''(s) - f'\big(U_0(s)\big)\Psi(s) - bU_0' = \Theta,$$

we have from condition (G_2) that b is uniquely determined by

$$b = -\frac{\int \Theta(s)\cdot p^*(s)\,ds}{\int U_0'(s)\cdot p^*(s)\,ds}$$

and $\lim_{s\to\pm\infty}\Psi(s) = 0$.

We now show that the linear operator $L : (W, b) \in C_0 \times \mathbf{R} \to C_0$ defined in (5.7) satisfies $R(L) = C_0$, that is, for each $Z(\cdot) \in C_0$, we have a $(W, b) \in C_0 \times \mathbf{R}$ so that

$$\begin{aligned} W_i(s) &- \frac{1}{\beta_i^0 - \alpha_i^0}\int_{-\infty}^{s} e^{\alpha_i^0(s-t)}\big\{W_i + \big[f'(U_0)W\big]_i + \big[bU_0'\big]_i\big\}\,dt \\ &- \frac{1}{\beta_i^0 - \alpha_i^0}\int_{s}^{\infty} e^{\beta_i^0(s-t)}\big\{W_i + \big[f'(U_0)W\big]_i + \big[bU_0'\big]_i\big\}\,dt \\ &= Z_i(s), \quad i = 1, 2, \ldots, n. \end{aligned}$$

To see this, we assume that $\xi = W - Z$ and obtain an equation for ξ as follows

$$\begin{aligned} \xi_i(s) = {} & \frac{1}{\beta_i^0 - \alpha_i^0}\Bigg(\int_{-\infty}^{s} e^{\alpha_i^0(s-t)}\big(\xi_i(t) + \big[f'\big(U_0(t)\big)\xi(t)\big]_i + \big[bU_0'\big]_i\big)\,dt \\ & + \int_{s}^{\infty} e^{\beta_i^0(s-t)}\big(\xi_i(t) + \big[f'\big(U_0(t)\big)\xi(t)\big]_i + \big[bU_0'\big]_i\big)\,dt\Bigg) \\ & + \frac{1}{\beta_i^0 - \alpha_i^0}\Bigg(\int_{-\infty}^{s} e^{\alpha_i^0(s-t)}\big(Z_i(t) + \big[f'\big(U_0(t)\big)Z(t)\big]_i\big)\,dt \\ & + \int_{s}^{\infty} e^{\lambda_2(s-t)}\big(Z_i(t) + \big[f'\big(U_0(t)\big)Z(t)\big]_i\big)\,ds\Bigg), \quad i = 1, 2, \ldots, n. \end{aligned}$$

Differentiating both sides twice yields

$$-c\xi' - D\xi''(z) - f'\big(U_0(z)\big)\xi(z) - bU_0' = \big(I + f'\big(U_0(z)\big)\big)Z(z). \tag{5.10}$$

Using the results about the operator T, we obtain that there exists a solution (ξ, b) satisfying (5.10) and $\xi(\pm\infty) = 0$. Returning to the variable W, we have $W = \xi + Z \in C_0$.

Let $N(L)$ be the null space of the operator L. As before, we set $N^{\perp}(L) = (C_0 \times \mathbf{R})/N(L)$ and $S = L|_{N^{\perp}(L)}$ to be the restriction of L. We have that S is invertible and Eq. (5.6) can be re-written as

$$(W, b) = S^{-1}\big[H(\tau, s, W, b)\big]. \tag{5.11}$$

Since the norm $\|S^{-1}\|$ is independent of τ, it follows from Lemmas 5.1–5.4 that there exist $\sigma > 0$, $\delta > 0$, and $\rho \in (0, 1)$ such that for all $\tau \in (0, \delta]$ and $(W, b), (\varphi, b_1), (\psi, b_2) \in B(\sigma) \subset N^{\perp}(L)$, we have

$$\big\|S^{-1}H(\tau, s, W, b)\big\| \leqslant \frac{1}{3}\big(\big\|(W, b)\big\| + \sigma\big)$$

and

$$\big\|S^{-1}H(\tau, s, \varphi, b_1) - S^{-1}H(\tau, s, \psi, b_2)\big\| \leqslant \rho\big\|(\varphi, b_1) - (\psi, b_2)\big\|,$$

where $\|(W, b)\| = \max\{\|W\|, |b|\}$. Hence $L^{-1}H$ is a uniform contractive mapping for $(W, b) \in N^{\perp}(L) \cap B(\sigma)$. By using the classical fixed point theorem, we conclude that for $\tau \in [0, \delta]$, (5.11) has a unique solution (W, b). Returning to the original variable, $(W + U_0, b + c_0)$ is a heteroclinic connection between the two equilibria $\mathbf{K}$ and $\mathbf{0}$. This completes the proof. $\square$

6. Applications

In this section, we apply our main result to some biological models: a reaction–diffusion model for a single species with age structure, a nonlocal Fisher model, a model of spatial spread of rabies among red foxes in Europe, a bio-reactor model and a hyperbolic model arising from the slow movement of biological species.

6.1. A reaction–diffusion model for a single species with age structure

The first application is about our model (1.2)

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - dw + \varepsilon \int_{-\infty}^{\infty} b\big(w(y, t - \tau)\big) f_\alpha(x - y)\, dy, \quad D, d > 0,\ x \in (-\infty, \infty), \tag{6.1}$$

where $0 \leqslant \varepsilon \leqslant 1$ and the kernel function $f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-x^2/4\alpha}$, $\alpha = \tau D_I > 0$, D_I is the diffusion rate for the immature population, and the function $b(\cdot)$ is the birth function of the species.

Our general result can be used to obtain the existence of traveling wave solutions even when the birth function $b(\cdot)$ is not monotone.

To illustrate this and as in [51], we consider first a particular birth function given by $b(w) = pwe^{-aw}$. This function has been used in the well-studied Nicholson's blowflies equation [25]. With this birth function, Eq. (6.1) becomes

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - dw + \varepsilon p \int_{-\infty}^{\infty} w(y, t-\tau)e^{-aw(y,t-\tau)} f_\alpha(x-y)\,dy. \tag{6.2}$$

It is easily seen that when

$$\varepsilon p/d > 1, \tag{6.3}$$

Eq. (6.2) has two spatially homogeneous equilibria

$$E_1 = 0, \qquad E_2 = \frac{1}{a}\ln\frac{\varepsilon p}{d}.$$

By using the monotone iteration scheme and the method of upper-lower solutions, So, Wu and Zou [51] proved the following

Theorem 6.1. *If $1 < \frac{\varepsilon p}{d} \leqslant e$, then there is a $c^* > 0$ such that for every $c \geqslant c^*$, Eq.* (6.2) *has a traveling wavefront solution which connects the trivial equilibrium $w_1 = 0$ to the positive equilibrium $w_2 = \frac{1}{a}\ln\frac{\varepsilon p}{d}$.*

Unfortunately, in the case when $\frac{\varepsilon p}{d} > e$, the method developed in [51] cannot apply, and so far it seems the only result is about the existence of wavefronts with large wave speeds recently proved in [16]. To be precise, we restate their result as follows.

Theorem 6.2. *If $\frac{\varepsilon p}{d} > e$, then there exist a $\tau^* > 0$ and a sufficiently large $c^* > 0$ such that if $\tau \in [0, \tau^*)$ then for every $c > c^*$, Eq.* (6.2) *has a traveling wavefront solution which connects the trivial equilibrium $w_1 = 0$ to the positive equilibrium $w_2 = \frac{1}{a}\ln\frac{\varepsilon p}{d}$.*

However, the existence of traveling waves with speeds close to the minimal speed is still left open. To address this issue, we first check the conditions (H_1)–(H_4) for Eq. (6.2). When $\tau = 0$, Eq. (6.2) reduces to

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - dw + \varepsilon pwe^{-aw}. \tag{6.4}$$

This gives

$$\Lambda_0^0(\lambda) = D\lambda^2 + c\lambda + \varepsilon p - d,$$

and we know, with $c_0^* = 2\sqrt{D(\varepsilon p - d)}$ that for every $c \geqslant c_0^*$, equation $\Lambda_0^0(\lambda) = 0$ has a real zero. Moreover, we find that $E_1 = 0$ is a stable node and $E_2 = \frac{1}{a}\ln\frac{\varepsilon p}{d}$ is a saddle point. By the classical phase-plane analysis (see [39]), it is easy to know that Eq. (6.4) has a traveling wavefront U_0 connecting E_1 to E_2. Thus all the conditions in Theorem 2.1 are satisfied.

Define $\Lambda_\tau^0(\lambda)$ and c_τ^* as

$$\Lambda_\tau^0(\lambda) = \varepsilon p e^{\alpha\lambda^2+\lambda c r} + \left[c\lambda - d + D\lambda^2\right]$$

and

$$c_\tau^* = \inf\left\{c > 0 \,\middle|\, \Lambda_\tau^0(\lambda) = 0 \text{ has a negative real zero}\right\}.$$

Applying Theorem 2.1, we have

Theorem 6.3. *For any given $c \geqslant c_0^* = 2\sqrt{D(\varepsilon p - d)} > 0$, there exists a constant $\delta = \delta(c) > 0$ so that for $\tau \in (0, \delta)$, Eq. (6.2) has a traveling wavefront $w = U(x - c_\tau^* ct/c_0^*)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = E_2, \qquad \lim_{s\to\infty} U(s) = E_1 = 0.$$

For the bistable case, we consider the birth function $b(m) = pm^2e^{-am}$ as an illustration. Then Eq. (6.1) becomes

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - dw + \varepsilon p \int_{-\infty}^{\infty} w^2(y, t-\tau)e^{-aw(y,t-\tau)} f_\alpha(x-y)\,dy. \tag{6.5}$$

When the positive constants ε, p, a satisfying the inequality

$$\varepsilon p > dae,$$

we have three constant solutions E_1, E_2 and E_3 with

$$E_1 = 0, \quad 0 < E_2 < E_3. \tag{6.6}$$

If $\tau = 0$, Eq. (6.5) reduces to

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - dw + \varepsilon p w^2 e^{-aw}. \tag{6.7}$$

By the classical result in [18] about the reaction–diffusion equation (6.7), it follows that there exists a unique (up to translation) traveling wave $w = U_0(x - c_0 t)$ satisfying $U_0' < 0$ and

$$\lim_{s\to-\infty} U_0(s) = E_3, \qquad \lim_{s\to\infty} U_0(s) = E_1 = 0.$$

Therefore, the condition (G_1) is satisfied. It remains to check the condition (G_2). It is easy to know that E_1 and E_3 are saddle points. Consider equation

$$-c_0u' - Du'' - f'(U_0)u = 0, \tag{6.8}$$

which has a bounded solution $u = U_0'$. The other linearly independent solution to (6.8) is unbounded since E_1 (or E_3) is a saddle point. This means that Eq. (6.8) has a unique bounded solution (up to scalar multiple). Similarly, we can know that the adjoint equation

$$c_0 u' - Du'' - f'(U_0)u = 0$$

has a unique bounded solution p^* so that (see also [15])

$$\int_{-\infty}^{\infty} U_0'(s)p^*(s)\,ds \neq 0.$$

The conditions for Theorem 3.1 are satisfied and we have the following

Theorem 6.4. *When $\varepsilon p > dae$, there exists a constant $\delta > 0$ so that for $\tau \in (0, \delta)$, Eq.* (6.5) *has a traveling wavefront $w = U(x - ct)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = E_3, \qquad \lim_{s\to\infty} U(s) = E_1 = 0$$

for some constant c.

6.2. A nonlocal Fisher model

Britton [9,10] used random walk arguments to derive an integro-differential reaction–diffusion population model

$$\frac{\partial u(x,t)}{\partial t} = \Delta u + u\big(1 + \alpha u - (1+\alpha)g * u\big), \quad x = (x_1, x_2, \ldots, x_m) \in R^m,\ t, u \in R, \tag{6.9}$$

where $\alpha > 0$ and $g * u$ represents a temporal-spatial convolution, and an example is

$$g * u = \int_{\mathbf{R}^m} \frac{e^{-\|y\|^2/4\tau}}{(4\pi\tau)^{m/2}} u(x - y, t - \tau)\,dy,$$

where m is the dimension of the space. According to [10], we have the following justification: *the term αu with $\alpha > 0$ represents an advantage in local aggregation, which could arise in many reasons. First, there is evidence that animals group together as a protective measure against predation, as in the case of grassland herds, schools of fish, or flock of birds, or Hamilton's* [26] *model of the aggregation of frogs. Second, there are advantages in grouping to optimize foraging benefits, by reducing per capita search time, by reducing the variance of the realized intakes, or by making available different food resources. Third, increased group size may result in increased reproductive success. Fourth, there may be social advantages, for example, in the utilization of the caste system by social insects. The term $-(1+\alpha)g * u$ with $\alpha > -1$ represents a disadvantage in global population levels being too high because of the resultant depletion of resources. The reason that this must be a global term is that the members of the population are moving* (*by diffusion*) *so that the force of intraspecies competition depends on population levels in a neighborhood of the original position.*

Due to the nonmonotonicity of the nonlinearity, the existence of traveling wavefronts connecting the equilibrium $u = 0$ and the equilibrium $u = 1$ has been an outstanding issue since 1990. When $\tau = 0$, we have $g * u = u$ and therefore Eq. (6.9) reduces to the classical Fisher equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u + u(1-u). \tag{6.10}$$

It is obvious that Eq. (6.10) has a traveling wavefront solution $U_0(\nu \cdot x - ct)$ for any $c \geqslant 2$. All the conditions in Theorem 2.1 are satisfied and the characteristic function $\Lambda_\tau^0(\lambda)$ is given by

$$\Lambda_\tau^0(\lambda) := \det\bigl[\lambda^2 I + \lambda c I + I\bigr] = \lambda^2 + \lambda c + 1$$

which is independent of τ. Thus we obtain $c_\tau^* = c_0^* = 2$ and the following

Theorem 6.5. *For any given $c \geqslant c_0^* = 2 > 0$, there exists a constant $\delta = \delta(c) > 0$ so that for $\tau \in (0, \delta)$, Eq. (6.9) has a traveling wavefront $u = U(\nu \cdot x - ct)$ which satisfies*

$$\lim_{s \to -\infty} U(s) = \mathbf{1}, \qquad \lim_{s \to \infty} U(s) = \mathbf{0}.$$

6.3. Spatial spread of rabies by red foxes

We now apply Theorem 2.1 to a delay reaction–diffusion model concerning the spatial spread of rabies in Europe carried by red foxes. For a related study of the spread of rabies worldwide, see Allen et al. [2] and Daszak et al. [14]. Motivated by Anderson's paper [3], we divide the fox population into two groups: the infective and the susceptible. The basic facts and assumptions of our proposed model can be found in [28] and [29]. Kallen et al. in [29] established a reaction–diffusion model

$$\begin{cases} \dfrac{\partial S}{\partial t} = -KIS, \\ \dfrac{\partial I}{\partial t} = D\dfrac{\partial^2 I}{\partial x^2} + KIS - \mu I, \end{cases} \tag{6.11}$$

where S and I are the susceptible and infective population densities, respectively. Here the parameters K is the transmission coefficient and μ is the death rate of the infective foxes. The diffusion term $D\frac{\partial^2 I}{\partial x^2}$, where $D > 0$ is the diffusive coefficient, represents the random motion of rabid foxes averaged out over the whole infective population. We should mention that the infective foxes I here consists of both rabid foxes and those in the incubation stage.

Suppose that S_0 is the initial (maximum) susceptible density. Kallen et al. [28,29] proved that if $\mu > KS_0$, that is, the mortality rate is greater than the rate of recruitment of new infectives, the infection dies out; otherwise, if $\mu < KS_0$, then the infective and susceptible foxes can coexist and after the outbreak of the disease, the population of the susceptible foxes will tend monotonically to a constant a, $0 < a < S_0$, where a is implicitly determined by the following equation

$$e^{-d\tau} a - \frac{\mu}{K} \log a = e^{-d\tau} S_0 - \frac{\mu}{K} \log S_0.$$

As the field observation in Europe showed the existence of periodic outbreak, model (6.11) deserves further refinement and Anderson et al. [3] speculated that oscillation may arise primarily from the incubation period. Motivated by this speculation, we now consider

$$\begin{cases} \dfrac{\partial S}{\partial t} = -KIS, \\ \dfrac{\partial I}{\partial t} = D\dfrac{\partial^2 I}{\partial x^2} + e^{-d\tau} KI(x, t-\tau)S(x, t-\tau) - \mu I, \end{cases} \tag{6.12}$$

where τ is the period of the incubation of rabies and the constant d is the death rate of susceptible foxes. Here I represents the density of rabid foxes excluding those in the incubation period. When $\tau = 0$, our delay model reduces to (6.11). We now look for a traveling wave solution $I(x,t) = f(z)$, $S(x,t) = g(z)$, where $z = x - ct$ and f and g are waveforms traveling to the right at speed c. Substituting these relations into (6.12) gives

$$\begin{cases} Df'' + cf' + e^{-d\tau} Kf(z + c\tau)g(z + c\tau) - \mu f = 0, \\ cg' - Kfg = 0 \end{cases} \tag{6.13}$$

subject to the boundary conditions

$$f(\pm\infty) = 0, \qquad g(-\infty) = a, \qquad g(+\infty) = S_0, \tag{6.14}$$

where a is a constant to be found. Substituting the second equation of (6.13) into the first yields

$$\frac{D}{c} f'' + f' + e^{-d\tau} g'(z + c\tau) - \frac{\mu g'}{Kg} = 0,$$

which on integration gives

$$\frac{D}{c} f' + f + e^{-d\tau} g(z + c\tau) - \frac{\mu}{K} \log g = A, \tag{6.15}$$

where A is a constant. Use the boundary condition at $z = \infty$ gives that $A = e^{-d\tau} S_0 - \frac{\mu}{K} \log S_0$, while the conditions at $z = -\infty$ show that

$$e^{-d\tau} a - \frac{\mu}{K} \log a = e^{-d\tau} S_0 - \frac{\mu}{K} \log S_0.$$

To obtain a lower bound for the wavespeed, we consider the system

$$\begin{cases} f' = \dfrac{c}{D}\left\{A - f - e^{-d\tau} g(z + c\tau) + \dfrac{\mu}{K} \log g\right\}, \\ g' = \dfrac{K}{c} fg, \end{cases} \tag{6.16}$$

which has equilibria points at $E_1 = (0, S_0)$ and $E_2 = (0, a)$. Linearizing at the point E_1, we have

$$\Lambda_\tau^{E_1}(\lambda) = \lambda^2 + \frac{c}{D}\lambda + \frac{KS_0}{D}\left(e^{-d\tau + \lambda c\tau} - \frac{\mu}{KS_0}\right).$$

So the minimal speed c_τ^* is given by

$$c_\tau^* = \inf\{c > 0 \mid \Lambda_\tau^{E_1}(\lambda) = 0 \text{ has a negative real zero}\}.$$

When $\tau = 0$, we get $c_0^* = 2\sqrt{DKS_0(1 - \mu/KS_0)}$. It is easy to know that when $\tau = 0$, $(0, S_0)$ is a stable node and all conditions in Theorem 2.1 are satisfied. Therefore, we have the following

Theorem 6.6. *For given $c \geqslant c_0^* = 2\sqrt{DKS_0(1 - \mu/KS_0)}$, there exist a constant $\delta > 0$ so that for $\tau \in (0, \delta)$, Eq.* (6.12) *has a traveling wavefront $u = U(x - c_\tau^* ct/c_0^*)$ which connects the equilibria E_1 and E_2.*

Remark 6.1. From (6.16), we know that $g' = \frac{K}{c} fg > 0$ as long as f and g are positive. This means that the density for the susceptible foxes is monotone in the wave spreading direction and it removes the possibility of oscillation of the solution patterns (or periodic outbreak of the disease). This is what we expect because we assume that in the disease free case, the deaths are equally balanced by births, and thus when the disease outbreaks, the density of susceptible foxes will decrease monotonically. We suspect that the occurrence of the periodic outbreak of rabies comes from the large birth rate of the susceptible foxes when their population density is low. This has been confirmed in our recent study [40].

6.4. A bio-reactor model

Traveling waves for bio-reactor models have been studied, see [6,7,27,30,50]. By using the argument in [34], we can establish the following delayed reaction–advection–diffusion system

$$\begin{cases} S_t = -vS_x - f(S)u, \\ u_t = u_{xx} - vu_x + e^{-d\tau} \displaystyle\int_{-\infty}^{\infty} f\big(S(y - v\tau, t - \tau)\big)u(y - v\tau, t - \tau) f_\alpha(x - y)\, dy - ku, \end{cases} \tag{6.17}$$

where $f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-x^2/4\alpha}$ with $\alpha = \tau D > 0$, $S(x, t)$ and $u(x, t)$ are the concentrations of nutrient and microbial populations at position x and time t, respectively. Parameter $v \geqslant 0$ is the flowing velocity, $k > 0$ is the cell death rate (or dilution rate). The constant $\tau \geqslant 0$ denotes the time delay involved in the conversion of nutrient to viable biomass. To form viable biomass, the incidence $f(S(y - v\tau, t - \tau))u(x - v\tau, t - \tau)$ experiences an intermediate state with average diffusion coefficient D which is between 0 (diffusion coefficient for the nutrient) and 1 (diffusion coefficient for the microbial biomass). The parameter d is the dilution rate of the intermediate state. In [58,60] the authors assumed that $k = d$. It should be noticed that if the intermediate state is inactive like nutrient, i.e., $D = 0$, then our model reduces to

$$\begin{cases} S_t = -vS_x - f(S)u, \\ u_t = u_{xx} - vu_x + e^{-d\tau} f\big(S(x - v\tau, t - \tau)\big)u(x - v\tau, t - \tau) - ku. \end{cases}$$

For the sake of simplicity, we suppose that

$$f(0) = 0, \qquad f'(s) > 0 \quad \text{for } s \geqslant 0,$$

though the monotonicity of f is not essential in our discussion below.

We also assume there is a S^* such that $e^{-d\tau} f(S^*) = k$. Define a function

$$G(S) = e^{-d\tau} S - k \int_{S^*}^{S} \frac{1}{f(s)}\, ds,$$

so that $G(0) = +\infty$ and for any $S^0 > S^*$, there is a unique $S_0 < S^*$ such that

$$G(S_0) = G\big(S^0\big). \tag{6.18}$$

When $\tau = 0$, Eq. (6.17) reduces to

$$\begin{cases} S_t = -vS_x - f(S)u, \\ u_t = u_{xx} - vu_x + f(S)u(t) - ku \end{cases}$$

which was studied by Smith and Zhao [50] and the special case when $v = 0$ has been studied by [30]. For the more general case without time delay, we refer to Huang [27]. We focus on traveling wave solution

$$S = S(x + ct), \qquad u = u(x + ct)$$

where we expect that $S(z)$ and $u(z)$ satisfy

$$S(-\infty) = S^0, \quad u(-\infty) = 0, \qquad S(+\infty) = S_0 < S^0, \quad u(+\infty) = 0,$$

and $S^0 > S^*$ is the initial fresh concentration of nutrient and S_0 is a positive constant determined by (6.18). A direct substitution of $S = S(z) = S(x + ct)$ and $u = u(z) = u(x + ct)$ into (6.17) gives

$$\begin{cases} 0 = -(v + c)S' - f(S)u, \\ 0 = u'' - (v + c)u' + e^{-d\tau} \displaystyle\int_{-\infty}^{\infty} f\big(S(y - v\tau - c\tau)\big)u(y - v\tau - c\tau) f_\alpha(z - y)\, dy - ku. \end{cases} \tag{6.19}$$

We are only interested in the nonnegative solution of Eq. (6.19). When $\tau = 0$, we have a result from [27] or [50]:

Theorem 6.7. *When $\tau = 0$, Eq. (6.19) has a positive traveling wave $(S(z), u(z))$ connecting $(S^0, 0)$ and $(S_0, 0)$ provided $c \geqslant c_0^* = \sqrt{4(f(S^0) - k)} - v$.*

Reversing the "direction of time" in (6.19) by the change $z \to -z$ gives

$$\begin{cases} 0=(v+c)S'-f(S)u, \\ 0=u''+(v+c)u'+e^{-d\tau}\displaystyle\int_{-\infty}^{\infty} f\big(S(y+v\tau+c\tau)\big)u(y+v\tau+c\tau)f_\alpha(z-y)\,dy-ku. \end{cases} \tag{6.20}$$

Inserting the first of (6.20) into the second and integrating it, we have a new system

$$\begin{cases} S'=\dfrac{f(S)u}{(v+c)}, \\ u'=(v+c)\left(-u-e^{-d\tau}\displaystyle\int_{-\infty}^{\infty} S(z+v\tau+c\tau)f_\alpha(z-y)\,dy+k\int_{S^0}^{S}\frac{ds}{f(s)}+e^{-d\tau}S^0\right). \end{cases} \tag{6.21}$$

The characteristic function of (6.21) at the point $E_1=(S^0,0)$ is given by

$$\Lambda_\tau^{E_1}(\lambda)=\lambda^2+(v+c)\lambda+\big(f\big(S^0\big)e^{\alpha\lambda^2+\lambda c\tau+v\tau-d\tau}-k\big),$$

and we thus define

$$c_\tau^*=\inf\big\{c>0 \,\big|\, \Lambda_\tau^{E_1}(\lambda)=0 \text{ has a negative real zero}\big\}.$$

It is easy to know that when $\tau=0$, the equilibrium $(S^0,0)$ of (6.21) is a stable node. Using Theorem 2.1, we have the following

Theorem 6.8. *For $c\geqslant c_0^*=\sqrt{4(f(S^0)-k)}-v$, there exist a constant $\delta>0$ so that for $\tau\in(0,\delta)$, Eq.* (6.17) *has a traveling wavefront $S=S(x+c_\tau^*ct/c_0^*)$, $u=u(x+c_\tau^*ct/c_0^*)$ which connects the equilibria $(S^0,0)$ and $(S_0,0)$.*

6.5. A hyperbolic model arising from the slow movement of individuals

We consider the following second order hyperbolic–parabolic equation

$$\begin{aligned} &\frac{\partial}{\partial t}m(x,t)+r_1\frac{\partial^2}{\partial t^2}m(x,t) \\ &\quad =D\frac{\partial^2}{\partial x^2}m(x,t)-dm(x,t)+u(t,\tau,x)-r_1\left(2\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)u(t,\tau,x) \end{aligned} \tag{6.22}$$

for the density of adult population $m(t,x)$ at time t and spatial location $x\in R$ of a given single species population with two age classes (the immature and the mature with maturation time $\tau>0$ being a constant) that moves randomly in space with a time lag $r_1>0$, where D, $d>0$ are constant diffusion and death rates of the adult at time t and location x. This equation can be obtained from the usual structured population model, see Raugel and Wu [44].

The maturation rate $u(t,\tau,x)$ is determined by the biological process during the maturation process. In [51], it was shown that if the immature moves instantaneously and if the birth rate is given by a function $b(m(x,t))$, then

$$u(t,\tau,x)=\varepsilon\int_{-\infty}^{\infty} b\big(m(y,t-\tau)\big) f_\alpha(x-y)\,dy \tag{6.23}$$

where

$$\varepsilon=\varepsilon(\tau)=e^{-\int_0^\tau d_2(a)\,da}\in(0,1]$$

is the survival rate during the maturation period and

$$f_\alpha(z)=\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{z^2}{4\alpha}}\quad \text{with } \alpha=\int_0^\tau D_I(a)\,da$$

is the probability that a new born at time $t-\tau$ and location 0 moves to the location z after maturation time τ.

We can show that

$$\left(2\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)u(t,\tau,x)=\frac{d}{d\theta}\int_{-\infty}^{\infty} f_\alpha(x-y)b\big(m(y,\theta)\big)\,dy\Big|_{\theta=t-\tau}.$$

Therefore, we obtain a closed system for the matured population

$$\begin{aligned}\frac{\partial}{\partial t}m(x,t)+r_1\frac{\partial^2}{\partial t^2}m(x,t)={}&D\frac{\partial^2}{\partial x^2}m(x,t)-dm(x,t)\\&+\varepsilon\int_{-\infty}^{\infty} f_\alpha(x-y)b\big(m(y,t-\tau)\big)\,dy\\&+r_1\frac{\partial}{\partial t}\left[\varepsilon\int_{-\infty}^{\infty} f_\alpha(x-y)b\big(m(y,t-\tau)\big)\,dy\right].\end{aligned} \tag{6.24}$$

This is a second-order hyperbolic equation. When $r=0$, Eq. (6.24) reduces to

$$\frac{\partial}{\partial t}m(x,t)=D\frac{\partial^2}{\partial x^2}m(x,t)-dm(x,t)+\varepsilon\int_{-\infty}^{\infty} f_\alpha(x-y)b\big(m(y,t-\tau)\big)\,dy \tag{6.25}$$

which is considered in Section 6.1. When $r_1\neq 0$ and the birth function is monotone, Eq. (6.24) has been studied in [44] and [41].

We now focus first on the birth function given by $b(w)=pwe^{-aw}$ as in Section 6.1. We assume that $\varepsilon p>d$ so that Eq. (6.24) has two equilibria $E_1=0$ and $E_2=\frac{1}{a}\ln\frac{\varepsilon p}{d}$.

Define the characteristic equation at the zero solution by

$$\Lambda_{\tau,r_1}^{E_1}(\lambda) = (1+\lambda r_1 c)\varepsilon p e^{\alpha\lambda^2+\lambda c r} + \big(c\lambda - d + \big(D - r_1 c^2\big)\lambda^2\big)$$

and

$$c_{\tau,r_1}^* = \inf\big\{c > 0 \,\big|\, \Lambda_{\tau,r_1}^{E_1}(\lambda) = 0 \text{ has a negative real zero}\big\}.$$

It is easy to check that the conditions (H_1)–(H_4) are satisfied. We have

Theorem 6.9. *For any given $c \geqslant c_{0,0}^* = 2\sqrt{D(\varepsilon p - d)} > 0$, there exist constants $\delta_1 = \delta_1(c)$ and $\delta_2 = \delta_2(c) > 0$ so that for $\tau \in (0, \delta_1)$ and $r \in (0, \delta_2)$, Eq.* (6.24) *has a traveling wavefront $m = U(x - c_{\tau,r_1}^* ct/c_{0,0}^*)$ which satisfies*

$$\lim_{s\to-\infty} U(s) = E_2, \qquad \lim_{s\to\infty} U(s) = E_1 = 0.$$

In the bistable case when the birth function is given by $b(w) = pw^2e^{-aw}$ with $\varepsilon p > dae$, the conditions (G_1), (G_2) and (H_4) are satisfied and we have the following

Theorem 6.10. *There exist constants δ_1 and $\delta_2 > 0$ so that for $\tau \in (0, \delta_1)$ and $r \in (0, \delta_2)$, Eq.* (6.24) *has a traveling wavefront $m = U(x - ct)$ for some c, which satisfies*

$$\lim_{s\to-\infty} U(s) = E_3, \qquad \lim_{s\to\infty} U(s) = E_1 = 0$$

where E_1 and E_3 are given in (6.6).

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