A non-local PDE model for population dynamics with state-selective delay: Local theory and global attractors[☆]

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Dedicated to Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

We propose a non-local PDE model for the evolution of a single species population that involves delayed feedback, where the delay such as the maturation time in the delayed birth rate, is selective and the selection depends on the status of the system. This delay selection, in contrast with the usual state-dependent delay widely used in ordinary delay differential equation, ensures the Lipschitz continuity of the nonlinear functional in the classical phase space. We also develop the local theory, and the existence and upper semi-continuity of the global attractor with respect to parameters.
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1. Introduction

By a delay differential equation (DDE), we mean an evolutionary system in which the (current) rate of change of the state depends on the historical status of the system. More precisely, if we let C_∞ be the

Banach space of bounded continuous functions $\phi : (-\infty, 0] \rightarrow \mathbb{R}^n$ with the norm $\|\phi\| = \sup_{s \leqslant 0} |\phi(s)|$ and if $F : C_\infty \rightarrow \mathbb{R}^n$ is a given mapping, then $\dot{x}(t) = F(x_t)$ is a DDE, where $x_t \in C_\infty$ is given by $x_t(s) = x(t + s)$ for $s \in (-\infty, 0]$. The simplest example is $\dot{x}(t) = f(x(t), x(t - r(x_t)))$, where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $r : C_\infty \rightarrow [0, \infty)$ is a given functional. In the case where r is non-constant, this equation is called a state-dependent DDE (SD-DDE) because the time retardation used to evaluate $\dot{x}$ depends on the state of the system as well.

DDEs arise in many applications. In situations where r is a constant, the solutions define a semiflow in $C_r = C([-r, 0]; \mathbb{R}^n)$. Therefore, similar to many evolutionary partial differential equations (PDEs), a DDE gives rise to a class of semiflows in a functional space. As such, DDEs have been one of the major sources of inspiration for the rapid development of infinite dimensional dynamical systems and nonlinear analysis. The basic framework and theory of DDEs with constant delays has already been developed, see the monographs of Diekmann et al. [16], Hale [27], Hale and Lunel [28], and Krisztin et al. [34].

A close look at most evolutionary systems shows that delays are often not constant, but rather depend on the system state and thus SD-DDEs are the only appropriate models. This was noted in the work of Driver [17] for a two-body problem (electron–proton interaction) of classical electrodynamics, where the motion of each particle is influenced by the electromagnetic fields of another. Driver noted that due to the finite speed of the propagation of these fields (the speed of light), the model involves time delays which are dependent on the electron–proton separation. Despite some earlier activities (see [26]), there has been no systematic investigation of SD-DDEs. This is perhaps due to the difficulty induced by the state-dependent delay $r = r(\phi)$: much of the geometric theory requires the estimation of the difference of two different solutions, and this requires the evaluation of $|F(\phi) - F(\psi)|$ that requires we consider not only how close ϕ and ψ are but also the effect of evaluating ϕ and ψ using slightly different lags. Consequently, it is natural to use $C_{\infty, L} = \{\phi$ is absolutely continuous and $\mathrm{esssup}_{(-\infty, 0]} |\dot{\phi}| \leqslant L\}$ as the phase space, but then the resulting semiflow is continuous only with respect to the (original) topology of C_∞.

There has been some recent interest in using SD-DDEs for modeling dynamical systems arising from remote control of objects (e.g. satellites or robot arms) [8,51], transmission dynamics of infectious diseases [20,29,35,36], hematological disorders [5,38], the dynamics of price adjustment in a single commodity market [5], neural nets [9,10,18,19,30,50].

In the context of population dynamics, delays arise frequently as the maturation time and this time is a function of the total population. A phenomenological model was formulated in [1] and Arino et al. [2,3] developed a model for the growth of a fish population in the larval stage to address the density dependence effects. Also, de Roos and Persson [15] investigated the consequence of size-dependent competition among the individuals by using SD-DDEs.

There has been some success in the fundamental and geometric theory of SD-DDEs. For a prototype scalar DDE with $r = r(x(t))$, Mallet-Paret and Nussbaum [39,40] gave a detailed discussion of the limiting profiles of periodic solutions and their boundary layer phenomenon, Krisztin and Arino [33] described a two-dimensional unstable manifold for a scalar DDE with SD-delayed negative feedback, Cooke and Huang [14] derived a linearization principle, Mallet-Paret et al. [41], Walther [51] and Arino et al. [3], obtained the existence of periodic solutions, and Krisztin [32] and Walther [52] developed the general theory of invariant manifolds.

Progress has been made for analogous partial differential equations with constant delay, arising from population biology/ecology due to the interaction between the spatial dispersal and the retarded reaction/feedback, in terms of both modeling and qualitative theory, see for example [6,13,45,47,49,53] and

references therein. Recent development leads to a new class of reaction–diffusion equations with the nonlinear reaction terms involving non-local (in space) and delayed (in time) nonlinearities, we refer this to the earlier work of Smith and Thieme [44], Britton [7], and Gourley and Britton [22], Gourley and Kuang [23], Gourley and Ruan [24], Gourley and Bartuccelli [21] and a recent survey of Gourley et al. [25] for relevant references.

To the best of our knowledge, nothing has been done in terms of partial functional differential equations with SD delayed terms, although these types of equations should arise very naturally from the related ordinary DDEs with SD delays and some non-local terms should be anticipated. In this paper, we introduce a new class of partial functional differential equations where the nonlinear term is non-local and the delayed term depends on the status of the system under consideration. The equation represents a philosophical formulation of what we believe a good approximation of a nonlinear evolution process involving spatial dispersal and time delay that depends on the system's status, and we hope this will inspire further discussions of a more realistic model that can be handled qualitatively.

The rest of this paper is organized as follows. In Section 2, we introduce the model and briefly discuss the advantage of our formulation of the SD term and its properties. The existence of weak solutions is discussed in Section 3, and the global attractor is discussed in Section 4 (existence of the attractor) and Section 5 (dependence of the attractor on the parameter).

2. Formulation of the model

Let us start with the following non-local PDE with *SD selective delay* (a more general equation will be introduced later in (6))

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = \int_{-r}^{0} \left\{ \int_{\Omega} b(u(t + \theta, y))f(x - y)\,\mathrm{d}y \right\} \xi(\theta, \|u(t)\|)\,\mathrm{d}\theta$$
$$\equiv (F(u_t))(x), \ x \in \Omega, \tag{1}$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, so $A : D(A) \to L^2(\Omega)$ generates an analytic semigroup, Ω is a smooth bounded domain in $\mathbb{R}^n$, $f : \Omega - \Omega \to \mathbb{R}$ is a bounded function to be specified later, $b : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz map and satisfies $|b(w)| \leqslant C_1|w| + C_2$ with $C_1 \geqslant 0$ and $C_2 \geqslant 0$, d is a positive constant. We notice that the form of Eq. (1) without the state-selective effect i.e. $\xi(\cdot, \cdot) \equiv$ const, has been used to discuss population models (for more details and references see the end of the article).

Also, in the above equation, the function $u(\cdot, \cdot) : [-r, +\infty) \times \Omega \to \mathbb{R}$, such that for any t the function $u(t) \equiv u(t, \cdot) \in L^2(\Omega)$, $\|\cdot\|$ is the $L^2(\Omega)$-norm. The function $\xi : [-r, 0] \times \mathbb{R} \to \mathbb{R}$ represents the *state-selective delay*. To illustrate this point, we first consider the case where $\xi(\theta, s) = \xi(\theta) = \mathrm{e}^{-\beta(c_1-\theta)^2}$ is independent of s, see Fig. 1, and we get the term similar to $\int_{-r}^{0} u(t + \theta)\xi(\theta, \|u(t)\|)\,\mathrm{d}\theta \equiv \int_{-r}^{0} u(t + \theta)\mathrm{e}^{-\beta(c_1-\theta)^2}\,\mathrm{d}\theta$ which is "a distributed delay analogue" to the term $u(t + c_1) = u(t - h)$ with the discrete delay $h = -c_1 > 0$.

We now consider the case where

$$\xi(\theta, s) = \mathrm{e}^{-\beta(g(s)-\theta)^2}, \quad \theta \in [-r, 0], \ s \in \mathbb{R}, \tag{2}$$

where $g(s)$ gives the coordinate of the maximum of ξ. Thus the system selects the maximal historical impact on the current change rate according to the system's current state.
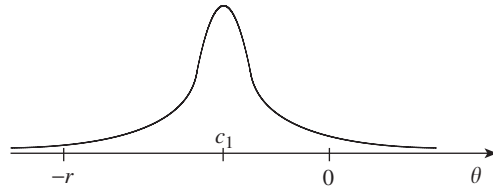
Fig. 1. The graph of a ξ that is independent of the state-dependent variable.

Our model is very much in the spirit of the so-called delay selection adopted in the neuroscience community, where much of the existing work related to delay adaptation in neural nets has concentrated on the fine tuning of a selected set of parameters in network architectures already endowed with a certain degree of structure. In [50], the successive parts of a spoken word are delayed differentially in order to arrive simultaneously onto a unit assigned to the recognition of that particular word. In [18], various examples of delay adaptation in distributed systems are provided: parallel computing machines, the auditory system of barn owls, echo location in bats, and the lateral line system of weakly electric fish. For these systems, a fundamental problem concerns how the flow of information from distinct, independent components can be best regulated to optimize a prescribed performance of the network. Delay shift and delay selection were investigated in [19] for the self-organized adaptation of delays. In [30], the self-organized adaptation of delays is incorporated into the projective adaptive resonance theory developed in [9,10]. This self-organized adaptation of delay, driven by the dissimilarity between input and stored patterns, is motivated by the recent electrophysiological study of transient instability to distinguish between faces and is supported by recent development regarding silent synapses.

It is easy to show that the function ξ in (2) is Lipschitz continuous in the second coordinate i.e.,

$$|\xi(\theta, s^1) - \xi(\theta, s^2)| \leqslant L_\xi \cdot |s^1 - s^2| \quad \forall \theta \in [-r, 0], \quad s^1, s^2 \in \mathbb{R}, \tag{3}$$

if the function g is Lipschitz (in fact, it is enough for ξ to be locally Lipschitz (see (9))).

Property (3) gives the estimate

$$\begin{aligned}
|\xi(\theta, \|u^1(t)\|) - \xi(\theta, \|u^2(t)\|)| &\leqslant L_\xi \cdot |\, \|u^1(t)\| - \|u^2(t)\| \,| \\
&\leqslant L_\xi \cdot \|u^1(t) - u^2(t)\| \quad \forall \theta \in [-r, 0].
\end{aligned} \tag{4}$$

This property is the main advantage of the proposed model over the models with the state-dependence of the form $u(t - r(\|u(t)\|))$. More precisely, estimating $u^1(t - r(\|u^1(t)\|)) - u^2(t - r(\|u^2(t)\|))$ requires some Lipschitz continuity (in time) of solution $u(t)$. We do *not* have such a property in the case of PDEs. On the other hand, property (4) is valid for merely $L^2(\Omega)$-functions u^i without requiring the Lipschitz continuity in time of solutions $u(t)$.

We now present a few remarks to conclude this section.

Remark 1. It is evident from our discussions in next sections that our approach remains valid for other choice of functions ξ, as long as the Lipschitz continuity in the second coordinate (3) (or more generally (9)) holds.

Remark 2. The function ξ chosen in (2) is continuous in the first coordinate θ, but we do *not* use this property in our study. In fact, we only need ξ to be L^2-integrable in θ on $[-r, 0]$ (see (10)). However, in

a particular case when $\xi(\theta, s)$ is a "movement" of an "etalon-function" of a fixed shape (for example, ξ in (2) is a movement of $\xi(\theta) = e^{-\beta(c_1-\theta)^2}$), then any discontinuity in the first coordinate (discontinuity of an "etalon-function") implies the discontinuity in the second one, so (3) cannot be satisfied. It is easy to realize if one considers a step-function as an "etalon-function".

Remark 3. If we are interested in a biological problem where $u(t)$ represents the density of a population, it is natural to assume that the delay depends on the total density or total number of individuals. In this case, we use $\|u(t)\|_{L^1(\Omega)}$ as the second coordinate $\xi(\theta, \|u(t)\|_{L^1(\Omega)})$. Property (3) gives

$$|\xi(\theta, \|u^1(t)\|_{L^1(\Omega)}) - \xi(\theta, \|u^2(t)\|_{L^1(\Omega)})| \leqslant L_\xi \cdot \|u^1(t) - u^2(t)\|_{L^1(\Omega)}$$
$$\leqslant L_\xi \sqrt{|\Omega|} \cdot \|u^1(t) - u^2(t)\|, \tag{5}$$

which is the desired property similar to (4). Here $|\Omega|$ denotes the measure of Ω.

Remark 4. We use the function ξ to represent non-local in time effect, and use function f to present non-local in space effect. Precise terms to describe such an interaction of non-local problems in both space and time remain to be an interesting and challenging task.

A slightly more general form than Eq. (1), that will be studied in the remaining part of this paper, is the following PDE with SD selective delay

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = \int_{-r}^0 \left\{ \int_\Omega b(u(t + \theta, y)) f(x - y) \, dy \right\} \xi(\theta, u(t), u_t) \, d\theta$$
$$\equiv (F(u_t))(x), \quad x \in \Omega, \tag{6}$$

where the function $\xi(\cdot, \cdot, \cdot) : [-r, 0] \times L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \to \mathbb{R}$ represents the SD delay. As discussed above, we will use the local Lipschitz property of ξ in the second and third coordinates (see (9)). In Section 4 we will construct the dynamical system associated with Eq. (6) in the space $H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$, so the pair $(u(t), u_t) \in H$ presents the state of the system and the dependence of ξ on its second and third coordinates reflects the nature of delay selection depending on the state.

3. Existence of solutions

We consider Eq. (6) with the following initial conditions:

$$u(0+) = u^0 \in L^2(\Omega), \quad u|_{(-r,0)} = \varphi \in L^2(-r, 0; L^2(\Omega)). \tag{7}$$

Definition 1. A function u is a weak solution of problem (6) subject to the initial conditions (7) on an interval $[0, T]$ if $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(-r, T; L^2(\Omega)) \cap L^2(0, T; D(A^{1/2}))$, $u(\theta) = \varphi(\theta)$ for $\theta \in (-r, 0)$ and

$$-\int_0^T \langle u, \dot{v} \rangle \, dt + \int_0^T \langle A^{1/2}u, A^{1/2}v \rangle \, dt + \int_0^T \langle du - F(u_t), v \rangle \, dt = -\langle u^0, v(0) \rangle \tag{8}$$

for any function $v \in L^2(0, T; D(A^{1/2}))$ with $\dot{v} \in L^2(0, T; D(A^{-(1/2)}))$ and $v(T) = 0$.

Theorem 1. *Assume that*

(i) $b : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz and there exist constants C_1 and C_2 so that $|b(w)| \leqslant C_1|w| + C_2$ for all $w \in \mathbb{R}$,*

(ii) $f : \Omega - \Omega \to \mathbb{R}$ *is bounded,*

(iii) $\xi : [-r, 0] \times L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \to \mathbb{R}$ *is locally Lipschitz with respect to the second and third coordinates, i.e., for any $M > 0$ there exists $L_{\xi,M}$ so that for all $\theta \in [-r, 0]$ and all $(v^i, \psi^i) \in H$ satisfying $\|v^i\|^2 + \int_{-r}^0 \|\psi^i(s)\|^2 \, ds \leqslant M^2$, $i = 1, 2$ one has*

$$|\xi(\theta, v^1, \psi^1) - \xi(\theta, v^2, \psi^2)| \leqslant L_{\xi,M} \cdot \left(\|v^1 - v^2\|^2 + \int_{-r}^0 \|\psi^1(s) - \psi^2(s)\|^2 \, ds \right)^{1/2}, \qquad (9)$$

and there exists $C_\xi > 0$ so that

$$\|\xi(\cdot, v, \psi)\|_{L^2(-r,0)} \leqslant C_\xi \quad \text{for all } (v, \psi) \in H. \qquad (10)$$

Then for any $(u^0, \varphi) \in H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$ problem (6) subject to the initial conditions (7) has a weak solution on every given interval $[0, T]$, which satisfies

$$u(t) \in C(0, T; L^2(\Omega)). \qquad (11)$$

Proof. Our proof is based on the Galerkin approximation. Let us denote by $\{e_k\}_{k=1}^\infty$ an orthonormal basis of $L^2(\Omega)$ such that $Ae_k = \lambda_k e_k, 0 < \lambda_1 < \cdots < \lambda_k \to +\infty$. We say that function $u_m(t, x) = \sum_{k=1}^m g_k(t)e_k(x)$ is a Galerkin approximate solution if

$$\langle \dot{u}_m + Au_m + du_m - F(u_{mt}), e_k \rangle = 0,$$

$$\langle u_m(0+), e_k \rangle = \langle u^0, e_k \rangle, \quad \langle u_m(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle \quad \forall \theta \in (-r, 0) \qquad (12)$$

$\forall k = 1, \ldots, m$. Here $g_k \in C^1(0, T; \mathbb{R}) \cap L^2(-r, T; \mathbb{R})$ with $\dot{g}_k(t)$ being absolutely continuous.

Eqs. (12) for a fixed m can be rewritten as the following system for the m-dimensional vector-function $v(t) = v_m(t) = (g_1(t), \ldots, g_m(t))^{\mathrm{T}}$:

$$\dot{v}(t) = \hat{f}(v(t)) + \int_{-r}^0 p(v(t + \theta))\widetilde{\xi}(\theta, v(t), v_t) \, d\theta, \qquad (13)$$

where function $\widetilde{\xi}$ satisfies properties similar to (9), (10) if one uses $|\cdot|_{\mathbb{R}^m}$ instead of $\|\cdot\|_{L^2(\Omega)}$. We notice that $\|u_m(t, \cdot)\|_{L^2(\Omega)}^2 = \sum_{k=1}^m g_k^2(t) = |v(t)|_{\mathbb{R}^m}^2$.

Under the assumptions of Theorem 1, the functions $\hat{f}$ and p are locally Lipschitz, $|p(s)| \leqslant c_1|s| + c_2$ for $s \in \mathbb{R}$, ξ is continuous and satisfies (9). Therefore, for any initial data $\varphi \in L^2(-r, 0; \mathbb{R}^m)$, $a \in \mathbb{R}^m$ there exists $\alpha > 0$ and a unique solution of (13) $v \in L^2(-r, \alpha; \mathbb{R}^m)$ such that $v_0 = \varphi$ and $v(0) = a$, and $v|_{[0,\alpha]} \in C([0, \alpha]; \mathbb{R}^m)$ (for more details see [43]).

We now claim that the nonlinear term

$$F(u_t; a; \psi) \equiv \int_{-r}^0 \left\{ \int_\Omega b(u(t + \theta, y)) f(x - y) \, dy \right\} \xi(\theta; a; \psi) \, d\theta$$

satisfies the following properties:

$$|\langle F(u_t, a, \psi), v \rangle_{L^2(\Omega)}|$$

$$\leqslant \frac{1}{2} M_f |\Omega| \left\{ \|v\|^2 + 2C_1^2 \int_{-r}^0 \|u(t+\theta)\|^2 \, d\theta + 2|\Omega| C_2^2 r \right\} \|\xi(\cdot, a, \psi)\|_{L^2(-r,0)} \tag{14}$$

and

$$\|F(u_t, a, \psi)\|^2 \leqslant 2M_f^2 |\Omega|^2 \left(C_1^2 \int_{-r}^0 \|u(t+\theta)\|^2 \, d\theta + C_2^2 r |\Omega| \right) \|\xi(\cdot, a, \psi)\|_{L^2(-r,0)}^2, \tag{15}$$

where $|\Omega| \equiv \int_\Omega dx$, $M_f \equiv \max\{|f(x)| : x \in \overline{\Omega}\}$ and u, v are arbitrary functions with $u \in L^2(-r, 0; L^2(\Omega))$, $v \in L^2(\Omega)$, $(a; \psi) \in H$.

To verify the claim, we first estimate

$$\langle F(u_t, a, \psi), v \rangle_{L^2(\Omega)} = \int_\Omega \left[\int_{-r}^0 \left\{ \int_\Omega b(u(t+\theta, y)) f(x-y) \, dy \right\} \xi(\theta; a; \psi) \, d\theta \right] v(x) \, dx$$

$$= \int_{-r}^0 \left[\int_\Omega \left\{ \int_\Omega b(u(t+\theta, y)) f(x-y) \, dy \right\} v(x) \, dx \right] \xi(\theta, a, \psi) \, d\theta. \tag{16}$$

We denote by $I(\theta, t) \equiv \int_\Omega \{\int_\Omega b(u(t+\theta, y)) f(x-y) \, dy\} v(x) \, dx$. Using $|f(s)| \leqslant M_f$, $\forall s \in \Omega$, we get

$$|I(\theta, t)| \leqslant M_f \int_\Omega |b(u(t+\theta, y))| \, dy \cdot \int_\Omega |v(x)| \, dx$$

$$\leqslant M_f \left(C_1 \int_\Omega |u(t+\theta, y)| \, dy + C_2 |\Omega| \right) |v|_{L^1(\Omega)}$$

$$\leqslant M_f |\Omega| \left(C_1 \|u(t+\theta)\|_{L^2(\Omega)} + \sqrt{|\Omega|} C_2 \right) \|v\|_{L^2(\Omega)}.$$

Hence

$$\left(\int_{-r}^0 |I(\theta, t)|^2 \, d\theta \right)^{1/2} \leqslant M_f |\Omega| \|v\| \left(\int_{-r}^0 \left(C_1 \|u(t+\theta)\|_{L^2(\Omega)} + \sqrt{|\Omega|} C_2 \right)^2 \, d\theta \right)^{1/2}$$

$$\leqslant \frac{1}{2} M_f |\Omega| \left\{ \|v\|^2 + \int_{-r}^0 \left(C_1 \|u(t+\theta)\| + \sqrt{|\Omega|} C_2 \right)^2 \, d\theta \right\}$$

$$\leqslant \frac{1}{2} M_f |\Omega| \left\{ \|v\|^2 + 2C_1^2 \int_{-r}^0 \|u(t+\theta)\|^2 \, d\theta + 2|\Omega| C_2^2 r \right\}.$$

This, together with

$$|\langle F(u_t, a, \psi), v \rangle_{L^2(\Omega)}| \leqslant \|\xi(\cdot, a, \psi)\|_{L^2(-r,0)} \cdot \|I(\cdot, t)\|_{L^2(-r,0)}$$

from (16), implies (14). Essentially in the same way, one obtains (15).

Now, we try to get an a priori estimate for the Galerkin approximate solutions. We multiply (12) by g_k and sum over $k = 1, \ldots, m$. Hence for $u(t) = u_m(t)$ and $t \in (0, \alpha] \equiv (0, \alpha(m)]$, the local existence interval for $u_m(t)$, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 + \|A^{1/2}u(t)\|^2 + d\|u(t)\|^2 \leqslant |\langle F(u_t, u(t), u_t), u(t)\rangle|. \tag{17}$$

Using (14) and (10), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 + \|A^{1/2}u(t)\|^2 \leqslant \tilde{k}_1 \|u(t)\|^2 + \tilde{k}_2 \int_{-r}^{0} \|u(t + \theta)\|^2 \, \mathrm{d}\theta + \tilde{k}_3. \tag{18}$$

As $[t - r, t] \subset [-r, t], \ t > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 + \|A^{1/2}u(t)\|^2 \leqslant \tilde{k}_1 \|u(t)\|^2 + \tilde{k}_2 \int_{-r}^{0} \|u(\theta)\|^2 \, \mathrm{d}\theta + \tilde{k}_2 \int_{0}^{t} \|u(s)\|^2 \, \mathrm{d}s + \tilde{k}_3.$$

Since $\mathrm{d}/\mathrm{d}t \|u(t)\|^2 + \|A^{1/2}u(t)\|^2 = \mathrm{d}/\mathrm{d}t (\|u(t)\|^2 + \int_{0}^{t} \|A^{1/2}u(\tau)\|^2 \, \mathrm{d}\tau + \tilde{k}_3)$, we denote by $\eta(t) \equiv \|u(t)\|^2 + \int_{0}^{t} \|A^{1/2}u(\tau)\|^2 \, \mathrm{d}\tau + \tilde{k}_3$ and rewrite the last estimate as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \eta(t) \leqslant \tilde{k}_2 \int_{-r}^{0} \|u(\theta)\|^2 \, \mathrm{d}\theta + \tilde{k}_4 \cdot \eta(t).$$

Multiplying it by $\mathrm{e}^{-\tilde{k}_4 t}$, one gets $\mathrm{d}/\mathrm{d}t (\mathrm{e}^{-\tilde{k}_4 t} \eta(t)) \leqslant \tilde{k}_2 \int_{-r}^{0} \|u(\theta)\|^2 \, \mathrm{d}\theta \cdot \mathrm{e}^{-\tilde{k}_4 t}$. Integrating from 0 to t and then multiplying by $\mathrm{e}^{\tilde{k}_4 t}$, we obtain

$$\eta(t) \leqslant \tilde{k}_5 \left(\|u(0)\|^2 + \int_{-r}^{0} \|u(\theta)\|^2 \, \mathrm{d}\theta + \tilde{k}_3 \right) \mathrm{e}^{\tilde{k}_4 t}.$$

So, we have the a priori estimate

$$\|u(t)\|^2 + \int_{0}^{t} \|A^{1/2}u(\tau)\|^2 \, \mathrm{d}\tau \leqslant \tilde{k}_5 \left(\|u(0)\|^2 + \int_{-r}^{0} \|u(\theta)\|^2 \, \mathrm{d}\theta + \tilde{k}_3 \right) \mathrm{e}^{\tilde{k}_4 t} - \tilde{k}_3 \tag{19}$$

for some $\tilde{k}_5 > 1$.

Estimate (19) gives that, for $u^0 \in L^2(\Omega)$, $\varphi \in L^2(-r, 0; L^2(\Omega))$, the family of approximate solutions $\{u_m(t)\}_{m=1}^{\infty}$ is uniformly (with respect to $m \in \mathbb{N}$) bounded in the space $L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; D(A^{1/2}))$, where $D(A^{1/2})$ is the domain of the operator $A^{1/2}$ and $[0, T]$ is the local existence interval. From (19) we also get the continuation of $u_m(t)$ on any interval, so (19) holds for all $t > 0$.

Using the definition of Galerkin approximate solutions (12) and their property (19), we can integrate over $[0, T]$ to obtain $\int_{0}^{T} \|A^{-1/2}\dot{u}_m(\tau)\|^2 \, \mathrm{d}\tau \leqslant C_T$ for any T. These properties of the family $\{u_m(t)\}_{m=1}^{\infty}$ give that $\{(u_m(t); \dot{u}_m(t))\}_{m=1}^{\infty}$ is a bounded sequence in the space

$$X \equiv L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; D(A^{1/2})) \times L^2(0, T; D(A^{-1/2})).$$

Then there exist a function $(u(t); \dot{u}(t))$ and a subsequence $\{u_{m_k}\} \subset \{u_m\}$ such that

$$(u_{m_k}; \dot{u}_{m_k}) \ \text{*-weakly converges to } (u; \dot{u}) \text{ in the space } X. \tag{20}$$

By a standard argument (using the strong convergence $u_{m_k} \to u$ in the space $L^2(0, T; L^2(\Omega))$ which follows from (20) and the Doubinskii's theorem (see e.g. [12,37,42]), one can show that any *-weak limit is a solution of (6) subject to the initial conditions (7).

To show that the weak solution is unique, we consider the difference of two solutions $w(t) = u^1(t) - u^2(t)$ which satisfies $\dot{w}(t) + Aw(t) + dw(t) = F(u_t^1) - F(u_t^2) \in L^2(0, T; D(A^{-1/2}))$. Multiplying it by $w(t)$ in $L^2(\Omega)$ and integrating from 0 to t and using (19), together with the property (14) of the nonlinear term F, one has for $t \in [0, T]$, the following:

$$\int_0^t \langle F(u_\tau^1) - F(u_\tau^2), w(\tau) \rangle \, \mathrm{d}\tau \leqslant C_T \int_{-r}^t \|w(\tau)\|^2 \, \mathrm{d}\tau$$
$$= C_T \int_{-r}^0 \|w(\tau)\|^2 \, \mathrm{d}\tau + C_T \int_0^t \|w(\tau)\|^2 \, \mathrm{d}\tau.$$

Hence, we get

$$\frac{1}{2}\|w(t)\|^2 + \int_0^t \|A^{1/2}w(\tau)\|^2 \, \mathrm{d}\tau + d \int_0^t \|w(t)\|^2 \, \mathrm{d}\tau$$
$$\leqslant \frac{1}{2}\|w(0)\|^2 + C_T \int_{-r}^0 \|w(\tau)\|^2 \, \mathrm{d}\tau + C_T \int_0^t \|w(\tau)\|^2 \, \mathrm{d}\tau. \tag{21}$$

Finally, the estimate $\|w(t)\|^2 \leqslant \|w(0)\|^2 + 2C_T \int_{-r}^0 \|w(\tau)\|^2 \, \mathrm{d}\tau + \hat{C}_T \int_0^t \|w(\tau)\|^2 \, \mathrm{d}\tau$ and Gronwall's inequality give

$$\|w(t)\|^2 \leqslant \left(\|w(0)\|^2 + 2C_T \int_{-r}^0 \|w(\tau)\|^2 \, \mathrm{d}\tau \right) \cdot \mathrm{e}^{\hat{C}_T t}, \quad t \in [0, T] \tag{22}$$

which implies the uniqueness of solutions.

To prove (11) we use standard argument (see e.g. [12,37]). More precisely, considering difference of two Galerkin approximate solutions u_k and u_m (with the same initial data), we can easily get the estimate similar to (22)

$$\max_{t \in [0,T]} \|u_k(t) - u_m(t)\|$$
$$\leqslant \left(\|(P_k - P_m)u^0\|^2 + 2C_T \int_{-r}^0 \|(P_k - P_m)\varphi(\tau)\|^2 \, \mathrm{d}\tau \right)^{1/2} \cdot \mathrm{e}^{\hat{C}_T t/2}, \tag{23}$$

where P_n is the orthogonal projection in $L^2(\Omega)$ onto the n-dimensional subspace spanned by $\{e_1, \ldots, e_n\}$. The property $P_n \to I, n \to \infty$ (strong convergence in $L^2(\Omega)$)) gives that $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $C(0, T; L^2(\Omega))$ which implies (11). The proof of Theorem 1 is complete. $\square$

4. Asymptotic properties of weak solutions

Theorem 1 allows us to define the evolution semi-group $S_t : H \to H$, with $H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$, by the formula $S_t(u^0; \varphi) \equiv (u(t); u(t + \theta))$, $\theta \in (-r, 0)$, where $u(t)$ is the weak solution of (6), (7). The continuity of the semi-group with respect to time follows from (11), and with respect to initial conditions from (22).

   Our goal in this section is to study the long-time asymptotic properties of the above evolution semi-group. Let us recall (see e.g. [4,48])

Definition 2. A global attractor of the semi-group S_t is a closed bounded set $\mathcal{U}$ in H, strictly invariant ($S_t \mathcal{U} = \mathcal{U}$ for any $t \geqslant 0$), such that for any bounded set $B \subset H$ we have $\lim_{t \to +\infty} \sup\{\mathrm{dist}_H(S_t y, \mathcal{U}), y \in B\} = 0$.

   The main results are as follows.

Theorem 2. *Under the assumptions of Theorem* 1, *there exists a constant* $\rho > 0$ *such that if* $C_1 r \leqslant \rho$ (*here* C_1 *is defined in Theorem* 1), *then the dynamical system* $(S_t; H)$ *has a compact global attractor* $\mathcal{U}$ *which is a bounded set in the space* $H_1 \equiv D(A^\alpha) \times W$, *where* $W = \{\varphi : \varphi \in L^\infty(-r, 0; D(A^\alpha)), \dot{\varphi} \in L^\infty(-r, 0; D(A^{\alpha-1}))\}$, $\alpha \leqslant \frac{1}{2}$.

   The proof is based on two lemmas.

Lemma 1. *For any* $0 < \alpha < 1$, $\varepsilon > 0$, $T > 0$ *and any bounded set* $B \subset H = L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$ *there exists a constant* $C_{\alpha,\varepsilon,T}(B)$ *such that for any weak solution of* (6) *with initial values in* B, *one has*

$$\|A^\alpha u(t)\| \leqslant C_{\alpha,\varepsilon,T}(B) \quad \text{for } t \in (\varepsilon, T]. \tag{24}$$

Proof. We rewrite (6) as $\dot{u}(t) + Au(t) = M(u(t); u_t)$, where $M(u(t); u_t) \equiv F(u_t) - du(t)$. The variation of constant formula gives

$$u(t) = \mathrm{e}^{-At} u(0) + \int_0^t \mathrm{e}^{-A(t-\tau)} M(u(\tau); u_\tau) \, \mathrm{d}\tau.$$

Using (see, e.g. [48]) for any $\beta \leqslant \alpha$ the estimate

$$\|A^\alpha \mathrm{e}^{-tA} u\| \leqslant \left(\frac{\alpha - \beta}{t} + \lambda_1 \right)^{\alpha - \beta} \mathrm{e}^{-t\lambda_1} \|A^\beta u\|,$$

we have

$$\|A^\alpha u(t)\| \leqslant \|A^\alpha \mathrm{e}^{-tA} u(0)\| + \int_0^t \|A^\alpha \mathrm{e}^{-(t-\tau)A} M(u(\tau); u_\tau)\| \, \mathrm{d}\tau$$

$$\leqslant \left(\frac{\alpha}{t} + \lambda_1 \right)^\alpha \mathrm{e}^{-\lambda_1 t} \|u(0)\| + \int_0^t \left(\frac{\alpha - \beta}{t - \tau} + \lambda_1 \right)^{\alpha - \beta} \mathrm{e}^{-\lambda_1(t-\tau)} \|A^\beta M(u(\tau); u_\tau)\| \, \mathrm{d}\tau.$$

Property (19) implies $\|M(u(\tau); u_\tau)\| \leqslant C_T(B)$ for $\tau \in [0, T]$. Then the last integral converges with $\beta = 0$ and $\alpha \in (0, 1)$. This gives (24). $\square$

Lemma 2. *There exists a constant* $\rho > 0$ *such that if* $C_1 r \leqslant \rho$, *then there exists a constant* C_d *such that for any bounded set* $B \subset H = L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$, *one has*

$$\|A^{1/2} u(t)\| \leqslant C_d, \quad t \geqslant t_1(B) > 0, \tag{25}$$

for some $t_1(B) > 0$, *where* $u(t)$ *is the weak solution of* (6), (7) *with initial data* $(u^0; \varphi) \in B$.

Proof. We first multiply (12) by $\dot{g}_k(t)$ and take the sum over $k = 1, \ldots, m$, and then we multiply (12) by $g_k(t)$ and take the sum again over $k = 1, \ldots, m$. The sum of the obtained equations is (for $u = u_m$)

$$
\langle F(u_t), \dot{u}(t) + u(t) \rangle
$$
$$
= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \{ \|A^{1/2}u(t)\|^2 + (d+1)\|u(t)\|^2 \} + \|\dot{u}(t)\|^2 + \|A^{1/2}u(t)\|^2 + d\|u(t)\|^2.
$$

Using the Cauchy–Schwartz inequality, (15) and (10) we obtain positive constants γ, k_1 and k_2 (independent of m) such that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \Psi(t) + \gamma \Psi(t) \leqslant k_1 \int_{-r}^{0} \|u(t+\theta)\|^2 \, \mathrm{d}\theta + k_2, \tag{26}
$$

where

$$
\Psi(t) \equiv \|A^{1/2}u(t)\|^2 + (d+1)\|u(t)\|^2
$$

and

$$
k_1 = 2 \left(1 + \frac{1}{2d} \right) |\Omega|^2 M_f^2 C_\xi^2 C_1^2.
$$

Using the Gronwall's inequality, we get

$$
\Psi(t) \leqslant 2 \left(\Psi(t_0) + \beta \int_{t_0 - r}^{t_0} \mathrm{e}^{\gamma(s - t_0)} \|u(s)\|^2 \, \mathrm{d}s \right) \mathrm{e}^{-\gamma(t - t_0)} + \frac{k_2 \beta \mathrm{e}^{\gamma t_0}}{\gamma(\gamma - \beta)} [\mathrm{e}^{\gamma t_0} - \mathrm{e}^{-\beta t_0} \mathrm{e}^{(\beta - \gamma)t}], \tag{27}
$$

where $\beta \equiv k_1 r \sqrt{2} \mathrm{e}^{\gamma(t_0 + r)}$. If we choose $t_0 \geqslant \varepsilon$, with ε being given in Lemma 1, then $\Psi(t_0)$ is uniformly bounded for all initial data $(u^0; \varphi) \in B$. Hence, (27) and condition $\beta - \gamma \equiv k_1 r \sqrt{2} \mathrm{e}^{\gamma(t_0 + r)} - \gamma < 0$ give (25). It is now easy to see that sufficiently small ρ and condition $C_1 r \leqslant \rho$ imply $\beta - \gamma < 0$, since $\beta = C_1 r \cdot$ const. This completes the proof.   □

We can now give a proof for Theorem 2. Lemmas 1, 2 and (6) give the estimate $\|A^{\alpha-1}\dot{u}(t)\|^2 \leqslant C(B)$ for $t \geqslant t_1(B)$, $\alpha \leqslant \frac{1}{2}$. This shows that there exists a compact set $K \subset H$ such that for any weak solution with initial value in a bounded set B, one has $(u(t); u_t) \subset K$ for $t \geqslant t_1(B)$. Here we set

$$
K \equiv \left\{ (v; \varphi) : \|A^\alpha v\|^2 + \mathrm{ess} \sup_{s \in [-r, 0]} (\|A^\alpha \varphi(s)\|^2 + \|A^{\alpha-1}\dot{\varphi}(s)\|^2) \leqslant \mathbb{R}_d \right\}, \quad \alpha \leqslant \frac{1}{2}.
$$

That means that the dynamical system $(S_t; H)$ is dissipative and asymptotically compact, hence (see, for example, [4,48]) it has a compact global attractor. The proof of Theorem 2 is complete.   □

5. Dependence on parameters

In this section, we consider the case where the function ξ may depend on a (not necessarily a scalar) parameter v varying in a compact set I. We have the following result about the upper semi-continuity of the global attractor.

Theorem 3. *Assume that estimate* (10) *is valid for all $v \in I$ with the constant C_ξ independent of v and*

$$\forall M > 0, \quad \text{ess} \sup_{|(v,\psi)|_H \leqslant M} \int_{-r}^{0} |\xi_v(\theta, v, \psi) - \xi_{v_0}(\theta, v, \psi)|^2 \, d\theta \to 0, \quad v \to v_0 \in I. \tag{28}$$

Denoted by $\mathscr{U}^v$ the global attractor for the dynamical system (S_t^v, H). Then

$$\lim_{v \to v_0} \sup\{\text{dist}_H(z, \mathscr{U}^{v_0}) : z \in \mathscr{U}^v\} = 0. \tag{29}$$

Proof. The proof uses standard arguments given in [4,11,31]. Since the global attractor $\mathscr{U}^v$ is closed in H, there exists an element $z^v \in \mathscr{U}^v$ such that

$$\text{dist}_H(z^v, \mathscr{U}^{v_0}) = \sup\{\text{dist}_H(z, \mathscr{U}^{v_0}) : z \in \mathscr{U}^v\}.$$

Denote by $z^v(t) = (u^v(t); u_t^v)) \subset \mathscr{U}^v$ the trajectory of (6) subject to the initial conditions (7) (with the parameter v) such that $z^v(0) = z^v$ (the property of invariantness of $\mathscr{U}^v$). The dynamical systems (S_t^v, H) have uniformly in $v \in I$ an absorbing set which is bounded in the space $H_1 \equiv D(A^\alpha) \times W, \alpha \leqslant \frac{1}{2}$ (see Theorem 2) as C_ξ (see (10)) is independent of v. Hence there exist a sequence $\{z^{v_k}(t)\}_1^\infty$ and an element $z^{v_0}(t) \in L^\infty(-\infty, +\infty; H_1)$ such that for any interval $[a, b]$ the sequence $\{z^{v_k}(t)\}_1^\infty$ converges to $z^{v_0}(t)$ in the *-weak topology of the space $L^\infty(a, b; H_1)$, when $k \to \infty$ ($v \to v_0$). Doubinskii's theorem gives the convergence $\lim_{k \to \infty} \max_{t \in [a,b]} \|u^{v_k}(t) - u^{v_0}(t)\| = 0$. This convergence together with property (28) allows us to pass to the limit $k \to \infty$ ($v \to v_0$) in (8) and prove that the function $u^{v_0}(\tau)$ is the weak solution of (6) subject to (7) with the parameter v_0. The function $u^{v_0}(\tau), \tau \in \mathbb{R}$ is bounded, hence belongs to the attractor $\mathscr{U}^{v_0}$. Hence the convergence $\lim_{k \to \infty} \max_{t \in [a,b]} \|u^{v_k}(t) - u^{v_0}(t)\| = 0$ (see [4,11,31]) completes the proof of Theorem 3. $\quad \square$

Remark 5. In the particular case when the function $\xi_{v_0}(\theta, v, \psi)$ (see (28)) is independent of (v, ψ) one has that the global attractor for problem (6) subject to (7) with the state-*selective* delay tends to the global attractor of the state-*independent* problem.

As an application we consider the diffusive Nicholson's blowflies equation (see e.g. [45,47]) with a state-selective delay. More precisely, we consider Eq. (6) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, the function f can be a constant as in [45,47] which leads to the local in space coordinate term or, for example,

$$f(s) = \frac{1}{\sqrt{4\pi\alpha}} e^{-s^2/4\alpha}$$

as in [46] which corresponds to the non-local term, the nonlinear function b is given by $b(w) = p \cdot we^{-w}$. Since b is bounded we have $C_1 = 0$ so the conditions of (Theorems 1–3) are satisfied (for any time delay $r > 0$). As a result, we conclude that for any function ξ satisfying conditions of Theorem 1 (see (9), (10)) the dynamical system (S_t, H) has a global attractor which is upper semicontinuous with respect to a parameter v if $\xi = \xi_v$ depends on v, provided ξ_v satisfies conditions of Theorem 3 (see (28)). Moreover, we emphasize that we can use $\|u(t)\|_{L^1(\Omega)}$ in (9) instead of $\|u(t)\|_{L^2(\Omega)}$ (see Remark 3) and the results remain true.

Acknowledgements

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