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Travelling waves for delayed reaction–diffusion equations with global response

By Teresa Faria[1], Wenzhang Huang[2] and Jianhong Wu[3,*]

[1] *Departamento de Matemática, Faculdade de Ciências/CMAF, Universidade de Lisboa, 1749-016 Lisboa, Portugal*

[2] *Department of Mathematical Sciences, University of Alabama in Huntsville Huntsville, AL 35899, USA*

[3] *Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada*

We develop a new approach to obtain the existence of travelling wave solutions for reaction–diffusion equations with delayed non-local response. The approach is based on an abstract formulation of the wave profile as a solution of an operational equation in a certain Banach space, coupled with an index formula of the associated Fredholm operator and some careful estimation of the nonlinear perturbation. The general result relates the existence of travelling wave solutions to the existence of heteroclinic connecting orbits of a corresponding functional differential equation, and this result is illustrated by an application to a model describing the population growth when the species has two age classes and the diffusion of the individual during the maturation process leads to an interesting non-local and delayed response for the matured population.

Keywords: delayed reaction–diffusion equation; travelling wave; heteroclinic orbit; monotone dynamical system; Nicholson's blowflies equation

1. Introduction

The purpose of this paper is to study the existence of travelling wave solutions for the following delayed reaction–diffusion equation with non-local interaction

$$\frac{\partial u(x,t)}{\partial t} = D\varDelta u(x,t) + F\left(u(x,t), \int_{-r}^{0}\int_{\Omega} \mathrm{d}\eta(\theta)\mathrm{d}\mu(y)g(u(x+y,t+\theta))\right), \tag{1.1}$$

where $x \in \mathbb{R}^m$ is the spatial variable, $t \geq 0$ is the time, $u(x,t) \in \mathbb{R}^n$, $D = \mathrm{diag}(d_1, \ldots, d_n)$ with positive constants d_i, $i = 1, \ldots, n$, $\varDelta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$ is the Laplacian operator, r is a positive constant, $\eta : [-r,0] \to \mathbb{R}^{n\times n}$ is of bounded variation, μ is a bounded measure on $\Omega \subset \mathbb{R}^m$ with values in $\mathbb{R}^{n\times n}$, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are given mappings with additional conditions to be specified later.

Equation (1.1) serves as a model for many physical, chemical, ecological and biological problems. In particular, as will be shown in §6, equation (1.1) includes a model for the population growth where the species has an age-structure and

* Author for correspondence (wujh@mathstat.yorku.ca).

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a non-monotone birth function, and the spatial diffusion of the individuals during the maturation period leads to an interesting non-local delayed response. See Britton (1990) and Gourley & Britton (1993) for some earlier work on non-local delayed reaction–diffusion equations.

Because of their significant role in governing the long time behaviour of dynamical systems with a diffusion process, travelling wave solutions have been one of lasting interests, and a variety of methods for studying the existence of travelling wave solutions have been developed. In this paper, we develop a new approach to study the existence of travelling wave solutions for equation (1.1). This approach reflects a natural connection between the existence of a travelling wave solution for equation (1.1) and the existence of a heteroclinic solution for the corresponding ordinary delay differential equation on $\mathbb{R}^n$

$$\dot{u}(t) = F\left(u(t), \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_\Omega g(u(t+\theta))\right), \tag{1.2}$$

where $\mu_\Omega = \int_\Omega \mathrm{d}\mu$.

Before giving a precise statement of our main result, we first formulate some assumptions about the nonlinearities F and g. Throughout the remaining part of this paper, we suppose that F and g are C^k-smooth functions, $k \geq 2$, and we let $F_u(u, v)$, $F_v(u, v)$ denote the partial derivatives of F with respect to the variables $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, respectively, and let $g_u(u)$ be the derivative of g with respect to the variable $u \in \mathbb{R}^n$. In addition, we suppose that equation (1.2) has two equilibria E_i, $i = 1, 2$, and we define

$$A_i = F_u\left(E_i, \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_\Omega g(E_i)\right), \quad B_i = F_v\left(E_i, \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_\Omega g(E_i)\right).$$

For a complex number λ we let

$$\Lambda_i(\lambda) = \det\left[\lambda I - A_i - B_i \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_\Omega g_u(E_i)\mathrm{e}^{\lambda\theta}\right].$$

We assume that the following hypotheses hold.

(H1) All eigenvalues corresponding to the equilibrium E_2 have negative real parts, that is, $\sup\{\Re\lambda : \Lambda_2(\lambda) = 0\} < 0$.

(H2) E_1 is hyperbolic and the unstable manifold at the equilibrium E_1 is M ($M \geq 1$) dimensional. In other words, $\Lambda_1(iv) \neq 0$ for all $v \in \mathbb{R}$ and $\Lambda_1(\lambda) = 0$ has exactly M roots with positive real parts, where the multiplicities are taken into account.

(H3) Equation (1.2) has a heteroclinic solution $u^* : \mathbb{R} \to \mathbb{R}^n$ from E_1 to E_2. Namely, equation (1.2) has a solution $u^*(t)$ defined for all $t \in \mathbb{R}$ such that

$$u^*(-\infty) := \lim_{t\to-\infty} u^*(t) = E_1, \quad u^*(\infty) := \lim_{t\to\infty} u^*(t) = E_2.$$

(H4) $\left\| \int_\Omega \mathrm{d}|\mu|(y) \|y\|_{\mathbb{R}^m} \right\|_{\mathbb{R}^{n\times n}} < \infty$, where $|\mu| = \mu^+ - \mu^-$ with μ^+ and μ^- the positive and negative parts of μ, respectively.

Our main result is as follows.

Theorem 1.1. *Under assumptions (H1)–(H4), there is a* $c^*>0$ *such that*

(i) *for each fixed unit vector* $\nu\in\mathbb{R}^m$ *and* $c>c^*$, *equation* (1.1) *has a travelling wave solution* $u(x,t)=U(\nu\cdot x+ct)$ *connecting* E_1 *to* E_2 *(that is,* $U(-\infty)=E_1$ *and* $U(-\infty)=E_2$*);*

(ii) *if restricted to a small neighbourhood of the heteroclinic solution* $u^*:\mathbb{R}\to\mathbb{R}^n$ *in the space* $C(\mathbb{R},\mathbb{R}^n)$ *of bounded continuous functions equipped with the sup-norm, then for each fixed* $c>c^*$ *and* $\nu\in\mathbb{R}^m$, *the set of all travelling wave solutions connecting* E_1 *to* E_2 *in this neighbourhood forms a M-dimensional manifold* $\mathcal{M}_\nu(c)$*;*

(iii) $\mathcal{M}_\nu(c)$ *is a* $C^{\mathrm{k}-1}$*-smooth manifold which is also* $C^{\mathrm{k}-1}$*-smooth with respect to c. More precisely, there is a* $C^{\mathrm{k}-1}$*-function* $h:U\times(c^*,\infty)\to C(\mathbb{R},\mathbb{R}^n)$, *where U is an open set in* $\mathbb{R}^M$, *such that* $\mathcal{M}_\nu(c)$ *has the form*

$$\mathcal{M}_\nu(c)=\{\psi:\psi=h(z,c),\ z\in U\}.$$

Let $\nu\cdot x+ct=s\in\mathbb{R}$ and $u(x,t)=U(\nu\cdot x+ct)$. Then, upon a straightforward substitution, a travelling wave $U(s)$ satisfies the second order equation

$$c\dot{U}(s)=D\ddot{U}(s)+F\left(U(s),\int_{-r}^{0}\int_{\Omega}\mathrm{d}\eta(\theta)\mathrm{d}\mu(y)g(U(s+\nu\cdot y+c\theta))\right),s\in\mathbb{R}. \quad (1.3)$$

Writing $V(s)=U(cs)$ and $\varepsilon=1/c^2$, then equation (1.3) leads to

$$\dot{V}(s)=\varepsilon D\ddot{V}(s)+F\left(V(s),\int_{-r}^{0}\int_{\Omega}\mathrm{d}\eta(\theta)\mathrm{d}\mu(y)g\left(V\left(s+\sqrt{\varepsilon}\nu\cdot y+\theta\right)\right)\right),s\in\mathbb{R}. \quad (1.4)$$

In the case where c is sufficiently large, ε is small and hence equation (1.4) is a singularly perturbed equation. Such an equation has been extensively investigated via both geometric and analytic methods where the main idea is to study the corresponding slow motion and fast motion. See, for example, Carpenter (1977), Fenichel (1971, 1979), Fife (1976), Hoppensteadt (1966), Jones (1995), Lin (1989) and Szmolyan (1991). The geometrical approach makes the connection of slow and fast motions by studying the intersection of the relevant invariant manifolds, while the analytic approach matches the slow and fast motion by using the asymptotic expansion of inner and outer layers. For both methods, to make a connection between slow and fast motions is far from being trivial. In addition, both methods work only on dynamical systems where the stable, unstable, and invariant manifolds play an essential role. It is very important to point out that the differential equation (1.4) does not generate a dynamical system, for there is no way an initial value problem can be formulated. In this paper, we take a different approach to avoid this difficulty. The central idea of our approach is to use a certain type of transformation to convert the singularly perturbed differential equation (1.4) into a regularly perturbed operational equation in a Banach space, that enables us to directly apply the Banach fixed point theorem and some existing results regarding the index of an associated Fredholm operator to prove the existence of travelling wave solutions. This approach also allows us to determine the number of travelling wave solutions as well as smooth dependence of travelling wave solutions on the wave speed c.

Theorem 1.1, relating the existence of travelling wave fronts for the reaction–diffusion equation (1.1) with delay and non-local interaction to the existence of a connecting orbit between two hyperbolic equilibria of the associated ordinary delay differential equation (1.2), enables us to apply some existing results for invariant curves of semiflows generated by ordinary delay differential equations to derive systematically sharp sufficient conditions for the existence of travelling wave fronts of delayed reaction–diffusion equations that, in turn, includes most of the existing results in the literature as special cases. In particular, as will be illustrated in §6 where a recently derived non-local delayed reaction–diffusion equation for the population growth of a single species when the delayed birth function is not monotone in the considered range is considered, theorem 1.1 allows us to apply the powerful monotone dynamical systems theory to obtain the existence of travelling waves.

This paper is organized as follows. In §2 we transform equation (1.4) into an operational integral equation involving a linear operator and a nonlinear perturbation. Section 3 is devoted to the study of the null space and range of the linear operator introduced in §2. The properties of the nonlinear function in the operational equation are studied in §4. The proof of our main theorem is given in §5. In the last section, we present applications of our main result to some population models, including a non-local delayed RD-system with non-monotone birth functions.

2. Operational equations for travelling wave solutions

In the sequel, we use more compact notations:

$$\zeta(\theta, y) = \eta(\theta)\mu(y), \quad \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) = \int_{-r}^{0}\int_{\Omega} \mathrm{d}\eta(\theta)\mathrm{d}\mu(y),$$

with $\Omega_r = [-r, 0] \times \Omega$. We will also let $C = C(\mathbb{R}, \mathbb{R}^n)$ be the space of continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}^n$ equipped with the standard norm $\|\psi\|_C = \sup\{\|\psi(t)\| : t \in \mathbb{R}\}$.

Our main approach to study the existence of travelling wave solutions is to convert the differential equation for a travelling wave into an equivalent operational equation in a suitable Banach space. For this purpose, we further transform equation (1.4) by introducing the variable $w(s) = V(s) - u^*(s)$ for $s \in \mathbb{R}$. Then we obtain the equation for w as

$$\begin{aligned}\dot{w}(s) &= \varepsilon D\ddot{w}(s) + \varepsilon D\ddot{u}^*(s) \\ &\quad + F\left(w(s) + u^*(s), \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g\big([w + u^*](s + \sqrt{\varepsilon\nu}\cdot y + \theta)\big)\right) \\ &\quad - F\left(u^*(s), \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g(u^*(s + \theta))\right) \\ &= \varepsilon D\ddot{w}(s) + P^0 w(s) + \mathcal{G}(\varepsilon, s, w), \quad s \in \mathbb{R},\end{aligned} \tag{2.1}$$

where $[w + u^*](t) = w(t) + u^*(t)$ for $t \in \mathbb{R}$, and the linear operator P^0: $C \to C$ is defined by

$$P^0 w(s) = A(s)w(s) + B(s)\int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g_u(u^*(s + \theta))w(s + \theta), \quad s \in \mathbb{R}, \tag{2.2}$$

with

$$A(s) = F_u\left(u^*(s), \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g(u^*(s+\theta))\right), \quad s \in \mathbb{R}, \tag{2.3}$$

$$B(s) = F_v\left(u^*(s), \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g(u^*(s+\theta))\right), \quad s \in \mathbb{R}, \tag{2.4}$$

and

$$\begin{aligned} \mathcal{G}(\varepsilon, s, w) = {} & F\left(w(s) + u^*(s), \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g([w+u^*](s+\sqrt{\varepsilon}\nu \cdot y + \theta))\right) \\ & - F\left(u^*(s), \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g(u^*(s+\theta))\right) - P^0 w(s) + \varepsilon D \ddot{u}^*(s). \end{aligned} \tag{2.5}$$

Next we transform equation (2.1) into an integral equation as follows. We first write equation (2.1) as

$$\varepsilon d_i \ddot{w}_i(s) - \dot{w}_i(s) - w_i(s) = -w_i(s) - P_i^0 w(s) - \mathcal{G}_i(\varepsilon, s, w), \quad s \in \mathbb{R}, \tag{2.6}$$

for $i=1, \dots, n$, where i denotes the ith component for the corresponding functions or operators. We observe that the equation

$$\varepsilon d_i z^2 - z - 1 = 0$$

has two real zeros α_i^ε and β_i^ε, with

$$-1 < \alpha_i^\varepsilon = \frac{1 - \sqrt{1+4\varepsilon d_i}}{2\varepsilon d_i} < 0, \quad \beta_i^\varepsilon = \frac{1 + \sqrt{1+4\varepsilon d_i}}{2\varepsilon d_i} > 0.$$

Moreover, it is easy to verify that

$$\lim_{\varepsilon \to 0^+} \alpha_i^\varepsilon = -1, \quad \lim_{\varepsilon \to 0^+} \beta_i^\varepsilon = +\infty. \tag{2.7}$$

It is well known that $w : \mathbb{R} \to \mathbb{R}^n$ is a bounded solution of equation (2.6) if and only if $w(s)$ is a bounded solution of the integral equation

$$\begin{aligned} w_i(s) = {} & \frac{1}{\varepsilon d_i(\beta_i^\varepsilon - \alpha_i^\varepsilon)} \int_{-\infty}^{s} \mathrm{e}^{\alpha_i^\varepsilon (s-t)} [w_i(t) + P_i^0 w(t)] \mathrm{d}t \\ & + \frac{1}{\varepsilon d_i(\beta_i^\varepsilon - \alpha_i^\varepsilon)} \int_{s}^{\infty} \mathrm{e}^{\beta_i^\varepsilon (s-t)} [w_i(t) + P_i^0 w(t)] \mathrm{d}t \\ & + \frac{1}{\varepsilon d_i(\beta_i^\varepsilon - \alpha_i^\varepsilon)} \left(\int_{-\infty}^{s} \mathrm{e}^{\alpha_i^\varepsilon (s-t)} \mathcal{G}_i(\varepsilon, t, w) \mathrm{d}t + \int_{s}^{\infty} \mathrm{e}^{\beta_i^\varepsilon (s-t)} \mathcal{G}_i(\varepsilon, t, w) \mathrm{d}t \right) \\ = {} & \frac{1}{\sqrt{1+4\varepsilon d_i}} \int_{-\infty}^{s} \mathrm{e}^{\alpha_i^\varepsilon (s-t)} [w_i(t) + P_i^0 w(t)] \mathrm{d}t \\ & + \frac{1}{\sqrt{1+4\varepsilon d_i}} \int_{s}^{\infty} \mathrm{e}^{\beta_i^\varepsilon (s-t)} [w_i(t) + P_i^0 w(t)] \mathrm{d}t \\ & + \frac{1}{\sqrt{1+4\varepsilon d_i}} \left(\int_{-\infty}^{s} \mathrm{e}^{\alpha_i^\varepsilon (s-t)} \mathcal{G}_i(\varepsilon, t, w) \mathrm{d}t + \int_{s}^{\infty} \mathrm{e}^{\beta_i^\varepsilon (s-t)} \mathcal{G}_i(\varepsilon, t, w) \mathrm{d}t \right), \\ & \qquad i = 1, \dots, n. \end{aligned} \tag{2.8}$$

Therefore, w is a bounded solution of equation (2.6) if and only if it solves

$$w(s) - \int_{-\infty}^{s} \mathrm{e}^{-(s-t)} [w(t) + P^0 w(t)] \mathrm{d}t = \mathcal{H}(s, w, \varepsilon), \tag{2.9}$$

where $\mathcal{H}(s,w,\varepsilon)=(\mathcal{H}_1(s,w,\varepsilon),\dots,\mathcal{H}_n(s,w,\varepsilon))$ is defined as

$$\begin{aligned}\mathcal{H}_i(s,w,\varepsilon) = &\int_{-\infty}^{s}\left[\frac{e^{\alpha_i^{\varepsilon}(s-t)}}{\sqrt{1+4\varepsilon d_i}}-e^{-(s-t)}\right]\left[w_i(t)+P_i^0 w(t)\right]dt\\ &+\frac{1}{\sqrt{1+4\varepsilon d_i}}\int_s^{\infty} e^{\beta_i^{\varepsilon}(s-t)}\left[w_i(t)+P_i^0 w(t)dt\right]\\ &+\frac{1}{\sqrt{1+4\varepsilon d_i}}\left(\int_{-\infty}^{s} e^{\alpha_i^{\varepsilon}(s-t)}\mathcal{G}_i(\varepsilon,t,w)dt+\int_s^{\infty} e^{\beta_i^{\varepsilon}(s-t)}\mathcal{G}_i(\varepsilon,t,w)dt\right),\\ &\text{for } i=1,\dots,n. \end{aligned}\tag{2.10}$$

In summary, we show that equation (1.3) has a solution $U:\mathbb{R}\to\mathbb{R}^n$ connecting E_1 to E_2 if and only if equation (2.9) has a solution w such that $\lim_{|s|\to\infty} w(s)=0$. Finally, we let L be the linear operator defined on the left hand side of equation (2.9), namely

$$[Lw](s) = w(s)-\int_{-\infty}^{s} e^{-(s-t)}[w(t)+P^0 w(t)]dt,\quad s\in\mathbb{R}.\tag{2.11}$$

Then we can write equation (2.9) as the operational equation

$$[Lw](s) = \mathcal{H}(s,w,\varepsilon),\quad s\in\mathbb{R}.\tag{2.12}$$

So our goal is to show the existence of solutions of equation (2.12). We shall achieve this by using the Banach fixed point theorem. For this purpose, we need further detailed properties of the nonlinear function $\mathcal{H}$ and the linear operator L. In the next section, we shall show that, with an appropriate choice of the Banach space, the operator L is surjective, an essential property required in the proof of our main theorem.

3. The kernel and range of the operator L

Let us first introduce some additional notations.

(i) For a vector $x\in\mathbb{R}^n$, $\|x\|=\|x\|_{\mathbb{R}^n}$, and for an $n\times n$ matrix A, $\|A\|=\|A\|_{\mathbb{R}^{n\times n}}$ denotes the norm of A as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$.
(ii) For a continuous function $w:[a-r,b]\to\mathbb{R}^n$, as usual we let $w_t\in C([-r,0],\mathbb{R}^n)$, $t\in[a,b]$, be defined by $w_t(\theta)=w(t+\theta)$ for $\theta\in[-r,0]$. Moreover, for $f\in C([-r,0],\mathbb{R}^n)$ we denote the norm of f by $\|f\|=\sup_{\theta\in[-r,0]}\|f(\theta)\|$.
(iii) In a similar fashion, for a function $h:[a,b+r]\to\mathbb{R}^n$ we define the function $h^t:[0,r]\to\mathbb{R}^n$ by $h^t(\theta)=h(t+\theta)$ for $\theta\in[0,r]$ and $t\in[a,b]$.
(iv) Let $C^1=C^1(\mathbb{R},\mathbb{R}^n)=\{\psi\in C:\dot{\psi}\in C\}$ be the Banach space equipped with the standard norm $\|\psi\|_{C^1}=\|\psi\|_C+\|\dot{\psi}\|_C$.
(v) Let $C_0=\{\psi\in C:\lim_{t\to\pm\infty}\psi(t)=0\}$ and $C_0^1=\{\psi\in C_0:\dot{\psi}\in C_0\}$ equipped with the same norms as C and C^1, respectively.

Let T: $C^1\to C$ be the linear operator obtained from the linearization of equation (1.2) around the heteroclinic solution u^*. That is,

$$(T\psi)(t) = \dot{\psi}(t)-P(t)\psi_t,\quad t\in\mathbb{R},\tag{3.1}$$

where for $t \in \mathbb{R}$ the linear operator $P(t): C([-r,0], \mathbb{R}^n) \to \mathbb{R}^n$ is defined by

$$P(t)\xi = A(t)\xi(0) + B(t)\int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_{\Omega} g_u(u^*(t+\theta))\xi(\theta), \tag{3.2}$$

with $A(t)$ and $B(t)$ defined in (2.3) and (2.4). We remark that $P^0\psi(t) = P(t)\psi_t$ for $\psi \in C$ and $t \in \mathbb{R}$. Since $u^*(t) \to E_1$ and E_2 as $t \to -\infty$ and $+\infty$, respectively, we have

$$\left.\begin{array}{ll} \lim\limits_{t\to\infty} A(t) = A_2, & \lim\limits_{t\to\infty} B(t) = B_2, \\ \lim\limits_{t\to-\infty} A(t) = A_1, & \lim\limits_{t\to-\infty} B(t) = B_1. \end{array}\right\} \tag{3.3}$$

Hypotheses (H1) and (H2) and (3.3) imply that the linear operator T is asymptotically hyperbolic as $t \to \pm\infty$ in the sense of Mallet-Paret (1999), p. 12. That is, the linear delay differential equations

$$\dot{\psi}(t) - P(+\infty)\psi_t = 0 \quad \text{and} \quad \dot{\psi}(t) - P(-\infty)\psi_t = 0,$$

where $P(+\infty)$, $P(-\infty)$ are the limiting operators defined in the obvious way, are hyperbolic. We define the formal adjoint equation of $T\psi = 0$ as

$$\dot{\phi}(t) = -P^*(t)\phi^t, \quad t \in \mathbb{R}, \tag{3.4}$$

where for $\xi \in C([0,r], \mathbb{R}^n)$

$$P^*(t)\xi = A^T(t)\xi(0) + \int_{-r}^{0} g_u^T(u^*(t))\mu_{\Omega}^T \mathrm{d}\eta^T(\theta) B^T(t-\theta)\xi(-\theta),$$

and for a matrix H, H^{T} denotes the transpose of H.

Lemma 3.1. *If $\phi \in C$ is a solution of equation* (3.4) *and ϕ is C^1-smooth, then* $\phi = 0$.

Proof. Let ϕ be a bounded solution of equation (3.4) and $h(t) = \phi(-t)$ for $t \in \mathbb{R}$. Then

$$\dot{h}(t) = A^T(-t)h(t) + \int_{-r}^{0} g_u^T(u^*(-t))\mu_{\Omega}^T \mathrm{d}\eta^T(\theta) B^T(-t-\theta)h(t+\theta) := Q(t)h_t. \tag{3.5}$$

The limiting equation of equation (3.5) as $t \to -\infty$ is

$$\dot{\xi}(t) = A_2^T\xi(t) + \int_{-r}^{0} g_u^T(E_2)\mu_{\Omega}^T \mathrm{d}\eta^T(\theta) B_2^T \xi(t+\theta) := Q(-\infty)\xi_t. \tag{3.6}$$

Since the linear delay differential equation (3.6) and the linear delay differential equation

$$\dot{\zeta}(t) = A_2\zeta(t) + B_2\int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_{\Omega} g_u(E_2)\zeta(t+\theta),$$

share the same eigenvalues, we conclude that all eigenvalues of equation (3.6) have negative real parts by assumption (H1). Let $\{J(t)\}_{t\geq 0}$ be the semigroup generated by the solutions of equation (3.6), that is, $J(t): C([-r,0], \mathbb{R}^n) \to C([-r,0], \mathbb{R}^n)$ and $J(t)\xi_0$ is the solution of equation (3.6) with initial condition

$\xi(\theta)=\xi_0(\theta)$ for $\theta\in[-r,0]$. Moreover, let $Z(t):[0,\infty)\to\mathbb{R}^{n\times n}$ be the matrix solution of equation (3.6) with initial condition

$$Z(\theta) = \begin{cases} I, & \text{for } \theta = 0, \\ 0, & \text{for } \theta\in[-r,0), \end{cases}$$

where I is the $n\times n$ identity matrix. Then there are positive constants $\gamma>0$ and $a>0$ such that

$$\|J(t)\xi_0\| \le \gamma \mathrm{e}^{-at}\|\xi_0\|, \quad \|Z(t)\| \le \gamma \mathrm{e}^{-at}, \quad t\ge 0, \xi_0\in C([-r,0],\mathbb{R}^n). \tag{3.7}$$

Let $\delta>0$ be such that $\delta\gamma\mathrm{e}^{ar}<a$. Since $Q(t)\to Q(-\infty)$ as $t\to-\infty$, there is a t^* such that

$$\|Q(t)-Q(-\infty)\|\le\delta, \quad t\le t^*. \tag{3.8}$$

Now we write equation (3.5) as

$$\dot{h}(t) = Q(-\infty)h_t + [Q(t)-Q(-\infty)]h_t. \tag{3.9}$$

By the variation of constants formula (see eqn (2.2) in Hale & Verduyn Lunel (1993)), solutions of equation (3.9) can be expressed as

$$h_t(\theta) = [J(t-s)h_s](\theta) + \int_s^{t+\theta} Z(t+\theta-\tau)[Q(\tau)-Q(-\infty)]h_\tau \mathrm{d}\tau, \quad s\le t, \tag{3.10}$$

for $\theta\in[-r,0]$. Note that $\theta\le 0$ and $Z(\tau)=0$ for $\tau<0$. From (3.7), (3.8) and (3.10) we obtain

$$\|h_t\| \le \gamma\mathrm{e}^{-a(t-s)}\|h_s\| + \delta\gamma\mathrm{e}^{ar}\int_s^t \mathrm{e}^{-a(t-\tau)}|h_\tau|\mathrm{d}\tau, \tag{3.11}$$

for $s\le t\le t^*$. Or equivalently,

$$\mathrm{e}^{at}\|h_t\| \le \gamma\mathrm{e}^{as}\|h_s\| + \delta\gamma\mathrm{e}^{ar}\int_s^t \mathrm{e}^{a\tau}\|h_\tau\|\mathrm{d}\tau. \tag{3.12}$$

The Gronwall inequality applied to (3.12) yields that

$$\mathrm{e}^{at}\|h_t\| \le \gamma\mathrm{e}^{as}\|h_s\|\mathrm{e}^{\delta\gamma\mathrm{e}^{ar}(t-s)}.$$

From the last inequality we have

$$\|h_t\| \le \gamma\mathrm{e}^{-(a-\delta\gamma\mathrm{e}^{ar})(t-s)}\|h_s\|, \quad s\le t\le t^*. \tag{3.13}$$

Note that h_s is bounded. By letting $s\to-\infty$ in (3.13), we immediately have

$$\|h_t\| = 0, \quad t\le t^*.$$

Then the uniqueness of the solution of equation (3.9) implies that $h_t=0$ for all $t\in\mathbb{R}$ and hence $\phi=0$. ■

Lemma 3.2. *$\mathcal{R}(T)=C$ and $\dim\mathcal{N}(T)=M$, where $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and null space of T, respectively.*

Proof. It follows from assumptions (H1)–(H2) that the operator T is Fredholm (see Chow *et al.* 1989, p. 7). Furthermore,

$$\begin{aligned}\operatorname{ind} T &= \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T)\\ &= \text{dimension of unstable manifold of } E_1\\ &\quad - \text{dimension of unstable manifold of } E_2\\ &= M - 0 = M.\end{aligned} \tag{3.14}$$

Moreover, we have $\mathcal{R}(T) = \{\psi \in C : \int_{-\infty}^{\infty} h(t)\psi(t)\mathrm{d}t = 0$ for every bounded solution $h(\cdot)$ of equation (3.4)$\}$. With the use of lemma 3.1, one concludes that $\mathcal{R}(T) = C$ and hence $\operatorname{codim} \mathcal{R}(T) = 0$. Therefore (3.14) implies that $\dim \mathcal{N}(T) = M$. ■

Lemma 3.3. *Let $y \in C_0$ be given. If ϕ is a bounded solution of the equation $T\phi = y$, then $\phi \in C_0^1$. In particular, $T\phi = 0$ implies that $\phi \in C_0^1$ and hence, $\mathcal{N}(T) \subset C_0^1$.*

Proof. We shall only prove $\lim_{t\to\infty} \phi(t) = 0$. The convergence of $\phi(t)$ to 0 as $t \to \infty$ can be proved analogously. By the definition of the operator T, $T\phi = y$ implies that

$$\dot{\phi}(t) = P(t)\phi_t + y(t), \quad t \in \mathbb{R},$$

or

$$\dot{\phi}(t) = P(-\infty)\phi_t + z(t), \quad t \in \mathbb{R}, \tag{3.15}$$

with $z(t) = [P(t) - P(-\infty)]\phi_t + y(t)$. Consider the homogeneous equation

$$\dot{\phi}(t) = P(-\infty)\phi_t. \tag{3.16}$$

Recall that for $\xi \in C([-r,0], \mathbb{R}^n)$,

$$P(-\infty)\xi = A_1\xi(0) + B_1\int_{-r}^{0} \mathrm{d}\tilde{\eta}(\theta)\xi(\theta),$$

where $\tilde{\eta}(\theta) = \eta(\theta)\mu_\Omega g_u(E_1)$, $\theta \in [-r,0]$. By assumption (H2), the generalized eigenfunction space U of equation (3.16) corresponding to eigenvalues with positive real part is M-dimensional. Let $\boldsymbol{\Phi} = (\boldsymbol{\Phi}^1, \dots, \boldsymbol{\Phi}^M)$ be a basis of U and $\boldsymbol{\Psi} = (\boldsymbol{\Psi}^1, \dots, \boldsymbol{\Psi}^M)^{\mathrm{T}}$ be a basis of the generalized eigenfunction space of the formal adjoint equation of equation (3.16) associated with U, satisfying

$$(\boldsymbol{\Psi}, \boldsymbol{\Phi}) = [(\boldsymbol{\Psi}^i, \boldsymbol{\Phi}^j)]_{M\times M} = I,$$

where for $\xi \in C([-r,0], \mathbb{R}^n)$ and $\psi \in C([0,r], \mathbb{R}^n)$, (ξ, ψ) is defined by

$$(\psi, \xi) = \psi^T(0)\xi(0) - \int_{-r}^{0} \psi^T(\tau - \theta)B_1 \mathrm{d}\tilde{\eta}(\theta)\xi(\tau)\mathrm{d}\tau.$$

Let $K(t) : C([-r,0], \mathbb{R}^n) \to C([-r,0], \mathbb{R}^n)$, $t \ge 0$, be the semigroup generated by solutions of equation (3.16). Define projections $K^U, K^S = (I - K^U) : C([-r,0], \mathbb{R}^n) \to C([-r,0], \mathbb{R}^n)$ with

$$K^U\xi = \boldsymbol{\Phi}(\boldsymbol{\Psi}, \xi), \quad \xi \in C([-r,0], \mathbb{R}^n).$$

Then there are positive constants $\alpha>0$ and $\beta>0$ such that for $\xi\in C([-r,0],\mathbb{R}^n)$

$$\|K(t)K^S\xi\|\le\beta e^{-\alpha t}\|\xi\|,\quad t\ge 0, \tag{3.17}$$

$$\|K(t)K^U\xi\|\le\beta e^{\alpha t}\|\xi\|,\quad t\le 0, \tag{3.18}$$

where for $t\le 0$, $K(t):\mathcal{R}(K^U)\to\mathcal{R}(K^U)$ is the inverse of $K(-t)|_{\mathcal{R}(K^U)}$. Now let $\phi(t)$ be a bounded solution of equation (3.15). Then $\phi_t=K^U\phi_t+K^S\phi_t$. By the variation-of-constants formula (see pp. 226–228 of Hale & Verduyn Lunel (1993)), we have

$$K^U\phi_t=K(t-s)K^U\phi_s+\int_s^t K(t-\tau)\Phi[\Psi(0)z(\tau)]\mathrm{d}\tau,\quad t\ge s, \tag{3.19}$$

$$K^S\phi_t=K(t-s)K^S\phi_s+\int_s^t \mathrm{d}_\tau[Y(t,\tau)^S]z(\tau),\quad t\ge s, \tag{3.20}$$

where $Y(t,\ \tau)^S$ is defined as follows (see eqn (9.10) in Hale & Verduyn Lunel (1993))

$$\begin{aligned} Y(t,\tau)^S&=\int_{t-r-\tau}^{t-r}K(\theta)[X_r-\Phi(\Psi,X_r)]\mathrm{d}\theta,\quad \text{if } \tau\le t-r,\\ Y(t,\tau)^S&=\int_0^{t-r}K(\theta)[X_r-\Phi(\Psi,X_r)]\mathrm{d}\theta\\ &\quad+\int_{t-r-\tau}^0 X_{r+\theta}\,\mathrm{d}\theta-\Phi\left(\Psi,\int_{t-r-\tau}^0 X_{r+\theta}\,\mathrm{d}\theta\right),\quad \text{if } \tau>t-r. \end{aligned} \tag{3.21}$$

Here we suppose $t-r\ge s$, and $X(t)$, $t\ge -r$, is the matrix solution of the homogeneous equation (3.16) with initial condition $X(0)=I$ and $X(\theta)=0$ for $\theta\in[-r,0)$. Applying $K(s-t)$, the inverse of $K(t-s)$ on $\mathcal{R}(K^U)$, to equation (3.19) we obtain

$$\begin{aligned} K(s-t)K^U\phi_t&=K^U\phi_s+\int_s^t K(s-t)K(t-\tau)\Phi[\psi(0)z(\tau)]\mathrm{d}\tau\\ &=K^U\phi_s+\int_s^t K(s-\tau)\Phi[\Psi(0)z(\tau)]\mathrm{d}\tau,\quad t\ge s, \end{aligned}$$

or

$$K^U\Phi_s=K(s-t)K^U\phi_t-\int_s^t K(s-\tau)\Phi[\Psi(0)z(\tau)]\mathrm{d}\tau,\quad t\ge s. \tag{3.22}$$

Therefore, (3.18) and (3.22) imply that

$$\begin{aligned} \|K^U\phi_s\|&\le\beta e^{\alpha(s-t)}\|\phi_t\|+\beta\int_s^t e^{\alpha(s-\tau)}\|\Phi[\Psi(0)z(\tau)]\|\mathrm{d}\tau\\ &\le\beta e^{\alpha(s-t)}\|\phi_t\|+\beta\int_s^t e^{\alpha(s-\tau)}\,\mathrm{d}\tau\sup_{s\le\tau\le t}\{\|\Phi[\Psi(0)z(\tau)]\|\}\mathrm{d}\tau\\ &=\beta e^{a(s-t)}\|\phi_t\|+\frac{\beta}{\alpha}(1-e^{\alpha(s-t)})\sup_{s\le\tau\le t}\{\|\Phi[\Psi(0)z(\tau)]\|\}. \end{aligned} \tag{3.23}$$

Since $\|\phi_t\|$ is bounded for $t\in\mathbb{R}$, by letting $s\to-\infty$ in (3.23), we obtain

$$\lim_{s\to-\infty}\|K^U\phi_s\|\le\frac{\beta}{\alpha}\sup_{-\infty\le\tau\le t}\{\|\Phi[\Psi(0)z(\tau)]\|\}. \tag{3.24}$$

Notice that, by the definition of $z(t)$, we have $\lim_{t\to\infty} z(t)=0$. Thus by letting $t\to-\infty$ in (3.24) we obtain

$$\lim_{s\to-\infty} \|K^U \phi_s\| = 0. \tag{3.25}$$

Next, we remark that for fixed $t\in\mathbb{R}$, $Y(t,\tau)$ is continuous with respect to the variable τ (see eqn (9.4) on p. 226 of Hale & Lunel (1993)). From expression (3.21), one sees that $Y(t,\tau)^S$ is continuously differentiable with respect to τ except for a finite jump at $\tau=t-r$, and

$$\left.\begin{aligned} \frac{\partial Y(t,\tau)^S}{\partial\tau} &= K(t-r-\tau)(X_r-\Phi(\Psi,X_r)), \quad \tau<t-r,\\ \frac{\partial Y(t,\tau)^S}{\partial\tau} &= X_{t-\tau}-\Phi(\Psi,X_{t-\tau}), \quad t-r<\tau<t. \end{aligned}\right\} \tag{3.26}$$

Therefore, (3.17), (3.20) and (3.26) yield that

$$\begin{aligned} \|K^S\phi_t\| \le{}& \|K(t-s)K^S\phi_s\| + \left\|\int_{t-r}^{t} \mathrm{d}_\tau[Y(t,\tau)^S]z(\tau)\right\| \\ &+ \left\|\int_{s}^{t-r} \mathrm{d}_\tau[Y(t,\tau)^S]z(\tau)\right\| \\ \le{}& \beta \mathrm{e}^{-\alpha(t-s)}\|\phi_s\| + \sup_{t-r\le\tau\le t}\{\|X_{t-\tau}-\Phi(\Psi,X_{t-\tau})\|\,\|z(\tau)\|\} \\ &+ \beta\left\|\int_s^{t-r} \mathrm{e}^{-\alpha(t-r-\tau)}\,\mathrm{d}\tau \sup_{s\le\tau\le t-r}\{\|X_r-\Phi(\Psi,X_r)\|\,\|z(\tau)\|\}\right\| \\ \le{}& \beta \mathrm{e}^{-\alpha(t-s)}\|\phi_s\| + \sup_{t-r\le\tau\le t}\{\|X_{t-\tau}-\Phi(\Psi,X_{t-\tau})\|\,\|z(\tau)\|\} \\ &+ \frac{\beta}{\alpha}\sup_{s\le\tau\le t-r}\{\|X_r-\Phi(\Psi,X_r)\|\,\|z(\tau)\|\}, \quad s\le t. \end{aligned} \tag{3.27}$$

By letting $s\to-\infty$ in (3.27), we conclude that

$$\begin{aligned} \|K^S\phi_t\| \le{}& \sup_{t-r\le\tau\le t}\{\|X_{t-\tau}-\Phi(\Psi,X_{t-\tau})\|\,\|z(\tau)\|\} \\ &+ \frac{\beta}{\alpha}\sup_{-\infty\le\tau\le t-r}\{\|X_r-\Phi(\Psi,X_r)\|\,\|z(\tau)\|\}. \end{aligned} \tag{3.28}$$

Since $\|z(\tau)\|\to 0$ as $\tau\to-\infty$, it immediately follows from (3.28) that

$$\lim_{t\to-\infty} \|K^s\phi_t\| = 0. \tag{3.29}$$

Combining (3.25) and (3.29), we have

$$\lim_{t\to-\infty}\phi_t = \lim_{t\to-\infty}(K^U\phi_t + K^S\phi_t) = 0.$$

From (3.15), we also have that $\lim_{t\to\infty}\dot\phi(t)=0$. ■

Let us return to the linear operator L defined in (2.11). It is obvious that if $w\in C_0$, then $Lw\in C_0$. Hence, we can consider L to be a linear operator from C_0 to C_0. For this operator, we have the following.

Theorem 3.4. *$\dim \mathcal{N}(L) = M$ and $\mathcal{R}(L) = C_0$.*

Proof. By definition, $w \in C_0$ and $Lw = 0$ if and only if

$$w(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[w(t) + P^0 w(t)]\mathrm{d}t, \quad s \in \mathbb{R}.$$

Hence, w is continuously differentiable. By differentiating the last equation one sees that $Lw = 0$ if and only if

$$\dot{w}(s) = P^0 w(s), \quad s \in \mathbb{R}.$$

Recall that for $z \in C_0$ and $s \in \mathbb{R}$

$$P^0 z(s) = A(s)z(s) + B(s)\int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_{\Omega} g_u(u^*(s+\theta))z(s+\theta) = P(s)z_s. \tag{3.30}$$

Thus, the above equation and lemma 3.3 imply that $w \in C_0^1$ and $Tw = 0$. That is, $w \in \mathcal{N}(L)$ if and only if $w \in \mathcal{N}(T)$. Therefore, lemmas 3.2 and 3.3 imply that $\dim \mathcal{N}(L) = \dim \mathcal{N}(T) = M$, with $\mathcal{N}(L) \subset C_0^1$. Next, we shall prove that $\mathcal{R}(L) = C_0$. That is, for each $z \in C_0$, we need to show that equation $Lw = z$, or equivalently,

$$w(s) - \int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[w(t) + P^0 w(t)]\mathrm{d}t = z(s), \quad s \in \mathbb{R}, \tag{3.31}$$

has a solution in C_0. To this end, we let $\xi(s) = w(s) - z(s)$, $s \in \mathbb{R}$. Upon a substitution, we obtain the equation for ξ as

$$\xi(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[\xi(t) + P^0\xi(t)]\mathrm{d}t + \int_{-\infty}^{s} \mathrm{e}^{-(s-t)}[z(t) + P^0 z(t)]\mathrm{d}t.$$

Differentiating the above equation yields that

$$\dot{\xi}(s) = P^0\xi(s) + z(s) + P^0 z(s), \quad s \in \mathbb{R}. \tag{3.32}$$

Thus, (3.30) implies that (3.32) is equivalent to the equation

$$(T\xi)(s) = z(s) + P^0 z(s). \tag{3.33}$$

From the expression of $P^0 z(s)$ it follows that $z \in C_0$ implies that $P^0 z(\cdot) \in C_0$, and hence $z + P^0 z \in C_0$. Thus lemmas 3.2 and 3.3 guarantee that equation (3.33) has a solution $\xi \in C_0^1$. Consequently, $w = \xi + z \in C_0$ is a solution of equation (3.31). ∎

4. Properties of the nonlinearity $\mathcal{H}$

In order to complete the proof of theorem 1.1, we need further information about the behaviour of the nonlinearity $\mathcal{H}(\cdot, \psi, \varepsilon)$ when $\varepsilon > 0$ is small and ψ is near the origin. To simplify the presentation, we let $R^{\varepsilon} : C \to C$ for small $\varepsilon \geq 0$ be defined by

$$R^{\varepsilon}\psi(s) = \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) g(\psi(s + \sqrt{\varepsilon}\nu \cdot y + \theta)), \quad s \in \mathbb{R}. \tag{4.1}$$

With the above notation, we can rewrite the nonlinear function $\mathcal{G}$ defined in (2.5) as

$$\begin{aligned}\mathcal{G}(\varepsilon,s,w) &= F\left(w(s)+u^*(s),\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)g([w+u^*](s+\sqrt{\varepsilon}\nu\cdot y+\theta))\right)\\ &\quad -F\left(u^*(s),\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)g(u^*(s+\theta))\right)-P^0w(s)+\varepsilon D\ddot{u}^*(s)\\ &= F(w(s)+u^*(s),R^\varepsilon[w+u^*](s))-F(u^*(s),R^\varepsilon u^*(s))\\ &\quad +F(u^*(s),R^\varepsilon u^*(s))-F(u^*(s),R^0u^*(s))\\ &\quad -P^0w(s)-\varepsilon D\ddot{u}^*(s)\\ &= P^\varepsilon w(s)-P^0w(s)+\varepsilon D\ddot{u}^*(s)\\ &\quad +F(w(s)+u^*(s),R^\varepsilon[w+u^*](s))-F(u^*(s),R^\varepsilon u^*(s))\\ &\quad -P^\varepsilon w(s)+F(u^*(s),R^\varepsilon u^*(s))-F(u^*(s),R^0u^*(s))\\ &= P^\varepsilon w(s)-P^0w(s)+G(\varepsilon,s,w)+\Theta(\varepsilon,s),\end{aligned} \tag{4.2}$$

where for $\varepsilon>0$ the linear operator $P^\varepsilon: C_0\to C$ is defined by

$$P^\varepsilon\psi(s)=A^\varepsilon(s)\psi(s)+B^\varepsilon(s)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)g_u(u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta))\psi(s+\sqrt{\varepsilon}\nu\cdot y+\theta), \tag{4.3}$$

for $s\in\mathbb{R}$, with

$$\left.\begin{aligned}A^\varepsilon(s)&=F_u(u^*(s),R^\varepsilon u^*(s)),\quad s\in\mathbb{R},\\ B^\varepsilon(s)&=F_v(u^*(s),R^\varepsilon u^*(s)),\quad s\in\mathbb{R},\end{aligned}\right\} \tag{4.4}$$

and

$$\begin{aligned}G(\varepsilon,s,\psi)&=F(\psi(s)+u^*(s),R^\varepsilon[\psi+u^*](s))\\ &\qquad -F(u^*(s),R^\varepsilon u^*(s))-P^\varepsilon\psi(s),\quad s\in\mathbb{R},\end{aligned} \tag{4.5}$$

$$\Theta(\varepsilon,s)=\varepsilon D\ddot{u}^*(s)+F(u^*(s),R^\varepsilon u^*(s))-F(u^*(s),R^0u^*(s)),\quad s\in\mathbb{R}. \tag{4.6}$$

From the above notations and (2.10), we can express $\mathcal{H}_i(s,w,\varepsilon)$ as

$$\begin{aligned}\mathcal{H}_i(s,w,\varepsilon) &= \int_{-\infty}^{s}\left[\frac{\mathrm{e}^{\alpha_i^\varepsilon(s-t)}}{\sqrt{1+4\varepsilon d_i}}-\mathrm{e}^{-(s-t)}\right]\left[w_i(t)+P_i^0w(t)\right]\mathrm{d}t\\ &\quad +\frac{1}{\sqrt{1+4\varepsilon d_i}}\int_s^\infty \mathrm{e}^{\beta_i^\varepsilon(s-t)}\left[w_i(t)+P_i^0w(t)\right]\mathrm{d}t\\ &\quad +\frac{1}{\sqrt{1+4\varepsilon d_i}}\int_{-\infty}^s \mathrm{e}^{\alpha_i^\varepsilon(s-t)}\left[P_i^\varepsilon w(t)-P_i^0w(t)\right]\mathrm{d}t\\ &\quad +\frac{1}{\sqrt{1+4\varepsilon d_i}}\int_s^\infty \mathrm{e}^{\beta_i^\varepsilon(s-t)}\left[P_i^\varepsilon w(t)-P_i^0w(t)\right]\mathrm{d}t\\ &\quad +\frac{1}{\sqrt{1+4\varepsilon d_i}}\left(\int_{-\infty}^s \mathrm{e}^{\alpha_i^\varepsilon(s-t)}G_i(\varepsilon,t,w)\mathrm{d}t+\int_s^\infty \mathrm{e}^{\beta_i^\varepsilon(s-t)}G_i(\varepsilon,t,w)\mathrm{d}t\right)\\ &\quad +\frac{1}{\sqrt{1+4\varepsilon d_i}}\left(\int_{-\infty}^s \mathrm{e}^{\alpha_i^\varepsilon(s-t)}\Theta_i(\varepsilon,t)\mathrm{d}t+\int_s^\infty \mathrm{e}^{\beta_i^\varepsilon(s-t)}\Theta_i(\varepsilon,t)\mathrm{d}t\right).\end{aligned} \tag{4.7}$$

Thus, we can rewrite $\mathcal{H}(s,w,\varepsilon)$ as

$$\mathcal{H}(s,w,\varepsilon) = W(s,\varepsilon) + \sum_{j=1}^{4} H^j(s,w,\varepsilon),$$

where for $i=1,2,\ldots,n$, $w\in C_0$, and $s\in\mathbb{R}$,

$$\begin{aligned}
H_i^1(s,w,\varepsilon) &= \int_{-\infty}^{s}\left[\frac{e^{\alpha_i^\varepsilon(s-t)}}{\sqrt{1+4\varepsilon d_i}} - e^{-(s-t)}\right]\left[w_i(t)+P_i^0 w(t)\right]dt \\
&\quad + \frac{1}{\sqrt{1+4\varepsilon d_i}}\int_s^\infty e^{\beta_i^\varepsilon(s-t)}\left[w_i(t)+P_i^0 w(t)\right]dt, \\
H_i^2(s,w,\varepsilon) &= \frac{1}{\sqrt{1+4\varepsilon d_i}}\int_{-\infty}^{s} e^{\alpha_i^\varepsilon(s-t)}\left[P_i^\varepsilon w(t)-P_i^0 w(t)\right]dt, \\
H_i^3(s,w,\varepsilon) &= \frac{1}{\sqrt{1+4\varepsilon d_i}}\int_s^\infty e^{\beta_i^\varepsilon(s-t)}\left[P_i^\varepsilon w(t)-P_i^0 w(t)\right]dt, \\
H_i^4(s,w,\varepsilon) &= \frac{1}{\sqrt{1+4\varepsilon d_i}}\left[\int_{-\infty}^{s} e^{\alpha_i^\varepsilon(s-t)} G_i(\varepsilon,t,w)dt + \int_s^\infty e^{\beta_i^\varepsilon(s-t)} G_i(\varepsilon,t,w)dt\right], \\
W_i(s,\varepsilon) &= \frac{1}{\sqrt{1+4\varepsilon d_i}}\left[\int_{-\infty}^{s} e^{\alpha_i^\varepsilon(s-t)}\Theta_i(\varepsilon,t)dt + \int_s^\infty e^{\beta_i^\varepsilon(s-t)}\Theta_i(\varepsilon,t)dt\right].
\end{aligned} \tag{4.8}$$

In what follows, we shall give a detailed analysis of the behaviour of functions $H^1(\cdot,w,\varepsilon),\ldots,H^4(\cdot,w,\varepsilon)$ and $W(\cdot,\varepsilon)$ for small $w\in C_0$ and $\varepsilon\geq 0$.

Lemma 4.1. *Let $\alpha\in C$ be given so that $\lim_{s\to\pm\infty}\alpha(s)=\alpha(\pm\infty)$ exist. Then for each $\varepsilon\geq 0$*

$$\lim_{s\to\pm\infty}\int_{\Omega_r} d\zeta(\theta,y)\alpha(s+\sqrt{\varepsilon}\nu\cdot y+\theta) = \int_{\Omega_r} d\zeta(\theta,y)\alpha(\pm\infty).$$

Proof. We shall prove lemma 4.1 only for the case when $s\to\infty$. The proof for the case where $s\to-\infty$ is analogous. For a positive integer j, let B_j be the open ball in $\mathcal{R}^m$ with radius j and centre at the origin. Then

$$\lim_{j\to\infty}\int_{B_j\cap\Omega} d|\mu| = \int_\Omega d|\mu|,$$

and hence the boundedness of $\int_\Omega d|\mu|$ implies that

$$\lim_{j\to\infty}\int_{(\mathbb{R}^m\setminus B_j)\cap\Omega} d|\mu| = 0.$$

Therefore, for any $\sigma>0$, there is a sufficiently large J such that

$$\left\|\int_{(\mathbb{R}^m\setminus B_J)\cap\Omega} d|\mu|\right\| < \sigma. \tag{4.9}$$

Now $\lim_{s\to\infty}\|\alpha(s)-\alpha(\infty)\|=0$ implies that there is a $t^*>0$ such that

$$\|\alpha(t)-\alpha(\infty)\| < \sigma, \quad t\geq t^*. \tag{4.10}$$

Note that if $s>t^*+\sqrt{\varepsilon}J+r$, then for all $y\in B_J\cap\Omega$ and $\theta\in[-r,0]$,

$$s+\sqrt{\varepsilon}\nu\cdot y+\theta > t^*+\sqrt{\varepsilon}J+r-\sqrt{\varepsilon}\|y\|-|\theta| \geq t^*.$$

Hence, for $s > t^* + \sqrt{\varepsilon}J + r$, we have

$$\left\| \alpha\left(s + \sqrt{\varepsilon}\nu \cdot y + \theta\right) - \alpha(\infty) \right\| < \sigma, \quad y \in B_J, \quad \theta \in [-r, 0]. \tag{4.11}$$

It follows from (4.9)–(4.11) that for all $s > t^* + \sqrt{\varepsilon}J + r$ and $\theta \in [-r, 0]$,

$$\begin{aligned} &\left\| \int_{\Omega} \mathrm{d}\mu(y) \left[\alpha(s + \sqrt{\varepsilon}\nu \cdot y + \theta) - \alpha(\infty)\right] \right\| \\ &\quad \le \left\| \int_{B_J \cap \Omega} \mathrm{d}\mu(y) \left[\alpha(s + \sqrt{\varepsilon}\nu \cdot y + \theta) - \alpha(\infty)\right] \right\| \\ &\qquad + \left\| \int_{(\mathbb{R}^m \setminus B_J) \cap \Omega} \mathrm{d}\mu(y) \left[\alpha(s + \sqrt{\varepsilon}\nu \cdot y + \theta) - \alpha(\infty)\right] \right\| \\ &\quad \le \sigma \left\| \int_{B_J \cap \Omega} \mathrm{d}|\mu| \right\| + 2\sigma \|\alpha\|_C \le \sigma \left(\left\| \int_{\Omega} \mathrm{d}|\mu| \right\| + 2\|\alpha\|_C \right). \end{aligned} \tag{4.12}$$

Since $\sigma > 0$ is arbitrary, (4.12) implies that

$$\lim_{s \to \infty} \left\| \int_{\Omega} \mathrm{d}\mu(y) \left[\alpha(s + \sqrt{\varepsilon}\nu \cdot y + \theta) - \alpha(\infty)\right] \right\| = 0, \tag{4.13}$$

uniformly for $\theta \in [-r, 0]$. Consequently, we have ■

$$\lim_{s \to \infty} \left\| \int_{\Omega_r} \mathrm{d}\zeta(\theta, y) \left[\alpha(s + \sqrt{\varepsilon}\nu \cdot y + \theta) - \alpha(\infty)\right] \right\| = 0.$$

Corollary 4.2. *For each $\varepsilon \ge 0$ and each $w \in C_0$, $\mathcal{H}(\cdot, w, \varepsilon) \in C_0$. In other words, $\mathcal{H}(\cdot, C_0, \varepsilon) \subseteq C_0$ for each $\varepsilon \ge 0$.*

Proof. For $w \in C_0$ and $\varepsilon \ge 0$, if we let $\alpha(t) = g_u(u^*(t))w(t)$, $t \in \mathbb{R}$, then $\alpha \in C$ and $\alpha(s) \to 0$ as $|s| \to \infty$. It follows from the definition of P^ε and lemma 4.1 that $P^\varepsilon w(s) \to 0$ as $|s| \to \infty$. Therefore, $H^i(s, w, \varepsilon) \to 0$ as $|s| \to \infty$ for $i = 1, 2, 3$. Next, by the definition of R^ε given in (4.1) and lemma 4.1 we have

$$\lim_{|s| \to \infty} R^\varepsilon[w + u^*](s) = \lim_{|s| \to \infty} R^\varepsilon u^*(s).$$

The above equality yields that $G(\varepsilon, s, w) \to 0$ as $|s| \to \infty$, and so does for the function $H^4(s, w, \varepsilon)$. Similarly, we obtain that $W(s, \varepsilon) \to 0$ as $|s| \to \infty$. ■

Proposition 4.3. *For $w \in C_0$ and small $\varepsilon \ge 0$, $H^1(\cdot, w, \varepsilon) = O(\varepsilon)\|w\|_{C_0}$.*

Proof. For $s \in \mathbb{R}$ and $\varepsilon \ge 0$ we have

$$\begin{aligned} &\int_{-\infty}^{s} |\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \sqrt{1 + 4\varepsilon d_i}\,\mathrm{e}^{-(s-t)}|\mathrm{d}t \\ &\quad = \int_{-\infty}^{s} |\mathrm{e}^{\alpha_i^\varepsilon(s-t)}(1 - \sqrt{1 + 4\varepsilon d_i}) + \sqrt{1 + 4\varepsilon d_i}\left(\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \mathrm{e}^{-(s-t)}\right)|\mathrm{d}t \\ &\quad \le |1 - \sqrt{1 + 4\varepsilon d_i}| \int_{-\infty}^{s} \mathrm{e}^{\alpha_i^\varepsilon(s-t)}\,\mathrm{d}t + \sqrt{1 + 4\varepsilon d_i} \int_{-\infty}^{s} |\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \mathrm{e}^{-(s-t)}|\mathrm{d}t. \end{aligned} \tag{4.14}$$

Since $\alpha_i^\varepsilon > -1$, for $t \le s$ we have

$$|\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \mathrm{e}^{-(s-t)}| = \mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \mathrm{e}^{-(s-t)}, \tag{4.15}$$

and (4.14) and (4.15) yield that

$$\begin{aligned}\int_{-\infty}^{s} &|\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \sqrt{1+4\varepsilon d_i}\,\mathrm{e}^{-(s-t)}|\mathrm{d}t \\ &\le |1-\sqrt{1+4\varepsilon d_i}|\left(-\frac{1}{\alpha_i^\varepsilon}\right) + \sqrt{1+4\varepsilon d_i}\left[-\frac{1}{\alpha_i^\varepsilon} - 1\right].\end{aligned}$$

Noticing that $\alpha_i^\varepsilon \to -1$ as $\varepsilon \to 0^+$, we obtain from the above inequality that

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{s} |\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \sqrt{1+4\varepsilon d_i}\,\mathrm{e}^{-(s-t)}|\mathrm{d}t = 0. \tag{4.16}$$

Next let $K = 1 + \|P^0\|_{\mathcal{L}(C_0,C_0)}$. Then

$$\begin{aligned}&\left|\int_{-\infty}^{s}\left[\frac{\mathrm{e}^{\alpha_i^\varepsilon(s-t)}}{\sqrt{1+4\varepsilon d_i}} - \mathrm{e}^{-(s-t)}\right]\left[w_i(t) + P_i^0 w(t)\right]\mathrm{d}t\right| \\ &\quad\le K\int_{-\infty}^{s}\left|\frac{\mathrm{e}^{\alpha_i^\varepsilon(s-t)}}{\sqrt{1+4\varepsilon d_i}} - \mathrm{e}^{-(s-t)}\right|\mathrm{d}t\|w\|_{C_0} \\ &\quad= \frac{K}{\sqrt{1+4\varepsilon d_i}}\int_{-\infty}^{s}|\mathrm{e}^{\alpha_i^\varepsilon(s-t)} - \sqrt{1+4\varepsilon d_i}\,\mathrm{e}^{-(s-t)}|\mathrm{d}t\|w\|_{C_0} \\ &\quad= O(\varepsilon)\|w\|_{C_0}.\end{aligned} \tag{4.17}$$

Next, since $\beta_i^\varepsilon \to \infty$ as $\varepsilon \to 0$, $1/\beta_i^\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0$. This yields that

$$\begin{aligned}&\left|\frac{1}{\sqrt{1+4\varepsilon d_i}}\int_{s}^{\infty}\mathrm{e}^{\beta_i^\varepsilon(s-t)}\left[w_i(t) + P_i^0 w(t)\right]\mathrm{d}t\right| \\ &\quad\le \frac{K}{\sqrt{1+4\varepsilon d_i}}\int_{s}^{\infty}\mathrm{e}^{\beta_i^\varepsilon(s-t)}\mathrm{d}t\|w\|_{C_0} \\ &\quad= \frac{K}{\beta_i^\varepsilon\sqrt{1+4\varepsilon d_i}}\|w\|_{C_0} = O(\varepsilon)\|w\|_{C_0}, \quad \text{as} \quad \varepsilon \to 0.\end{aligned} \tag{4.18}$$

The proposition therefore follows from estimates (4.17) and (4.18). ■

Lemma 4.4. *For $\varepsilon > 0$ and $(s, y, \theta) \in \mathbb{R}\times\mathbb{R}^m\times[-r,0]$,*

$$\left\|\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)\left[g(u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta)) - g(u^*(s+\theta))\right]\right\|$$

$$\le \sqrt{\varepsilon}\|\eta\|\left\|\int_{\Omega}\mathrm{d}|\mu|(y)\|y\|\right\|\|g_u\|\ \|\dot{u}^*\|_C,$$

where $\|\eta\| = V_{[-r,0]}\eta$ and

$$\|g_u\| = \sup\{\|g_u(\lambda u^*(t) + (1-\lambda)u^*(\tau))\| : (\lambda, t, \tau) \in [0,1]\times\mathbb{R}\times\mathbb{R}\}.$$

Proof. Since g is differentiable, for $(s,y,\theta)\in\mathbb{R}\times\mathbb{R}^m\times[-r,0]$, we have

$$\begin{aligned} g(u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta))-g(u^*(s+\theta)) \\ &= \int_0^1 g_u(\lambda u^*(s+\theta)+(1-\lambda)u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta))\mathrm{d}\lambda \\ &\quad\times\left[u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta)-u^*(s+\theta)\right], \end{aligned}$$

The above equality yields that for $(s,y,\theta)\in\mathbb{R}\times\mathbb{R}^m\times[-r,0]$,

$$\|g(u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta))-g(u^*(s+\theta))\|\leq\sqrt{\varepsilon}\|y\|\,\|g_u\|\dot{u}^*\|_C. \tag{4.19}$$

Recalling that $\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)=\int_{-r}^0\int_\Omega \mathrm{d}\eta(\theta)\,\mathrm{d}\mu(y)$, as an immediate consequence of (4.19) we have, for $s\in\mathbb{R}$ and $\varepsilon>0$,

$$\begin{aligned} &\left\|\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)\left[g(u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta))-g_u(u^*(s+\theta))\right]\right\| \\ &\quad\leq\sqrt{\varepsilon}\|\eta\|\left\|\int_\Omega \mathrm{d}|\mu|(y)\|y\|\right\|\|g_u\|\,\|\dot{u}^*\|_C. \end{aligned}$$

■

Proposition 4.5. *There exist* $\varepsilon_0>0$ *and* $M_0>0$ *such that for all* $\varepsilon\in[0,\varepsilon_0]$, *and* $\psi\in C_0$,

$$\|H^2(\cdot,\psi,\varepsilon)\|_{C_0}\leq\sqrt{\varepsilon}M_0\|\psi\|_{C_0}.$$

Proof. Let $h(s,y,\theta)=g_u(u^*(s+\sqrt{\varepsilon}\nu\cdot y+\theta))$. From the definitions of $P^\varepsilon\psi$ and $P^0\psi$, we have

$$\begin{aligned} [P^\varepsilon-P^0]\psi(s) &= [A^\varepsilon(s)-A(s)]\psi(s) \\ &\quad+[B^\varepsilon(s)-B(s)]\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(s,y,\theta)\psi(s+\sqrt{\varepsilon}\nu\cdot y+\theta) \\ &\quad+B(s)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)[h(s,y,\theta)-h(s,0,\theta)]\psi(s+\sqrt{\varepsilon}\nu\cdot y+\theta) \\ &\quad+B(s)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(s,0,\theta)\left[\psi(s+\sqrt{\varepsilon}\nu\cdot y+\theta)-\psi(s+\theta)\right]. \end{aligned} \tag{4.20}$$

For $\varepsilon>0$, let

$$E^\varepsilon(s)=\mathrm{diag}\left(\frac{\mathrm{e}^{\alpha_1^\varepsilon s}}{\sqrt{1+4\varepsilon d_1}},\frac{\mathrm{e}^{\alpha_2^\varepsilon s}}{\sqrt{1+4\varepsilon d_2}},\cdots,\frac{\mathrm{e}^{\alpha_n^\varepsilon s}}{\sqrt{1+4\varepsilon d_n}}\right),\quad s\in\mathbb{R}.$$

Then

$$\dot{E}^\varepsilon(s)=\mathrm{diag}\left(\frac{\alpha_1^\varepsilon\mathrm{e}^{\alpha_1^\varepsilon s}}{\sqrt{1+4\varepsilon d_1}},\frac{\alpha_2^\varepsilon\mathrm{e}^{\alpha_2^\varepsilon s}}{\sqrt{1+4\varepsilon d_2}},\cdots,\frac{\alpha_n^\varepsilon\mathrm{e}^{\alpha_n^\varepsilon s}}{\sqrt{1+4\varepsilon d_n}}\right),\quad s\in\mathbb{R}.$$

Since $\alpha_i^\varepsilon\to-1$ as $\varepsilon\to0$ for $i=1,\ldots,n$, there are $\varepsilon_0>0$ and $K_0>0$ such that for $\varepsilon\in[0,\varepsilon_0]$,

$$\|E^\varepsilon(0)\|\leq K_0,\quad\int_{-\infty}^s\|E^\varepsilon(s-t)\|\mathrm{d}t\leq K_0,\quad\int_{-\infty}^s\|\dot{E}^\varepsilon(s-t)\|\mathrm{d}t\leq K_0. \tag{4.21}$$

By the definition of H^2 and (4.20), we have

$$\begin{aligned}
&H^2(s,\psi,\varepsilon)\\
&\quad=\int_{-\infty}^{s} E^{\varepsilon}(s-t)[P^{\varepsilon}-P^{0}]\psi(t)\,\mathrm{d}t\\
&\quad=\int_{-\infty}^{s} E^{\varepsilon}(s-t)[A^{\varepsilon}(t)-A(t)]\psi(t)\,\mathrm{d}t\\
&\qquad+\int_{-\infty}^{s} E^{\varepsilon}(s-t)[B^{\varepsilon}(t)-B(t)]\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(t,y,\theta)\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)\,\mathrm{d}t\\
&\qquad+\int_{-\infty}^{s} E^{\varepsilon}(s-t)B(t)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)[h(t,y,\theta)-h(t,0,\theta)]\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)\,\mathrm{d}t\\
&\qquad+\int_{-\infty}^{s} E^{\varepsilon}(s-t)B(t)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(t,0,\theta)\big[\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)-\psi(t+\theta)\big]\mathrm{d}t.
\end{aligned}\tag{4.22}$$

Let

$$\hat{u}^*_{\varepsilon}(t)=R^{\varepsilon}u^*(t)=\int_{-r}^{0}\int_{W}\mathrm{d}\eta(\theta)\,\mathrm{d}\mu(y)g(u^*(t+\sqrt{\varepsilon}\nu\cdot y+\theta)),\quad t\in\mathbb{R}.\tag{4.23}$$

Then, from the definitions of $A^{\varepsilon}(t)$, $A(t)$ and $\hat{u}^*_{\varepsilon}(t)$, it follows that

$$\begin{aligned}
A^{\varepsilon}(t)-A(t)&=F_u(u^*(t),\hat{u}^*_{\varepsilon}(t))-F_u(u^*(t),\hat{u}^*_0(t))\\
&=\int_0^1 F_{uv}(u^*(t),\hat{u}^*_0(t)+\tau[\hat{u}^*_{\varepsilon}(t)-\hat{u}^*_0(t)])\mathrm{d}\tau[\hat{u}^*_{\varepsilon}(t)-\hat{u}^*_0(t)].
\end{aligned}\tag{4.24}$$

Since F is C^2-smooth, there is a constant $K_1>0$ such that

$$\left\|\int_0^1 F_{uv}(u^*(t),\hat{u}^*_0(t)+\tau[\hat{u}^*_{\varepsilon}(t)-\hat{u}^*_0(t)])\mathrm{d}\tau\right\|\le K_1,\quad t\in\mathbb{R},\ \varepsilon\in[0,\varepsilon_0].\tag{4.25}$$

Lemma 4.4 and (4.25) therefore yield that

$$\|A^{\varepsilon}(t)-A(t)\|\le\sqrt{\varepsilon}K_1\|\eta\|\left\|\int_{\Omega}\mathrm{d}|\mu|(y)\|y\|\right\|\|g_u\|\dot{u}^*\|_{C_0},\quad t\in\mathbb{R},\ \varepsilon\in[0,\varepsilon_0].\tag{4.26}$$

For all $s\in\mathbb{R}$ and $\varepsilon\in[0,\varepsilon_0]$, (4.21) and (4.26) imply that

$$\begin{aligned}
&\left\|\int_{-\infty}^{s} E^{\varepsilon}(s-t)[A^{\varepsilon}(t)-A(t)]\psi(t)\,\mathrm{d}t\right\|\\
&\quad\le\sqrt{\varepsilon}K_1\|\eta\|\left\|\int_{\Omega}\mathrm{d}|\mu|(y)\|y\|\right\|\|g_u\|\dot{u}^*\|_{C_0}\int_{-\infty}^{s}\|E^{\varepsilon}(s-t)\|\,\mathrm{d}t\|\psi\|_{C_0}\\
&\quad\le\sqrt{\varepsilon}M_1\|\psi\|_{C_0},
\end{aligned}\tag{4.27}$$

where

$$M_1=K_0K_1\|\eta\|\left\|\int_{\Omega}\mathrm{d}|\mu|(y)\|y\|\right\|\|g_u\|\;\|\dot{u}^*\|_{C_0},$$

and K_0 is defined in (4.21).

Arguing in the same way as above, we obtain that there is a constant $K_2>0$ such that

$$\|B^\varepsilon(t)-B(t)\|\leq\sqrt{\varepsilon}K_2\|\eta\|\left\|\int_\Omega \mathrm{d}|\mu|(y)\|y\|\right\|\|g_u\|\ \|\dot{u}^*\|_{C_0},\quad s\in\mathbb{R}.$$

Thus, for all $s\in\mathbb{R}$,

$$\left\|\int_{-\infty}^{s}E^\varepsilon(s-t)[B^\varepsilon(t)-B(t)]\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(t,y,\theta)\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)\ \mathrm{d}t\right\|$$
$$\leq\sqrt{\varepsilon}M_2\|\psi\|_{C_0},\tag{4.28}$$

with $M_2=K_0K_2\|\eta\|\int_\Omega\mathrm{d}|\mu|(y)\|y\|\ \|\|g_u\|\dot{u}^*|_{C_0}$.

It is also clear that for $s\in\mathbb{R}$,

$$\left\|\int_{-\infty}^{s}E^\varepsilon(s-t)B(t)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)[h(t,y,\theta)-h(t,0,\theta)]\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)\ \mathrm{d}t\right\|$$
$$\leq\sqrt{\varepsilon}M_3\|\psi\|C_0,\tag{4.29}$$

with

$$M_3=2K_0\sup_{t\in\mathbb{R}}\{\|B(t)\|\}\|\eta\|\left\|\int_\Omega\mathrm{d}|\mu|(y)\right\|\|g_u\|.$$

Next, if $\psi\in C_0^1$, by exchanging the order of integration and integration by parts we have

$$\begin{aligned}
&\int_{-\infty}^{s}E^\varepsilon(s-t)B(t)\left[\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(t,0,\theta)\left[\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)-\psi(t+\theta)\right]\right]\mathrm{d}t\\
&=\int_{-\infty}^{s}E^\varepsilon(s-t)B(t)\left[\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(t,0,\theta)\int_0^1\dot{\psi}(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\sqrt{\varepsilon}(\nu\cdot y)\ \mathrm{d}\tau\right]\mathrm{d}t\\
&=\sqrt{\varepsilon}\int_0^1\left[\int_{-\infty}^{s}E^\varepsilon(s-t)B(t)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)h(t,0,\theta)\dot{\psi}(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\mathrm{d}t\right]\mathrm{d}\tau\\
&=\sqrt{\varepsilon}\int_0^1\left(\left[E^\varepsilon(s-t)B(t)\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)h(t,0,\theta)\psi(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\right]\Bigg|_{t=-\infty}^{t=s}\right)\mathrm{d}\tau\\
&\quad+\sqrt{\varepsilon}\int_0^1\left\{\int_{-\infty}^{s}\left[\dot{E}^\varepsilon(s-t)B(t)-E^\varepsilon(s-t)\dot{B}(t)\right]\right.\\
&\qquad\left.\times\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)h(t,0,\theta)\psi(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\ \mathrm{d}t\right\}\mathrm{d}\tau\\
&\quad-\sqrt{\varepsilon}\int_0^1\left[\int_{-\infty}^{s}E^\varepsilon(s-t)B(t)\right.\\
&\qquad\left.\times\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)\frac{\partial h(t,0,\theta)}{\partial t}\psi(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\right]\mathrm{d}t\,\mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{-\infty}^{s} E^{\varepsilon}(s-t)B(t)\left[\int_{\Omega_r} \mathrm{d}\zeta(\theta,y)h(t,0,\theta)\left[\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)-\psi(t+\theta)\right]\right]\mathrm{d}t \\
&= \sqrt{\varepsilon}E^{\varepsilon}(0)B(s)\int_0^1\left[\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)h(s,0,\theta)\psi(s+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\right]\mathrm{d}\tau \\
&\quad+\sqrt{\varepsilon}\int_0^1\left\{\int_{-\infty}^{s}\left[\dot{E}^{\varepsilon}(s-t)B(t)-E^{\varepsilon}(s-t)\dot{B}(t)\right]\right. \\
&\qquad\left.\times\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)h(t,0,\theta)\psi(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\mathrm{d}t\right\}\mathrm{d}\tau \\
&\quad-\sqrt{\varepsilon}\int_0^1\left[\int_{-\infty}^{s}E^{\varepsilon}(s-t)B(t)\right. \\
&\qquad\left.\times\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)(\nu\cdot y)\frac{\partial h(t,0,\theta)}{\partial t}\psi(t+\tau\sqrt{\varepsilon}\nu\cdot y+\theta)\,\mathrm{d}t\right]\mathrm{d}\tau,\quad s\in\mathbb{R}.
\end{aligned} \tag{4.30}$$

Recalling that $h(t,0,\theta)=g_u(u^*(t+\theta))$, we have

$$\frac{\partial h(t,0,\theta)}{\partial t}=\frac{\partial g_u(u^*(t+\theta))}{\partial t}=g_{uu}(u^*(t+\theta))\dot{u}^*(t+\theta).$$

Therefore, (4.30) implies that for all $s\in\mathbb{R}$,

$$\begin{aligned}
&\left\|\int_{-\infty}^{s}E^{\varepsilon}(s-t)B(t)\left[\int_{\Omega_r}\mathrm{d}\zeta(\theta,y)h(t,0,\theta)[\psi(t+\sqrt{\varepsilon}\nu\cdot y+\theta)-\psi(t+\theta)]\right]\mathrm{d}t\right\| \\
&\le\sqrt{\varepsilon}M_4\|\psi\|_{C_0},
\end{aligned} \tag{4.31}$$

where

$$M_4=2K_0\sup_{t\in\mathbb{R}}\{\|B(t)\|+\|\dot{B}(t)\|\}\|(\|g_u\|+\|\tilde{g}_{uu}\|\,\|\dot{u}^*\|_C)\eta\|\left\|\int_{\Omega}\mathrm{d}\mu(y)|y|^+\right\|_{\mathbb{R}^m}$$

and $\|\tilde{g}_{uu}\|=\sup_{t\in\mathbb{R}}\|g_{uu}(u^*(t))\|\}$. It, therefore, follows from (4.27)–(4.29) and (4.31) that for $\varepsilon\in[0,\varepsilon_0]$ and $\psi\in C_0$

$$\|H^2(\cdot,\psi,\varepsilon)\|_{C_0}\le\sqrt{\varepsilon}M_0\|\psi\|_{C_0},\quad\text{with}\quad M_0=\sum_{j=1}^{4}M_j. \tag{4.32}$$

Since $H^2(\cdot,\cdot,\varepsilon):C_0\to C_0$ is a bounded linear operator and C_0^1 is dense in C_0, the inequality equation (4.32) holds for all $\psi\in C_0$. ■

Proposition 4.6. *For* $\varepsilon>0$ *and* $\psi\in C_0$, $H^3(\cdot,\psi,\varepsilon)=O(\varepsilon)\|\psi\|_{C_0}$ *as* $\varepsilon\to 0$.

Proof. Since $\beta_i^{\varepsilon}\to\infty$ as $\varepsilon\to 0$ for $i=1,\ldots,n$, one obtains that for all $s\in\mathbb{R}$

$$\int_s^{\infty}\mathrm{e}^{\beta_i^{\varepsilon}(s-t)}\,\mathrm{d}t=\frac{1}{\beta_i^{\varepsilon}}\to 0\quad\text{as}\quad\varepsilon\to 0,\quad i=1,\ldots,n. \tag{4.33}$$

Thus proposition 4.6 follows from (4.33) and the definition of H^3. ■

Proposition 4.7. *$H^4(\cdot, 0, \varepsilon) = 0$ and for each $\delta > 0$, there is a $\sigma > 0$ such that*

$$\|H^4(\cdot, \phi, \varepsilon) - H^4(\cdot, \psi, \varepsilon)\|_{C_0} \le \delta \|\phi - \psi\|_{C_0},$$

for all $\varepsilon \in [0,1]$ and all ϕ, $\psi \in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and centre at the origin.

Proof. It is apparent that, from the definition of $G(\varepsilon, \cdot, \psi)$ (see equation (4.5)), $G_\psi(\varepsilon, \cdot, \psi)$ and $G_{\psi\psi}(\varepsilon, \cdot, \psi)$ are continuous for $\varepsilon \in [0,1]$ and for ψ in a neighbourhood of the origin in C_0. Moreover, we have $G_\psi(\varepsilon, \cdot, 0) \equiv 0$ for $\varepsilon \in [0,1]$. It therefore follows that

$$\|G(\varepsilon, \cdot, \psi)\|_{C_0} = O(\|\psi\|_{C_0}^2) \quad \text{as} \quad \|\psi\|_{C_0} \to 0, \tag{4.34}$$

uniformly for $\varepsilon \in [0,1]$, and the proposition follows from the definition of H^4 and (4.34). ■

Proposition 4.8. $\|W(\cdot, \varepsilon)\|_{C_0} = O(\varepsilon)$ *as* $\varepsilon \to 0$.

Proof. We note that $\ddot{u}^*(\cdot)$ is bounded in C_0 and

$$F(u^*(\cdot), R^\varepsilon u^*(\cdot)) - F(u^*(\cdot), R^0 u^*(\cdot)) = O(\varepsilon)\|\dot{u}^*\|_{C_0} \quad \text{as} \quad \varepsilon \to 0,$$

by lemma 4.4. Therefore, we obtain proposition 4.8 from the expression of $W(\cdot, \varepsilon)$ given in (4.8).

5. Proof of the main theorem

We shall complete the proof of our main theorem 1.1 in this section. To do so we need a final auxiliary result. By theorem 3.4 we have $\dim \mathcal{N}(L) = M$. Therefore, there are functions $w_1, \ldots, w_M \in C_0$ which give a basis of $\mathcal{N}(L)$. Hence there exist linear functionals $h_1, \ldots, h_M : C_0 \to \mathbb{R}$, such that

$$h_i(w_i) = 1, \quad h_i(w_j) = 0, \quad i \neq j, \quad i, j = 1, \ldots, M.$$

Lemma 5.1. *Let $X = \{\phi \in C_0 : h_i(\phi) = 0,\ i = 1, \ldots, M\}$. Then*

$$C_0 = X \oplus \mathcal{N}(L).$$

Proof. We note first that this result is not new. Nevertheless, we give a short proof here for the sake of completion. For each $\psi \in C_0$, let $\phi = \psi - \sum_{i=1}^M h_i(\psi) w_i$. Then we have $h_i(\phi) = 0$, $i = 1, \ldots, M$, and $\psi = \phi + \sum_{i=1}^M h_i(\psi) w_i$. That is, each $\psi \in C_0$ can be expressed as the sum of an element of X and an element of $\mathcal{N}(L)$. Moreover, let $\psi \in X \cap \mathcal{N}(L)$. Thus there are constants c_i, $i = 1, \ldots, M$, such that

$$\psi = \sum_{i=1}^M c_i w_i.$$

The definition of X and h_i imply that

$$0 = h_i(\psi) = c_i h_i(w_i) = c_i, \quad i = 1, \ldots, M.$$

Hence $\psi = 0$ and thus $X \cap \mathcal{N}(L) = 0$. This proves the lemma. ∎

It is clear that $X \subset C_0$ is a Banach space. If we let $S = L|_X$ be the restriction of L on X, then S: $X \to C_0$ is one-to-one and onto, since $\mathcal{R}(L) = C_0$ by theorem 3.4. Therefore, S has an inverse S^{-1}: $C_0 \to X$ which is a bounded linear operator.

Proof of theorem 1.1. For each $\psi \in C_0$, there are unique $\xi \in \mathcal{N}(L)$ and $\phi \in X$ such that $\psi = \phi + \xi$. Hence ψ is a solution of equation (2.12) if and only if

$$L\phi = \mathcal{H}(\cdot, \xi + \phi, \varepsilon) \tag{5.1}$$

or, equivalently, if and only if ϕ is a solution of the equation

$$\phi = S^{-1}\mathcal{H}(\cdot, \phi + \xi, \varepsilon). \tag{5.2}$$

Let $\|S^{-1}\| = \|S^{-1}\|_{\mathcal{L}(C_0, X)}$. It follows from propositions 4.3 and 4.5–4.8 that there are $\sigma > 0$, $\varepsilon^* > 0$, and $0 < \rho < 1$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and $\psi, \varphi \in \overline{B(\sigma)} \subset C_0$,

$$\|\mathcal{H}(\cdot, \psi, \varepsilon)\|_{C_0} \le \frac{1}{3\|S^{-1}\|}(\|\psi\|_{C_0} + \sigma), \tag{5.3}$$

$$\|\mathcal{H}(\cdot, \psi, \varepsilon) - \mathcal{H}(\cdot, \varphi, \varepsilon)\|_{C_0} \le \frac{\rho}{\|S^{-1}\|}\|\psi - \varphi\|_{C_0}. \tag{5.4}$$

For each fixed $\xi \in \mathcal{N}(L) \cap \overline{B(\sigma)}$, (5.3) implies that

$$\|S^{-1}\mathcal{H}(\cdot, \phi + \xi, \varepsilon)\|_{C_0} \le \frac{1}{3}(\|\phi + \xi\|_{C_0} + \sigma) \le \sigma \quad \text{for} \quad \varepsilon \in (0, \varepsilon^*], \quad \phi \in X \cap \overline{B(\sigma)}. \tag{5.5}$$

Hence, together with (5.4) we see that the mapping

$$\mathcal{F} : (X \cap \overline{B(\sigma)}) \times (\mathcal{N}(L) \cap \overline{B(\sigma)}) \times (0, \varepsilon^*) \to X \cap \overline{B(\sigma)},$$

given by

$$\mathcal{F}(\phi, \xi, \varepsilon) = S^{-1}\mathcal{H}(\cdot, \phi + \xi, \varepsilon),$$

is a uniform contraction mapping of $\phi \in X \cap \overline{B(\sigma)}$. Hence, for each $(\xi, \varepsilon) \in (\mathcal{N}(L) \cap \overline{B(\sigma)}) \times (0, \varepsilon^*)$ there is a unique fixed point $\phi_{(\xi,\varepsilon)} \in X \cap \overline{B(\sigma)}$ of the mapping $\mathcal{F}(\cdot, \xi, \varepsilon)$. In other words, $\phi_{(\xi,\varepsilon)}$ is the unique solution in $X \cap \overline{B(\sigma)}$ of equation (5.2). Thus, for $\varepsilon \in (0, \varepsilon^*)$ fixed, $\psi_{(\xi,\varepsilon)} = \phi_{(\xi,\varepsilon)} + \xi$ is a solution of equation (2.12). Notice that $\mathcal{N}(L) \cap \overline{B(\sigma)}$ is M-dimensional. It follows that for each $\varepsilon \in (0, \varepsilon^*)$ and for each unit vector $\nu \in \mathbb{R}^m$, the set

$$\Gamma_\nu(\varepsilon) = \{\psi_{(\xi,\varepsilon)} : \xi \in \mathcal{N}(L) \cap \overline{B(\sigma)}\}$$

is an M-dimensional manifold. This proves claims (i) and (ii) in the statement of the theorem.

To prove claim (iii), we first note that if F, g are $C^k(k\geq 2)$, then $\mathcal{H}(\cdot,\psi,\varepsilon)$ is continuous on (ψ,ε) and C^{k-1}-smooth with respect to ψ. Hence $\mathcal{F}(\phi,\xi,\varepsilon)$ is continuous on (ϕ,ξ,ε) and C^{k-1}-smooth with respect to ϕ and ξ. The uniform contraction mapping principle (see pp. 25–26 of Chow & Hale (1982)) implies that the fixed point $\phi_{(\xi,\varepsilon)}$ is a continuous mapping on (ξ,ε) and C^{k-1} on ξ. Therefore, in addition we conclude that for each $\varepsilon\in(0,\varepsilon^*)$ and for each unit vector $\nu\in\mathbb{R}^m$, $\Gamma_\nu(\varepsilon)$ is a C^{k-1} manifold. It is locally given as the graph of a C^{k-1} mapping that is also continuous with respect to c.

Let $c=1/\sqrt{\varepsilon}$ with $\varepsilon\in(0,\varepsilon^*)$ and

$$\mathcal{M}_\nu(c)=\{U: U(s)=\psi_\xi(s/c)+u^*(s/c), s\in\mathbb{R}, \psi_\xi\in\Gamma_\nu(s/c^2)\}.$$

Then $\mathcal{M}_\nu(c)$ is an M-dimensional manifold in a neighbourhood of u^* consisting of travelling wave solutions of equation (1.1) with wave speed c and direction ν. Moreover, for each $c>c^*$ and each unit vector $\nu\in\mathbb{R}^m$, $\mathcal{M}_\nu(c)$ is a C^{k-1} manifold that is given by the graph of a C^{k-1}-mapping that is continuous on c.

It remains to prove that the above fixed point $\phi_{(\xi,\varepsilon)}$ is also C^{k-1}-smooth on ε. We will achieve this in several steps.

Assume the functions F, g in equation (1.1) are $C^k(k\geq 2)$. For $p\in\mathbb{N}$, define $\mathcal{X}_0^p$ as the space of the functions $\phi:\mathbb{R}\to\mathbb{R}^n$ such that $\phi\in C_0$ and ϕ is C^p-smooth.

Claim 1 From the definition of P^0 in (2.2), it is clear that P^0: $C_0\to C_0$ is linear bounded and that $P_0(\mathcal{X}_0^p)\subset\mathcal{X}_0^p$, for $1\leq p\leq k-1$.

Claim 2 From the definition of L in (2.11), L: $C_0\to C_0$ is linear bounded and $L(\mathcal{X}_0^p)\subset\mathcal{X}_0^p$, for $1\leq p\leq k-1$.

Claim 3 From the definition of $\mathcal{H}$ in (2.10) and (2.5), we have $\mathcal{H}(\cdot,\mathcal{X}_0^{p-1},\varepsilon)\subset\mathcal{X}_0^p$ for $\varepsilon>0$, $p=1,\dots,k-1$, where $\mathcal{X}_0^0=C_0$.

Claim 4 $\mathcal{N}(L)\subset\mathcal{X}_0^{k-1}$.

In fact, from theorem 3.4 we have $\mathcal{N}(L)=\mathcal{N}(T)=\{\phi\in C^1:\dot{\phi}(t)=P^0\phi(t), t\in\mathbb{R}\}$. From claim 1, by induction we conclude that $\mathcal{N}(T)\subset\mathcal{X}_0^{k-1}$.

Claim 5 For each $(\xi,\varepsilon)\in(\mathcal{N}(L)\cap\overline{B(\sigma)})\times(0,\varepsilon^*)$, the fixed point $\phi^*:=\phi_{(\xi,\varepsilon)}\in\mathcal{X}_0^1$.

To prove this claim, we fix $(\xi,\varepsilon)\in(\mathcal{N}(L)\cap\overline{B(\sigma)})\times(0,\varepsilon^*)$, and define $\psi^*=\phi^*+\xi$. From $\phi^*=\mathcal{F}(\phi^*,\xi,\varepsilon)$, we obtain

$$L\psi^*=\mathcal{H}(\cdot,\psi^*,\varepsilon),$$

or equivalently,

$$\psi^*(s)=\mathcal{H}(s,\psi^*,\varepsilon)+\int_{-\infty}^s \mathrm{e}^{-(s-t)}[\psi^*(t)+P^0\psi^*(t)]\mathrm{d}t,\quad s\in\mathbb{R}.$$

Hence $\psi^*\in\mathcal{X}_0^1$. From claim 4, we conclude that $\phi^*\in\mathcal{X}_0^1$.

Claim 6 The fixed point $\phi^*=\phi_{(\xi,\varepsilon)}$ is C^1-smooth with respect to $>\varepsilon$.

Consider $\mathcal{F}$ restricted to $\phi\in X\cap\overline{B(\sigma)}\cap\mathcal{X}_0^1$; more precisely, using claims 2 and 3 we consider

$$\begin{gathered}\mathcal{F}^1:(X\cap\overline{B(\sigma)}\cap\mathcal{X}_0^1)\times(\mathcal{N}(L)\cap\overline{B(\sigma)})\times(0,\varepsilon^*)\to X\cap\overline{B(\sigma)}\cap\mathcal{X}_0^1,\\ \mathcal{F}^1(\phi,\xi,\varepsilon)=\mathcal{F}(\phi,\xi,\varepsilon).\end{gathered}$$

Notice that $\mathcal{F}^1$ is a uniform contraction of $\phi \in X \cap \overline{B(\sigma)} \cap \mathcal{X}_0$ for the norm $\|\cdot\|_{C_0}$, and that $\mathcal{F}^1$ is a C^1-mapping on (ϕ,ξ,ε). In fact, for $\psi(s)=\phi(s)+\xi(s)$ C^1-smooth on s, from the definition of $\mathcal{H}$ and $\mathcal{G}$ in equations (2.10) and (2.5), we conclude that $\frac{\partial \mathcal{H}}{\partial \varepsilon}(s,\psi,\varepsilon)$ exists and is continuous. In claim 5, we have proven that there exists a fixed point $\phi^* = \phi_{(\xi,\varepsilon)}$ of $\mathcal{F}^1$. By repeating the arguments used to prove the differentiability of the fixed point in the uniform contraction principle (see e.g. pp. 25–26 of Chow & Hale (1982)), we conclude that $\phi_{(\xi,\varepsilon)}$ is a C^1-smooth mapping on (ξ, ε).

Claim 7 The fixed point $\phi^* = \phi_{(\xi,\varepsilon)}$ is C^{k-1}-smooth with respect to ε.

As in claim 5, by induction we prove that $\phi_{(\xi,\varepsilon)}(\cdot) \in \mathcal{X}_0^p$, $p=2,\ldots,k-1$. By using claims 2 and 3, we consider now

$$\mathcal{F}^p : (X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^p) \times (\mathcal{N}(L) \cap \overline{B(\sigma)}) \times (0,\varepsilon^*) \to X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^p,$$
$$\mathcal{F}^p(\phi,\xi,\varepsilon) = \mathcal{F}(\phi,\xi,\varepsilon),\, p = 2,\ldots,k-1.$$

As in the proof of the uniform contraction principle, by an inductive argument we conclude that $\phi^* = \phi_{(\xi,\varepsilon)}$ is C^{k-1}-smooth with respect to ε. ■

Remark 5.2. In some applications, the diffusion process does not apply to all state variables and thus the model is of a mixed type such as

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + F\left(W, \displaystyle\int_{\Omega_r} \mathrm{d}\alpha(\theta,y) f(W(x+y,t+\theta))\right), \\ \dfrac{\partial v}{\partial t} = G\left(W, \displaystyle\int_{\Omega_r} \mathrm{d}\beta(\theta,y) g(W(x+y,t+\theta))\right), \end{cases} \tag{5.6}$$

with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$, and $W=(u,v)^{\mathrm{T}}$. This system can be regarded as a special case of equation (1.1) if we allow some of the diffusion coefficients d_i to be zero. We remark that under the same assumptions (H1)–(H4) on the nonlinearities F, f, G and g, theorem 1.1 remains true for system (5.6). In fact, if for some index i, the diffusion coefficient d_i is zero in equation (1.1), then we have $\alpha_i^\varepsilon = -1$ and $\beta_i^\varepsilon = \infty$. Consequently, the nonlinear function $\mathcal{H}_i$ (see equation (2.10)) in the equation for the variable w_i will be reduced to

$$\mathcal{H}_i(s,w,\varepsilon)(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-t)} \mathcal{G}_i(\varepsilon,t,w)\,\mathrm{d}t.$$

It is apparent that all results presented so far remain valid without any change.

6. Applications to a non-local delayed RD-system with non-monotone birth functions

Our main result, theorem 1.1, relates the existence of travelling wave fronts for the reaction–diffusion equation (1.1) with delay and non-local interaction to the existence of a connecting orbit between two hyperbolic equilibria of the associated ordinary delay differential equation (1.2). This enables us to apply some existing results for invariant curves of order-preserving semiflows generated by ordinary delay differential equations to derive systematically sharp sufficient conditions for the existence of travelling wave fronts of delayed reaction–diffusion equations that, in turn, include most of the existing results in the literature as special cases. In this section, we illustrate this by a recently derived non-local delayed reaction–diffusion equation for the population growth of a single species when the delayed birth function is not monotone in the considered range.

We start with a short review of relevant results for the existence of heteroclinic orbits in monotone dynamical systems. Let X be an ordered Banach space with a closed cone K. For u, $v \in X$ we write $u \geq v$ if $u-v \in K$, and $u > v$ if $u \geq v$ but $u \neq v$.

Lemma 6.1. *Let U be a subset of X and $\Phi:[0,\infty) \times U \to U$ be a semiflow such that*

(i) *Φ is strictly order-preserving, i.e. $\Phi(t,u) > \Phi(t,v)$ for $t \geq 0$ and for all u, $v \in U$ with $u > v$;*
(ii) *for some $t_0 > 0$, $\Phi(t_0,\cdot)$: $U \to U$ is set-condensing with respect to a measure of non-compactness.*

Suppose $u_2 > u_1$ are two equilibria of Φ and assume $[u_1,u_2] := \{u : u_2 \geq u \geq u_1\}$ contains no other equilibria. Then there exists a full orbit connecting u_1 and u_2. Namely, there is a continuous function $\phi : \mathbb{R} \to U$ such that $\Phi(t,\phi(s)) = \phi(t+s)$ for all $t \geq 0$ and all $s \in \mathbb{R}$, and either (a) $\phi(t) \to u_1$ as $t \to \infty$ and $\phi(t) \to u_2$ as $t \to -\infty$ or (b) $\phi(t) \to u_1$ as $t \to -\infty$ and $\phi(t) \to u_2$ as $t \to \infty$.

In applications, one can easily distinguish the above cases (a) and (b) by looking at the stability of the equilibria. For detailed discussions and related references, see Wu *et al.* (1995), Matano (1984), Polacik (1990), Dance & Hess (1991) and Smith (1986, 1995).

Returning to equations (1.1) and (1.2), we use the standard phase space for equation (1.2). In this section, C will denote the Banach space $C = C([-r,0];\mathbb{R}^n)$ of continuous $\mathbb{R}^n$-valued functions on $[-r,0]$ with the usual supremum norm. Under the smoothness condition on F, system (1.2) generates a (local) semiflow on C given by

$$\Phi(t,\phi) = u(\phi)(t+\cdot), \quad t \geq 0, \phi \in C,$$

for all those t for which a unique solution $u(\phi)$ of equation (1.2) with $u(\phi)(\theta) = \phi(\theta)$ for $\theta \in [-r,0]$ is defined. Let B be an $n \times n$ quasipositive matrix, that is, $B + \lambda I \geq 0$ for all sufficiently large λ. Here and in what follows, we write $A \geq B$ for $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ if and only if $a_{ij} \geq b_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Define

$$K_B = \{\phi \in C : \phi \geq 0, \phi(t) \geq \mathrm{e}^{B(t-s)}\phi(s), -r \leq s \leq t \leq 0\}.$$

Then K_B is a closed cone in C and this induces a partial order on C, denoted by $\geq_B$. Namely, $\phi \geq_B \psi$ if and only if $\phi - \psi \in K_B$.

We will need the following conditions.

(O_B) $\hat{E}_2 \geq_B \hat{E}_1$, here $\hat{E}_i$ is the constant mapping on $[-r,0]$ with the value E_i, $i=1,2$.

(M_B) Whenever ϕ, $\psi \in C$ with $\phi \geq_B \psi$, then

$$F\left(\phi(0), \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_\Omega g(\phi(\theta))\right) - F\left(\psi(0), \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_\Omega g(\psi(\theta))\right) \geq B[\phi(0)-\psi(0)],$$

Under the above assumptions, Smith & Thieme (1991) proved the following.

Lemma 6.2. *Assume that there exists an $n\times n$ quasipositive matrix B such that (O_B) and (M_B) are satisfied. Then*

(i) $[E_1, E_2]_B := \{\phi \in C : \hat{E}_2 \geq B\phi \geq {}_B \hat{E}_1\}$ *is positively invariant for the semiflow* Φ*;*
(ii) *the semiflow* Φ: $[0,\infty)\times[E_1, E_2]_B \to [E_1, E_2]_B$ *is strictly monotone with respect to* $\geq_B$ *in the sense that if* $\phi, \psi \in [E_1, E_2]_B$ *with* $\phi >_B \psi$*, then* $\Phi(t,\psi) >_B \Phi(t,\psi)$ *for all* $t \geq 0$.

In Smith & Thieme (1991), it was also shown that (M_B) holds if for all u, $v \in \mathbb{R}^n$ with $\hat{u}$, $\hat{v} \in [E_1, E_2]_B$ the following is satisfied:

$$\begin{cases} F_u\left(u, \int_{-r}^0 \mathrm{d}\eta(\theta)\mu_\Omega g(v)\right) \geq B, \\ \left[F_u\left(u, \int_{-r}^0 \mathrm{d}\eta(\theta)\mu_\Omega g(v)\right) - B\right]\mathrm{e}^{Br} + F_v\left(u, \int_{-r}^0 \mathrm{d}\eta(\theta)\mu_\Omega g(v)\right) g'(v) \geq 0. \end{cases}$$

In the case where $n=1$, it was shown in Smith & Thieme (1990) that (M_B) holds for some $B<0$ if

$$(S_B)\ L_2 < 0, \quad L_1 + L_2 < 0, \quad r|L_2| < 1, \quad rL_1 - \ln(r|L_2|) > 1,$$

where

$$L_1 = \inf_{E_1 \leq u, v \leq E_2} F_u\left(u, \int_{-r}^0 \mathrm{d}\eta(\theta)\mu_\Omega g(v)\right)$$

and

$$L_2 = \inf_{E_1 \leq u, v \leq E_2} F_v\left(u, \int_{-r}^0 \mathrm{d}\eta(\theta)\mu_\Omega g(v)\right) g'(v).$$

Note also that $[E_1, E_2]_B$ is a bounded set in C and that $\Phi(t,\cdot)$: $C \to C$ is compact for $t>r$. Therefore, for $t_0 > r$, the mapping $\Phi(t_0,\cdot)$: $[E_1, E_2]_B \to [E_1, E_2]_B$ is compact, and hence is set-condensing. This observation allows us to derive from lemmas 6.1 and 6.2 and theorem 1.1 the following general result.

Theorem 6.3. *Assume that*

(i) *(H1), (H2) and (H4) are satisfied;*
(ii) *there exists an $n\times n$ quasipositive matrix B such that (O_B) and (M_B) are satisfied;*
(iii) *there exist no other equilibria in* $[E_1, E_2]_B$.

Then the conclusions of theorem 1.1 hold.

We now apply theorem 6.3 to a reaction–diffusion equation with time delay and non-local effect, recently derived by So *et al.* (2001), for the total mature population of a single species population with two age classes and a fixed maturation period living in a spatially unbounded environment. In So *et al.* (2001), the existence of a travelling wave front was established for the special case when the birth function is the one which appears in the well-known Nicholson's blowflies equation and when the birth function remains monotonically increasing in the interval between the trivial equilibrium and the positive equilibrium representing the maximal capacity of the environment. However, as will be shown below, in a wide range of parameter values, this monotonicity condition is not satisfied and the method developed there cannot be applied. Theorem 6.3 enables us to address the existence of travelling waves when this monotonicity is not satisfied.

Let $u(t, a, x)$ denote the density of the population of the species under consideration at time $t \geq 0$, age $a \geq 0$ and location $x \in R$. It is natural to assume

$$|u(t, a, \pm\infty)| < \infty, \quad \text{for} \quad t \geq 0, \quad a \geq 0. \tag{6.1}$$

A standard argument on population dynamics with age structure and diffusion (cf. Metz & Diekmann 1986) gives

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u, \tag{6.2}$$

where $D(a)$ and $d(a)$ are the diffusion rate and death rate respectively, at age a. Let $r \geq 0$ be the maturation time for the species. Then the total matured population at time t and location x is given by

$$w(t, x) = \int_r^\infty u(t, a, x)\mathrm{d}a,$$

and using equation (6.2) and the biologically realistic assumption

$$u(t, \infty, x) = 0, \tag{6.3}$$

we can get

$$\frac{\partial w}{\partial t} = u(t, r, x) + \int_r^\infty \left[D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u\right]\mathrm{d}a.$$

We assume that the diffusion and death rates for the mature population are age independent, that is, $D(a) = D_m$ and $d(a) = d_m$ for $a \in [r, \infty)$, where D_m and d_m are constants. Furthermore, since only the mature can reproduce, we have

$$u(t, 0, x) = b(w(t, x)), \tag{6.4}$$

where $b(\cdot)$ is the birth function. Then

$$\frac{\partial w}{\partial t} = u(t, r, x) + D_m\frac{\partial^2 w}{\partial x^2} - d_m w. \tag{6.5}$$

Denote by D_I and d_I the diffusion and death rates of the immature, respectively i.e. $D(a) = D_I(a)$ and $d(a) = d_I(a)$ for $a \in [0, r]$. In So *et al.* (2001), it was shown that, provided

$$\alpha := \int_0^r D_I(a)\,\mathrm{d}a > 0, \tag{6.6}$$

the term $u(t, r, x)$ can be explicitly written, using a combination of integration along characteristics, method of separation of variables and Fourier transformation, as

$$u(t, r, x) = \frac{\mathrm{e}^{-\int_0^r d_I(a)\mathrm{d}a}}{\sqrt{4\pi\alpha}}\int_{-\infty}^\infty b(w(t - r, y))\mathrm{e}^{\frac{-(x-y)^2}{4\alpha}}\,\mathrm{d}y. \tag{6.7}$$

Hence $w(t, x)$ satisfies

$$\frac{\partial w}{\partial t} = D_m\frac{\partial^2 w}{\partial x^2} - d_m w + \frac{\mathrm{e}^{-\int_0^r d_I(\theta)\,\mathrm{d}\theta}}{\sqrt{4\pi\alpha}}\int_{-\infty}^\infty b(w(t - r, y))\mathrm{e}^{\frac{-(x-y)^2}{4\alpha}}\mathrm{d}y, \quad \text{for } t > r. \tag{6.8}$$

Let

$$\varepsilon = \mathrm{e}^{-\int_0^r d_I(a)\mathrm{d}a} \quad \text{and} \quad f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{\frac{-x^2}{4\alpha}}.$$

Then, $0<\varepsilon\leq 1$ and equation (6.8) becomes

$$\frac{\partial w}{\partial t}=D_m\frac{\partial^2 w}{\partial x^2}-d_m w+\varepsilon\int_{-\infty}^{\infty} b(w(t-r,y))f_\alpha(x-y)\mathrm{d}y. \tag{6.9}$$

Equation (6.9) is a reaction–diffusion equation with time delays and non-local effects, with ε reflecting the impact of the death rate for immature and α representing the effect of the dispersal rate of the immature on the matured population.

When $\alpha\to 0$, that is, as the immature become immobile, equation (6.9) reduces to

$$\frac{\partial w}{\partial t}=D_m\frac{\partial^2 w}{\partial x^2}-d_m w+\varepsilon b(w(t-r,x)), \tag{6.10}$$

and the non-local effect disappears. If we further let $\varepsilon\to 1$, that is, all immatures live to maturity, then equation (6.10) becomes

$$\frac{\partial w}{\partial t}=D_m\frac{\partial^2 w}{\partial x^2}-d_m w+b(w(t-r,x)), \tag{6.11}$$

which has been widely studied for different choices of the birth function $b(\cdot)$. In particular, So *et al.* (2001) considered a particular birth function for equation (6.9) given by $b(w)=pw\mathrm{e}^{-aw}$. This function has been used in the well-studied Nicholson's blowflies equation (see Gurney *et al.* 1980). In the discrete case, it is commonly known as the Ricker's model (cf. Ricker 1954). With this birth function, equation (6.9) becomes

$$\frac{\partial w}{\partial t}=D_m\frac{\partial^2 w}{\partial x^2}-d_m w+\varepsilon p\int_{-\infty}^{\infty} w(t-r,y)\mathrm{e}^{-aw(t-r,y)}f_\alpha(x-y)\,\mathrm{d}y. \tag{6.12}$$

For the case when $D_I(\theta)\equiv 0$ and $d_I(\theta)\equiv 0$, i.e. $\alpha=0$, $\varepsilon=1$, equation (6.12) reduces to

$$\frac{\partial w}{\partial t}=D_m\frac{\partial^2 w}{\partial x^2}-d_m w+pw(t-r,y)\mathrm{e}^{-aw(t-r,x)}, \tag{6.13}$$

which was studied in So & Zou (2001), where the monotone iteration scheme and the method of upper–lower solutions in Wu & Zou (1997, 2001) were used to show that a travelling wave front exists when $1<\varepsilon p/d_m\leq e$. This result was extended to equation (6.12). More precisely, So *et al.* (2001) proved the following.

Theorem 6.4. *If* $1<\varepsilon p/d_m\leq e$, *then there exists a* $c^*>0$ *such that for every* $c>c^*$, *equation* (6.12) *has a travelling wave front solution, which connects the trivial equilibrium* $w_1=0$ *to the positive equilibrium* $w_2=\frac{1}{a}\ln\frac{\varepsilon p}{d_m}$.

Unfortunately, in the case when $\varepsilon p/d_m>e$, the method developed in So *et al.* (2001) cannot be used as the involved iteration scheme is no longer monotone. It is suspected that the method developed in Wu & Zou (2001) for travelling waves of reaction–diffusion equations without local effects and based on a non-standard exponential ordering could be utilized to this case but the construction of a pair of upper-lower solutions seems to be a highly nontrivial task. We are now in the position to confirm this existence by using theorem 6.3.

We first notice that the associated ordinary differential equation of (6.12) is

$$\frac{\mathrm{d}w}{\mathrm{d}t}=-d_m w(t)+\varepsilon b(w(t-r)), \tag{6.14}$$

with $b(w) = pw\mathrm{e}^{-aw}$. If $\varepsilon p/d_m > 1$, then equation (6.14) has exactly two nonnegative equilibria:

$$E_1 = 0, \quad E_2 = \frac{1}{a}\ln\frac{\varepsilon p}{d_m}.$$

The corresponding characteristic equations are

$$\Lambda_1(\lambda) := \lambda + d_m - \varepsilon p\,\mathrm{e}^{-\lambda r} = 0$$

and

$$\Lambda_2(\lambda) := \lambda + d_m - \varepsilon b'(E_2)\mathrm{e}^{-\lambda r} = 0,$$

where

$$b'(E_2) = \frac{d_m}{\varepsilon}\left(1 - \ln\frac{\varepsilon p}{d_m}\right).$$

As $\varepsilon p > d_m$, we can easily show that the unstable manifold for E_1 is at least one-dimensional. Furthermore, E_1 is hyperbolic for $r \neq r_n$, $n \in \mathbb{N}_0$, where

$$r_n = \frac{2\pi - \arccos\left(\dfrac{d_m}{\varepsilon p}\right)}{\sqrt{\varepsilon^2 p^2 - d_m^2}} + 2n\pi.$$

We now claim that if $e < \varepsilon p/bd_m \leq e^2$, then E_2 is asymptotically stable. In fact, in this case,

$$|\varepsilon b'(E_2)| = |d_m\left(1 - \ln\frac{\varepsilon p}{d_m}\right)| \leq d_m,$$

and hence all zeros of $\Lambda_2(\lambda)$ have negative real parts.

In the case where $\varepsilon p/d_m > e^2$, the asymptotical stability of E_2 holds only when the delay r is sufficiently small. Namely, in $\Lambda_2(\lambda) = 0$, we let $\lambda = \mathrm{i}\omega$ to get

$$\mathrm{i}\omega = -d_m + d_m\left(1 - \ln\frac{\varepsilon p}{d_m}\right)[\cos(\omega r) - \mathrm{i}\sin(\omega r)], \tag{6.15}$$

from which we can find the minimal $\hat{r} > 0$ so that (6.15) has a solution $\omega > 0$. This is given by

$$\hat{r} = \frac{\pi - \arccos\dfrac{1}{\ln\frac{\varepsilon p}{d_m} - 1}}{d_m\sqrt{\left(\ln\frac{\varepsilon p}{d_m} - 1\right)^2 - 1}}. \tag{6.16}$$

It then follows that if $\varepsilon p/d_m > e^2$ and $0 \leq r < \hat{r}$ then E_2 is asymptotically stable.

We now choose $B < 0$ so that (S_B) holds. Recall that

$$b'(w) = p\,\mathrm{e}^{-aw}(1 - aw) \text{ and } b''(w) = pa\,\mathrm{e}^{-aw}(aw - 2).$$

Therefore, $b'(w)$ is decreasing on $[0,\frac{2}{a})$ and increasing on $[\frac{2}{a},\infty)$. Consequently, on $[E_1,E_2]$, we have

$$b'(w) \geq b'_{\min} = \begin{cases} b'(E_2) = \dfrac{d_m}{\varepsilon}\left(1 - \ln\dfrac{\varepsilon p}{d_m}\right), & E_2 < \dfrac{2}{a}, \\ b'\left(\dfrac{2}{a}\right) = -\dfrac{p}{e^2}, & E_2 \geq \dfrac{2}{a}. \end{cases} \tag{6.17}$$

For equation (1.2), we have

$$F(u,v) = -d_m u + \varepsilon v, g(w) = b(w) = pw\mathrm{e}^{-aw}, \quad \int_{-r}^{0} \mathrm{d}\eta(\theta) = 1, \quad \mu_\Omega = 1.$$

Therefore, for L_1, L_2 as in (S_B)

$$L_1 = \inf_{0 \leq u,v \leq E_2} F_u(u, b(v)) = -d_m < 0$$

and

$$L_2 = \inf_{0 \leq u,v \leq E_2} F_v(u, b(v))b'(v) = \varepsilon b'_{\min} < 0.$$

Therefore, (S_B) (and hence (M_B)) holds if

$$r\varepsilon|b'_{\min}| < 1 \tag{6.18}$$

and

$$\frac{\mathrm{e}^{-rd_m}}{r\varepsilon|b'_{\min}|} > e. \tag{6.19}$$

The latter is equivalent to

$$r\mathrm{e}^{rd_m} e\varepsilon|b'_{\min}| < 1. \tag{6.20}$$

Clearly, if equation (6.20) holds so does equation (6.18). Therefore, we conclude that (M_B) holds if $0<r<\hat{r}$, where $\hat{r}$ is the unique solution of

$$r\mathrm{e}^{rd_m} e\varepsilon|b'_{\min}| = 1 \tag{6.21}$$

As $B<0$, we also have that (O_B) holds. Therefore, from theorem 6.3, we have

Theorem 6.5. *If $\varepsilon p/d_m > e$, then there exist $r^* > 0$ and $c^* > 0$ such that if $r \in [0,r^*)$ then for every $c > c^*$, equation (6.12) has a travelling wave, which connects the trivial equilibrium $w_1 = 0$ to the positive equilibrium $w_2 = \frac{1}{a}\ln(\varepsilon p/d_m)$, where*

$$r^* = \begin{cases} \min\{\hat{r}, \tilde{r}, r_0\}, & \dfrac{\varepsilon p}{d_m} > e^2, \\ \min\{\tilde{r}, r_0\}, & \dfrac{\varepsilon p}{d_m} \leq e^2. \end{cases}$$

As a final remark, we note that in order to apply theorem 6.3 for specific systems (1.1), all we need to do is to choose the quasipositive matrix B and to verify the hyperbolicity of the two equilibria. It turns out that much of the known results can be obtained as a special case of theorem 6.3. For example, consider the following Fisher–KPP equation with delay

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t-r)]. \tag{6.22}$$

Using theorem 6.3, we can get

Corollary 6.6. *There exists* $c^*>0$ *such that if* $0\le r\le e^{-1}$ *then for any* $c>c^*$, *equation* (6.22) *has a travelling wave front with wave speed* c.

To prove the corollary, we note that the corresponding ordinary delay differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = u(t)[1-u(t-r)] := F(u,u(t-r)], \tag{6.23}$$

for which $E_1=0$ and $E_2=1$. When u, $v\in[0,1]$ we have $F_u(u,v)=1-v\ge 0$ and $F_v(u,v)=-u$. Therefore,

$$\begin{aligned}[F_u(u,v)-B]e^{Br}+F_v(u,v) &= [1-v-B]e^{B}-u \\ &= (1-v)e^{Br}-Be^{Br}-u\ge -Be^{r}-1\ge 0.\end{aligned}$$

as long as $f(B):=-Be^{Br}\ge 1$. This is possible if $r\le e^{-1}$. In this case, we can choose $B=-r^{-1}$ so that $f(B)=r^{-1}e^{-1}=1$. This verifies (M_B). (O_B) follows from $1-e^{B(t-s)}\ge 0$ if $-r\le s\le t\le 0$. Note that $\Lambda_1(\lambda)=\lambda-1$ and $\Lambda_2(\lambda)=\lambda+e^{-\lambda r}$. Thus, E_1 is hyperbolic and its unstable manifold is one-dimensional, and all eigenvalues corresponding to E_2 have negative real parts if $r\le e^{-1}<\pi/2$. This proves corollary 6.6.

In Wu & Zou (2001), it was shown that for any $c>2$, there exists $r^*(c)>0$ such that if $0\le r\le r^*(c)$, then equation (6.22) has a travelling wave front with wave speed c. Their argument was based on an iterative scheme, coupled with the construction of a pair of upper and lower solutions. Note that our claim above gives an explicit form for r^*.

There is another way to incorporate the time delay to a logistic equation, such as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t-r)[1-u(x,t)], \tag{6.24}$$

which was also derived by Kobayshi (1977) from a branching process. The existence of travelling wave of equation (6.24) can be obtained by using the general theory of Schaaf (1987) or the general monotone iteration technique developed in Wu & Zou (1997, 2001). It is interesting to note that this existence result becomes a trivial application of our Theorem 6.3 by choosing $B=-1$, since the corresponding $F(u,v)=v(1-u)$ satisfies $F_u(u,v)=-v\ge -1$ and $F_v(u,v)=1-u\ge 0$ for all u, $v\in[0,1]$. It is also clear that $E_2=1$ is asymptotically stable, and that $E_1=0$ is hyperbolic for $r\ne r_n$, where $r_n=(2n-1/2)\pi$, $n\in\mathbb{N}$, and its unstable manifold is at least one-dimensional.

Remark 6.7. We consider the nonlinear reaction term F to be of the form given in equation (1.1) in order to cover sufficiently large classes of equations and, at the same time, to keep the notations relatively in a minimum of complexity. A straightforward extension of the reaction term that has its application can be of the form

$$F(u(x,t), \int_{-r}^{0}\int_{\Omega} \mathrm{d}\eta(\theta)\mathrm{d}\mu(y)K(\theta,y)g(u(x+y,t+\theta))),$$

where K is a continuous and bounded function from $[-r,0]\times\Omega$ to $\mathbb{R}^{n\times n}$. In this case, the corresponding reaction equation (1.2) becomes

$$\dot{u}(t) = F(u(t), \int_{-r}^{0} \mathrm{d}\eta(\theta)\mu_{\Omega}(\theta)g(u(t+\theta))),$$

with $\mu_{\Omega}(\theta)=\int_{\Omega}\mathrm{d}\mu(y)K(\theta,y)$. One can see that all arguments developed in the paper are still valid and theorem 1.1 remains true for this more general form.

Remark 6.8. Our focus in this paper is on the existence of travelling waves for the delayed reaction–diffusion equation (1.1) in the neighbourhood of a heteroclinic orbit of the corresponding ordinary delay differential equation (1.2). Whether some qualitative properties of the heteroclinic orbits such as monotonicity can be inherited by the travelling waves remains to be an interesting problem. We note, however, that if equation (1.2) is a monotone system that has a monotone heteroclinic solution u^* connecting E_1 and E_2, then we are able to use a travelling wave solution $V(t)$ of equation (1.4) near u^* to construct a monotone increasing lower and a monotone increasing upper solution for an integral equation equivalent to equation (1.4). Thus a further monotone iteration argument (see Wu & Zou 2001, 1997) can be applied to obtain a monotone travelling wave.

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