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# Existence and uniqueness of a wavefront in a delayed hyperbolic–parabolic model[☆]

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**Abstract**

We consider a class of hyperbolic–parabolic equation with time delay and a non-local feedback due to the maturation for the adult population density of a single species population, and we show that the wave profile is described by a hybrid system that consists of an integral transformation and an ordinary differential equation. We show the existence and uniqueness of a travelling wavefront of the hyperbolic–parabolic system in the so-called bistable case, by considering the same problem for a properly parametrized parabolic system, and then by considering the continuous dependence of the wave speed on the parameter involved.


*Keywords:* Hyperbolic–parabolic problem; Delay; Wavefront; Hybrid system

## 1. Introduction and the model

We consider the following second-order hyperbolic–parabolic equation:

$$\frac{\partial}{\partial t} m(t,x) + r\frac{\partial^2}{\partial t^2} m(t,x) = D\frac{\partial^2}{\partial x^2} m(t,x) - d_1 m(t,x) + u(t,\tau,x) - r\left(2\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) u(t,\tau,x) \tag{1.1}$$

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for the density of adult population $m(t, x)$ at time $t$ and spatial location $x \in R$ of a given single species population with two age classes (the immature and mature with maturation time $\tau > 0$ being a constant) that moves randomly in space with a time lag $r > 0$, where $D$ and $d_1 > 0$ are constant diffusion and death rates of the adult at time $t$ and location $x$. This equation can be obtained from the usual structured population model, see Raugel and Wu [8]. See also [1,2,4,5,9,12,13] for discussions of the interaction of diffusion and delay in ecological systems.

The maturation rate $u(t, \tau, x)$ is determined by the biological process during the maturation process. In So et al. [11], it was shown that if the immature moves instantaneously and if the birth rate is given by a function $b(m(t, x))$, then

$$u(t, \tau, x) = \varepsilon \int_{-\infty}^{\infty} b(m(t - \tau, y) f(x - y)\, \mathrm{d}y, \tag{1.2}$$

where

$$\varepsilon = \varepsilon(\tau) = \mathrm{e}^{-\int_0^\tau d_2(a)\, \mathrm{d}a} \in (0, 1]$$

is the survival rate during the maturation period and

$$f(z) = \frac{1}{\sqrt{4\pi\alpha}} \mathrm{e}^{-z^2/4\alpha} \quad \text{where } \alpha = \int_0^\tau D_I(a)\, \mathrm{d}a$$

is the probability that a new born at time $t - \tau$ and location 0 moves to the location $z$ after maturation time $\tau$.

We can show that

$$\left(2 \frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) u(t, \tau, x) = \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{-\infty}^{\infty} f_\alpha(x - y) b(m(\theta, y))\, \mathrm{d}y|_{\theta = t - \tau}.$$

Therefore, we obtain a closed system for the matured population

$$\begin{aligned} &\frac{\partial}{\partial t} m(t, x) + r \frac{\partial^2}{\partial t^2} m(t, x) \\ &\quad = D \frac{\partial^2}{\partial x^2} m(t, x) - d_1 m(t, x) + \varepsilon \int_{-\infty}^{\infty} f(x - y) b(m(t - \tau, y))\, \mathrm{d}y \\ &\qquad + r \frac{\partial}{\partial t} \left[\varepsilon \int_{-\infty}^{\infty} f(x - y) b(m(t - \tau, y))\, \mathrm{d}y\right]. \end{aligned} \tag{1.3}$$

This is a second-order hyperbolic equation. When $r = 0$, Eq. (1.3) reduces to

$$\frac{\partial}{\partial t} m(t, x) = D \frac{\partial^2}{\partial x^2} m(t, x) - d_1 m(t, x) + \varepsilon \int_{-\infty}^{\infty} f(x - y) b(m(t - \tau, y))\, \mathrm{d}y. \tag{1.4}$$

The existence of travelling wave front of Eq. (1.4) in the so-called monostable case (where there is no other zero of $du = \varepsilon b(u)$ in $(0, K)$ ) was established in [11] by using the standard techniques involving super- and subsolutions. When the function $b(\cdot)$ exhibits the so-called bistable nonlinearity that the equation

$$d_1 u = \varepsilon b(u), \quad u \geqslant 0$$

has three zeros, $u_1 = 0$, $u_2 = \bar{u}$, and $u_3 = K$ such that

(H1) $0 < \bar{u} < K$,
(H2) $b'(\eta) \geqslant 0$, for $\eta \in [0, K]$,
(H3) $d_1 > \max\{\varepsilon b'(0), \varepsilon b'(K)\}$,
(H4) $d_1 < \varepsilon b'(\bar{u})$.

The existence, uniqueness and asymptotic stability of a travelling wavefront to Eq. (1.4) were recently studied in [6] by using the comparison and squeezing techniques.

The purpose of this paper is to study the existence of travelling wavefront for the more complicated equation (1.3) in the case when the birth function $b(u)$ possesses the property of bistable nonlinearity. Difficulty arises here since for Eq. (1.3) the standard comparison principle does not hold. We overcome this difficulty by introducing an associated parameterized system that has a unique wavefront with a wave speed $C$ that depends on the involved speed $c$ as the parameter. Our key point is to show that the wavefront of such an associated system gives a wavefront of model (1.3) if $C(c) = c$ has a solution and a major technique step is to investigate the continuity of $C(c)$ and the possibility that $C(c) = c$ does have a solution.

## 2. Patterns of travelling wavefronts and an associated parabolic system

A travelling wave front for Eq. (1.3) is of the form

$$m(t, x) = u(s) = u(x + ct), \quad s = x + ct.$$

Substituting this into (1.3), we get

$$c\dot{u}(s) + rc^2\ddot{u}(s) = D\ddot{u}(s) - d_1 u(s) + \varepsilon \int_{-\infty}^{\infty} f(z) b(u(s - c\tau - z))\, \mathrm{d}z + r\varepsilon \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\infty}^{\infty} f(z) b(u(s - c\tau - z))\, \mathrm{d}y$$

or equivalently

$$c\frac{\mathrm{d}}{\mathrm{d}s}\left[u(s) - r\varepsilon \int_{-\infty}^{\infty} f(z) b(u(s - c\tau - z))\, \mathrm{d}z\right] = (D - rc^2)\frac{\mathrm{d}^2 u(s)}{\mathrm{d}s^2} - d_1 u(s) + \varepsilon \int_{-\infty}^{\infty} f(z) b(u(s - c\tau - z))\, \mathrm{d}z. \tag{2.1}$$

It is natural to introduce the following transformation:

$$v(s) = u(s) - r\varepsilon \int_{-\infty}^{\infty} f(s - c\tau - x) b(u(x))\, \mathrm{d}x. \tag{2.2}$$

Note that

$$f(x)=\frac{1}{\sqrt{4\pi\alpha}}\,\mathrm{e}^{-x^2/(4\alpha)},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=-\frac{x}{2\alpha}f(x)=:-\frac{1}{2\alpha}g(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x)=\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)]=f(x)-\frac{x^2}{2\alpha}f(x).$$

Therefore, we obtain from (2.1)

$$c\,\frac{\mathrm{d}}{\mathrm{d}s}v(s)=(D-rc^2)\,\frac{d^2}{ds^2}v(s)-d_1v(s)+\varepsilon\int_{-\infty}^{\infty}G(s-c\tau-x)b(u(x)\,\mathrm{d}x, \quad (2.3)$$

where

$$G(x)=\left[1+r\left(\frac{D-rc^2}{(2\alpha)^2}x^2-\left(d_1+\frac{D-rc^2}{2\alpha}\right)\right)\right]f(x) \quad (2.4)$$

and

$$\int_{-\infty}^{\infty}G(x)\,\mathrm{d}x=1-rd_1. \quad (2.5)$$

In other words, the wave profile Eq. (2.1) with delay is now equivalent to a hybrid system (2.2)–(2.3), where (2.2) is an integral transformation and (2.3) is a differential equation.

In what follows, we shall look for a wavefront for systems (2.2) and (2.3) with wave speed $c$ such that

$$0<D-rc^2. \quad (2.6)$$

We also assume that $r$ is sufficiently small so that

$$r_1\left(d+\frac{D-rc^2}{2\alpha}\right)<r\left(d_1+\frac{D}{2\alpha}\right)<1. \quad (2.7)$$

Under these conditions, we have

$$G(x)>0 \quad \text{for all } x\in R,$$

and

$$\int_{-\infty}^{\infty}G(x)\,\mathrm{d}x=1-rd_1>0.$$

Our approach towards the proof of the existence of wavefront for systems (2.2) and (2.3) is to consider the following *associated parabolic system*:

$$\frac{\partial}{\partial t}w(t,x)=D(c)\,\frac{\partial^2 w(t,x)}{\partial x^2}-d_1w(t,x)+\varepsilon\int_{-\infty}^{\infty}G(x-c\tau-y)b(\varphi(t,y)\,\mathrm{d}x \quad (2.8)$$

and

$$\varphi(t,x) = w(t,x) + r\varepsilon \int_{-\infty}^{\infty} f(x - c\tau - y) b(\varphi(t,y))\,\mathrm{d}x, \tag{2.9}$$

where $c$ is a real parameter satisfying (2.6) and (2.7) and $D(c) = D - rc^2$. The central idea is to establish the existence of travelling wavefront $w(x,t) = V(x + C(c)t)$, $\varphi(t,x) = U(x + C(c)t)$ to (2.8) and (2.9) such that

$$V(-\infty) = 0, \quad V(\infty) = v_{\max} = (1 - rd_1)K, \quad U(-\infty) = 0, \quad U(\infty) = K, \tag{2.10}$$

where $K > 0$ is the maximal positive solution of equation $d_1 u = \varepsilon b(u)$. Here and in what follows we assume that the above equation has only three solutions: $u_1 = 0$, $u_2 = \bar{u}$ and $u_3 = K$. As shall be shown for any $c$, $r$, $d_1$ and $D$ satisfying (2.6), (2.7) and another technical condition (3.5) to be given in the next section, there exists a wave speed $C(c)$ so that the functions $U(s)$ and $V(s)$ satisfy

$$C(c)\frac{\mathrm{d}}{\mathrm{d}s}V(s) = D(c)\frac{\mathrm{d}^2}{\mathrm{d}s^2}V(s) - d_1 V(s) + \varepsilon \int_{-\infty}^{\infty} G(s - c\tau - x) b(U(x))\,\mathrm{d}x, \tag{2.11}$$

$$U(s) = V(s) + r\varepsilon \int_{-\infty}^{\infty} f(s - c\tau - y) b(U(y))\,\mathrm{d}y, \tag{2.12}$$

as well as the boundary condition (2.10). If we can find a point $c$ such that $c = C(c)$, then for such a solution $c > 0$ we obtain a travelling wavefront for the original Eqs. (2.2) and (2.3).

## 3. Uniqueness of wavefronts for associated system

For $x \in R$, we start with the following more general system:

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) = D(c)\dfrac{\partial^2 w(t,x)}{\partial x^2} - d_1 w(t,x) \\ \qquad\qquad + \varepsilon \int_{-\infty}^{\infty} G(x - cr_2 - y) b(\varphi(t - r_1, y))\,\mathrm{d}y, \\ \varphi(t,x) = w(t,x) + r\varepsilon \int_{-\infty}^{\infty} f(x - cr_2 - y) b(\varphi(t,y))\,\mathrm{d}y, \end{cases} \tag{3.1}$$

with the following initial data:

$$\begin{cases} w(s,x) = \phi(s,x), \quad s \in [-r_1, 0], \\ \varphi(s,x) = w(s,x) + r\varepsilon \int_{-\infty}^{\infty} f(x - cr_2 - y) b(\varphi(s,y))\,\mathrm{d}y, \quad s \in [-r_1, 0], \end{cases} \tag{3.2}$$

where $r_1$ and $r_2$ are nonnegative numbers. In this section, we shall prove that Eq. (3.1) has at most one travelling wavefront (up to translation) $w(t,x) = V(x + C(c)t)$, $\varphi = U(x + C(c)t)$ with the wave speed $C(c)$ dependent on $c$. Note that when $r_1 = 0$ and $r_2 = \tau$, system (3.1) reduces to (2.8) and (2.9).

Let $X = BUC(R, R)$ be the Banach space of bounded and uniformly continuous functions from $R$ to $R$ with the usual supremum norm $|\cdot|_X$, and let $X^+ = \{\phi \in X : \phi(x) \geqslant 0, \ x \in R\}$. It is easy to see that $X^+$ is a closed cone of $X$ and $X$ is a Banach Lattice under the partial ordering induced by $X^+$.

The heat equation

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D\dfrac{\partial^2 w(t,x)}{\partial x^2}, & t>0, x\in R, \\ w(0,x)=\phi(x), & x\in R \end{cases}$$

has the solution

$$T(t)\phi(x) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4Dt}\right)\phi(y)\,\mathrm{d}y, \quad t>0, x\in R, \phi\in X, \tag{3.3}$$

and $T(t): X\to X$ is an analytic semigroup on $X$ with $T(t)X^+\subset X^+$ for all $t\geqslant 0$.

Let $C=C([-r_1,0],X)$ be the Banach space of continuous functions from $[-r_1,0]$ into $X$ with the supremum norm $\|\cdot\|$ and let $C^+=\{\phi\in C:\phi(s)\in X^+,\forall s\in[-r_1,0]\}$. Then $C^+$ is a closed cone of $C$. As usual, we identify an element $\phi\in C$ as a function from $[-r_1,0]\times R$ into $R$ defined by $\phi(s,x)=\phi(s)(x)$. For any continuous function $y:[-r_1,b)\to X$, where $b>0$, we define $y_t\in C$, $t\in[0,b)$, by $y_t(s)=y(t+s)$, $s\in[-r_1,0]$. Then $t\mapsto y_t$ is a continuous function from $[0,b)$ to $C$.

We also assume that

$$b'(u)\geqslant 0 \quad \text{for } u\in[-2\delta_0, K+2\delta_0], \tag{3.4}$$

where $\delta_0$ is a small positive constant. In addition, since $b$ is the birth function, it is natural to assume that $b'(u)$ is a bounded function, i.e., the bound

$$b'_{\max} = \sup\{|b'(u)|, -\infty\leqslant u<\infty\}$$

exists.

To show the existence and positiveness of solutions to (3.1) and (3.2) when $\phi\in C^+$, we need some order preserving property for the second equation in (3.1). For any $v\in X$, we consider an operator $\mathfrak{R}: X\to X$ defined by

$$(\mathfrak{R}u)(x) = v(x) + r\varepsilon\int_{-\infty}^{\infty} f(x-cr_2-y)b(u(y))\,\mathrm{d}y.$$

Let

$$X_{\delta_0} = \{\phi\in X: -\delta_0\leqslant\phi(x)\leqslant K+\delta_0,\ x\in R\}.$$

We have the following:

**Lemma 3.1.** *Suppose that r is sufficiently small so that*

$$r < \frac{\delta_0}{(K+2\delta_0)\varepsilon b'_{\max}}. \tag{3.5}$$

*Then*

(i) *for every* $v\in X$, *equation*

$$u(x) = v(x) + r\varepsilon\int_{-\infty}^{\infty} f(x-cr_2-y)b(u(y))\,\mathrm{d}y \tag{3.6}$$

*has one and only one solution* $u=F(v)$ *in* $X$. *In particular, if* $v \in X^+$, *then* $u=F(v) \in X^+$;

(ii) *for any* $v \in X_{\delta_0}$, $F(v) \in X_{2\delta_0}$;

(iii) $F(v) - F(\bar{v}) \in X^+$, *if* $v - \bar{v} \in X^+$ *and* $v, \bar{v} \in X_{\delta_0}$;

(iv) *if* $v \in X_{\delta_0}$ *is non-decreasing on R then so is* $F(v)$;

(v) *for any* $v, \bar{v} \in X_{\delta_0}$, *we have*

$$\|F(v) - F(\bar{v})\| \leqslant 2\|v - \bar{v}\|; \tag{3.7}$$

(vi) *if* $v$ *is a constant function of value* $\tilde{v}$, *then Eq.* (3.6) *reduces to the* algebraic equation

$$\tilde{u} = \tilde{v} + r\varepsilon b(\tilde{u}), \tag{3.8}$$

*which has a unique solution* $\tilde{u}$ such that $u = F(v)$. *Moreover,*

$$\frac{d\tilde{u}}{d\tilde{v}} = F'(\tilde{v}) = \frac{1}{1 - r\varepsilon b'(\tilde{u})}. \tag{3.9}$$

**Proof.** (i) Note that by (3.5) we have $r\varepsilon b'_{\max} < 1$ and

$$|\Re u - \Re\bar{u}| \leqslant r\varepsilon b'_{\max} \int_{-\infty}^{\infty} f(y + cr_2 - x)|u - \bar{u}|\, dy \leqslant r\varepsilon b'_{\max}\|u - \bar{u}\|.$$

We then conclude that $\Re$ is a contraction on Banach space $X$ and thus $\Re$ has a unique fixed point $u = F(v)$. When $u \in X^+$, it is obvious that $\Re u \in X^+$. So $\Re$ is also a contraction on the closed cone $X^+$ of Banach space $X$ and has a unique fixed point $u = F(v) \in X^+$.

(ii) Since we know that for given $v \in X_{\delta_0}$ the contractive operator $\Re$ has a unique fixed point that is the limit of the sequence $\{u_n\}$ given by $u_0 = 0$, $u_1 = \Re u_0 = v(x)$, and

$$u_{n+1} = \Re u_n = v(x) + r\varepsilon \int_{-\infty}^{\infty} f(x - cr_2 - y) b(u_n(y))\, dy.$$

Since the fixed point has the convergent series expansions

$$F(v) = u_1 + (u_2 - u_1) + \cdots + (u_{n+1} - u_n) + \cdots,$$

by (3.5) we obtain

$$\begin{aligned} F(v) &\leqslant v(x) + r\varepsilon b'_{\max}\|u_1 - u_0\| + \cdots r\varepsilon b'_{\max}\|u_n - u_{n-1}\| + \cdots \\ &\leqslant v(x) + r\varepsilon b'_{\max}\|v\| + \cdots \|v\|(r\varepsilon b'_{\max})^{n-1} + \cdots \\ &\leqslant \frac{K + \delta_0}{1 - r\varepsilon b'_{\max}} < K + 2\delta_0. \end{aligned} \tag{3.10}$$

Similarly, we have $F(v) > -2\delta_0$.

(iii) Suppose that $v - \bar{v} \in X^+$. Note

$$F(\bar{v}) = \bar{v}(x) + r\varepsilon \int_{-\infty}^{\infty} f(x - cr_2 - y) b(F(\bar{v})(y))\, dy$$

and

$$F(v) = v(x) + r\varepsilon \int_{-\infty}^{\infty} f(x - cr_2 - y) b(F(v)(y))\, dy.$$

A subtraction of the above two equations gives

$$w(x) = \bar{v}(x) - v(x) + r\varepsilon \int_{-\infty}^{\infty} f(y + cr_2 - x) \times \{b(F(v)(y) + w(y)) - b(F(v)(y))\}\, dy, \tag{3.11}$$

where $w(x) = F(\bar{v}) - F(v)$. We can use the same argument as above for (i) to conclude that (3.11) has a unique fixed point in $X^+$. Thus (iii) holds.

(iv) In a similar manner, we can deduce that $F(v)(x + y) - F(v)(x) \geqslant 0$ with $y \geqslant 0$ if $v$ is non-decreasing.

(v) By (3.5) and (3.11) we have that

$$\|F(\bar{v}) - F(v)\| \leqslant \frac{\|v - \bar{v}\|}{1 - r\varepsilon b'_{\max}} < 2\|v - \bar{v}\|.$$

(vi) The proof of this part is trivial, and is thus omitted. □

**Remark 3.1.** From this lemma we know that if $0 \leqslant u < K$, then $0 \leqslant v \leqslant v_{\max} = K - r\varepsilon b(K)$. Note that $d_1 K = \varepsilon b(K)$. We find that $v_{\max} = K(1 - d_1 r)$.

**Remark 3.2.** For any fixed $t$ in system (3.1), we can solve the second equation by Lemma 3.1 to obtain

$$\varphi(t, x) = F(w)(t, x).$$

Thus systems (3.1) and (3.2) can be transformed into

$$\begin{cases} \dfrac{\partial}{\partial t} w(t, x) = D(c) \dfrac{\partial^2 w(t, x)}{\partial x^2} - d_1 w(t, x) \\ \qquad\qquad + \varepsilon \int_{-\infty}^{\infty} G(x - cr_2 - y) b(F(w)(t - r_1, y))\, dx \\ w(s, x) = \phi(s, x). \end{cases} \tag{3.12}$$

There are three constant solutions $w_1 = 0$, $w_2 = F(\bar{u})$ and $w_3 = v_{\max} = F^{-1}(K)$ satisfying the first equation in (3.12). In other words, the equation

$$d_1 w - \varepsilon b(F(w)) \int_{-\infty}^{\infty} G(x)\, dx = 0$$

has three real constant solutions $w_1 = 0$, $w_2 = F(\bar{u})$ and $w_3 = v_{\max} = F^{-1}(K)$.

In what follows, we assume further the following conditions hold:

(H3′) $d_1 > \frac{1}{1 - r\varepsilon b'(0)} \max\{\varepsilon b'(0), \varepsilon b'(K)\} \int_{-\infty}^{\infty} G(x)\, dx$, (H4′) $d_1 < \varepsilon b'(\bar{u}) \int_{-\infty}^{\infty} G(x)\,\mathrm{d}x$.

Under assumption (H3′), we can choose a positive constant $\delta_0$ sufficiently small so that

$$d_1 > \varepsilon b'(u) \int_{-\infty}^{\infty} G(x)\,\mathrm{d}x \quad \text{for } u \in [-\delta_0, 0) \tag{3.13}$$

and

$$d_1 > \varepsilon b'(u) \int_{-\infty}^{\infty} G(x)\,\mathrm{d}x \quad \text{for } u \in (K, K+\delta_0].$$

Indeed, by (H3′), this can be achieved by modifying (if necessary) the definition of $b$ outside the closed interval $[0, K]$.

From Lemma 3.1, we also know that when $r$ is sufficiently small and $v(s) \in X_{\delta_0}$,

$$F(v)(s) = v(s) + O(r) \tag{3.14}$$

and

$$F^{-1}(v(s)) = v(s) + O(r)$$

uniformly for any $v \in X_{\delta_0}$. Therefore, we shall assume throughout this paper that $r$ is sufficiently small so that for any $\delta < \min\{w_2/2, (v_{\max} - w_2)/2, \delta_0/2\}$, we have

$$M_1(r,\delta) := \min\left\{ b(F(v)(s)) \int_{-\infty}^{\infty} G(y)\,\mathrm{d}y - d_1 v(s);\ s \in \Omega_1 \right\} > 0 \tag{3.15}$$

and

$$M_2(r,\delta) := \min\left\{ d_1 v(s) - b(F(v)(s)) \int_{-\infty}^{\infty} G(y)\,\mathrm{d}y;\ s \in \Omega_2 \right\} > 0, \tag{3.16}$$

where $\Omega_1$ and $\Omega_2$ are defined by

$$\Omega_1 = \{s;\ v_{\max} - \tfrac{3}{2}\delta < v(s) < v_{\max} - \delta\}, \quad \Omega_2 = \{s;\ \delta < v(s) < \tfrac{3}{2}\delta\}.$$

These two assumptions (3.15) and (3.16) are reasonable because when $r = 0$, we have $M_1(0,\delta) > 0$ and $M_2(0,\delta) > 0$ provided that (H1)–(H4) hold.

Now we return to system (3.1). For any $\phi \in [-\delta_0, K+\delta_0]_C \triangleq \{\phi \in C;\ \phi(s,x) \in [-\delta_0, K+\delta_0],\ s \in [-r, 0],\ x \in R\}$, define

$$F_1(\phi)(x) = -d_1\phi(0,x) + \varepsilon \int_{-\infty}^{\infty} G(y + cr_2 - x) b(F(\phi)(-r_1, y))\,\mathrm{d}y, \quad x \in R.$$

Then $F_1(\phi) \in X$ and $F_1 : [-\delta_0, K+\delta_0]_C \to X$ is globally Lipschitz continuous.

**Definition 3.1.** A pair $(w, \varphi) = (w, F(w))$ in $C[(-r_1, b), X] \times C[(-r_1, b), X]$ is called a supersolution (subsolution) of (3.1) if

$$\begin{cases} w(t) \geqslant (\leqslant) T(t-t_0)w(t_0) + \int_{t_0}^{t} T(t-s)F_1(w_s)\,\mathrm{d}s, \\ \varphi(t) = F(w)(t), \quad t \geqslant t_0 - r_1 \end{cases} \tag{3.17}$$

for all $b > t > t_0 > 0$. If $(w, F(w))$ is both a supersolution and a subsolution on $[0, b)$, then it is said to be a mild solution of (3.1).

**Remark 3.3.** Assume that there is a bounded and continuous function pair $(w, \varphi)$ of functions defined on $R \times [-r_1, b)$ that are $C^2$ with respect to $x \in R$, and $C^1$ with respect to $t \in (0, b)$, and

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) \geqslant (\leqslant) D(c) \dfrac{\partial^2 w(t,x)}{\partial x^2} - d_1 w(t,x) \\ \qquad\qquad + \varepsilon \int_{-\infty}^{\infty} G(x - cr_2 - y) b(\varphi(t - r_1, y)\, \mathrm{d}y, \quad t \geqslant 0, \\ \varphi(t,x) = w(t,x) + r\varepsilon \int_{-\infty}^{\infty} f(x - cr_2 - y) b(\varphi(t,y))\, \mathrm{d}y, \quad t \geqslant -r_1 \end{cases}$$

for $x \in R$. Then by the fact that $T(t)X^+ \subset X^+$, it follows that (3.17) holds, and hence $(w, \varphi)$ is a supersolution (subsolution) of (3.1) on $[0, b)$.

Define

$$\Theta(J,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-d_1 t - \frac{(J+1)^2}{4Dt}\right), \quad J \geqslant 0,\ t > 0.$$

We first establish the following existence of solution and comparison result for later use.

**Lemma 3.2.** *For any initial value $\phi \in [-\delta_0, K + \delta_0]_C$, systems* (3.1) *and* (3.2) *have a mild solution for all $t \in [0, \infty)$ with $(-\delta_0, F(-\delta_0)) \leqslant (w(t,x,\phi), F(w)(t,x,\phi)) \leqslant (K + \delta_0, F(K + \delta_0))$ in the sense*

$$-\delta_0 \leqslant w(t,x) \leqslant K + \delta_0, \quad F(-\delta_0) \leqslant F(w)(t,x,\phi) \leqslant F(K + \delta_0),$$

*and $(w(t,x,\phi), F(w)(t,x,\phi))$ is a classical solution to* (3.1) *and* (3.2) *for $(t,x) \in (-r_1, \infty) \times R$. Moreover, for any pair of supersolution $(w^+, F(w^+))$ and subsolution $(w^-, F(w^-))$ of* (3.1) *and* (3.2) *with $-\delta_0 \leqslant w^+(t,x), w^-(t,x) \leqslant K + \delta_0$ for $t \in [-r_1, +\infty)$ and $x \in R$, and $w^+(s,x) \geqslant w^-(s,x)$ for $s \in [-r_1, 0]$ and $x \in R$, there holds $w^+(t,x) \geqslant w^-(t,x)$ for all $t \geqslant 0$ and $x \in R$, and*

$$w^+(t,x) - w^-(t,x) \geqslant \Theta(|x - z|, t - t_0) \int_z^{z+1} [w^+(t_0, y) - w^-(t_0, y)]\, \mathrm{d}y \quad (3.18)$$

*for every $z \in R$ and $t \geqslant t_0 \geqslant 0$.*

**Proof.** Note that $\varphi(t,x) = F(w)(t,x)$. From the abstract setting in [7], it follows that a mild solution $(w, F(w))$ of (3.1) and(3.2) is a solution to the associated integral equation

$$\begin{cases} w(t) = T(t - t_0) w(t_0) + \int_{t_0}^{t} T(t-s) F_1(w_s)\, \mathrm{d}s, \\ w_0 = \phi \in [-\delta_0, K + \delta_0]_C. \end{cases}$$

Clearly, $\mathbf{v}^+ = (K + \delta_0, F(K + \delta_0))$ and $\mathbf{v}^- = (-\delta_0, F(-\delta_0))$ are supersolution and subsolution of (3.1) and (3.2), respectively. As aforementioned, $F_1 : [-\delta_0, K + \delta_0]_C$ is globally

Lipschitz continuous. It also satisfies the quasi-monotone condition in the sense that

$$\lim_{h\to 0^+}\frac{1}{h}\,dist(\psi(0)-\varphi(0)+h[F_1(\psi_s)-F_1(\varphi_s)];\,X^+)=0 \tag{3.19}$$

for all $\psi,\varphi\in[-\delta_0,K+\delta_0]_C$ with $\psi\geqslant\varphi$. To see this, we have by (3.3) that

$$\begin{aligned}F_1(\psi_s)-F_1(\varphi_s)=&-d_1[\psi(0,x)-\varphi(0,x)]\\&+\int_{-\infty}^{\infty}G(y+cr_2-x)b((F\psi)(-r_1,y))\\&-G(y+cr_2-x)b((F\varphi)(-r_1,y))\,\mathrm{d}y\\ \geqslant&-d_1[\psi(0,x)-\varphi(0,x)],\end{aligned}$$

and hence, for any $h>0$ with $d_1h<1$

$$\psi(0)-\varphi(0)+h[F_1(\psi_s)-F_1(\varphi_s)]\geqslant(1-d_1h)(\psi(0)-\varphi(0))\geqslant 0$$

from which (3.19) holds. Therefore, the existence, uniqueness of $w$ follows from [7, Corollary 5]. Moreover, by a semigroup theory argument as in the proof of [7, Theorem 1], we conclude that $(w,F(w))$ is a classical solution for $t\geqslant r_1$.

Since $(w^+(t,x),F(w^+)(t,x))\geqslant(w^-(t,x),F(w^-)(t,x))$, it follows from Corollary 5 in [7] that

$$-\delta_0\leqslant w^-(t,x)\leqslant w(t,x,w^-)\leqslant w(t,x,w^+)\leqslant w^+(t,x)\leqslant K+\delta_0,\quad t\geqslant 0,x\in R,$$

thus $w^-(t,x)\leqslant w^+(t,x)$ for all $t\geqslant 0$ and $x\in R$.

We next prove the last inequality of the lemma. Let $v(t,x)=w^+(t,x)-w^-(t,x)$. For any given $t_0\geqslant 0$, using again the abstract setting in [7], we have

$$\begin{aligned}v(t)&\geqslant T(t-t_0)v(t_0)+\int_{t_0}^{t}T(t-s)[F_1(w_s^+)-F_1(w_s^-)]\,\mathrm{d}s\\&\geqslant T(t-t_0)v(t_0)-d_1\int_{t_0}^{t}T(t-s)v(s)\,\mathrm{d}s.\end{aligned} \tag{3.20}$$

Note that $z(t)=\exp(-d_1(t-t_0))\;T(t-t_0)v(t_0)$ satisfies the following equation:

$$v(t)=T(t-t_0)v(t_0)-d_1\int_{t_0}^{t}T(t-s)v(s)\,\mathrm{d}s.$$

Then we can directly solve (3.20) to obtain

$$w^+(t)-w^-(t)\geqslant \mathrm{e}^{-d_1(t-t_0)}T(t-t_0)(w^+(t_0)-w^-(t_0)),\quad t\geqslant t_0. \tag{3.21}$$

Combining (3.3), (3.21) and the definition of $\Theta(J,t)$, we have for all $t\geqslant t_0\geqslant 0$ and $x\in R$ that

$$w^+(t,x)-w^-(t,x)\geqslant\Theta(|x-z|,t-t_0)\int_z^{z+1}[w^+(t_0,y)-w^-(t_0,y)\,\mathrm{d}y.$$

This completes the proof. □

**Remark 3.4.** From this lemma we know that if $w^+(0,x) \not\equiv w^-(0,x)$, then for any $t>0$,

$$w^+(t,x) - w^-(t,x) \geqslant \Theta(|x-z|, t-t_0) \int_z^{z+1} [w^+(t_0,y) - w^-(t_0,y)\,\mathrm{d}y > 0.$$

In particular, if $(w(t,x,\phi), F(w)(t,x,\phi))$ is a solution of (3.1) with the initial data $\phi \in [-\delta_0, K+\delta_0]_C$ and $\phi(\neq$ constant) is a non-decreasing function on $R$, then for any fixed $t>0$, $w(t,x)$ is strictly increasing in $x \in R$.

Using this property, we can now establish the estimate of the derivative for the travelling wavefront.

**Lemma 3.3.** *Let* $(V(x+C(c)t), F(V)(x+C(c)t))$ *be a non-decreasing travelling wavefront of* (3.1). *Then*

$$0 < V'(\xi) < \frac{b(K)}{2\sqrt{D(c)d_1}}(1-rd_1) \tag{3.22}$$

*and*

$$\lim_{|\xi|\to\infty} V'(\xi) = 0. \tag{3.23}$$

**Proof.** Using Lemma 3.2, we have that for $\xi = x + C(c)t$ and every $h>0$,

$$V(\xi+h) - V(\xi) \geqslant \max_{z\in R} \Theta(|x-z|, t) \int_z^{z+1} [V(y+h) - V(y)]\,\mathrm{d}y > 0,$$

which implies that

$$V'(\xi) \geqslant \max_{z\in R} \Theta(|x-z|, t)[V(z+1) - V(z)] > 0.$$

Next let

$$\lambda_1 = \frac{C(c) - \sqrt{C^2(c) + 4D(c)d_1}}{2D(c)} < 0, \quad \lambda_2 = \frac{C(c) + \sqrt{C^2(c) + 4D(c)d_1}}{2D(c)} > 0.$$

We then have

$$V(\xi) = \frac{1}{D(c)(\lambda_2-\lambda_1)} \left[ \int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(\xi-s)} H(V)(s)\,\mathrm{d}s + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-s)} H(V)(s)\,\mathrm{d}s \right]$$

and

$$\begin{aligned} V'(\xi) &= \frac{1}{D(c)(\lambda_2-\lambda_1)} \\ &\quad \times \left[ \lambda_1 \int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(\xi-s)} H(V)(s)\,\mathrm{d}s + \lambda_2 \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-s)} H(V)(s)\,\mathrm{d}s \right] \\ &\leqslant \frac{1}{D(c)(\lambda_2-\lambda_1)} \left[ \lambda_2 \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-s)} H(V)(s)\,\mathrm{d}s \right], \end{aligned} \tag{3.24}$$

where

$$H(V)(s)=\int_{-\infty}^{\infty} b(F(V)(z))G(s-cr_2-C(c)r_1-z)\,\mathrm{d}z. \tag{3.25}$$

Since $\lambda_2-\lambda_1 \geqslant 2\sqrt{d_1/D(c)}$, it follows from (3.24), (3.25) and (2.5) that

$$V'(\xi)\leqslant \frac{\lambda_2 b(K)}{2\sqrt{D(c)d_1}}\int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-s)}\,\mathrm{d}s\int_{-\infty}^{\infty} G(x)\,\mathrm{d}x=\frac{b(K)}{2\sqrt{D(c)d_1}}(1-rd_1).$$

Finally, by (2.10), (3.24), (3.25) and the dominant convergence theorem, we have (3.23). □

**Lemma 3.4.** *Let $(V(x+C(c)t), F(V)(x+C(c)t))$ be a non-decreasing travelling wave-front of* (3.1). *Then there exist three positive numbers $\beta_0$ (which is independent of V), $\sigma_0$ and $\bar{\delta}$ such that for any $\delta \in (0, \bar{\delta}]$ and every $\xi_0 \in R$, the functions $(w^+, F(w^+))$ and $(w^-, F(w^-))$ defined by*

$$w^{\pm}(t,x):=V(x+C(c)t+\xi_0 \pm \sigma_0\delta(\mathrm{e}^{\beta_0 r_1}-\mathrm{e}^{-\beta_0 t}))\pm \delta \mathrm{e}^{-\beta_0 t}$$

*are a supersolution and a subsolution of* (3.1) *and* (3.2) *on $t \in [0,+\infty)$, respectively.*

**Proof.** By (H′3), we can choose a $\beta_0>0$ and an $\varepsilon^*>0$ such that

$$d_1>\beta_0+\varepsilon \mathrm{e}^{\beta_0 r_1}\left(\max\{b'(0), b'(K)\}\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y+\varepsilon^*\right).$$

By (3.4), there exists a $\delta^*>0$ such that

$$\begin{aligned}&0\leqslant b'(\eta)\leqslant b'(0)+\varepsilon^* \quad \text{for all } \eta \in [-\delta^*, \delta^*],\\ &0\leqslant b'(\eta)\leqslant b'(K)+\varepsilon^* \quad \text{for all } \eta \in [K-\delta^*, K+\delta^*].\end{aligned} \tag{3.26}$$

Let $c_0(c,r_1,r_2)=|C(c)|r_1+|c|r_2+(\mathrm{e}^{\beta_0 r_1}-1)$. Since $\lim_{\xi\to\infty} V(\xi)=v_{\max}$ and $\lim_{\xi\to-\infty} V(\xi)=0$, we have $\lim_{\xi\to\infty} F(V)(\xi)=K$ and $\lim_{\xi\to-\infty} F(V)(\xi)=0$. Therefore, there exists a sufficiently large constant $M_0=M_0(V,\beta_0,\varepsilon^*,\delta^*)>0$ such that

$$F(V)(\xi)\leqslant \delta^* \quad \text{for all } \xi\leqslant -M_0/2+c_0(c,r_1,r_2), \tag{3.27}$$

$$F(V)(\xi)\geqslant K-\delta^* \quad \text{for all } \xi\geqslant M_0/2-c_0(c,r_1,r_2), \tag{3.28}$$

and

$$\begin{aligned}d_1>{}&\beta_0+\varepsilon \mathrm{e}^{\beta_0 r_1}(\max\{b'(0), b'(K)\}+\varepsilon^*)\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y\\ &+\varepsilon \mathrm{e}^{\beta_0 r_1} b'_{\max}\left[\int_{M_0/2}^{\infty}+\int_{-\infty}^{-M_0/2} G(y)\,\mathrm{d}y\right].\end{aligned} \tag{3.29}$$

In view of Lemma 3.3, we have $m_0:=\min\{V'(\xi);\, |\xi|\leqslant M_0\}>0$. Define

$$\sigma_0=\frac{1}{\beta_0 m_0}\left[\varepsilon \mathrm{e}^{\beta_0 r_1} b'_{\max}\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y-d_1+\beta_0\right]>0 \tag{3.30}$$

and

$$\bar{\delta} = \min\left\{\frac{1}{\sigma_0}, \delta^* e^{-\beta_0 r_1}, (F^{-1}(K+\delta^*) - v_{\max})e^{-\beta_0 r_1}\right\}. \tag{3.31}$$

We now prove that $(w^+, F(w^+))$ is a supersolution. The proof of $(w^-, F(w^-))$ *being a subsolution* can be dealt with similarly. By translation, we can also assume that $\xi_0 = 0$. For any given $\delta \in (0, \bar{\delta}]$, let $\xi(t) = x + C(c)t + \sigma_0\delta(e^{\beta_0 r_1} - e^{-\beta_0 t})$. Then we have

$$\begin{aligned}
&S(w^+)(t,x)\\
&\quad \triangleq \frac{\partial w^+(t,x)}{\partial t} - D(c)\frac{\partial^2 w^+}{\partial x^2} + d_1 w^+(t,x)\\
&\qquad - \varepsilon\int_{-\infty}^{\infty} b(F(w^+(t-r_1, x-y)) G(y-cr_2)\,dy\\
&\quad = V'(\xi(t))(C(c)+\sigma\beta_0\delta e^{-\beta_0 t}) - \beta_0\delta e^{-\beta_0 t} - D(c)V''(\xi) + d_1 V(\xi) + d_1\delta e^{-\beta_0 t}\\
&\qquad - \varepsilon\int_{-\infty}^{\infty} b\{F(V[\xi(t) - C(c)r_1 - y + \sigma_0\delta(1-e^{\beta_0 r_1})e^{-\beta_0 t}]\\
&\qquad + \delta e^{-\beta_0(t-r_1)})\}G(y-cr_2)\,dy\\
&\quad = \sigma_0\beta_0\delta V'(\xi(t))e^{-\beta_0 t} - \beta_0\delta e^{-\beta_0 t} + d_1\delta e^{-\beta_0 t}\\
&\qquad - \varepsilon\int_{-\infty}^{\infty} b\{F(V[\xi(t) - C(c)r_1 - y - cr_2 + \sigma_0\delta(1-e^{\beta_0 r_1})e^{-\beta_0 t}]\\
&\qquad + \delta e^{-\beta_0(t-r_1)})\}G(y)\,dy\\
&\qquad + \varepsilon\int_{-\infty}^{\infty} b\{F(V[\xi(t) - C(c)r_1 - y - cr_2])\}G(y)\,dy\\
&\quad \geqslant (\sigma_0\beta_0\delta V'(\xi(t)) - \beta_0\delta + d_1\delta)e^{-\beta_0 t} - \varepsilon\int_{-\infty}^{\infty} b'(\eta_1)\{\delta e^{-\beta_0(t-r_1)}\}G(y)\,dy\\
&\quad = \left\{\sigma_0\beta_0\delta V'(\xi(t)) - \beta_0\delta + d_1\delta - \varepsilon\delta e^{\beta_0 r_1}\int_{-\infty}^{\infty} b'(\eta_1)G(y)\,dy\right\}e^{-\beta_0 t},
\end{aligned}$$

where

$$\begin{aligned}
\eta_1 &= \theta_1 F(V[\xi(t) - C(c)r_1 - y - cr_2 + \sigma_0\delta(1-e^{\beta_0 r_1})e^{-\beta_0 t}] + \delta e^{-\beta_0(t-r_1)})\\
&\quad + (1-\theta_1)F(V[\xi(t) - C(c)r_1 - y - cr_2]).
\end{aligned} \tag{3.32}$$

By (3.31) and (3.32), we obtain that

$$0 \leqslant \eta_1 \leqslant F(v_{\max} + \delta e^{\beta_0 r_1}) \leqslant K + \delta^*$$

and hence $b'(\eta_1) \geqslant 0$. Therefore, we have

$$S(w^+)(t,x) \geqslant \left\{\sigma_0\beta V'(\xi(t)) - \beta_0 + d_1 - \varepsilon e^{\beta_0 r_1}\int_{-\infty}^{\infty} b'(\eta_1)G(y)\,dy\right\}\delta e^{-\beta_0 t}. \tag{3.33}$$

To estimate the right-hand side of the above inequality, we should consider $\xi(t)$ in three cases: (i) $|\xi(t)| \leqslant M_0$, (ii) $\xi(t) \geqslant M_0$ and (iii) $\xi(t) \leqslant -M_0$.

Case (i): $|\xi(t)| \leqslant M_0$. By virtue of (3.30) and (3.33), we have

$$S(w^+)(t,x) \geqslant \left\{ \sigma_0 \beta V'(\xi(t)) - \beta_0 + d_1 - \varepsilon e^{\beta_0 r_1} b'_{\max} \int_{-\infty}^{\infty} G(y)\,dy \right\} \delta e^{-\beta_0 t} \geqslant 0.$$

Case (ii): $\xi(t) \geqslant M_0$. For $y \in [-\frac{1}{2}\xi(t), \frac{1}{2}\xi(t)]$, we have

$$\tfrac{1}{2} M_0 \leqslant \tfrac{1}{2}\xi(t) \leqslant \xi(t) - y \leqslant \tfrac{3}{2}\xi(t).$$

Furthermore, for any $\delta \in (0, \bar{\delta}]$, due to $\sigma_0 \delta \leqslant 1$, we also obtain

$$\begin{aligned} &\xi(t) - C(c) r_1 - y - c r_2 + \sigma_0 \delta (1 - e^{\beta_0 r_1}) e^{-\beta_0 t} \\ &\quad \geqslant \tfrac{1}{2} M_0 - C(c) r_1 - c r_2 + \sigma_0 \delta (1 - e^{\beta_0 r_1}) \\ &\quad \geqslant \tfrac{1}{2} M_0 - c_0(c, r_1, r_2) \end{aligned}$$

and

$$\xi(t) - C(c) r_1 - y - c r_2 \geqslant \tfrac{1}{2} M_0 - C(c) r_1 - c r_2 \geqslant \tfrac{1}{2} M_0 - c_0(c, r_1, r_2).$$

Therefore, by (3.26)–(3.28), we have from (3.32) that

$$K - \delta^* \leqslant \eta_1 \leqslant K + \delta^*$$

and

$$b'(\eta_1) \leqslant b'(K) + \varepsilon^*.$$

Consequently, by (3.29) and (3.33), we have

$$\begin{aligned} &S(w^+)(t,x) \\ &\quad \geqslant \left\{ \sigma_0 \beta V'(\xi(t)) - \beta_0 + d_1 - \varepsilon e^{\beta_0 r_1} \int_{-\infty}^{\infty} b'(\eta_1) G(y)\,dy \right\} \delta e^{-\beta_0 t} \\ &\quad \geqslant \left\{ -\beta_0 + d_1 - \varepsilon e^{\beta_0 r_1} \int_{-\infty}^{\infty} b'(\eta_1) G(y)\,dy \right\} \delta e^{-\beta_0 t} \\ &\quad \geqslant \left\{ -\beta_0 + d_1 - \varepsilon e^{\beta_0 r_1} \int_{-\xi(t)/2}^{\xi(t)/2} b'(\eta_1) G(y)\,dy - \varepsilon e^{\beta_0 r_1} \int_{\xi(t)/2}^{\infty} b'(\eta_1) G(y)\,dy \right. \\ &\qquad \left. - \varepsilon e^{\beta_0 r_1} \int_{-\infty}^{-\xi(t)/2} b'(\eta_1) G(y)\,dy \right\} \delta e^{-\beta_0 t} \\ &\quad \geqslant -\beta_0 + d_1 - \varepsilon e^{\beta_0 r_1} (b'(K) + \varepsilon^*) \int_{-\infty}^{\infty} G(y)\,dy \\ &\qquad - \varepsilon e^{\beta_0 r_1} b'_{\max} \left[ \int_{M_0/2}^{\infty} + \int_{-\infty}^{-M_0/2} G(y)\,dy \right] \\ &\quad \geqslant 0. \end{aligned}$$

Case (iii): $\xi(t) \leqslant -M_0$. The proof in this case is similar to that in case (ii) and hence is omitted. □

**Theorem 3.1** (*Uniqueness*). *Assume that* (3.1) *has a non-decreasing travelling wavefront* $(V(x+C(c)t),\ F(V)(x+C(c)t))$. *Then for any travelling wavefront* $(\bar{V}(x+\bar{C}(c)t),\ F(\bar{V})(x+\bar{C}(c)t))$ *with* $0 \leqslant \bar{V} \leqslant v_{\max}$, *we have* $\bar{C}(c) = C(c)$ *and* $\bar{V}(\cdot) = V(\cdot + \xi_0)$ *for some* $\xi_0 \in R$.

**Proof.** We extend the standard proof in [3,6,10] to our case. Since $V$ and $\bar{V}$ have the same limit as $\xi \to \pm\infty$, there exist $\bar{\xi} \in R$ and a sufficiently large $h > 0$ such that for every $s \in [-r_1, 0]$ and $x \in R$,

$$V(x + C(c)s + \bar{\xi}) - \bar{\delta} < \bar{V}(x + \bar{C}(c)s) < V(x + C(c)s + \bar{\xi} + h) + \bar{\delta},$$

and

$$\begin{aligned} & V(x + C(c)s + \bar{\xi} - \sigma_0 \bar{\delta}(\mathrm{e}^{\beta_0 r_1} - \mathrm{e}^{-\beta_0 s}) - \bar{\delta}\mathrm{e}^{-\beta_0 s} \\ & \quad < \bar{V}(x + \bar{C}(c)s) \\ & \quad < V(x + C(c)s + \bar{\xi} + h + \sigma_0 \bar{\delta}(\mathrm{e}^{\beta_0 r_1} - \mathrm{e}^{-\beta_0 s}) + \bar{\delta}\mathrm{e}^{-\beta_0 s}, \end{aligned}$$

where $\beta_0$, $\sigma_0$ and $\bar{\delta}$ are constants given in Lemma 3.4. Noting that the operator $F(v)(\cdot)$ defined in Lemma 3.1 is non-decreasing if $v$ is non-decreasing, we can still use the comparison result to obtain that for all $t \geqslant 0$ and $x \in R$,

$$\begin{aligned} & V(x + C(c)t + \bar{\xi} - \sigma_0 \bar{\delta}(\mathrm{e}^{\beta_0 r_1} - \mathrm{e}^{-\beta_0 t}) - \bar{\delta}\mathrm{e}^{-\beta_0 t} \\ & \quad < \bar{V}(x + \bar{C}(c)t) \\ & \quad < V(x + C(c)t + \bar{\xi} + h + \sigma_0 \bar{\delta}(\mathrm{e}^{\beta_0 r_1} - \mathrm{e}^{-\beta_0 t}) + \bar{\delta}\mathrm{e}^{-\beta_0 t}. \end{aligned}$$

Keeping $\xi = x + C(c)t$ fixed and letting $t \to \infty$, we have from the first inequality that $C(c) \leqslant \bar{C}(c)$ and from the second inequality that $C(c) \geqslant \bar{C}(c)$. This yields $C(c) = \bar{C}(c)$. Moreover, we get

$$V(\xi + \bar{\xi} - \sigma_0 \bar{\delta}\mathrm{e}^{\beta_0 r_1}) < \bar{V}(\xi) < V(\xi + \bar{\xi} + h + \sigma_0 \bar{\delta}\mathrm{e}^{\beta_0 r_1}) \quad \text{for } \xi \in R. \tag{3.34}$$

Define

$$\xi^* = \inf\{\xi;\ \bar{V}(\cdot) \leqslant V(\cdot + \xi)\}, \qquad \xi_* = \sup\{\xi;\ \bar{V}(\cdot) \geqslant V(\cdot + \xi)\}.$$

By (3.34), we find that both $\xi^*$ and $\xi_*$ are well defined. In particular, since $V(\cdot + \xi_*) \leqslant \bar{V}(\cdot) \leqslant V(\cdot + \xi^*)$, we have $\xi_* \leqslant \xi^*$.

To complete the proof, we show now $\xi_* \geqslant \xi^*$. Assume to the contrary that $\xi_* < \xi^*$ and $\bar{V}(\cdot) \not\equiv V(\cdot + \xi^*)$. Since $\lim_{\xi \to \infty} V'(\xi) = 0$, it follows that there exists a large constant $\bar{M} = \bar{M}(V) > 0$ such that

$$V'(\xi) \leqslant 1 \quad \text{if } |\xi| \geqslant \bar{M}.$$

By the fact that $\bar{V}(\cdot) \leqslant V(\cdot + \xi^*)$ and $\bar{V}(\cdot) \not\equiv V(\cdot + \xi^*)$, we can conclude from Remark 3.4 that $\bar{V}(\cdot) < V(\cdot + \xi^*)$ on $R$. Therefore, by the continuity of $V$ and $\bar{V}$, there exists a small $\bar{h} > 0$ such that

$$\bar{V}(\xi) < V(\xi + \xi^* - \bar{h}) \tag{3.35}$$

provided that $\xi \in [-\bar{M} - 1 - \xi^*, \bar{M} + 1 - \xi^*]$. When $|\xi + \xi^*| \geqslant \bar{M} + 1$, we have

$$\begin{aligned} V(\xi + \xi^* - \bar{h}) - \bar{V}(\xi) &> V(\xi + \xi^* - \bar{h}) - V(\xi + \xi^*) \\ &= -\bar{h}V'(\xi + \xi^* - \theta\bar{h}) \\ &\geqslant -\bar{h}, \end{aligned}$$

which, together with (3.35), implies that for any $s \in [-r_1, 0]$ and $x \in R$,

$$V(x + C(c)s + \xi^* - \bar{h} + \sigma_0\bar{h}(e^{\beta_0 r_1} - e^{-\beta_0 s})) + \bar{h}e^{-\beta_0 s} \geqslant \bar{V}(x + C(c)s).$$

Therefore, we have by the comparison principle that

$$V(x + C(c)t + \xi^* - \bar{h} + \sigma_0\bar{h}(e^{\beta_0 r_1} - e^{-\beta_0 t})) + \bar{h}e^{-\beta_0 t} \geqslant \bar{V}(x + C(c)t). \tag{3.36}$$

In (3.36), as before we keep $\xi = x + C(c)t$ fixed and let $t \to \infty$ to obtain

$$V(\xi + \xi^* - \bar{h}) \geqslant \bar{V}(\xi).$$

This contradicts the definition of $\xi^*$ since $\bar{h} > 0$. Hence $\xi_* = \xi^*$ and the proof is complete. □

**Remark 3.5.** In particular, when $r = 0$ and $r_2 = 0$, system (3.1) is independent of $c$. Therefore, if taking $r_1 = \tau$, then by Theorem 3.1 we know that the travelling wavefront to (3.1) (or to (1.3)) is unique (up to translation). This result can also be seen in [6].

## 4. Existence of travelling wavefront for Eq. (1.3)

To give the existence of travelling wavefront for Eq. (1.3), we first consider system (3.1) with delay $r_1 = 0$ and $r_2 = \tau$, namely,

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) = D(c)\dfrac{\partial^2 w(t,x)}{\partial x^2} - d_1 w(t,x) \\ \qquad\qquad + \varepsilon\int_{-\infty}^{\infty} G(x - c\tau - y)b(\varphi(t,y)\,dy, \quad t \geqslant 0, \\ \varphi(t,x) = w(t,x) + r\varepsilon\int_{-\infty}^{\infty} f(x - c\tau - y)b(\varphi(t,y))\,dy, \quad t \geqslant 0, \end{cases} \tag{4.1}$$

where $c \in R$ is a parameter.

Let $\zeta \in C^\infty(R, R)$ be a fixed function with the following properties:

$$\zeta(s) = 0 \quad \text{if } s \leqslant -2; \qquad \zeta(s) = 1 \quad \text{if } s \geqslant 2;$$

$$0 < \zeta'(s) < 1; \qquad |\zeta''(s)| \leqslant 1 \quad \text{if } s \in (-2, 2).$$

Then we have the following result.

**Lemma 4.1.** *Assume that the parameters $c$ and $r$ satisfy* (2.6), (2.7) *and* (3.5). *Then there exist two small constants $\delta^* > 0$, $\varepsilon_0 > 0$ and a large constant $C_0 > 0$, which are independent of $c$ and $\tau$ such that $(v_1^+(t,x), F(v_1^+))$ and $(v_1^-(t,x), F(v_1^-))$ defined by*

$$v_1^+(t,x) = v_{\max} + \delta^* - v_{\max}\zeta(-\varepsilon_0(x + C_0 t))$$

*and*

$$v_1^-(t,x) = -\delta^* + v_{\max}\zeta(\varepsilon_0(x - C_0 t))$$

*are a supersolution of* (4.1) *for* $c \geqslant 0$, *and a subsolution of* (3.1) *for* $c \leqslant 0$, *respectively*.

**Proof.** From (3.9), we have $0 < F'(0) = 1/(1 - r\varepsilon b'(0))$. By (H3′), we can find three constants $\rho \in [1/2, 1)$, $l > 0$ and $\delta^* < \min\{u^+/2, (K - u^+)/2, \delta_0/2\}$ such that

$$\rho d_1 > \frac{F(-\delta^*)}{-\delta^*}\,\varepsilon \int_{-\infty}^{\infty} G(y)\,\mathrm{d}y \times [\max\{b'(0), b'(K)\} + l],$$

$$\left(\frac{1}{\rho} - \rho\right)\delta^* < K,$$

$$0 \leqslant b'(\eta) < b'(0) + l \quad \text{for } \eta \in [-2\delta^*, 2\delta^*], \tag{4.2}$$

and

$$0 \leqslant b'(\eta) < b'(K) + l \quad \text{for } \eta \in [K - 2\delta^*, K + 2\delta^*].$$

We can choose two positive constants $\varepsilon^* > 0$ and $M_0 > 0$, with $\varepsilon^*$ sufficiently small and $M_0$ sufficiently large, such that

$$v_{\max}\varepsilon^* < 2(1 - \rho)\delta^*$$

and

$$-\min\{M_1(r,\delta^*), M_2(r,\delta^*)\} + K\varepsilon b'_{\max}\varepsilon^* + 2K\varepsilon b'_{\max}\left[\int_{M_0}^{\infty} + \int_{-\infty}^{-M_0} G(y)\,\mathrm{d}y\right] < 0.$$

Take $\kappa \in (0,1)$ sufficiently small such that

$$0 \leqslant \zeta(s) < \frac{\varepsilon^*}{2} \quad \text{if } s < -2 + \kappa,$$

$$1 \geqslant \zeta(s) > 1 - \frac{\varepsilon^*}{2} \quad \text{if } s > 2 - \kappa. \tag{4.3}$$

Take $\bar{\omega} > 0$ small enough such that

$$(1 - \bar{\omega})(2 - \kappa/2) > 2 - \kappa.$$

Finally, take $\varepsilon_0 > 0$ small enough such that

$$\varepsilon_0 M_0 \leqslant \bar{\omega}(2 - \kappa),$$

$$D\varepsilon_0^2 v_{\max} - \delta^*\left\{d_1\rho + \frac{F(-\delta^*)}{\delta^*}\varepsilon(b'(0) + l)\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y\right\} < 0 \tag{4.4}$$

and

$$D(c)\varepsilon_0^2 v_{\max} + 2\varepsilon b'_{\max} v_{\max}\varepsilon^* \int_{-\infty}^{\infty} G(y)\,\mathrm{d}y + 4\varepsilon v_{\max}\left[\int_{-\infty}^{-M_0} + \int_{M_0}^{\infty} G(y)\,\mathrm{d}y\right] < \min\{M_1(r,\delta^*), M_2(r,\delta^*)\}. \tag{4.5}$$

We also set

$$\tilde{M} = \min\{\zeta'(s); -2 + \kappa/2 \leqslant s \leqslant 2 - \kappa/2\} > 0$$

and take

$$C_0 = \frac{1}{\varepsilon_0 v_{\max}\tilde{M}}\left[D\varepsilon_0^2 v_{\max} + d_1 v_{\max} + \varepsilon b(F(v_{\max}))\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y + 4\varepsilon v_{\max} b' \max\right] > 0. \tag{4.6}$$

Next we shall prove that $(v_1^-, F(v_1^-))$ is a subsolution of (4.1) for $c \leqslant 0$. The proof that $(v_1^+, F(v_1^+))$ is a supersolution of (4.1) for $c \geqslant 0$ is analogous and hence is omitted. Set $\xi = x - C_0 t$. Substituting $(v_1^-, F(v_1^-))$ into (4.1), we find that the second equation is naturally satisfied. For the first equation, we have

$$\begin{aligned} S(v_1^-(t,x)) := {} & \frac{\partial v_1^-(t,x)}{\partial t} - D(c)\,\frac{\partial^2 v_1^-(t,x)}{\partial x^2} + d_1 v_1^-(t,x) \\ & - \varepsilon \int_{-\infty}^{\infty} G(x - c\tau - y) b(F(v_1^-))\,\mathrm{d}y \qquad (4.7) \\ = {} & - C_0\varepsilon_0 v_{\max}\zeta'(\varepsilon_0\xi) - D(c)\varepsilon_0^2 v_{\max}\zeta''(\varepsilon_0\xi) + d_1 v_1^-(t,x) \\ & - \varepsilon \int_{-\infty}^{\infty} G(x - c\tau - y) b(F(v_1^-))\,\mathrm{d}y \\ = {} & - C_0\varepsilon_0 v_{\max}\zeta'(\varepsilon_0\xi) - D(c)\varepsilon_0^2 v_{\max}\zeta''(\varepsilon_0\xi) + d_1 v_1^-(t,x) \\ & - \varepsilon b(F(v_1^-))(t,x)\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y \\ & - \varepsilon \int_{-\infty}^{\infty} [b(F(v_1^-)(t, x - c\tau - y)) - b(F(v_1^-)(t,x))]G(y)\,\mathrm{d}y \\ = {} & - C_0\varepsilon_0 v_{\max}\zeta'(\varepsilon_0\xi) - D(c)\varepsilon_0^2 v_{\max}\zeta''(\varepsilon_0\xi) \\ & + d_1 v_1^-(t,x) - \varepsilon b(F(v_1^-))(t,x)\int_{-\infty}^{\infty} G(y)\,\mathrm{d}y \\ & - \varepsilon \int_{-\infty}^{\infty} b'(\eta)[F(v_1^-)(t, x - c\tau - y) - F(v_1^-)(t,x)]G(y)\,\mathrm{d}y, \end{aligned} \tag{4.8}$$

where $\eta = \theta F(v_1^-))(t,x) + (1-\theta)F(v_1^-))(t, x - c\tau - y)$.

Since $\varepsilon_0\xi \in (-\infty, \infty)$, for our proof we shall split the interval $(-\infty, \infty)$ into three parts: $(-\infty, -2 + \kappa/2]$, $[2 - \kappa/2, \infty)$ and $(-2 + \kappa/2, 2 - \kappa/2)$.

Case (i): $\varepsilon_0\xi \in (-\infty, -2 + \kappa/2]$.

In this case we have $\varepsilon_0\xi \leqslant -2 + \kappa$, $0 \leqslant \zeta(\varepsilon_0\xi) \leqslant \varepsilon^*/2$,

$$-\delta^* \leqslant v_1^-(t,x) \leqslant -\delta^* + v_{\max}\varepsilon^*/2 < -\delta^* + (1-\rho)\delta^* = -\rho\delta^* \leqslant -\tfrac{1}{2}\delta^*,$$

and

$$F(-\delta^*) \leqslant F(v_1^-)(t,x).$$

Set $E_i(t,x)=\{y\in R;\ F(v_1^-)(t,x-c\tau-y)\leqslant 0\}$. It then follows from (4.2), (4.4) and (4.7) that

$$\begin{aligned}S(v_1^-)(t,x)&\leqslant D(c)\varepsilon_0^2 v_{\max}+d_1 v_1^-(t,x)-\varepsilon\int_{E_i(t,x)} b'(\bar\eta)F(v_1^-)\\&\quad\times(t,x-c\tau-y)G(y)\,\mathrm{d}y\\&\leqslant D(c)\varepsilon_0^2 v_{\max}-d_1\rho\delta^*-F(-\delta^*)(b'(0)+l)\int_{-\infty}^{\infty}G(y)\,\mathrm{d}y\\&=D(c)\varepsilon_0^2 v_{\max}-\delta^*\left\{d_1\rho+\frac{F(-\delta^*)}{\delta^*}\varepsilon(b'(0)+l)\int_{-\infty}^{\infty}G(y)\,\mathrm{d}y\right\}\\&\leqslant 0,\end{aligned}$$

where $\bar\eta=\theta F(v_1^-)(t,x-c\tau-y)\in[F(-\delta^*),0]\in[-2\delta^*,0]$.

Case (ii): $\varepsilon_0\xi\geqslant 2-\kappa/2$.

In this case, $\varepsilon_0\xi\geqslant 2-\kappa/2$, $1-\varepsilon^*/2\leqslant\zeta(\varepsilon_0\xi)\leqslant 1$,

$$\begin{aligned}v_{\max}-\delta^*\geqslant v_1^-(t,x)&\geqslant-\delta^*+v_{\max}(1-\varepsilon^*/2)\geqslant v_{\max}-\delta^*-(1-\rho)\delta^*\\&\geqslant v_{\max}-\tfrac{3}{2}\delta^*,\end{aligned}$$

and thus from (3.15)

$$d_1 v_1^-(t,x)-\varepsilon b(F(v_1^-))(t,x)\int_{-\infty}^{\infty}G(y)\,\mathrm{d}y\leqslant -M_1(r,\delta^*).$$

By the choice of $\varepsilon_0$ and $\bar\omega$, we have

$$\bar\omega\xi\geqslant\frac{\bar\omega(2-\kappa/2)}{\varepsilon_0}\geqslant M_0.$$

For $c\leqslant 0$ and $y\in[-\bar\omega\xi,\bar\omega\xi]$, it follows

$$\varepsilon_0(\xi-c\tau-y)\geqslant\varepsilon_0(1-\bar\omega)\xi-\varepsilon_0 c\tau\geqslant(1-\bar\omega)\varepsilon_0\xi\geqslant(1-\bar\omega)(2-\kappa/2)\geqslant 2-\kappa,$$

and hence from (3.7) and (4.3) we obtain

$$\begin{aligned}&\int_{-\bar\omega\xi}^{\bar\omega\xi}|F(v_1^-)(t,x-c\tau-y)-F(v_1^-)(t,x)|G(y)\,\mathrm{d}y\\&\quad\leqslant 2\int_{-\bar\omega\xi}^{\bar\omega\xi}|v_1^-(t,x-c\tau-y)-v_1^-(t,x)|G(y)\,\mathrm{d}y\\&\quad\leqslant 2v_{\max}\int_{-\bar\omega\xi}^{\bar\omega\xi}|\zeta(\varepsilon_0\xi-c\tau-y)-\zeta(\varepsilon_0\xi)|G(y)\,\mathrm{d}y\\&\quad\leqslant 2v_{\max}\varepsilon^*\int_{-\infty}^{\infty}G(y)\,\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{-\bar{\omega}\xi} + \int_{\bar{\omega}\xi}^{\infty} |F(v_1^-)(t, x - c\tau - y) - F(v_1^-)(t, x)|G(y)\,\mathrm{d}y \\
&\quad \leqslant \int_{-\infty}^{-\bar{\omega}\xi} + \int_{\bar{\omega}\xi}^{\infty} 2\|v_1^-(t, x - c\tau - y) - v_1^-(t, x)\|G(y)\,\mathrm{d}y \\
&\quad \leqslant 4v_{\max}\left(\int_{-\infty}^{-\bar{\omega}\xi} + \int_{\bar{\omega}\xi}^{\infty} G(y)\,\mathrm{d}y\right).
\end{aligned}$$

Therefore, we have from (4.5) and (4.8) that

$$\begin{aligned}
S(v_1^-)(t, x) &\leqslant D(c)\varepsilon_0^2 v_{\max} - M_1(r, \delta^*) \\
&\quad + b'_{\max}\varepsilon \int_{-\bar{\omega}\xi}^{\bar{\omega}\xi} |F(v_1^-)(t, x - c\tau - y) - F(v_1^-)(t, x)|G(y)\,\mathrm{d}y \\
&\quad + 4\varepsilon b'_{\max} v_{\max}\left[\int_{-\infty}^{-\bar{\omega}\xi} + \int_{\bar{\omega}\xi}^{\infty} G(y)\,\mathrm{d}y\right] \\
&\leqslant D(c)\varepsilon_0^2 v_{\max} - M_1(r, \delta^*) + 2\varepsilon b'_{\max} v_{\max}\varepsilon^* \int_{-\infty}^{\infty} G(y)\,\mathrm{d}y \\
&\quad + 4\varepsilon b'_{\max} v_{\max}\left[\int_{-\infty}^{-M_0} + \int_{M_0}^{\infty} G(y)\,\mathrm{d}y\right] \\
&< 0.
\end{aligned}$$

Case (iii): In this case, we have from (4.6) and (4.8) that

$$\begin{aligned}
S(v_1^-)(t, x) &\leqslant -C_0\varepsilon_0 v_{\max}\tilde{M} + D\varepsilon_0^2 v_{\max} + d v_{\max} + b(F(v_{\max})) \\
&\quad \times \int_{-\infty}^{\infty} G(y)\,\mathrm{d}y + 4v_{\max} b'_{\max} \\
&= 0.
\end{aligned}$$

Combining cases (i)–(iii), we obtain the desired result and hence the proof is complete. □

**Remark 4.1.** In a similar manner, we can also prove that $v_2^+(t, x) = v_{\max} + \delta^* - v_{\max}\zeta(-\varepsilon_c(x + C_c t))$ and $v_2^-(t, x) = -\delta^* + v_{\max}\zeta(\varepsilon_c(x - C_c t))$ are a supersolution and subsolution of (4.1) for all $c$ satisfying (2.6), respectively. Here $\varepsilon_c = \varepsilon_0/(1 + |c|\tau)$ and $C_c = (1 + |c|\tau)C_0$.

**Lemma 4.2.** *For every c and r satisfying* (2.6), (2.7) *and* (3.5), *there exists a unique strictly monotonic travelling wavefront solution* $(V(x + C(c)t), F(V)(x + C(c)t))$ *for system* (4.1) *with speed* $C(c)$ *being a continuous function of* $c$.

**Proof.** The proof for the existence of travelling wavefront is similar to that of Appendix in [6]; see also the proof of Theorem 3.1 in [3]. The monotonicity of this wavefront can be obtained from Lemma 3.2 (or Remark 3.4). Furthermore, we can obtain the uniqueness result directly from Theorem 3.1.

Next we shall show that $C(c)$ is a continuous function of $c$.

Suppose that $(V_c(x+C(c)t), F(V_c)(x+C(c)t))$ is the travelling wavefront with speed $C(c)$. Without loss of generality, we assume that $0 < V_c(0) = F(\bar{u}) < v_{\max}$. Then we have

$$\begin{cases} C(c)V_c' = D(c)V_c''(\xi) - d_1 V(\xi) + \varepsilon\int_{-\infty}^{\infty} G(\xi - c\tau - y)b(U(y))\,\mathrm{d}y, \\ U(\xi) = V(\xi) + r\varepsilon\int_{-\infty}^{\infty} f(\xi - c\tau - y)b(U(\xi))\,\mathrm{d}y, \end{cases}$$

where $\xi = x + C(c)t$, or equivalently,

$$C(c)V_c' = D(c)V_c''(\xi) - d_1 V(\xi) + \int_{-\infty}^{\infty} G(\xi - c\tau - y)b(F(V)(y))\,\mathrm{d}y.$$

Hence re-writing the above equation into an integral one, we obtain

$$V_c(\xi) = \frac{1}{D(c)(\lambda_2(C(c)) - \lambda_1(C(c)))}\left[\begin{array}{l}\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(C(c))(\xi - s)} H_c(V_c)(s)\,\mathrm{d}s \\ \quad + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(C(c))(\xi - s)} H_c(V_c)(s)\,\mathrm{d}s\end{array}\right], \tag{4.9}$$

where

$$\lambda_1(C(c)) = \frac{C(c) - \sqrt{C^2(c) + 4D(c)d_1}}{2D(c)} < 0,$$
$$\lambda_2(C(c)) = \frac{C(c) + \sqrt{C^2(c) + 4D(c)d_1}}{2D(c)} > 0,$$

and

$$H_c(V_c)(s) = \int_{-\infty}^{\infty} G(s - c\tau - y)b(F(V)(y))\,\mathrm{d}y.$$

Since $0 \leqslant V \leqslant v_{\max}$, $0 \leqslant F(V) \leqslant K$ and

$$\lambda_2(C(c)) - \lambda_1(C(c)) = \frac{\sqrt{C^2(c) + 4D(c)d_1}}{D(c)} \geqslant 2\sqrt{\frac{d_1}{D(c)}},$$

using a similar argument as that in Lemma 3.5, we can obtain

$$|V_c'(\xi)| \leqslant \frac{b(K)}{2\sqrt{D(c)d_1}}.$$

Suppose that $c_n$ satisfies (2.6) and $c_n \to c$, but $C(c_n)$ does not converge to $C(c)$, then there exists a subsequence $c_{n_k} \to c$ so that $C(c_{n_k}) \to b \neq C(c)$. By the Arzela–Ascoli theorem, we can choose a subsequence of $\{c_{n_k}\}$, also denoted by $\{c_{n_k}\}$, such that $V_{c_{n_k}}(.)$ converges to a continuous function $\bar{V}(\cdot)$ in $R$. Let $H^* = \sup\{|c_n|\}$. Since $V_{c_{n_k}}(\cdot)$ is non-decreasing, $V_{c_{n_k}}(0) = F(\bar{u})$ and by (A.3) and (A.4) in the Appendix of [6], there exist a small positive constant $\delta_*$, two large positive constants $M^*$ and $L^*$, which are independent of $c$ and $\tau$ so that

$$V_{c_{n_k}}(x) \leqslant F(\bar{u}) - \delta_* \quad \text{if } x \leqslant -M^* - L^*H^*\tau \leqslant -M^* - L^*|c_{n_k}|\tau,$$

and

$$V_{c_{n_k}}(x) \geqslant F(\bar{u}) - \delta_* \quad \text{if } x \geqslant M^* + L^*H^*\tau \geqslant M^* + L^*|c_{n_k}|\tau.$$

It then follows that $\bar{V}(\cdot)$ is non-decreasing, $0 \leqslant \bar{V}(\cdot) \leqslant v_{\max}$ and

$$\limsup_{x\to-\infty} \bar{V}(x) \leqslant F(\bar{u}) - \delta_*, \quad \liminf_{x\to\infty} \bar{V}(x) \geqslant F(\bar{u}) - \delta_*.$$

In Eq. (4.9) with $c$ being replaced by $c_{n_k}$, we let $k \to \infty$ and apply the dominant convergence theorem to get

$$\begin{aligned}\bar{V}(\xi) = &\frac{1}{D(c)(\lambda_2(b) - \lambda_1(b))} \\ &\times \left[ \int_{-\infty}^{\xi} e^{\lambda_1(b)(\xi-s)} H_c(V_c)(s)\, ds + \int_{\xi}^{\infty} e^{\lambda_2(b)(\xi-s)} H_c(V_c)(s)\, ds \right],\end{aligned}$$

which means that $\bar{V}(x + bt)$ is also a solution of (4.1). But for the given parameter $c$, by the uniqueness of travelling wavefront of Eq. (4.1) we obtain that $b = C(c)$, which is a contradiction. This completes the proof. □

**Theorem 4.1.** *Assume that r satisfies*

$$D - r^2 C_0 > 0. \tag{4.10}$$

*Then* (1.3) *admits a strictly monotonic travelling wavefront* $(V(x + c^*t), F(V)(x + c^*t))$ *with* $|c^*| \leqslant C_0$, *where* $C_0$ *is provided in Lemma* 4.1.

**Proof.** By (4.10), we know that for any $c$ satisfying $|c| \leqslant C_0$, (2.6) and (2.7) hold. Therefore, we have from Lemma 4.2 that there exists a strictly monotonic travelling wavefront $(V(x + C(c)t), F(V)(x + C(c)t))$ for Eq. (4.1). Next we will show that there exists at least one $c^*$ so that $C(c^*) = c^*$ and $|c^*| \leqslant C_0$.

To this end, it suffices to prove that the curves $y=c$ and $y=C(c)$ have at least one common point in region $|y| \leqslant C_0$ of the $(c, y)$ plane. For $c \leqslant 0$, let $v_1^-(t, x)$ be the subsolution of (4.1) given in Lemma 4.1. Then there exists a large positive constant $X$ such that $V(\cdot) \geqslant v_1^-(0, \cdot - X)$. Therefore, by the comparison, it follows that $V(x + C(c)t) \geqslant v_-^-(t, x - X)$ for all $t \geqslant 0$ and $x \in R$. Hence by the choice of $\delta^*$ (letting $\delta^* \to 0$), it is easy to deduce that $C(c) \geqslant -C_0$. Similarly, we can show that $C(c) \leqslant C_0$ for $c \geqslant 0$. We know from Lemma 4.2 that $C(c)$ is a continuous function of $c$ for any $|c| < C_0 < \sqrt{D/r}$. Thus by (4.10) we conclude that there is at least one common point $c^*$ so that $C(c^*) = c^*$ in the region $|c| < C_0 < \sqrt{D/r}$ and $|y| \leqslant C_0$. This completes the proof. □

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