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A CONTINUOUS-TIME GARCH MODEL FOR STOCHASTIC VOLATILITY WITH DELAY

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ABSTRACT. We consider a (B, S)-security market with standard riskless asset $B(t) = B_0 e^{rt}$ and risky asset $S(t)$ with stochastic volatility depending on time t and the history of stock price over the interval $[t - \tau, t]$. The stock price process $S(t)$ satisfies a stochastic delay differential equation (SDDE) with past-dependent diffusion coefficient. We state some results on option pricing in such a market and its completeness. We derive a continuous-time analogue of GARCH(1,1) model for our past-dependent volatility. We then show that the equation for the expected squared volatility under the risk-neutral measure is a deterministic delay differential equation, and we construct the solutions for such an equation. We also construct numerical solutions and develop estimation procedures to the option pricing problem, and show the comparison of numerical results.

1 Introduction An assumption made implicitly by Black and Scholes in [**6**] is that the historical performance of the (B, S)-security markets can be ignored. In particular, the so-called Efficient Market Hypothesis implies that all information available is already reflected in the present price of the stock and the past stock performance gives no information that can aid in predicting future perfromance. However, some statistical studies of stock prices (see [**3**] and [**32**]) indicate the dependence on past returns.

The issue of market's delayed response was raised by Bernard and Thomas in [**5**]. They analyzed the drift of estimated cumulative ab-

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normal returns after earnings are announced. They observed that the returns continue to drift up for good news firms and down for bad news firms. They provided two possible explanations for this. The first explanation suggests that at least a portion of the price response to new information is delayed. They explain that the delay might occur either because traders fail to assimilate available information, or because certain costs (such as transaction costs) exceed gains from immediate exploitation of information for a sufficiently large number of traders. The second explanation suggests that, because the capital-asset-pricing model used to calculate the abnormal returns is either incomplete or misestimated, researchers fail to adjust raw returns fully for risk. They came to a conclusion that their results are consistent with a delayed response to information. This is summarized in [**4**]: "The results of this paper cast serious doubt on any belief that asset pricing model misspecifications might explain post-earnings-announcement drift. An understanding of this anomaly appears to require either some model of inefficient markets, or identification of some cost (other than transactions costs) that impede the impounding of public information in prices." See [**8**], [**13**] and [**20**] for more evidence and analysis of past-dependence of stock returns.

There were some attempts to model the past-dependence. For example, in [**23**] a diffusion approximation result was obtained for processes satisfying some equations with past-dependent coefficients, and this result was applied to a model of option pricing, in which the underlying asset price volatility depends on the past evolution to obtain a generalized (asymptotic) Black-Scholes formula. It was shown that the volatility is a deterministic function of time, which is determined by the initial stock price path. This implies that the option price is given by the Black-Sholes formula with some parameter of volatility. Therefore, the implied volatility plot for their model is flat with respect to the strike price.

A new class of nonconstant volatility models was suggested in [**16**], which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. In the model suggested in [**16**], the past-dependence of the stock price process was introduced through volatility given as a function of exponentially weighted moments of historic log-price. This was done in such a way that the price and volatility form a multi-dimensional Markov process. The model produced implied volatility skews of convex and concave shapes. The direction of the skew was determined by whether the asset price was below or above its recent average value.

In [**18**], the model of [**16**] was extended by analysing the discrete-time model that is convergent to the continuous-time model in [**16**]. The author showed that his model shares many common features with GARCH(1,1) model and that the pseudo maximum likelihood method can be applied to estimate the parameters involved.

Chang and Yoree [**9**] studied the pricing of a European contingent claim for the (B, S)-securities markets with a hereditary price structure. The price dynamics for the bank account and the evolution of the stock account are described by a linear functional differential equation and a linear stochastic functional differential equation, respectively. They showed that the rational price for a European contingent claim is given by an expectation of the discounted final payoff, and that it is independent of the mean growth rate of the stock. Later in [**10**], they showed that Black-Scholes formula can be generalized to include the (B, S)-securities market with affine hereditary price structure.

Mohammed et al. [**29**] derived a delayed option price formula by solving a PDE similar to that of Black and Scholes. In their work, the volatility has the form $\sigma(S(t-b))$ for some delay parameter $b > 0$. When deriving the PDE, they assumed that the option price is a function of the time and the current value of the stock only.

The purpose of this paper is to introduce a general framework for modeling the past-dependence of a stock price process and to develop the necessary numerical scheme for solving the model equation and estimation procedures for indentifying parameters involved.

A general theory of stochastic delay differential equations (SDDE) can be found in [**28**]. The problem of delay estimation for SDDE was considered in [**24**]. Discrete-time approximation schemes of SDDE were studied in [**17**] and [**25**]. The GARCH option pricing models were considered in [**11**] and [**19**]. A delayed Black and Scholes formula was derived in [**29**].

2 The model In our model, the *bond* (riskless asset) is represented by the price function $B(t)$ given by

$$B(t) = B_0 e^{rt}, \quad t \in [0, T], \tag{2.1}$$

where $r > 0$ is the risk-free rate of return, and the *stock* (risky asset) is the stochastic process $(S(t))_{t\in[-\tau,T]}$ which satisfies the following SDDE

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \tag{2.2}$$

where $S_t(\theta) := S(t+\theta)$, $\theta \in [-\tau, 0]$, $\mu \in R$, $\sigma : [0,T] \times C \to R$ is a continuous mapping, C is the Banach space of continuous functions from $[-\tau, 0]$ into R, equipped with the supremum norm, and $W(t)$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ for which the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$. To specify a solution, we need to give the initial data of S on $[-\tau, 0]$. In this paper, we assume this initial data is deterministic, that is, the *initial data* for (2.2) is given by $S(\theta) = \varphi(\theta)$ with $\theta \in [-\tau, 0]$ for some $\varphi \in C$.

The existence and uniqueness of a solution of (2.2) are guaranteed if the volatility coefficient in (2.2) satisfies the following local Lipschitz and growth conditions (see [**28**]):

$$\begin{gathered} \forall\, n \geq 1, \quad \exists\, L_n > 0, \quad \forall\, t \in [0,T], \quad \forall\, \eta_1, \eta_2 \in C, \\ \|\eta_1\| \leq n, \quad \|\eta_2\| \leq n : \\ |\sigma(t,\eta_1)\,\eta_1(0) - \sigma(t,\eta_2)\,\eta_2(0)| \leq L_n \|\eta_1 - \eta_2\| \end{gathered} \tag{2.3}$$

and

$$\exists\, K > 0, \quad \forall\, t \in [0,T], \quad \eta \in C: \quad |\sigma(t,\eta)\,\eta(0)| \leq K(1 + \|\eta\|), \tag{2.4}$$

where the norm is $\|\eta\| := \max_{\theta\in[-\tau,0]} |\eta(\theta)|$.

The *discounted stock price* is defined by

$$Z(t) := \frac{S(t)}{B(t)}\,. \tag{2.5}$$

Using Girsanov's theorem (see [**26**]), we obtain the following result concerning the change of probability measure in the above market.

Lemma 2.1. *For a given process $(S(t))_{[-\tau,T]}$, under the assumption*

$$\int_0^T \left(\frac{r-\mu}{\sigma(t,S_t)}\right)^2 dt < \infty, \quad a.s. \tag{2.6}$$

the following holds:

1) *There is a probability measure $\mathcal{P}^*$ equivalent to $\mathcal{P}$ such that*

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} := \exp\left\{ \int_0^T \frac{r-\mu}{\sigma(s,S_s)}\, dW(s) - \frac{1}{2}\int_0^T \left(\frac{r-\mu}{\sigma(s,S_s)}\right)^2 ds \right\} \tag{2.7}$$

is its Radon-Nikodym density;

2) *The discounted stock price $Z(t)$ is a positive local martingale with respect to $\mathcal{P}^*$, and is given by*

$$Z(t) = Z_0 \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(s, S_s)\, ds + \int_0^t \sigma(s, S_s)\, dW^*(s)\right\}, \tag{2.8}$$

where

$$W^*(t) := \int_0^t \frac{\mu - r}{\sigma(s, S_s)}\, ds + W(t)$$

is a standard Wiener process with respect to $\mathcal{P}^*$.

Remarks.

1. The process $Z(t)$ can also be written as

$$dZ(t) = Z(t)\sigma(t, S_t)\, dW^*(t),$$

and, in particular, we have

$$d \ln Z(t) = -\frac{1}{2}\sigma^2(t, S_t)dt + \sigma(t, S_t)\, dW^*(t).$$

2. A sufficient condition for the right-hand side of (2.7) to be martingale with t in place of T is

$$E \exp\left\{\frac{1}{2}\int_0^T \left(\frac{r - \mu}{\sigma(t, S_t)}\right)^2 dt\right\} < \infty.$$

3. The condition (2.6) in the lemma is satisfied if there exists $\delta > 0$ such that $\sigma(t, \phi) \geq \delta$ for all $t \in [0, T]$ and $\phi \in C$.

Accordingly, the only source of randomness in our model for the market consisting of the stock $S(t)$ and the bond $B(t)$ is a standard Wiener process $W(t)$, $t \in [0, T]$, with T denoting the terminal time. This Wiener process generates the filtration $\mathcal{F}_t := \sigma\{W(s) : 0 \leq s \leq t\}$. It can be shown that the $\mathcal{P}^*$-completed filtrations generated by either W, W^*, S or Z all coincide. This is useful since S is the observed process. See [**15**] and [**19**] for details.

Recall that a process $\pi = (\alpha_t, \beta_t)_{t\in[0,T]}$ is called a *trading strategy* if π is predictable and $\left(\int_0^t \beta_s^2 d[Z, Z]_s\right)^{\frac{1}{2}}$, $t \in [0, T]$, is locally integrable under $\mathcal{P}^*$, where $[Z, Z]_t$ is the quadratic variation. Recall also that

π is *admissible* if it is *self-financing*, i.e., the *discounted value process* $X_t(\pi) := \alpha_t + \beta_t Z(t)$ solves

$$X_t(\pi) = X_0(\pi) + \int_0^t \beta \, dZ,$$

and if, in addition, $X_t(\pi)$ is a nonnegative martingale under $\mathcal{P}^*$. A *contingent claim* $\mathcal{C}$ is a positive $\mathcal{F}_T$ measurable random variable. We call a contingent claim *attainable* if there exists an admissible strategy π that *generates* $\mathcal{C}$, i.e., $X_T(\pi) = e^{-rT}\mathcal{C}$. For such a claim $\mathcal{C}$, $\phi_0 := X_0(\pi) = E_{\mathcal{P}^*}(e^{-rT}\mathcal{C})$ is called a *price associated with* $\mathcal{C}$ and this is the only reasonable price for $\mathcal{C}$ at time 0 if we assume the absence of arbitrage opportunities. For times t between 0 and T, the fair price of the claim is given by $\phi_t = e^{rt}X_t(\pi) = e^{rt}E_{\mathcal{P}^*}(e^{-rT}\mathcal{C}|\mathcal{F}_t)$. A market is said to be *complete* if every $\mathcal{P}^*$-integrable claim is attainable.

Theorem 2.1. (Completeness)

(i) *If the discounted stock price process $Z(t)$ is a martingale under $\mathcal{P}^*$, the model* (2.2) *is complete;*
(ii) *Under condition* (2.6) *for every given $S(t)$, the model* (2.2) *is complete and the initial price of any integrable claim $\mathcal{C}$ is given by*

$$\phi_0 = E_{\mathcal{P}^*}(e^{-rT}\mathcal{C}), \tag{2.9}$$

and the price of the claim at any time $0 \leq t \leq T$ is given by $\phi_t = e^{rt}E_{\mathcal{P}^*}(e^{-rT}\mathcal{C}|\mathcal{F}_t)$.

The proof of this theorem is standard and is similar, for example, to the proof of corresponding theorems in [**15**] and [**19**].

Let the so-called *market price of risk* process be given by $\lambda(t) := (\mu - r)/\sigma(t, S_t)$ for $t \geq 0$. Changing the probability measure in equation (2.2) for stock price and using Ito's lemma lead to

$$\ln S(t) = \ln S(0) + \int_0^t (r - \frac{1}{2}\sigma^2(u, S_u)) \, du + \int_0^t \sigma(u, S_u) \, dW^*(u),$$

or, equivalently,

$$\ln \frac{S(t)}{S(t-\tau)} = r\tau - \frac{1}{2}\int_{t-\tau}^t \sigma^2(u, S_u) \, du + \int_{t-\tau}^t \sigma(u, S_u) \, dW^*(u), \tag{2.10}$$

where $W^*(t) = \int_0^t \lambda(s)ds + W(t)$. The expression (2.10), as well as the following expressions in terms of the physical measure $\mathcal{P}$, will be needed later for deriving a continuous-time analogue of GARCH(1,1)-model for stochastic volatility:

$$(2.11) \quad \ln \frac{S(t)}{S(t-\tau)} = r\tau + \int_{t-\tau}^{t} \left[\lambda(u)\sigma(u,S_u) - \frac{1}{2}\sigma^2(u,S_u)\right] du + \int_{t-\tau}^{t} \sigma(u,S_u)\, dW(u).$$

We conclude this section by showing that $S(t) > 0$ a.s. for all $t \in [0,T]$, when $\varphi(0) > 0$. Define the following process:

$$N(t) := \mu t + \int_0^t \sigma(s,S_s)\, dW(s), \quad t \in [0,T].$$

This is a semimartingale with the quadratic variation

$$[N,N]_t = \int_0^t \sigma^2(s,S_s)\, ds.$$

Then, from equation (2.2) we get

$$dS(t) = S(t)dN(t), \quad S(0) = \varphi(0).$$

This equation has a solution:

$$\begin{aligned} S(t) &= \varphi(0) \exp\left\{ N(t) - \frac{1}{2}[N,N]_t \right\} \\ &= \varphi(0) \exp\left\{ \mu t + \int_0^t \sigma(u,S_u)\, dW(u) - \frac{1}{2}\int_0^t \sigma^2(u,S_u)\, du \right\}. \end{aligned}$$

From this we see that if $\varphi(0) > 0$, then $S(t) > 0$ a.s. for all $t \in [0,T]$.

3 A continuous-time GARCH In this section, we show that a model of (B,S)-security market with delayed response arises as a continuous-time equivalent of the GARCH(1,1)-model. The GARCH models are proved consistent with stock market data and are widely used in equity modeling (see [**7**]).

We continue to consider the risk-neutral world where the stock price $S(t)$ has the dynamics given by

$$dS(t) = rS(t)\,dt + \sigma(t, S_t)S(t)\,dW^*(t),$$

where $W^*(t)$ is defined in Lemma 2.1 and $S_t(\theta) = S(t+\theta), \theta \le 0$. We consider the following equation for the variance $\sigma^2(t, S_t)$:

$$\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s, S_s)\,dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t, S_t). \tag{3.1}$$

Here, all the parameters α, γ, τ and V are positive constants. The Wiener process $W(t)$ is the same as in (2.2).

Note that our model is different from the continuous-time analogue of GARCH model given in [**30**]. The latter one is sometimes called GARCH diffusion, mainly because of another Wiener process appearing in the equation for volatility. However, ours is more in line with the original spirit of GARCH, since it has a longer "memory" in the volatility term. And most importantly, our model contains only one source of randomness, i.e., the Wiener process in the equation for stock price (for derivation see Appendix A).

Taking into account (2.11), we note that equation (3.1) is equivalent to

$$\begin{aligned}\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\Bigg[\ln\frac{S(t)}{S(t-\tau)} - r\tau \\ - \int_{t-\tau}^{t}(\lambda(u)\sigma(u, S_u) - \frac{1}{2}\sigma^2(u, S_u))du\Bigg]^2 \\ - (\alpha+\gamma)\sigma^2(t, S_t).\end{aligned} \tag{3.2}$$

Using the definition of risk-neutral measure, we obtain from (3.1) that

$$\begin{aligned}\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\Bigg[\int_{t-\tau}^{t} \sigma(s, S_s)\,dW^*(s) \\ &\qquad - \int_{t-\tau}^{t} \lambda(u)\sigma(u, S_u)\,du\Bigg]^2 - (\alpha+\gamma)\sigma^2(t, S_t) \\ &= \gamma V + \frac{\alpha}{\tau}\Bigg[\int_{t-\tau}^{t} \sigma(s, S_s)\,dW^*(s) - (\mu - r)\tau\Bigg]^2 \\ &\qquad - (\alpha+\gamma)\sigma^2(t, S_t).\end{aligned}$$

Taking the expectations under the risk-neutral measure $\mathcal{P}^*$ on both sides of the equation above, and denoting $v(t) := E_{\mathcal{P}^*}[\sigma^2(t, S_t)]$, we obtain the following deterministic delay differential equation

$$\frac{dv(t)}{dt} = \gamma V + \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau}\int_{t-\tau}^{t} v(s)ds - (\alpha + \gamma)v(t). \tag{3.3}$$

Both the stochastic process $\sigma^2(t, S_t)$ and the deterministic process $v(t)$ have the same initial data $\sigma_0^2(t)$ on the interval $[-\tau, 0]$:

$$\sigma^2(t) = v(t) = \sigma_0^2(t), \quad t \in [-\tau, 0].$$

Note that (3.3) has a stationary solution $v(t) \equiv X = V + \alpha\tau(\mu - r)^2/\gamma$.

An unusual result is that equation (3.3) for the expectation of the squared volatility under the risk-neutral measure $\mathcal{P}^*$ contains the drift parameter μ. In standard equity option pricing problems the drift parameter plays no role, having disappeared through the Girsanov transformation (see Lemma 2.1). However, our model inherited this property from the discrete-time GARCH(1,1) model where the drift parameter enters the equation for volatility:

$$\begin{aligned}\ln(S_n/S_{n-1}) &= m + \sigma_n\xi_n, \quad \{\xi_n\} \sim \text{i.i.d. } N(0,1),\\ \sigma_n^2 &= \gamma V + \alpha\left(\sigma_{n-1}\xi_{n-1}\right)^2 + (1 - \alpha - \gamma)\sigma_{n-1}^2\\ &= \gamma V + \alpha\left(\ln(S_{n-1}/S_{n-2}) - m\right)^2 + (1 - \alpha - \gamma)\sigma_{n-1}^2.\end{aligned}$$

It seems nontrivial to have an explicit formula for a solution of (3.3) with arbitrarily given initial data. However we can describe asymptotic behaviors of solutions of (3.3) by substituting $v(t) = X + Ce^{\rho t}$ into (3.3) to obtain the so-called characteristic equation for ρ (see [**14**])

$$\rho^2 = \frac{\alpha}{\tau} - \frac{\alpha}{\tau}e^{-\rho\tau} - (\alpha + \gamma)\rho.$$

The only non-zero solution to this equation is $\rho \approx -\gamma$. Then, we have $v(t) \approx X + Ce^{-\gamma t}$ for large values of t, and X is asymptotically stable.

These observations can be directly checked using numerical simulations for equation (3.3). The numerical scheme is defined as follows:

$$\begin{aligned}v_i = \gamma X\Delta t + \left(1 + \frac{\alpha(\Delta t)^2}{\tau} - (\alpha + \gamma)\Delta t\right)v_{i-1}&\\ + \frac{\alpha(\Delta t)^2}{\tau}(v_{i-2} + \ldots + v_{i-l}),&\end{aligned}$$

where $v_i = v(t_i)$ and $\{t_i\}$ is a time grid with a mesh of constant size Δt. A typical solution is shown in Figure 1. Figure 2 shows the dependence of the terminal expected variance $v(T)$ on delay τ for a typical constant initial value.

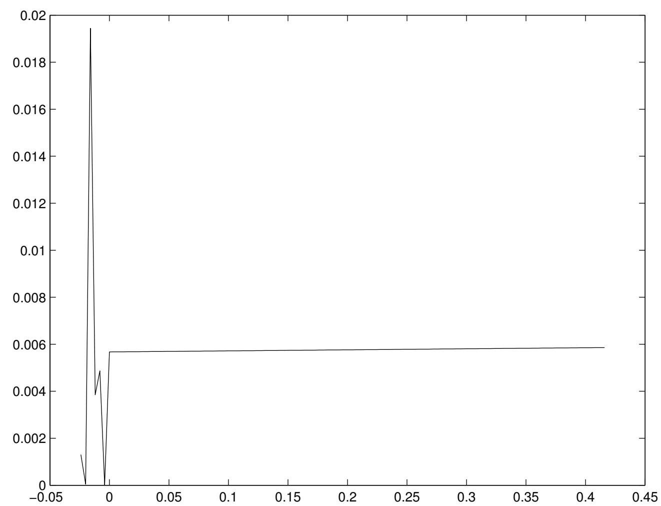

FIGURE 1: Solution of fde (3.3) with $V = 0.01406$, $\alpha = 0.0575$, $\gamma = 0.0539$ and delay $\tau = 0.028$.

4 Parameter estimation We now develop an estimation technique for some parameters involved in the analogue of GARCH(1,1) model introduced in Section 3. These are the drift coefficient μ, time delay τ, and weights α, β and γ. The technique involves Maximum Likelihood (ML) method in combination with the unbiased Akaike information criterion (AICC).

4.1 Drift estimation The parameter μ is unobservable, but it can be easily estimated from observations of $S(u)$, $u \in [0, t]$. The maximum likelihood estimator of μ is given by (see [**1**], [**2**])

$$\tilde{\mu}(t) = \frac{1}{t}\int_0^t S^{-1}(u)\, dS(u).$$

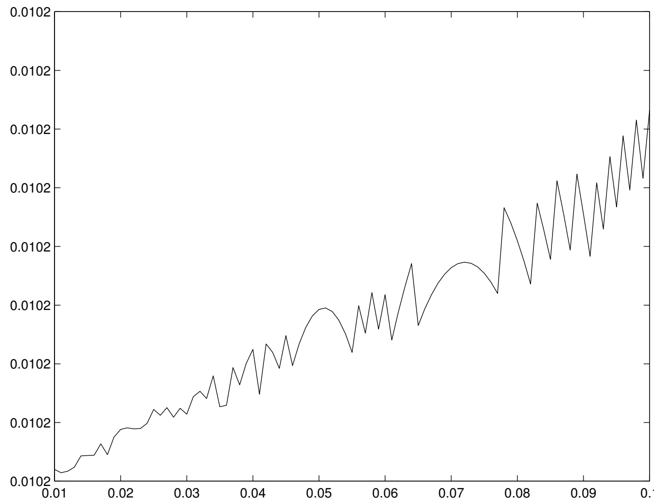

FIGURE 2: Dependence of variance $v(T)$ on delay τ.

Or, in terms of discrete-time observations over an increasing time-grid:

$$\frac{1}{t}\int_0^t S^{-1}(u)\,dS(u) = \lim_{n\to+\infty}\sum_{j=1}^{2^n} S^{-1}((j-1)t2^{-n}) \times \left[S(jt2^{-n}) - S((j-1)t2^{-n})\right].$$

The statistical properties of $\tilde{\mu}(t)$ can be easily derived. Namely, since

$$\frac{dS(t)}{S(t)} = \mu\,dt + \sigma(t, S_t)\,dW(t),$$

we have

$$\tilde{\mu}(t) = \mu + \frac{1}{t}\int_0^t \sigma(s, S_s)\,dW(s),$$

and hence $\tilde{\mu}(t)$ is normal $N\left(\mu, \frac{1}{t^2}\int_0^t E_{\mathcal{P}}\sigma^2(u, S_u)du\right)$, where the expectation $E_{\mathcal{P}}\sigma^2(u, S_u)$ can be found explicitly using (3.1). This means that $\tilde{\mu}(t)$ is unbiased and mean-square consistent at the sampling interval $[0, t]$ as $t \to +\infty$.

We note that t plays the role of the "sample size" in its numerical meaning, while 2^n is the numerical "computational size."

4.2 Time delay and other parameters estimation In this subsection, we show that the maximum likelihood (ML) method can be used to estimate parameters α, β, γ and V. Parameter l takes discrete values and has to be treated differently. We show that the unbiased Akaike information criterion (AICC) can be used to select l.

4.2.1 *Consistency and asymptotic normality of the ML estimators* For simplicity of presentation, in this section we will use notation x_t meaning $x(t)$. Suppose that we observe the sequence $\{y_t\}$ with

$$\begin{aligned} y_t &= \mu_0 + \varepsilon_{0t}, \quad \varepsilon_{0t} = \xi_t h_{0t}^{1/2}, \\ h_{0t} &= \omega_0 + \frac{\alpha_0}{l}\mathcal{E}_{0t-1}^2 + \beta_0 h_{0t-1}, \end{aligned} \tag{4.1}$$

where $\mathcal{E}_{0t-1} = \sum_{k=1}^{l} \varepsilon_{0t-k}$, $l \geq 1$ is a fixed integer, $\{\xi_t\}_{t \in Z}$ is i.i.d. N(0,1). Let $\mathcal{F}_t$ be the σ-algebra generated by $\{y_t, y_{t-1}, \dots\}$. We define the compact parameter space

$$\begin{aligned} \Theta \equiv \big\{\theta = (\mu, \omega, \alpha, \beta) \in [-m, m] \times [w^{-1}, w] \\ \times [a, 1-a] \times [b, 1-b] : \ \alpha + \beta \leq 1\big\} \end{aligned}$$

for some positive constants m, w, a and b. We assume that the true parameter $\theta_0 = (\mu_0, \omega_0, \alpha_0, \beta_0)$ is in the interior of Θ. For any parameter $\theta \in \Theta$, we define

$$\begin{aligned} & y_t = \mu + \varepsilon_t, \\ & \hat{h}_t = \omega + \frac{\alpha}{l}\mathcal{E}_{t-1}^2 + \beta \hat{h}_{t-1}, \quad \mathcal{E}_{t-1} = \sum_{k=1}^{l} \varepsilon_{t-k}, \end{aligned} \tag{4.2}$$

with the initial data given by $\hat{h}_0 = \omega/(1-\beta)$ and $\{\hat{h}_t\}_{-l+1 \leq t \leq -1}$, chosen arbitrarily. This gives the convenient expression for the variance process

$$\hat{h}_t = \frac{\omega}{1-\beta} + \frac{\alpha}{l}\sum_{i=0}^{t-1} \beta^i \mathcal{E}_{t-i-1}^2.$$

Since the conditional distribution of $\{\xi_t\}$ is the standard normal, the log-likelihood function takes the form (ignoring constants)

$$\widehat{L}_T(\theta) = \frac{1}{2T}\sum_{t=1}^{T} \hat{l}_t(\theta), \quad \text{where} \quad \hat{l}_t(\theta) \equiv -\left(\ln \hat{h}_t(\theta) + \frac{\varepsilon_t^2}{\hat{h}_t(\theta)}\right).$$

It will be convenient to work with the unobserved variance process

$$h_t = \frac{\omega}{1-\beta} + \frac{\alpha}{l}\sum_{i=0}^{\infty}\beta^i \mathcal{E}_{t-i-1}^2$$

and the unobserved log-likelihood function

$$L_T(\theta) = \frac{1}{2T}\sum_{t=1}^{T} l_t(\theta), \quad \text{where} \quad l_t(\theta) \equiv -\left(\ln h_t(\theta) + \frac{\varepsilon_t^2}{h_t(\theta)}\right).$$

The process $h_t(\theta)$ is the model of the conditional variance when the infinite past history of the data is observed.

Lemma 4.1. *For all $\theta \in \Theta$, the expectation*

$$E\left[\frac{\partial h_t}{\partial \theta}\frac{\partial h_t}{\partial \theta'}\, h_t^{-2}\right]$$

exists and is a positive definite matrix.

We refer to [**27**] for the proof.

Theorem 4.1. *$E[L_T(\theta)]$ is uniquely maximized at θ_0.*

Proof. Consider

$$E[l_t(\theta)] - E[l_t(\theta_0)] = E\left[\ln\frac{h_{0t}}{h_t} - \frac{\varepsilon_t^2}{h_t} + \frac{\varepsilon_{0t}^2}{h_{0t}}\right].$$

Since $\varepsilon_t^2 = \varepsilon_{0t}^2 + 2(\mu_0 - \mu)\varepsilon_{0t} + (\mu_0 - \mu)^2$, using the law of iterated expectations we obtain

$$E[l_t(\theta)] - E[l_t(\theta_0)] = E\left[\ln\frac{h_{0t}}{h_t} - \frac{h_{0t}}{h_t} + 1 - \frac{(\mu_0-\mu)^2}{h_t}\right] \le 0,$$

where the equality takes place when $\ln(h_{0t}/h_t) = 0$ a.s. and $\mu = \mu_0$. The former expression is equivalent to

$$(\theta-\theta_0)'\left(\frac{\partial h_t}{\partial \theta}h_t^{-1}\right)_{\theta=\theta^*} = 0 \quad \text{a.s.}$$

for some $\theta^* \in \Theta$, which occurs if and only if $\theta = \theta_0$ by Lemma 4.1. Therefore, $E[L_T(\theta)]$ is uniquely maximized at θ_0. $\square$

We can then state two theorems, and we again refer to [**27**] for details.

Theorem 4.2. *Let θ_T be the solution to $\max_{\theta\in\Theta} L_T(\theta)$ and $\hat{\theta}_T$ the corresponding solution to $\max_{\theta\in\Theta} \widehat{L}_T(\theta)$. Then $\theta_T \to \theta_0$ and $\hat{\theta}_T \to \theta_0$ in probability as $T \to \infty$.*

We define the following matrices:

$$
\begin{aligned}
A_0 &= -E\left[\frac{\partial^2 l_t(\theta_0)}{\partial\theta\partial\theta'}\right],\\
\widehat{A}_T &= \frac{1}{2T}\sum_{t=1}^{T}\hat{h}_t^{-2}\frac{\partial\hat{h}_t}{\partial\theta}\frac{\partial\hat{h}_t}{\partial\theta'} + \frac{1}{T}\sum_{t=1}^{T}\hat{h}_t^{-1}\frac{\partial\varepsilon_t}{\partial\theta}\frac{\partial\varepsilon_t}{\partial\theta'},\\
A_T &= \frac{1}{2T}\sum_{t=1}^{T}h_t^{-2}\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'} + \frac{1}{T}\sum_{t=1}^{T}h_t^{-1}\frac{\partial\varepsilon_t}{\partial\theta}\frac{\partial\varepsilon_t}{\partial\theta'},\\
A &= \frac{1}{2}E\left[h_t^{-2}\frac{\partial h_t}{\partial\theta}\frac{\partial h_t}{\partial\theta'}\right] + E\left[h_t^{-1}\frac{\partial\varepsilon_t}{\partial\theta}\frac{\partial\varepsilon_t}{\partial\theta'}\right].
\end{aligned}
$$

Theorem 4.3. *The following statements hold:*

(a) $\sqrt{T}(\hat{\theta}_T - \theta_0) \sim N(0, A_0^{-1})$ *asymptotically as* $T \to \infty$*;*
(b) *Consistent estimator of* A_0 *is given by* $\widehat{A}_T$ *evaluated at* $\hat{\theta}_T$*.*

4.2.2 *Numerical results* We now show how to use the maximum likelihood method to estimate the time delay. In particular, we choose values for the parameters that maximize the chance (or likelihood) of the data occurring, and then use chosen values of parameters to test how our model (of market with delayed response) works.

Recall that the discrete-time model for volatility in the market with delayed response is

$$
\begin{aligned}
\varepsilon_n &= y_n - \mu \equiv \ln\frac{S_n}{S_{n-1}} - \mu,\\
\sigma_n^2 &= \omega + \frac{\alpha}{l}\left(\sum_{i=1}^{l}\varepsilon_{n-i}\right)^2 + \beta\sigma_{n-1}^2,
\end{aligned}
\tag{4.3}
$$

where $\alpha+\beta+\gamma = 1$ and $\omega = \gamma V$. The parameter μ can be eliminated by assigning $\mu = (\sum_{k=1}^{N} y_k)/N$. The parameter $l \geq 1$ represents the delay. For $l = 1$, we obtain the GARCH(1,1) model. The correspondence between continuous-time parameter of delay τ and its discrete-time analogue l is given by $\tau = l\Delta$, where Δ is the size of a mesh of the discrete-time grid. The probability distribution of ε_n conditional on information up to time $n-1$ is assumed to be normal.

The likelihood function is given by

$$L(\alpha, \beta, \omega, l) = \prod_{n=1}^{N} \left[\frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(\frac{-\varepsilon_n^2}{2\sigma_n^2} \right) \right],$$

where σ_n is the function of α, β, ω and l (parameter γ can be eliminated due to equality above). Our task is to maximize the product subject to constraints:

$$\alpha \geq 0, \quad \beta \geq 0, \quad l \geq 1,$$
$$\alpha + \beta < 1.$$

Taking logarithms, we see that this is equivalent to maximizing (l is fixed for now)

$$f(\alpha, \beta, \omega, l) = \sum_{n=1}^{N} \left[-\ln(\sigma_n^2) - \frac{\varepsilon_n^2}{\sigma_n^2} \right]$$

with σ_n^2, $n \geq l+1$, explicitly given by

$$\sigma_n^2 = \omega A_n(\beta) + \alpha B_n(\beta) + R_n(\beta),$$

where

$$\begin{aligned} A_n(\beta) &= 1 + \beta + \beta^2 + ... + \beta^{n-l-1}, \\ B_n(\beta) &= v_{n-1} + v_{n-2}\beta + ... + v_l \beta^{n-l-1}, \\ R_n(\beta) &= v_l \beta^{n-l}, \\ v_n &= \frac{1}{l} \left(\sum_{i=0}^{l-1} \varepsilon_{n-i} \right)^2, \end{aligned}$$

and $\sigma_n^2 = \varepsilon_n^2$ for $n = 1, \ldots, l$.

For each fixed l, we maximize the likelihood function with respect to the other parameters. Thus, we obtain $\widehat{\alpha}(l)$, $\widehat{\beta}(l)$ and $\widehat{\omega}(l)$ for

$l = 1, \ldots, l_{\max}$. Then, we minimize AICC function to choose order $l \in [1, l_{\max}]$:

$$\mathrm{AICC}(\widehat{\alpha}(l), \widehat{\beta}(l), \widehat{\omega}(l), l) = -2\ln L(\widehat{\alpha}(l), \widehat{\beta}(l), \widehat{\omega}(l), l) + \frac{2(l+3)N}{(N-l-4)}.$$

This function is an AICC function for ARMA(l+1,1) model. Note that the discrete-time model (4.3) is very similar to GARCH(l,1) model, the only difference is the presence of the cross-product terms in the equation for volatility. And as it was mentioned in [**7**], any GARCH(p,q) model can be considered as an ARMA(p+q,q) model. Therefore, it is reasonable to assume that AICC function for our model is similar to the one for ARMA(l+1,1) model.

We search iteratively to find parameters that maximize the likelihood using a combination of direct search method and variable metric method, known as the Broyden-Fletcher-Goldfarb-Shanno variant of Davidon-Fletcher-Powell maximization algorithm (see [**31**]). Table 1 shows the results and performance of the algorithm applied to collections of daily observations of S&P500 index price during 1990–1993.

Year	l	$\sqrt{V}$	α	β	γ
1990	1	0.1873	0.0620	0.8443	0.0937
1991	15	0.1603	0.5663	0.1131	0.3206
1992	–	–	–	–	–
1993	1	0.0714	0.0403	0.8073	0.1524
1992–93	4	0.0857	0.0446	0.8505	0.1049
1990–93	7	0.1186	0.0575	0.8886	0.0539

TABLE 1: Results of ML-AICC method of parameters estimation applied to S&P500 data.

The algorithm seems to be stable in almost all cases, except for the year 1992 where the maximum of likelihood function was achieved on the boundary of the feasible region, defined by the constraints and, therefore, cannot be accepted as a local extremum.

It is interesting to compare estimated parameters for different years. The annual pools of data showed little similarity, on the contrary to the results for 1992–93 and 1990–93, where the estimated parameter values were very close. This is a strong argument in favor of the results for larger datasets.

These results can be checked by looking at the autocorrelation structure of $\{\varepsilon_n\}$, i.e., correlation of series $\{\varepsilon_n\}$ and $\{\varepsilon_{n+k}\}$ for each lag $k \geq 1$ (see Table 2). Really, as the table shows, the highest by absolute value autocorrelation for $\{\varepsilon_n\}$ is at the lag 7, which indicates the consistency with ML-AICC method results.

Lag k	η_k, autocorr. $\{u_n\}$	ε_k, autocorr. $\{u_n^2\}$	θ_k, autocorr. $\{u_n^2/\sigma_n^2\}$
1	0.0310	0.0429	−0.0346
2	−0.0454	0.1325	0.0188
3	0.0084	0.0762	0.0553
4	−0.0053	0.1225	0.0045
5	0.0188	0.0779	0.0231
6	−0.0305	0.0971	−0.0001
7	−0.0957	0.0604	−0.0279
8	−0.0021	0.0369	0.0038
9	0.0494	0.0961	0.0148
10	−0.0242	0.1009	0.0301
11	0.0280	0.0566	−0.0254
12	0.0439	0.0074	−0.0336
13	0.0360	0.2219	0.0708
14	0.0204	0.0746	−0.0054
15	−0.0087	0.1402	0.0071

TABLE 2: Autocorrelation structure in the dataset for 1990–1993.

Another test of consistency of our results is to look at how our model for σ_n^2 removes autocorrelations in $\{\varepsilon_n^2\}$. For that purpose, we consider autocorrelations for $\{\varepsilon_n^2\}$ and $\{\varepsilon_n^2/\sigma_n^2\}$. There is an efficient way to check it by using Ljung-Box statistic for both series. Its value is defined by

$$N \sum_{k=1}^{15} \frac{N+2}{N-k}\, \phi_k^2 = 160.64,$$

$$N \sum_{k=1}^{15} \frac{N+2}{N-k}\, \theta_k^2 = 14.18,$$

where $N = 1006$ is the total number of observations, k is the index for lag and ϕ_k, θ_k are the autocorrelations of $\{\varepsilon_n^2\}$ and $\{\varepsilon_n^2/\sigma_n^2\}$ resp. For 15 lags in total, the zero autocorrelation hypothesis can be rejected with 95% confidence when the Ljung-Box statistic is greater than 25.

From these values, we see that there is a strong evidence for autocorrelation in $\{\varepsilon_n^2\}$, since its Ljung-Box statistic is over 160. And for the $\{\varepsilon_n^2/\sigma_n^2\}$ series the Ljung-Box statistic is about 14, suggesting that the autocorrelation has been largely removed by our model (4.3) with parameters obtained by ML-AICC method.

5 Numerical approximation for the option pricing problem

Here we are going to introduce a numerical approximation method for the general equation for the evaluation function, which will allow us to find European call option price. In the first subsection, we give a summary of the results from [**21**], where the general equation was derived.

5.1 Summary of the previous results In our previous work [**21**] we derived a general integro-differential equation for evaluation function H given by

$$F(t, S_t) = \int_{-\tau}^{0} e^{-r\theta} H(S(t+\theta), S(t), t)\, d\theta,$$

where $F(t, S_t)$ is the option price. The general equation, that was obtained using an analogue of Ito's lemma and argument based on a construction of a certain risk-free portfolio, has the following form

$$0 = H|_{\theta=0} - e^{-r\theta} H|_{\theta=-\tau} + \int_{-\tau}^{0} e^{-r\theta} \left(H_3' + rS(t)H_2' + \frac{1}{2}\sigma^2(t, S_t) S^2(t) H_{22}'' \right) d\theta, \tag{5.1}$$

where H_i', $i = 1, 2, 3$, represents the derivative of $H(S(t+\theta), S(t), t)$ with respect to the i-th argument.

Also, in the previous work [**21**], we derived an explicit pricing formula for call option price with some simplifying assumptions. The stock price model is

$$(5.2)\qquad \begin{cases} dS(t) = rS(t)\,dt + \sigma(t, S_t)S(t)\,dW(t), \\ \sigma^2(t, S_t) = \sigma^2(0)e^{-(\alpha+\gamma)t} \\ \qquad\qquad + \left[\gamma V + \dfrac{\alpha}{\tau}\ln^2\left(\dfrac{S(t)}{S(t-\tau)}\right)\right]\dfrac{1-e^{-(\alpha+\gamma)t}}{\alpha+\gamma}, \end{cases}$$

where $S(t) = \varphi(t), t \in [-\tau, 0]$, is given. The expression for volatility came from the continuous-time analogue of GARCH-model. Under some minor assumptions, a formula for European call option price written on the stock (5.2) has the following form

$$(5.3)\qquad F(S_0) = h_1(S(0), 0) + (\sigma^2(0) - \Sigma(S_0)) \\ \left[\mathcal{I}(r + \alpha + \gamma, 0, S(0)) - \mathcal{I}(r, 0, S(0))\right],$$

where $h_1(S, t)$ is the Black-Scholes call option price with the variance $\sigma^2(0)$ and

$$\Sigma(S_0) = \frac{\alpha}{\tau(\alpha+\gamma)}\ln^2\left(\frac{S(0)}{S(-\tau)}\right) + \frac{\gamma V}{\alpha+\gamma}\,,$$

$$\mathcal{I}(p, t, S) = \frac{1}{2}S^2\int_t^T e^{p(t-\xi)}\frac{\partial^2 h_1}{\partial S^2}(S, \xi)\,d\xi \quad \text{for } p \geq 0.$$

Several assumptions were needed for us to derive the closed-form pricing formula (5.3) from the general equation (5.1). Therefore, it is useful to justify the above formula through different numerical schemes. For this purpose, we use the well-known Monte Carlo simulation of solutions of stock price equation (5.2), and also solve the general equation (5.1) by the finite-difference approximation method (see next subsection).

Table 3 shows the simulation results compared to values obtained by the formula (5.3), finite-difference method and Black-Scholes pricing formula.

5.2 Finite-difference method for general equation In this section we will show how to solve the general equation (5.1) for evaluation function H, which will allow us to find the value of European call option price in *any* (B, S)-market with delayed response (2.1)–(2.2).

Strike price	Simulation	Formula	FDM
375	0.1040	0.1031	0.1021
415	0.1036	0.1031	0.1044
435	0.1035	0.1026	0.1041
450	0.1035	0.1031	0.1040
475	0.1036	0.1034	0.1032

TABLE 3: Implied volatility for stochastic volatility model (5.2) for $\alpha = 0.0575$ and $\gamma = 0.0539$: a comparison of simulation results with the formula (5.3) and the finite difference method (FDM) for general equation.

Let us consider a continuous function $\varphi \in C[-\tau, 0]$ with $\varphi(0) = x$, $\varphi(-\tau) = y$, $x, y \in R$. Then, the system (5.1) in terms of continuous function $S_t = \varphi$ will have the following form

$$
\begin{aligned}
(5.4) \qquad 0 = {} & H(x,x,t) - e^{r\tau} H(y,x,t) \\
& + \int_{-\tau}^{0} e^{-r\theta} \left(H_3' + rxH_2' + \frac{1}{2}\sigma^2(t,\varphi)x^2 H_{22}'' \right) d\theta, \\
& \int_{-\tau}^{0} e^{-r\theta} H(\varphi(\theta), x, T)\, d\theta = \max(x - K, 0).
\end{aligned}
$$

Our main objective now is to solve system (5.4) for the function $H(y, x, t)$. Let us consider the function $\varphi_{xy}(\theta) = x + (y - x)(e^{-r\theta} - 1)/(e^{r\tau} - 1)$, $\theta \in [-\tau, 0]$, which connects y and x. After substituting φ_{xy} into (5.4) and changing the integration over the variable θ to the variable $s = \varphi_{xy}(\theta)$, we obtain the following:

$$
\begin{aligned}
(5.5) \qquad 0 = {} & \frac{x-y}{\widehat{\tau}} \left(H(x,x,t) - e^{r\tau} H(y,x,t) \right) \\
& + \int_{y}^{x} \left(H_3' + rxH_2' + \frac{1}{2}\sigma^2(t,\varphi_{xy})x^2 H_{22}'' \right)(s,x,t)\, ds, \\
& \int_{y}^{x} H(s,x,T)\, ds = \frac{1}{\widehat{\tau}}(x-y)\max(x-K,0),
\end{aligned}
$$

where $\widehat{\tau} = (e^{r\tau} - 1)/r$. This is an integro-differential equation and it can be reduced to a PDE using the following substitution:

$$f(x,y,t) = \int_y^x H(s,x,t)\,ds,$$

and the PDE has the following form:

$$(5.6)\quad 0 = \frac{x-y}{\widehat{\tau}}\left(-\frac{\partial f}{\partial y}\bigg|_{y=x} + e^{r\tau}\frac{\partial f}{\partial y}\right) + \frac{\partial f}{\partial t} + rx\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\bigg|_{y=x}\right) + \frac{1}{2}\sigma^2(x,y,t)x^2\left(\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y}\bigg|_{y=x} + \frac{\partial^2 f}{\partial y^2}\bigg|_{y=x}\right)$$

subject to boundary conditions

$$f\,\Big|_{t=T} = \frac{1}{\widehat{\tau}}(x-y)\max(x-K,0),$$

$$f\,\Big|_{y=x} = 0.$$

An analytic solution to equation (5.6) seems hard to find. One way to solve it is to consider the finite-difference numerical approximation scheme for derivatives in (5.6). We obtain the following iterative updating scheme as we move back in time from T:

$$(5.7)\quad f_{i,j}^{\text{new}} = (1-2a_{i,j})f_{i,j} + c_{i,j}[f_{i,j+1} - f_{i,j-1}] + (b_i + a_{i,j})f_{i+1,j} + (-b_i + a_{i,j})f_{i-1,j} + (-d_{i,j} + b_i + a_{i,j})f_{i,i+1} + (d_{i,j} - b_i + a_{i,j})f_{i,i-1} + (-a_{i,j}/2)[f_{i+1,i-1} + f_{i-1,i+1}],$$

where the coefficients are defined by

$$a_{i,j} = \sigma_{i,j}^2 x_i^2 \frac{\Delta t}{2(\Delta x)^2}, \qquad b_i = r x_i \frac{\Delta t}{2\Delta x},$$

$$c_{i,j} = e^{r\tau}\frac{x_i - x_j}{\widehat{\tau}}\frac{\Delta t}{2\Delta x}, \qquad d_{i,j} = \frac{x_i - x_j}{\widehat{\tau}}\frac{\Delta t}{2\Delta x},$$

$f_{i,j} = f(x_i, x_j, t)$ and $f_{i,j}^{\text{new}} = f(x_i, x_j, t-\Delta t)$. Based on our numerical simulations, the scheme (5.7) seems stable as $(\Delta t, \Delta x) \to 0$ if the following condition holds:

$$\sigma_{i,j}^2\, x_i^2\, \frac{\Delta t}{(\Delta x)^2} < 1.$$

See Tables 3 and 4 for numerical results on the finite-difference method (5.7) applied to continuous-time GARCH model (5.2).

Strike price	Simulation	Formula	FDM
375	0.1277	0.1270	0.1207
415	0.1122	0.1089	0.1068
435	0.1104	0.0922	0.1090
450	0.1113	0.1088	0.1039
475	0.1163	0.1217	0.0839

TABLE 4: Implied volatility for stochastic volatility model (5.2) for $\alpha = 14.375$ and $\gamma = 13.475$: a comparison of simulation results with the formula (5.3) and the finite difference method (FDM) for general equation.

Strike price	375	415	435	450	450
Implied volat.	0.1238	0.1124	0.1106	0.1110	0.1142

TABLE 5: Simulation results for continuous-time analogue (3.1) of GARCH(1,1) for $\alpha = 14.375$ and $\gamma = 13.475$.

Conclusion In this paper we considered completeness of (B, S)-security market where stock price volatility depends on the history of stock price over a finite interval of time. An option pricing approach in such a market was introduced. We assumed that option price depends on the history of stock price and the time. We showed that the option price satisfies an integro-differential equation with boundary conditions specified according to the option payoff function. A numerical scheme was provided to construct the solution of this equation.

Also we considered a continuous-time limit of GARCH(1,1) model for stochastic volatility. The resulting volatility perfectly fit into our model for the market with delayed response. This introduced a new unobservable parameter of time delay. A procedure of estimating the delay and other parameters involved in the model for volatility was provided.

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Appendix: derivation of continuous-time analogue of GARCH
The discrete-time model has the following form

$$
\begin{aligned}
&Y_n = Y_{n-1} + \left(\mu - \frac{1}{2}\sigma_n^2\right) + \sigma_n \varepsilon_n, \\
&\sigma_n^2 = \gamma V + \frac{\alpha}{l}\left(\sum_{k=1}^{l} \sigma_{n-k}\, \varepsilon_{n-k}\right)^2 + (1-\alpha-\gamma)\sigma_{n-1}^2, \\
&\{\varepsilon_n\}_{n\geq 1} \sim i.i.d.\ N(0,1),
\end{aligned} \tag{5.8}
$$

where an initial data is given by $(Y_i, \sigma_i^2) = (y_i, v_i)$ with $i = -l, \ldots, 0$. For any fixed $l \geq 1$ define a partition $\pi = \{nh \mid n \geq -l,\ h = \tau/l\}$. Then discrete-time model (5.8) defined over π takes a form

$$
\begin{aligned}
Y_{nh}^{\pi} &= Y_{(n-1)h}^{\pi} + \left(\mu - \frac{1}{2}(\sigma_{nh}^{\pi})^2\right) h + \sigma_{nh}^{\pi}\ \varepsilon_{nh}^{\pi}, \\
(\sigma_{nh}^{\pi})^2 &= \gamma^{\pi} V + \frac{\alpha^{\pi}}{l}\left(\sum_{k=1}^{l} \sigma_{(n-k)h}^{\pi}\ h^{-\frac{1}{2}}\ \varepsilon_{(n-k)h}^{\pi}\right)^2 \\
&\qquad + (1-\alpha^{\pi}-\gamma^{\pi})\ (\sigma_{(n-1)h}^{\pi})^2, \\
\{\varepsilon_{nh}^{\pi}\}_{n\geq 1} &\sim i.i.d.\ N(0,h),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
Y_{nh}^{\pi} &= Y_{(n-1)h}^{\pi} + \left(\mu - \frac{1}{2}(\sigma_{nh}^{\pi})^2\right) h + \sigma_{nh}^{\pi}\ \varepsilon_{nh}^{\pi}, \\
(\sigma_{nh}^{\pi})^2 &= (\sigma_{(n-1)h}^{\pi})^2 + \gamma^{\pi} V \\
&\quad + \frac{\alpha^{\pi}}{\tau}\left(Y_{(n-1)h}^{\pi} - Y_{(n-l-1)h}^{\pi} - \sum_{k=1}^{l}\left(\mu - \frac{1}{2}(\sigma_{(n-k)h}^{\pi})^2\right) h\right)^2 \\
&\qquad - (\alpha^{\pi}+\gamma^{\pi})\ (\sigma_{(n-1)h}^{\pi})^2.
\end{aligned}
$$

Let us take $\gamma^\pi = \gamma h$, $\alpha^\pi = \alpha h$ and define $(Y^\pi(t), \sigma^\pi(t))$ by

$$
\begin{aligned}
Y^\pi(t) &= Y^\pi_{(n-1)h} + \left(\mu - \frac{1}{2}(\sigma^\pi_{nh})^2\right)(t-(n-1)h) \\
&\qquad + \sigma^\pi_{nh}\left(W(t) - W((n-1)h)\right), \\
(\sigma^\pi(t))^2 &= (\sigma^\pi_{(n-1)h})^2 + \Bigg[\gamma V + \frac{\alpha}{\tau}\Bigg(Y^\pi_{(n-1)h} - Y^\pi_{(n-l-1)h} \\
&\qquad - \sum_{k=1}^{l}\left(\mu - \frac{1}{2}(\sigma^\pi_{(n-k)h})^2\right)h\Bigg)^2 \\
&\qquad - (\alpha+\gamma)\,(\sigma^\pi_{(n-1)h})^2\Bigg](t-(n-1)h)
\end{aligned}
$$

for $(n-1)h \le t < nh$ with $n \ge 1$, where $W(t)$ is a Wiener process defined on our probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathcal{P})$. Notice that $(Y^\pi(t), \sigma^\pi(t))$ is a continuous mapping from $[-\tau, \infty) \times \Omega$ to R^2 and its values coincide with $(Y^\pi_{nh}, \sigma^\pi_{nh})$ for $t = nh$ with $n \ge 0$. We define $v^\pi(t) = v_i + (v_{i+1} - v_i)(t-ih)h^{-1}$ and $y^\pi(t) = y_i + (y_{i+1} - y_i)(t-ih)h^{-1}$ for $ih \le t < (i+1)h$ with $i = -l, \cdots -l+1$.

Let us consider a SDDE

$$
\begin{aligned}
&dY(t) = \left(\mu - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,dW(t), \qquad (5.9)\\
&\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left(Y(t) - Y(t-\tau) - \int_{t-\tau}^{t}(\mu - \frac{1}{2}\sigma^2(u))du\right)^2 \\
&\qquad - (\alpha+\gamma)\sigma^2(t),
\end{aligned}
$$

with the initial data given by $(Y(t), \sigma^2(t)) = (y(t), v(t))$ for $t \in [-\tau, 0]$. By defining $S(t) = \exp(Y(t))$ with $\varphi(t) = \exp(y(t))$ and applying the Ito's lemma, we conclude that $S(t)$ coincides with the process introduced in Section 3.

Now if the initial data of (5.8) and (5.9) are close in the sense that

$$
\|y^\pi - y\|^2 + \|v^\pi - v\|^2 \le Ch \qquad (5.10)
$$

for some constant $C > 0$ then $(Y^\pi(t), (\sigma^\pi(t))^2)$ and $(Y(t), \sigma^2(t))$ are close in the following sense (see [**17**])

$$
E\int_0^T |Y^\pi(t) - Y(t)|^2\,dt + E\int_0^T |(\sigma^\pi(t))^2 - \sigma^2(t)|^2\,dt < C'h
$$

for some constant $C' > 0$, under the regularity conditions for coefficients of (5.9). Namely,

$$\begin{aligned} &|G(0)| + |H(0)| < \infty, \\ &|G(\eta) - G(\xi)| + |H(\eta) - H(\xi)| \le L\|\eta - \xi\|, \end{aligned} \tag{5.11}$$

for some $L > 0$ and for all $\eta, \xi \in C([-\tau, 0], R^2)$, where

$$H(\eta) = \begin{bmatrix} \mu - \eta_2(0)/2 \\ \gamma V + \frac{\alpha}{\tau}\left(\eta_1(0) - \eta_1(-\tau) - \int_{-\tau}^{0}(\mu - \frac{\eta_2(\theta)}{2})\, d\theta\right)^2 - (\alpha + \gamma)\eta_2(0) \end{bmatrix},$$

$$G(\eta) = \begin{bmatrix} \sqrt{\eta_2(0)} & 0 \\ 0 & 0 \end{bmatrix},$$

and $|\cdot|$, $\|\cdot\|$ are Euclidean norm and supremum norm, respectively. Note that the convergence result still holds when condition (5.11) is satisfied locally in $C([-\tau, 0], R^2)$.

In other words, by choosing continuous functions $y(t)$ and $v(t)$ such that (5.10) is satisfied for the partition π defined by every small $h > 0$ we ensure the convergence of the solution of discrete-time model (5.8) to the solution of continuous-time model (5.9) in the L^2-norm as h tends to zero.

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