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A PARTIAL DIFFERENTIAL EQUATION WITH DELAYED DIFFUSION

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Abstract. We consider a simple diffusion equation with a delayed Laplacian operator to model the time required for spatial movement. Such an equation defines a semiflows on a Frechét space and the associated infinitesimal generator has a quite interesting spectral property so that the associated unstable and stable subspaces are both infinitely dimensional, and there is a sequence of eigenvalues of the generator that approaches the imaginary axis.
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1 Introduction

Motivated by a simple looking linear parabolic equation with delay in the Laplacian operator, we consider here semiflows on general Frechét spaces. The need for such a framework instead of C_0-semigroups on Banach spaces is due to the fact that solutions of the aforementioned equation become less and less smooth (with respect to space) as time increases, and hence the space of smooth initial conditions is required in order for the abstract Cauchy initial value problem to be well-posed.

Here we present a case study of the simplest possible diffusion equation where diffusion occurs with a delay, and we show that such an equation generates a semigroup of bounded operators in a carefully chosen Frechét space whose metric is induced by a family of semi-norms. The generator of such a semiflow can be calculated, and we show its spectrum contains a sequence of points with unbounded positive real part, and hence both stable subspaces and unstable subspaces of the semiflow must be of infinite dimensions. Our analysis shows the lack of exponential dichtomy as there is a sequence of eigenvalues of the generator that approach the imaginary axis.

The prototype equation is $\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2}{\partial x^2} u(t-1,x)$ subject to a certain boundary condition. This arises when we consider random movement of a biological species and when we assume the spatial movement of the species is delayed (with the delay normalized to 1), as noted in [2]. Naturally, more realistic biological models should incorporate the term describing the birth and death processes that are usually nonlinear, see [4], but the aforementioned prototype equation shows the complexity of the study and our work here should be regarded as a preliminary work towards a more comprehensive investigation of the dynamics of reaction diffusion equations with time delay in the diffusion term.

Remark: It appears that every time when the delay appears in the highest spatial derivative in the evolution equation, that term seems to be considered as a singular perturbation, if the delay is near zero. We noticed some similar behaviour in the wave equation when the delay, $r > 0$, is present in the term $\frac{\partial^2}{\partial x^2} u(t-r,x)$.

2 Linear C_0-semigroups on Frechét spaces.

Let X be a Frechét space with distance d.

2.1 The semigroup.

Let $t \in \mathbb{R}_+$. We say that $T(t)$ is a C_0-semigroup on Frechét space X, if the following conditions are satisfied:

1. $T(t) : X \to X$, is a continuous linear operator, for each $t \geq 0$;
2. If $x \in X$ and $s \in [0, \infty)$,then $\lim_{t\to s} T(t)x = T(s)x$;
3. If $t, s \geq 0$, $x \in X$, then $T(t+s)x = T(t)T(s)x$.

2.2 The infinitesimal generator.

If $T(t)$ is a co-semigroup on Frechét space X, we define:

$$(A) := \{x \in X : \lim_{h\to 0^+} \frac{T(h)x - x}{h} \text{ exists in} X\}.$$

If $x \in (A)$, we define $Ax := \lim_{h\to 0^+} \frac{T(h)x-x}{h}$ and we say that A is the infinitesimal generator of $T(t)$.

It follows from Komura [3] that A is a closed operator and that (A) is dense in X.

We now discuss a special class of Frechét spaces that will be helpful to treat our examples.

We suppose that for each $p \in \mathbb{Z}_+$, X_p is a Banach space with norm $\|\cdot\|_p$, $X_{p+1} \subset X_p$ and for every $x \in X_{p+1}$ we have that $\|x\|_p \leq \|x\|_{p+1}$.

Let $X := \cap_{p=0}^{\infty} X_p$. In X we define the following distance: for $x, y \in X$

$$d(x,y) = \sum_{p=0}^{\infty} \frac{1}{2^p} \frac{\|x-y\|_p}{1+\|x-y\|_p}.$$

We remark that this distance is invariant by translations.

The following lemma is obvious but we present a short proof of it.

Lemma 2.1. *With the above distance, X is a complete space and so it is a Frechét space.*

Proof: If (x_n) is a Cauchy sequence in X then it is also a Cauchy sequence in X_p for every $p \in \mathbb{Z}_+$. Therefore, this sequence will converge in X_p and the above assumptions imply that the limit x is independent of p. Therefore, $x \in X$ and $x_n \to x$ in X.

Exemple 2.1. *We are interested in the following boundary value problem:*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2}{\partial x^2} u(t-1,x), \\ u_x(t,0) = u_x(t,\pi) = 0. \end{cases} \tag{2.1}$$

Consider the following spaces

$$\begin{aligned} X := \{\phi(t,x) &= \sum_{n=0}^{\infty} a_n(t)\cos(nx);\ a_n : [-1,1] \to \mathbb{R} \text{ is continuous for each} \\ n \in \mathbb{Z}_+,\ \|\phi\|_p : &= \sup_{t\in[-1,0]} [\sum_{n=0}^{\infty} (n^p |a_n(t)|)^2]^{1/2} < \infty, \forall p \in \mathbb{Z}_+\} \end{aligned}$$

and for each $p \in \mathbb{Z}_+$

$$\begin{aligned} X_p := \{\phi(t,x) &= \sum_{n=0}^{\infty} a_n(t)\cos(nx); a_n : [-1,1] \to \mathbb{R} \text{ is continuous for each} \\ n \in \mathbb{Z}_+,\ \|\phi\|_p : &= \sup_{t\in[-1,0]} [\sum_{n=0}^{\infty} (n^p |a_n(t)|)^2]^{1/2} < \infty\}. \end{aligned}$$

Then $X = \cap_{p=0}^{\infty} X_p$. Morever, we observe that if $\phi \in X$, then $\|\phi\|_p \leq \|\phi\|_{p+1}$ for every $p \in \mathbb{Z}_+$.

In the space X, we consider the following distance, for $\phi,\ \psi \in X$:

$$d(\phi,\psi) := \sum_{p=0}^{\infty} \frac{1}{2^p} \frac{\|\phi-\psi\|_p}{1+\|\phi-\psi\|_p}.$$

By Lemma 2.1, X is a Frechét space.

If we substitute $\phi(t,x) = \sum_{n=0}^{\infty} y_n(t)\cos(nx)$ to the equation (2.1), we see that $y_n(t)$ must be a solution of the ordinary retarded equation:

$$\frac{d}{dt}y_n(t) = -n^2 y_n(t-1). \tag{2.2}$$

This equation can be solved for all $t \geq 0$, as long as the initial condition on the space $C([-1,0],\mathbb{R})$ is given.

Let us suppose that $\phi \in X$ is given with $\phi(t,x) = \sum_{n=0}^{\infty} a_n(t)\cos(nx)$, where $a_n \in C([-1,0],\mathbb{R})$. Let $y_n(t)$ be the solution of (2.2) such that $y_n = a_n$ on $[-1,0]$.

Clearly, $T(t)\phi = \sum_{n=0}^{\infty}(y_n)_t\cos(nx)$ is the mild solution of (2.1), with initial condition ϕ.

We point out that if $\phi \in X$ then $T(t)\phi$ belongs to the space X. In fact, if $\phi(t,x) = a_0 + \sum_{n=1}^{\infty} a_n(t)\cos(nx)$ for $t \in [-1,0]$, the solution for $t \in [0,1]$ is given by

$$u(t,x) = a_0(0) + \sum_{n=1}^{\infty} y_n(t)\cos(nx),$$

where $y_n(t) = y_n(0) - n^2\int_0^t a_n(s-1)ds$. Therefor, $T(t)\phi \in X$ for $t \in [0,1]$, because

$$\|u_t\|_p \leq \left[\|a_0\|_\infty^2 + \sum_{n=1}^{\infty}(n^p(1+n^2)\|a_n\|_\infty)^2\right]^{1/2} = 2\|\phi\|_{p+2} < \infty.$$

We can proceed step by step to obtain the $u(t,x)$ in the subsequent interval $[1,2]$, $[2,3]$,.... We now note that for each fixed $t \in [0,\infty)$, $T(t)$ is a continuous operator from X to X. In fact, since the distance is invariant by translations, if $\phi \in X$ and $t \in [0,1]$, then

$$\begin{aligned} d(T(t)\phi, 0)) &= \sum_{p=0}^{\infty}\frac{1}{2^p}\frac{\|T(t)\phi\|_p}{1+\|T(t)\phi\|_p} \leq \sum_{p=0}^{\infty}\frac{1}{2^p}\frac{2\|\phi\|_{p+2}}{1+2\|\phi\|_{p+2}} \\ &\leq 2^3\sum_{p=0}^{\infty}\frac{1}{2^{(p+2)}}\frac{\|\phi\|_{p+2}}{1+\|\phi\|_{p+2}} \leq 2^3 d(\phi,0). \end{aligned}$$

Similarly, if $t \geq 0$, then there exists a $q \in \mathbb{Z}_+$ such that $t \in [q, q+1]$ and hence

$$\|T(t)\phi\|_p \leq 2^{2(q+1)}\|\phi\|_{p+2(q+1)}.$$

$$\begin{aligned} d(T(t)\phi,0) &= \sum_{p=0}^{\infty}\frac{1}{2^p}\frac{\|T(t)\phi\|_p}{1+\|T(t)\phi\|_p} \\ &\le \sum_{p=0}^{\infty}\frac{1}{2^p}\frac{2^{2(q+1)}\|\phi\|_{p+2(q+1)}}{1+2^{2(q+1)}\|\phi\|_{p+2(q+1)}} \\ &\le 2^{4(q+1)}\sum_{p=0}^{\infty}\frac{1}{2^{(p+2(q+1))}}\frac{\|\phi\|_{p+2(q+1)}}{1+\|\phi\|_{p+2(q+1)}} \\ &\le 2^{4(q+1)}d(\phi,0)\le 2^{4(t+1)}d(\phi,0) \end{aligned}$$

Therefore, for $t \ge 0$ we have

$$d(T(t)\phi,0) \le 2^4 e^{(4\log 2)t} d(\phi,0).$$

Finally, we remark that the mapping $t \in [0,\infty) \to u_t(\cdot,x) = a_0(0) + \sum_{n=1}^{\infty} y_{nt}\cos(nx)$, is continuous. This follows from the fact that the series $u(t,x) = a_0(0) + \sum_{n=1}^{\infty} y_n(t)\cos(nx)$ converges uniformly for t on bounded closed intervals of $[-1,\infty)$ and that $t \in [-1,\infty) \to a_0(0)+\sum_{n=1}^{\infty} y_n(t)cos(nx) \in \mathbb{R}$ is continuous.

2.3 Infinite Dimensional Stable and Unstable Subspaces and Non-Splitting

The characteristic equation is given by

$$\lambda = -n^2 e^{-\lambda}$$

for every positive integer n.

Let $\lambda = x + iy$ with reals x and y. We get

$$\begin{aligned} e^x(x\cos y - y\sin y) &= -n^2, \\ e^x(x\sin y + y\cos y) &= 0. \end{aligned}$$

In the above system, $\sin y = 0$ implies that $y = 0$ and the unique solution will be $(x,y) = (0,0)$, obtained by taking $n = 0$.

For $\sin y \ne 0$, we have,

$$x = h(y) := -y\frac{\cos y}{\sin y} \tag{2.3}$$

and

$$f(y) := e^{-y\frac{\cos y}{\sin y}}\frac{y}{\sin y} = n^2. \tag{2.4}$$

Both h and f are even functions on y.

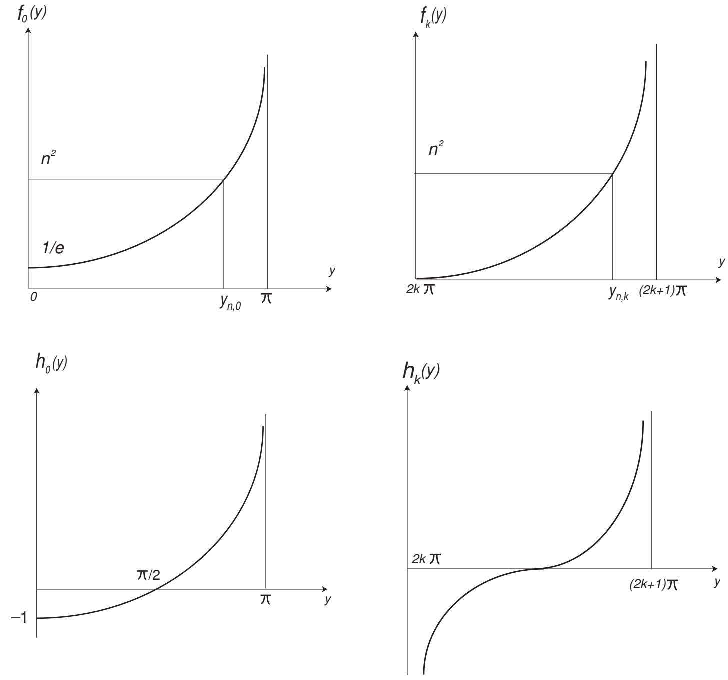

Figure 1. The graph of h and f.

It is thus necessary to consider the graph of $z = f(y)$ above the y-axis. This part of the graph has infinite number of branches: the branch on the interval $[0, \pi)$ (denoted by $z = f_0(y)$) starts at $1/e$ and monotonically increases to infinity, and the branch (denoted by $z = f_k(y)$) for each fixed interval $(2k\pi, 2k\pi + \pi)$, with $k = 1, 2, \cdots$, starts from 0 and monotonically increases to infinity. As such, for each nonnegative integer k and for each positive integer n, the equation (2.4) has a unique solution $y_{n,k} \in (2k\pi, 2k\pi + \pi)$. Moreover, we have $y_{n,k} \to 2k\pi + \pi$ as $n \to \infty$.

Similarly we denote by $x = h_k(y)$ for each fixed interval $(2k\pi, 2k\pi + \pi)$ with $k = 0, 1, 2, \cdots$.

Substituting $y_{n,k}$ into (2.3), we then get $x_{n,k} = -y_{n_k} \frac{\cos y_{n,k}}{\sin y_{n,k}}$ and $x_{n,k} \to +\infty$ as $n \to \infty$, and $x_{n,k} \to -\infty$ as $k \to \infty$.

We now consider the possibility that if we can get a sequence of (x, y) that solves (2.3), (2.4) and $x + iy$ approaches the imaginary axis. For this purpose, we write $f_k(y)$ in terms of a shifted variable $y = 2k\pi + s$ as follows

$$g_k(s) = f_k(y) := e^{-(2k\pi+s)\frac{\cos s}{\sin s}} \frac{2k\pi + s}{\sin s}.$$

Note that $g_k(\frac{\pi}{2}) = 2k\pi + \frac{\pi}{2}$, we conclude that if N_k is the largest integer that is less than or equal to $\sqrt{2k\pi + \frac{\pi}{2}}$ and if $n \geq N_k + 1$, then equation (3.2) has a solution $y_{n,k}$ on the interval $(2k\pi + \frac{\pi}{2}, 2k\pi + \pi)$. In particular, letting $n_k = N_K + 1$ and $s_{n,k} = y_{n,k} - 2k\pi$, we get

$$\frac{(n_k+1)^2}{2k\pi + s_{n_k,k}} = e^{-(2k\pi + s_{n_k,k})\frac{\cos s_{n_k,k}}{\sin s_{n_k,k}}} \frac{1}{\sin s_{n_k,k}}. \tag{2.5}$$

It now becomes clear that $s_{n_k,k} \to \frac{\pi}{2}$ as $k \to \infty$. For otherwise, if there is a subsequence that converges $s^* \in (\frac{\pi}{2}, \pi]$, we will get a contradiction in (2.5): the left-hand side converges to 1 while the right-hand side converges to ∞.

It also follows from (2.5) that

$$e^{-(2k\pi + s_{n_k,k})\frac{\cos s_{n_k,k}}{\sin s_{n_k,k}}} \to 1$$

from which we conclude that $x_{n_k,k} \to 0+$ as $k \to \infty$.

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