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Regular dynamics in a delayed network of two neurons with all-or-none activation functions

Shangjiang Guo[a,*], Lihong Huang[a], Jianhong Wu[b]

[a] *College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China*
[b] *Department of Mathematics and Statistics, York University, Toronto, Ont., Canada M3J 1P3*

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Abstract

We consider a delayed network of two neurons with both self-feedback and interaction described by an all-or-none threshold function. The model describes a combination of analog and digital signal processing in the network and takes the form of a system of delay differential equations with discontinuous nonlinearity. We show that the dynamics of the network can be understood in terms of the iteration of a one-dimensional map, and we obtain simple criteria for the convergence of solutions, the existence, multiplicity and attractivity of periodic solutions.

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1. Introduction

We consider the following model for an artificial network of two neurons

$$\begin{cases} \dot{x} = -\mu x + a_{11} f(x(t-\tau)) + a_{12} f(y(t-\tau)), \\ \dot{y} = -\mu y + a_{21} f(x(t-\tau)) + a_{22} f(y(t-\tau)), \end{cases} \tag{1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $x(t)$ and $y(t)$ denote the state variables associated with the neurons, $\mu > 0$ is the interact decay rate, $\tau > 0$ is the synaptic transmission delay, a_{11}, a_{12}, a_{21} and a_{22} are the synaptic weights, and $f : \mathbb{R} \to \mathbb{R}$ is the

* Corresponding author.
E-mail address: shangjguo@etang.com (S. Guo).

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activation function. Such a model describes the evolution of the so-called Hopfield net [6,7,15] where each neuron is represented by a linear circuit consisting of a resistor and a capacitor, and where each neuron is connected to another via the nonlinear activation function *f* multiplied by the synaptic weights $a_{ij}(i \neq j)$. We also assume that each neuron has self-feedback and signal transmission is delayed due to the finite switching speed of neurons.

We focus here on the computational performance described by the asymptotical behaviors of model (1), where the signal transmission is of digital nature: a neuron is either fully active or completely inactive. Namely, the signal transmission is of McCulloch–Pitts type [8–10,16,17,21] and we have

$$f(\xi) = \begin{cases} -\delta, & \text{if } \xi > 0, \\ \delta, & \text{if } \xi \leq 0, \end{cases} \tag{2}$$

where $\delta > 0$ is a given constant. Therefore, the model describes a combination of analog and digital signal processing. Differential equations of this type usually occur in control systems, e.g., in heating systems and the pupil light reflex, if the controlling function is determined by a constant delay $\tau > 0$ and the switch recognizes only the positions "*on*" $[f(\xi) = \delta]$ and "*off*" $[f(\xi) = -\delta]$. Because each variable changes continuously but depends on the signs of other variables, such a system retains a continuous-time framework and can be proposed as a useful simplification to gain analytical insight (see, for example [4]). In addition, a rather confusing variety of names have been applied to this system, such as "Glass networks"(see, for example [4,5]), "piecewise-linear equations", "switching networks", "nonlinear chemical reaction networks", "gene networks", "Boolean kinetic equations" and variants of these. Here we avoid this confusion by calling it "McCulloch–Pitts networks". By the discontinuous nonlinearity, the differential equation allows detailed analysis. It turns out that there is a rich solution structure. To simplify our presentation, we first rescale the variables by

$$t^* = \mu t, \quad \tau^* = \mu\tau, \quad x^*(t^*) = \frac{\mu}{\delta}x(t), \quad y^*(t^*) = \frac{\mu}{\delta}y(t), \quad f^*(\xi) = \frac{1}{\delta}f\left(\frac{\delta}{\mu}\xi\right),$$

and then drop the $*$ to get

$$\begin{cases} \dot{x} = -x + a_{11}f(x(t-\tau)) + a_{12}f(y(t-\tau)), \\ \dot{y} = -y + a_{21}f(x(t-\tau)) + a_{22}f(y(t-\tau)) \end{cases} \tag{3}$$

with

$$f(\xi) = \begin{cases} -1, & \text{if } \xi > 0, \\ 1, & \text{if } \xi \leq 0. \end{cases} \tag{4}$$

It is natural to have the phase space $X = C([-\tau, 0]; \mathbb{R}^2)$ as the Banach space of continuous mappings from $[-\tau, 0]$ to $\mathbb{R}^2$ equipped with the sup-norm, see [13]. Note that for each given initial value $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X$, one can solve system (3) on intervals $[0, \tau]$, $[\tau, 2\tau]$, ... successively to obtain a unique mapping $(x^{\Phi}, y^{\Phi})^{\mathrm{T}} : [-\tau, \infty) \to \mathbb{R}^2$ such that $x^{\Phi}|_{[-\tau,0]} = \varphi$, $y^{\Phi}|_{[-\tau,0]} = \psi$, $(x^{\Phi}, y^{\Phi})^{\mathrm{T}}$ is continuous for all $t \geq 0$, piecewise differentiable and satisfies (3) for $t > 0$. This gives a unique solution of (3) defined for all $t \geq -\tau$. In applications, a network usually starts from a constant (or nearly constant) state. Therefore, we shall concentrate on the case where each component of Φ has no sign change and has at most finitely many zeros on $[-\tau, 0]$. More precisely, we consider $\Phi \in X^{+,+} \bigcup X^{+,-} \bigcup X^{-,+} \bigcup X^{-,-} = X_0$, where

$$C^{\pm} = \{\pm\varphi;\ \varphi : [-\tau, 0] \to [0, \infty) \text{ is continuous and has only finitely many zeros on } [-\tau, 0]\}$$

and

$$X^{\pm,\pm} = \{\Phi \in X; \quad \Phi = (\varphi, \psi)^{\mathrm{T}}, \varphi \in C^{\pm} \text{ and } \psi \in C^{\pm}\}.$$

Clearly, all constant initial values (except for 0) are contained in X_0.

Obviously, the synaptic weights have a fundamental effect on the dynamics of the networks. For a particular connection topology, Guo and Huang [8] have shown that all solutions starting from nonoscillatory initial states will be eventually synchronized and stabilized at a unique limit cycle, and hence such a network can be used as a synchronized oscillator. In this paper, however, we consider the following connection topology:

(H1). $a_{11} + a_{12} = 0, a_{11} > 0; \quad a_{21} < 0, a_{21} < a_{22} \leq -a_{21}.$

In other words, we assume that the self-feedback to neuron 1 is inhibitory, the interaction from neuron 2 to neuron 1 is excitatory and this self-feedback and interaction have equal weights. We also assume that the interaction from neuron 1 to 2 is excitatory, but the self-feedback weight is denominated by the interaction.

By using (4) and some simple changes of variables, we can see that the semiflow defined by system (3) under the condition (H1) is topologically equivalent to that of (3) and (4) while one of the following two conditions is satisfied:

(H2). $a_{11} = a_{12} > 0; \quad a_{21} > 0, -a_{21} < a_{22} \leq a_{21};$

(H3). $a_{12} < 0, a_{12} < a_{11} \leq -a_{12}; \quad a_{21} = -a_{22} < 0.$

Despite the low number of units, two-neuron networks with delay often display the same dynamical behaviors as large networks and, can thus be used as prototypes for us to understand the dynamics of large networks with delayed feedback. Much has been done when the function *f* is smooth, see for example [1–3,11,12,22,24,28,29]. When *f* is discontinuous, however, results in the aforementioned work can not be verified as the dynamical systems theory which usually requires the continuity and smoothness of nonlinear functions involved. Recently, in [8,16,17,23], model Eq. (1) with piecewise constant activation was studied when the synaptic connection topology satisfies either $[|a_{12}| < a_{11}, |a_{22}| < a_{21}, a_{11}a_{22} - a_{12}a_{21} = 0]$, or $[a_{11} = a_{22} = 0, a_{12} = a_{21} = 1]$, or $[a_{11} = a_{22} = 0, a_{12} = -a_{21} = 1]$. Here, we consider only the case where (H1) is satisfied, and we will show that the dynamics of model Eq. (1) is quite regular, and is fully determined by the size of the parameters of the system and the ratio $\varphi(0)/\psi(0)$ of the initial value, via the connection with the interaction of a one-dimensional map.

2. Preliminary results

In this section, we establish several technical lemmas which will be used for the description of the dynamics of (3) under the condition (H1).

First, we further rescale variables in (3) by

$$u(t) = \frac{x(t)}{a_{11} - a_{12}}, \quad v(t) = \frac{y(t)}{a_{22} - a_{21}}, \quad B = \frac{a_{21} + a_{22}}{a_{21} - a_{22}}.$$

Then the rescaled variables satisfy

$$\begin{cases} \dot{u} = -u + \dfrac{1}{2} f(u(t-\tau)) - \dfrac{1}{2} f(v(t-\tau)), \\ \dot{v} = -v - \dfrac{1+B}{2} f(u(t-\tau)) + \dfrac{1-B}{2} f(v(t-\tau)). \end{cases} \tag{5}$$

The discontinuity of *f* makes it difficult to apply directly dynamical system theory to system (5). But, the simple form of (5) and (4) enables us to carry out a direct elementary analysis of the dynamics of the network due to its obvious connection with the following systems of linear nonhomogeneous ordinary differential equations:

$$\begin{cases} \dot{u} = -u, \\ \dot{v} = -v + B; \end{cases} \tag{6}$$

$$\begin{cases} \dot{u} = -u + 1, \\ \dot{v} = -v - 1; \end{cases} \tag{7}$$

$$\begin{cases} \dot{u} = -u, \\ \dot{v} = -v - B; \end{cases} \tag{8}$$

$$\begin{cases} \dot{u} = -u - 1, \\ \dot{v} = -v + 1. \end{cases} \tag{9}$$

For the sake of simplicity, in the remaining part of this paper, for a given $s \in [0, \infty)$ and a continuous function z: $[-\tau, \infty) \to \mathbb{R}$, we define a mapping $z_s : [-\tau, 0] \to \mathbb{R}$ by $z_s(\theta) = z(s + \theta)$ for $\theta \in [-\tau, 0]$.

First, we have the following observation:

Lemma 2.1. *If $(u(t), v(t))^{\mathrm{T}}$ is a solution of system (5) with initial value $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X_0$, then the solution of (5) with the initial value $\Phi = (-\varphi, -\psi)^{\mathrm{T}} \in X_0$* is $(-u(t), -v(t))^{\mathrm{T}}$.

Lemma 2.1 means that it suffices to investigate the asymptotical behaviors of solutions of system (5) with initial value $\Phi \in X_0$ satisfying $\varphi(0) + \psi(0) \geq 0$. In what follows, let $(u(t), v(t))^{\mathrm{T}}$ be a solution of (5) with initial value in X_0. For now we make an observation that we will need several times later.

Lemma 2.2. *If there exists some $t_0 \geq 0$ such that $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{-,+}$ and that $\varphi(t_0) + \psi(t_0) \geq 0$, then the first zero of $u(t) \cdot v(t)$ in $[t_0, \infty)$ is $t_1 = t_0 + \ln(1 - u(t_0))$. Moreover, we have $u(t_1) = 0$ and $v(t_1) = (u(t_0) + v(t_0))/(1 - u(t_0)) \geq 0$.*

Lemma 2.3. *If there exists some $t_0 \geq 0$ such that $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{+,-}$ and $u(t_0) + v(t_0) > 0$, then either*

(i) *there exists some $t_0^* \geq t_0$ such that $(u_t, v_t)^{\mathrm{T}} \in X^{+,+}$ for $t \geq t_0^* + \tau$; or*
(ii) *there exists some $t_0^* \geq t_0$ such that $(u_{t_0^*+\tau}, v_{t_0^*+\tau})^{\mathrm{T}} \in X^{-,+}$ and $u(t_0^* + \tau) + v(t_0^* + \tau) > 0$.*

Proof. We distinguish two cases.

Case 1. $[1 + u(t_0)]/[1 - v(t_0)] \geq \mathrm{e}^{\tau}$. From (5) and $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{+,-}$, $(u(t), v(t))^{\mathrm{T}}$ satisfies (9) for $t \in (t_0, t_0 + \tau)$. By the continuity of solutions, for $t \in [t_0, t_0 + \tau]$, we have

$$u(t) = (u(t_0) + 1)\,\mathrm{e}^{t_0 - t} - 1, \quad v(t) = (v(t_0) - 1)\,\mathrm{e}^{t_0 - t} + 1. \tag{10}$$

Let t_1 be the first zero of $u(t) \cdot v(t)$ in $[t_0, \infty)$. Then (10) holds for all $t \in [t_0, t_1 + \tau]$. On the other hand, $u(t) \cdot v(t) = 0$ implies

$$t = t_0 + \ln(1 + u(t_0)) \quad \text{or} \quad t = t_0 + \ln(1 - v(t_0)).$$

In view of $u(t_0) + v(t_0) > 0$, we have

$$t_1 = t_0 + \ln(1 - v(t_0)).$$

This, together with (10), implies that

$$u(t_1+\tau)=\frac{1+u(t_0)}{1-v(t_0)}\,\mathrm{e}^{-\tau}-1\geq 0,\qquad v(t_1+\tau)=1-\mathrm{e}^{\tau}>0.$$

Moreover, it is easy to see that $(u_{t_1+\tau}, v_{t_1+\tau})^{\mathrm{T}}\in X^{+,+}$.

Case 2. $1<[1+u(t_0)]/[1-v(t_0)]<\mathrm{e}^{\tau}$. Using a similar argument, we get

$$u(t_1+\tau)=\frac{1+u(t_0)}{1-v(t_0)}\,\mathrm{e}^{-\tau}-1<0\quad\text{and}\quad v(t_1+\tau)=1-\mathrm{e}^{\tau}>0.$$

Note that $u(t_0)<0$ and (10) holds, we have some $t_2\in[t_1, t_1+\tau]$ such that $u(t_2)=0$. In fact, by (10), we have

$$t_2=t_0+\ln(1+u(t_0)),$$

and we can easily show that $u(t-\tau)>0$ and $v(t-\tau)>0$ for $t\in(t_1+\tau, t_2+\tau)$. Therefore, $(u(t), v(t))^{\mathrm{T}}$ satisfies (6) for $t\in(t_1+\tau, t_2+\tau)$. Thus, for $t\in[t_1+\tau, t_2+\tau]$, we have

$$\begin{aligned}u(t)&=u(t_1+\tau)\,\mathrm{e}^{t_1+\tau-t}=\left[\frac{1+u(t_0)}{1-v(t_0)}\,\mathrm{e}^{-\tau}-1\right]\mathrm{e}^{t_1+\tau-t},\\ v(t)&=[v(t_1+\tau)-B]\,\mathrm{e}^{t_1+\tau-t}+B=(1-\mathrm{e}^{\tau}-B)\,\mathrm{e}^{t_1+\tau-t}+B.\end{aligned}\tag{11}$$

It follows that

$$\begin{aligned}u(t_2+\tau)&=\mathrm{e}^{-\tau}-\frac{1-v(t_0)}{1+u(t_0)}<0,\\ v(t_2+\tau)&=(1-\mathrm{e}^{-\tau}-B)\cdot\frac{1-v(t_0)}{1+u(t_0)}+B>0,\end{aligned}$$

and hence

$$u(t_2+\tau)+v(t_2+\tau)=(B+\mathrm{e}^{-\tau})\left[1-\frac{1-v(t_0)}{1+u(t_0)}\right]=(B+\mathrm{e}^{-\tau})\frac{u(t_0)+v(t_0)}{1+u(t_0)}>0.$$

From (10) and (11), it is easy to see that $(u_{t_2+\tau}, v_{t_2+\tau})^{\mathrm{T}}\in X^{-,+}$ and $u(t_2+\tau)+v(t_2+\tau)>0$. □

By Lemma 2.3, in order to discuss the asymptotical behaviors of $(u(t), v(t))^{\mathrm{T}}$ with initial value $\Phi\in X^{+,-}$ satisfying $\varphi(0)+\psi(0)>0$, it is sufficient to investigate the limiting behaviors of solution $(u(t), v(t))^{\mathrm{T}}$ as $t\to\infty$ for any initial value $\Phi\in X^{+,+}\bigcup X^{-,+}$ satisfying $\varphi(0)+\psi(0)>0$.

For the sake of convenience, we introduce two parameters M and m as follows

$$M=(1-\mathrm{e}^{-\tau})\left(\mathrm{e}^{\tau}-\frac{B}{B+1}\right),\qquad m=\frac{1-\mathrm{e}^{-\tau}}{B+\mathrm{e}^{-\tau}}.$$

In Section 3, we will show that the dynamics of the network can be understood in terms of the iteration of function F given by

$$F(x) = \begin{cases} f_1(x), & \text{if } x \in (m, M), \\ f_2(x), & \text{if } x \in (0, m], \\ 0, & \text{if } x = 0, \end{cases} \tag{12}$$

where

$$f_1(x) = \frac{B(B + \mathrm{e}^{-\tau})(\mathrm{e}^{\tau} + 1)x + B(\mathrm{e}^{\tau} - 1)(B - 1)}{(B\,\mathrm{e}^{\tau} - B - 1)x + (1 + 3B)(\mathrm{e}^{\tau} - 1) + B\,\mathrm{e}^{-\tau}}, \tag{13}$$

and

$$f_2(x) = \frac{(B + \mathrm{e}^{-\tau})^2 x}{(2 - 2\,\mathrm{e}^{-\tau} - B)x + (2 - \mathrm{e}^{-\tau})^2}. \tag{14}$$

Thus, we need to investigate the properties of function F. Obviously, $f_1(x)$ defined on (m, M) is continuous, monotonically increasing, and satisfies

$$x - f_1(x) = \frac{g(x)}{(B\,\mathrm{e}^{\tau} - B - 1)x + (1 + 3B)(\mathrm{e}^{\tau} - 1) + B\,\mathrm{e}^{-\tau}},$$

where

$$g(x) = (B\,\mathrm{e}^{\tau} - B - 1)x^2 + [(1 + 3B)(\mathrm{e}^{\tau} - 1) + B\,\mathrm{e}^{-\tau} - B(B + \mathrm{e}^{-\tau})(\mathrm{e}^{\tau} + 1)]x - B(B - 1)(\mathrm{e}^{\tau} - 1).$$

It is an easy exercise to show that $x - f_1(x)$ and $g(x)$ have the same sign on (m, M). Obviously, $y = g(x)$ is a quadratic function with respect to x. Moreover,

$$g(-B) = 2B^2\,\mathrm{e}^{\tau}[B + 2(\mathrm{e}^{-\tau} - 1)], \quad g(M) = \frac{B(\mathrm{e}^{\tau} - 1)}{(B + 1)^2} \cdot h(B),$$

where

$$\begin{aligned} h(B) = {} & B^3(\mathrm{e}^{-\tau} - 1 - \mathrm{e}^{\tau}) + B^2(\mathrm{e}^{2\tau} - 3\mathrm{e}^{\tau} + \mathrm{e}^{-\tau} + \mathrm{e}^{-2\tau} - 3) \\ & + B(2\,\mathrm{e}^{2\tau} - \mathrm{e}^{\tau} + \mathrm{e}^{-2\tau} - 4) + \mathrm{e}^{2\tau} + \mathrm{e}^{\tau} - \mathrm{e}^{-\tau} - 1. \end{aligned}$$

Our motivation here is to determine the sign of $g(x)$ on (m, M) by investigating the properties of its graph. Thus, it is necessary to consider the zeros of the cubic function $h(B)$. In fact, we have

Lemma 2.4. *The polynomial $h(B)$ has one and only one positive zero. Moreover, if $\tau < \ln 2$, then the positive zero, denoted by B_1^*, belongs to $(0, 2(1 - \mathrm{e}^{-\tau}))$; and if $\tau \geq \ln 2$, then the positive zero, denoted by B_2^*, belongs to $[2(1 - \mathrm{e}^{-\tau}), \infty)$.*

Proof. Assume that the three zeros of $h(B)$ are λ_1, λ_2 and λ_3, respectively. Then

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{\mathrm{e}^{2\tau} - 3\,\mathrm{e}^{\tau} + \mathrm{e}^{-\tau} + \mathrm{e}^{-2\tau} - 3}{\mathrm{e}^{\tau} + 1 - \mathrm{e}^{-\tau}}, \quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \frac{\mathrm{e}^{2\tau} + \mathrm{e}^{\tau} - \mathrm{e}^{-\tau} - 1}{\mathrm{e}^{\tau} + 1 - \mathrm{e}^{-\tau}} > 0,$$

$$\lambda_1 \cdot \lambda_2 + \lambda_2 \cdot \lambda_3 + \lambda_3 \cdot \lambda_1 = -\frac{2\,\mathrm{e}^{2\tau} - \mathrm{e}^{\tau} + \mathrm{e}^{-2\tau} - 4}{\mathrm{e}^{\tau} + 1 - \mathrm{e}^{-\tau}}.$$

If $\tau < \ln 2$, then $\lambda_1 + \lambda_2 + \lambda_3 < 0$. It is easy to know that one of λ_1, λ_2 and λ_3 must have a negative part. This, together with $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$, implies that one and only one of λ_1, λ_2 and λ_3 is positive, denoted by B_1^*. Using the fact that $h(2(1 - \mathrm{e}^{-\tau})) < 0$ and that $h(0) > 0$, we have $B_1^* \in (0, 2(1 - \mathrm{e}^{-\tau}))$. On the other hand, if $\tau \geq \ln 2$, then $\lambda_1 \cdot \lambda_2 + \lambda_2 \cdot \lambda_3 + \lambda_3 \cdot \lambda_1 < 0$. Therefore, one of λ_1, λ_2 and λ_3 must have a negative part. This, together with $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 > 0$, implies that one and only one of λ_1, λ_2 and λ_3 is positive, denoted by B_2^*. In view of $h(2(1 - e^{-\tau})) \geq 0$, we have $B_2^* \in [2(1 - e^{-\tau}), \infty)$. □

Now, by means of the properties of parabola $y = g(x)$, we distinguish several cases to discuss the sign of $g(x)$ on (m, M).

Case 1. $B = 2(1 - e^{-\tau})$. It follows from the definition of $g(x)$ that $g(m) = 0$ and $g(-B) = 0$. If $\tau > \ln 2$ (or $\tau < \ln 2$), then $B > 1$ (*resp.* $B < 1$) and hence $B\,e^{\tau} - B - 1 > B - 1 > 0$ (*resp.* $B\,e^{\tau} - B - 1 < B - 1 < 0$). Thus, the parabola $y = g(x)$ is open upward (*resp.* downward). Notice that $m < M$, then $g(x) > 0$ (*resp.* $g(x) < 0$) for $x \in (m, M)$. If $\tau = \ln 2$, then $B = 1$ and hence $g(x) = 0$ for all $x \in (m, M)$.

Case 2. $0 \leq B < 2(1 - e^{-\tau})$ and $\tau \geq \ln 2$. It is easy to see that $g(m) > 0$ and $g(-B) < 0$. If $B > 1/(e^{\tau} - 1)$, then the parabola $y = g(x)$ is open upward, and $x = -B$ lies between the two zeros. As $g(m) > 0$, $x = m$ lies to the right of the right zero. Hence, $g(x) > 0$ for all $x \in (m, M)$. If $B < 1/(e^{\tau} - 1)$, then the parabola $y = g(x)$ is open downward, and the two zeros are separated by the line $x = m$. From $h(0) > 0$ and $h(1/(e^{\tau} - 1)) > 0$ and the fact that $h(B)$ has one and only one positive zero, it follows that $h(B) > 0$ for $B \in (0, 1/(e^{\tau} - 1))$. That is, $g(M) > 0$. Therefore, $g(x) > 0$ for $x \in (m, M)$.

Case 3. $0 \leq B \leq B_1^*$ and $\tau < \ln 2$. In view of Lemma 2.4, we have $Be^{\tau} - B - 1 < B - 1 < 0$, which implies that the parabola $y = g(x)$ is open downward, and that the two zeros are separated by the line $x = m$. By the definition of B_1^* in Lemma 2.4, $h(B) \geq 0$ for all $B \in [0, B_1^*]$. That is, $g(M) \geq 0$. Therefore, $g(x) > 0$ for $x \in (m, M)$.

Case 4. $B_1^* < B < 2(1 - e^{-\tau})$ and $\tau < \ln 2$. Using a similar argument as above, we have that the parabola $y = g(x)$ is open downward. By the definition of B_1^*, $h(B) < 0$. Namely, $g(M) < 0$. This, together with the fact that $g(m) > 0$, implies that the right zero of $g(x)$, denoted by x_1^*, lies between $x = m$ and $x = M$. Therefore, $g(x) > 0$ for $x \in (m, x_1^*)$ and $g(x) < 0$ for $x \in (x_1^*, M)$.

Case 5. $B > 2(1 - e^{-\tau})$ and $\tau \leq \ln 2$. By some simple computation, we have $g(m) < 0$ and $g(-B) > 0$. If $B > 1/(e^{\tau} - 1)$, then the parabola $y = g(x)$ is open upward. In view of Lemma 2.4 and the fact that $1/(e^{\tau} - 1) \geq 2(1 - e^{-\tau})$, we have $h(B) < 0$. Namely, $g(M) < 0$. Thus, both $x = m$ and $x = M$ lie between the two zeros of $g(x)$. Therefore, $g(x) < 0$ for all $x \in (m, M)$. If $2(1 - e^{-\tau}) < B < 1/(e^{\tau} - 1)$, then the parabola $y = g(x)$ is open downward and its two zeros are separated by the line $x = -B$. In view of $g(m) < 0$, we have $g(x) < 0$ for all $x \in (m, M)$.

Case 6. $2(1 - e^{-\tau}) < B < B_2^*$ and $\tau > \ln 2$. It is easy to see that $g(m) < 0$, $g(-B) > 0$ and $h(2(1 - e^{-\tau})) > 0$. Then, by the definition of B_2^*, we obtain $h(B) > 0$, i.e., $g(M) > 0$. Thus, the right zero of $g(x)$, denoted by x_2^*, lies between $x = m$ and $x = M$. Moreover, $g(x) < 0$ for $x \in (m, x_2^*)$ and $g(x) > 0$ for $x \in (x_2^*, M)$.

Case 7. $B \geq B_2^*$ and $\tau > \ln 2$. By the definition of B_2^*, we have $h(B) \leq 0$. Namely, $g(M) \leq 0$. Thus, both $x = m$ and $x = M$ lie between the two zeros of $g(x)$. Therefore, $g(x) < 0$ for all $x \in (m, M)$.

From Cases (1)–(7), we can summarize the properties of $g(x)$ as follows.

Lemma 2.5.

(i) *Let* $B = 2(1 - e^{-\tau})$. *Then,* $g(x) > 0$ *if* $\tau > \ln 2$; $g(x) < 0$ *if* $\tau < \ln 2$; $g(x) = 0$ *if* $\tau = \ln 2$.
(ii) *Let* $B_1^* < B < 2(1 - e^{-\tau})$. *If* $\tau \geq \ln 2$, *then* $g(x) > 0$; *and if* $\tau < \ln 2$, *then there exists* $x_1^* \in (m, M)$ *such that* $g(x_1^*) = 0$, $g(x) > 0$ *for* $x \in (m, x_1^*)$, *and* $g(x) < 0$ *for* $x \in (x_1^*, M)$.

(iii) *Let* $0 \le B \le B_1^*$. *Then* $g(x) > 0$ *for all* $x \in (m, M)$.
(iv) *Let* $B \ge B_2^*$. *Then* $g(x) < 0$ *for all* $x \in (m, M)$.
(v) *Let* $2(1-e^{-\tau}) < B < B_2^*$. *If* $\tau \le \ln 2$, *then* $g(x) < 0$ *for all* $x \in (m, M)$; *if* $\tau > \ln 2$, *then there exists* $x_2^* \in (m, M)$ *such that* $g(x_2^*) = 0$, $g(x) < 0$ *for* $x \in (m, x_2^*)$, *and* $g(x) > 0$ *for* $x \in (x_2^*, M)$.

As stated before, $x - f_1(x)$ and $g(x)$ have the same sign on (m, M). Therefore, applying Lemma 2.5, we have

Lemma 2.6. *The properties of function* $f_1(x)$ *defined on* (m, M) *are as follows*:

(i) *Let* $B = 2(1-e^{-\tau})$. *Then for all* $x \in (m, M)$, $f_1(x) < x$ *if* $\tau > \ln 2$; $f_1(x) > x$ *if* $\tau < \ln 2$; $f_1(x) = x$ *if* $\tau = \ln 2$.
(ii) *Let* $B_1^* < B < 2(1-e^{-\tau})$. *If* $\tau \ge \ln 2$, *then* $f_1(x) < x$ *for all* $x \in (m, M)$; *if* $\tau < \ln 2$, *then there exists* $x_1^* \in (m, M)$ *such that* $f_1(x_1^*) = x_1^*$, $f_1(x) < x$ *for* $x \in (m, x_1^*)$, *and* $f_1(x) > x$ *for* $x \in (x_1^*, M)$.
(iii) *Let* $0 \le B \le B_1^*$. *Then* $f_1(x) < x$ *for all* $x \in (m, M)$.
(iv) *Let* $B \ge B_2^*$. *Then* $f_1(x) > x$ *for all* $x \in (m, M)$.
(v) *Let* $2(1-e^{-\tau}) < B < B_2^*$. *If* $\tau \le \ln 2$, *then* $f_1(x) > x$ *for all* $x \in (m, M)$; *if* $\tau > \ln 2$, *then there exists* $x_2^* \in (m, M)$ *such that* $f_1(x_2^*) = x_2^*$, $f_1(x) > x$ *for* $x \in (m, x_2^*)$, *and* $f_1(x) < x$ *for* $x \in (x_2^*, M)$.

On the other hand, by using some simple arguments, we can obtain some results about the properties of function $f_2(x)$ defined as (14).

Lemma 2.7. $f_2(x)$ *defined on* $(0, m]$ *is continuous, monotonically increasing, and satisfies*:

(i) *If* $B = 2(1-e^{-\tau})$, *then* $f_2(x) = x$ *for all* $x \in (0, m]$.
(ii) *If* $B < 2(1-e^{-\tau})$, *then* $f_2(x) < x$ *for all* $x \in (0, m]$.
(iii) *If* $B > 2(1-e^{-\tau})$, *then* $f_2(x) > x$ *for all* $x \in (0, m]$.

Thus, by combining Lemmas 2.6 and 2.7 and the definition of function F in (12), we see that the following is true.

Lemma 2.8. $F(x)$ *is continuous, monotonically increasing on the interval* $[0, M)$, *and satisfies*:

(I) *In case where* $\tau < \ln 2$;
  (i) *If* $B = 2(1-e^{-\tau})$, *then* $F(x) > x$ *for* $x \in (m, M)$ *and* $F(x) = x$ *for* $x \in [0, m]$.
  (ii) *If* $B > 2(1-e^{-\tau})$, *then* $F(x) > x$ *for* $x \in (0, M)$ *and* $F(0) = 0$.
  (iii) *If* $B_1^* < B < 2(1-e^{-\tau})$, *then* $F(x) < x$ *for* $x \in (0, x_1^*)$, $F(x) > x$ *for* $x \in (x_1^*, M)$, $F(0) = 0$ *and* $F(x_1^*) = x_1^*$, *where* x_1^* *is defined as in* Lemma 2.6.
  (iv) *If* $0 \le B \le B_1^*$, *then* $F(x) < x$ *for* $x \in (0, M)$ *and* $F(0) = 0$.
(II) *In case where* $\tau > \ln 2$;
  (i) *If* $B = 2(1-e^{-\tau})$, *then* $F(x) < x$ *for* $x \in (m, M)$, *and* $F(x) = x$ *for* $x \in [0, m]$.
  (ii) *If* $0 \le B < 2(1-e^{-\tau})$, *then* $F(x) < x$ *for* $x \in (0, M)$ *and* $F(0) = 0$.
  (iii) *If* $2(1-e^{-\tau}) < B < B_2^*$, *then* $F(x) > x$ *for* $x \in (0, x_2^*)$, $F(x) < x$ *for* $x \in (x_2^*, M)$, $F(0) = 0$ *and* $F(x_2^*) = x_2^*$, *where* x_2^* *is defined as in* Lemma 2.6.
  (iv) *If* $B \ge B_2^*$, *then* $F(x) > x$ *for* $x \in (0, M)$ *and* $F(0) = 0$.
(III) *In case where* $\tau = \ln 2$.
  (i) *If* $B = 2(1-e^{-\tau}) = 1$, *then* $F(x) = x$ *for* $x \in [0, M)$.
  (ii) *If* $B < 2(1-e^{-\tau}) = 1$, *then* $F(x) < x$ *for* $x \in (0, M)$ *and* $F(0) = 0$.
  (iii) *If* $B > 2(1-e^{-\tau}) = 1$, *then* $F(x) > x$ *for* $x \in (0, M)$ *and* $F(0) = 0$.

3. Convergence and periodicity

Once the connection weights are given, the limiting behaviors of solution $(u(t), v(t))^{\mathrm{T}}$ of (5) as $t \to \infty$ are completely determined by the initial value $\Phi \in X_0$. Based on the several lemmas established in Section 2, we now distinguish some theorems to describe the asymptotical behaviors of $(u(t), v(t))^{\mathrm{T}}$. We state the first result about the convergence.

Theorem 3.1. *Suppose that there exists some $t_0 \geq 0$ such that $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{+,+}$, then $(u(t), v(t))^{\mathrm{T}} \to (0, B)^{\mathrm{T}}$ as $t \to \infty$. Similarly, if there exists some $t_0 \geq 0$ such that $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{-,-}$, then $(u(t), v(t))^{\mathrm{T}} \to (0, -B)^{\mathrm{T}}$ as $t \to \infty$.*

Proof. We only consider the case where $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{+,+}$ for some $t_0 \geq 0$. The case where $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{-,-}$ can be dealt with analogously.

Clearly, $u(t)$ and $v(t)$ satisfy system (6) for $t \in [t_0, t_0 + \tau]$. Therefore, for $t \in [t_0, t_0 + \tau]$ we have

$$u(t) = u(t_0)\,\mathrm{e}^{t_0 - t}$$

and

$$v(t) = (v(t_0) - B)\,\mathrm{e}^{t_0 - t} + B$$

Therefore, $u_{t_0+\tau}(\theta) = u(t_0 + \tau + \theta) > 0$ and $v_{t_0+\tau}(\theta) = v(t_0 + \tau + \theta) > 0$ for $\theta \in (-\tau, 0)$, and so $u_{t_0+\tau} \in C^+$ and $v_{t_0+\tau} \in C^+$. Repeating this argument on $[t_0 + \tau, t_0 + 2\tau]$, $[t_0 + 2\tau, t_0 + 3\tau]$, ..., consecutively, we ensure that $u_t \in C^+$ and $v_t \in C^+$ for all $t \geq 0$. Therefore, (6) holds for almost all $t > 0$. It follows that

$$u(t) = u(t_0)\,\mathrm{e}^{t_0 - t}$$

and

$$v(t) = (v(t_0) - B)\,\mathrm{e}^{t_0 - t} + B.$$

This shows that $(u(t), v(t))^{\mathrm{T}} \to (0, B)^{\mathrm{T}}$ as $t \to \infty$. □

Theorem 3.1 means that the points $(0, B)^{\mathrm{T}}$ and $(0, -B)^{\mathrm{T}}$ attract each point of $X^{+,+}$ and $X^{-,-}$, respectively. Next, we have an important result about the existence of periodic solutions. Namely,

Theorem 3.2. *Suppose that there exists some $t_0 \geq 0$ such that $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{-,+}$ (or $X^{+,-}$) and $u(t_0) + v(t_0) = 0$, then $u(t) + v(t) = 0$ for all $t \geq t_0$ and $u(t) = -v(t) = q(t)$ for $t \geq t_0 + \tau + \ln(1 - u(t_0))$ (respectively, for $t \geq t_0 + \tau + \ln(1 + u(t_0))$), where $q : \mathbb{R} \to \mathbb{R}$ is a periodic function with the minimal period $\omega = 2\ln(2\,\mathrm{e}^{\tau} - 1)$.*

Proof. We only consider the case where $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{-,+}$. The case where $(u_{t_0}, v_{t_0})^{\mathrm{T}} \in X^{+,-}$ can be dealt with analogously.

By using Eq. (5), we can easily obtain that $u(t) + v(t) = 0$ for all $t \geq t_0$. Therefore, it suffices to show that the solution $u(t)$ of the equation

$$\dot{u} = -u + f(u(t - \tau))$$

with the initial condition $u_{t_0} \in C^-$ is eventually periodic with the minimal period $2\ln(2\,\mathrm{e}^{\tau} - 1)$. Let t_1 be the first nonnegative zero of $u(t)$ on $[t_0, \infty)$. Then for $t \in (t_0, t_1 + \tau)$ except at most finitely many t, we have

$$\dot{u} = -u + 1, \tag{15}$$

from which and the continuity of the solution it follows that

$$u(t) = \mathrm{e}^{t_0 - t}[u(t_0) - 1] + 1, \quad t \in [t_0, t_1 + \tau]$$

and, in particular,

$$u(t_1) = \mathrm{e}^{t_0 - t_1}[u(t_0) - 1] + 1 = 0.$$

This implies

$$t_1 = t_0 + \ln[u(t_0) - 1]$$

and

$$u(t_1 + \tau) = \mathrm{e}^{t_0 - (t_1 + \tau)}[u(t_0) - 1] + 1 = 1 - \mathrm{e}^{-\tau} < 0.$$

Also

$$u_{t_1+\tau}(\theta) := u(t_1 + \tau + \theta) = \mathrm{e}^{t_0 - (t_1 + \tau + \theta)}[u(t_0) - 1] + 1 = 1 - \mathrm{e}^{-(\tau + \theta)} > 0$$

for $\theta \in (-\tau, 0]$. Therefore, $u_{t_1+\tau} \in C^+$.

To construct a solution of (15) beyond $[t_0, t_1 + \tau]$, we consider the solution of (15) with the new initial value defined by $\varphi^* = u_{t_1+\tau}$. Let t_2 be the first zero after t_1 of u. Then $t_2 > t_1 + \tau$ and on $(t_1 + \tau, t_2 + \tau)$, we have

$$\dot{u} = -u - 1 \tag{16}$$

and hence

$$u(t) = \mathrm{e}^{-(t - t_1 - \tau)}[u(t_1 + \tau) + 1] - 1 = (2 - \mathrm{e}^{-\tau})\,\mathrm{e}^{-(t - t_1 - \tau)} - 1$$

for $t \in [t_1 + \tau, t_2 + \tau]$. In particular,

$$u(t_2) = (2 - \mathrm{e}^{-\tau})\,\mathrm{e}^{t_1 - t_2 + \tau} - 1 = 0,$$

which implies

$$t_2 = t_1 + \tau + \ln(2 - \mathrm{e}^{-\tau}).$$

Also,

$$u_{t_2+\tau}(\theta) = u(t_2 + \tau + \theta) = (2 - \mathrm{e}^{-\tau})\,\mathrm{e}^{t_1 - t_2 - \theta} - 1 = \mathrm{e}^{-(\tau + \theta)} - 1 < 0$$

for $\theta \in (-\tau, 0]$. Therefore, $u_{t_2+\tau} \in C^-$.

Repeating the above arguments and letting t_3 be the first zero after t_2 of u, we have that $t_3 > t_2 + \tau$ and that (15) holds on $(t_2 + \tau, t_3 + \tau)$. Consequently,

$$u(t) = \mathrm{e}^{-(t-t_2-\tau)}[u(t_2 + \tau) - 1] + 1 = (\mathrm{e}^{-\tau} - 2)\,\mathrm{e}^{-(t-t_2-\tau)} + 1$$

for $t \in [t_2 + \tau, t_3 + \tau]$. In particular,

$$u(t_3) = (\mathrm{e}^{-\tau} - 2)\,\mathrm{e}^{t_2-t_3+\tau} + 1 = 0,$$

which implies

$$t_3 = t_2 + \tau + \ln(2 - \mathrm{e}^{-\tau}).$$

Also,

$$u_{t_3+\tau}(\theta) = u(t_3 + \tau + \theta) = (\mathrm{e}^{-\tau} - 2)\,\mathrm{e}^{t_2-t_3-\theta} + 1 = 1 - \mathrm{e}^{-(\tau+\theta)} > 0$$

for $\theta \in (-\tau, 0]$. Therefore, $u_{t_3+\tau} \in C^+$.

This also shows that $u_{t_3+\tau}(\theta) = u_{t_1+\tau}(\theta)$ for $\theta \in (-\tau, 0]$. Due to the uniqueness of the Cauchy initial value problem (see Hale and Verduyn Lunel [13]), we have $u(t + t_3 + \tau) = u(t + t_1 + \tau)$ for $t \geq 0$. Namely, for $t \geq t_1 + \tau$, $u(t)$ is periodic with the minimal period $\omega = t_3 - t_1 = 2\ln(2\,\mathrm{e}^{\tau} - 1)$. □

As a consequence, if the initial value $\Phi = (\varphi, \psi)^{\mathrm{T}}$ is neutralizing, (i.e., $\varphi = -\psi$) then the solution $(u, v)^{\mathrm{T}}$: $[-\tau, \infty) \to \mathbb{R}^2$ is neutralized, that is, $u(t) = -v(t)$ for all $t \geq 0$, due to the uniqueness of the Cauchy initial value problem of (5) (see Hale and Verduyn Lunel [13]). The above result shows that the solution $(u(t), v(t))^{\mathrm{T}}$ of (5) is neutralized even if the initial value Φ is not neutralizing but $\varphi(0) = -\psi(0)$ and $(\varphi, \psi)^{\mathrm{T}} \in X^{+,-} \bigcup X^{-,+}$. Moreover, it is interesting to note that the periodic function q and its minimal period ω are both independent of the choice of the initial value of $\Phi \in X^{+,-} \bigcup X^{-,+}$ with $\varphi(0) = -\psi(0)$. Moreover, note that $\omega/(2\tau) \to 1$ as $\tau \to \infty$.

It remains to discuss the case where the initial value $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X^{-,+}$ and $\varphi(0) + \psi(0) \geq 0$. In view of Lemma 2.2, the first zero of $u(t) \cdot v(t)$ in $[0, +\infty)$ is $t_1 = \ln(1 - \varphi(0))$. Moreover, $u(t_1) = 0$ and $v(t_1) = (\varphi(0) + \psi(0))/(1 - \varphi(0)) \geq 0$. It is easy to see that the values of $[t_1, u(t_1), v(t_1)]$ are completely determined by $\varphi(0)$ and $\psi(0)$. Without loss of generality, we let $u(0) = \varphi(0) = 0$ and $v(0) = \psi(0) = v \geq 0$. We will show that the behavior of $(u(t), v(t))^{\mathrm{T}}$ as $t \to +\infty$ is completely determined by the value v. Recall that if $v = 0$, then by Theorem 3.2, $(u(t), v(t))^{\mathrm{T}}$ is eventually periodic and coincides with the periodic solution $(q(t), -q(t))^{\mathrm{T}}$. Our analysis below shows that the behavior of $(u(t), v(t))^{\mathrm{T}}$ as $t \to +\infty$ can be understood in terms of the iteration of a one-dimensional map in case $v > 0$.

We start with

Case 1. $v \geq \mathrm{e}^{\tau} - 1$. In view of (5) and $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X^{-,+}$, $(u(t), v(t))^{\mathrm{T}}$ satisfies system (7). By the continuity of the solution, for $t \in [0, \tau]$, we have

$$\begin{aligned} u(t) &= [u(0) - 1]\,\mathrm{e}^{t_0-t} + 1 = 1 - \mathrm{e}^{-t}, \\ v(t) &= [v(0) + 1]\,\mathrm{e}^{t_0-t} - 1 = (1 + v)\,\mathrm{e}^{-t} - 1. \end{aligned} \tag{17}$$

It follows that $u(\tau) = 1 - \mathrm{e}^{-\tau} > 0$ and $v(\tau) = (1 + v)\,\mathrm{e}^{-\tau} - 1 \geq 0$. From (17), it follows that $(u_\tau, v_\tau)^{\mathrm{T}} \in X^{+,+}$. Thus, by Theorem 3.1, we have $(u(t), v(t))^{\mathrm{T}} \to (0, B)^{\mathrm{T}}$ as $t \to \infty$.

Case 2. $M \leq v < \mathrm{e}^{\tau} - 1$. Using a similar argument as above, we have that (17) holds for $t \in [0, \tau]$. Moreover, $u(\tau) = 1 - \mathrm{e}^{-\tau} > 0$ and $v(\tau) = (1 + v)\mathrm{e}^{-\tau} - 1 < 0$. Recall that $v(0) = v > 0$, there exists $t_2 \in [0, \tau]$ such that

$v(t_2) = 0$. From (17), we have

$$t_2 = \ln(1 + v).$$

Moreover, from (17), it follows that $u(t - \tau) > 0$ and $v(t - \tau) > 0$ for $t \in (\tau, t_2 + \tau)$. Thus, $(u(t), v(t))^{\mathrm{T}}$ satisfies (6) for $t \in (\tau, t_2 + \tau)$. Namely, for $t \in [\tau, t_2 + \tau]$, we have

$$\begin{aligned} u(t) &= u(\tau)\,\mathrm{e}^{\tau - t} = (1 - \mathrm{e}^{-\tau})\,\mathrm{e}^{\tau - t}, \\ v(t) &= [v(\tau) - B]\,\mathrm{e}^{\tau - t} + B = [(1 + v)\,\mathrm{e}^{-\tau} - 1 - B]\,\mathrm{e}^{\tau - t} + B. \end{aligned} \tag{18}$$

It follows that $u(t_2 + \tau) = (1 - \mathrm{e}^{-\tau})(1 + v) > 0$ and $v(t_2 + \tau) = B + \mathrm{e}^{-\tau} - (1 + B)(1 + v) > 0$. Also since $u(\tau) = 1 - \mathrm{e}^{-\tau} > 0$ and $v(\tau) = (1 + v)\,\mathrm{e}^{-\tau} - 1 < 0$, there exists $t_3 \in (\tau, t_2 + \tau)$ such that $v(t_3) = 0$. From (18), we see that

$$t_3 = \tau + \ln[B + 1 - (1 + v)\,\mathrm{e}^{-\tau}] - \ln B.$$

Moreover, from (17) and (18), we have $u(t - \tau) > 0$ and $v(t - \tau) < 0$ for $t \in (t_2 + \tau, t_3 + \tau)$. Thus, $(u(t), v(t))^{\mathrm{T}}$ satisfies system (9) for $t \in (t_2 + \tau, t_3 + \tau)$. It follows that for $t \in [t_2 + \tau, t_3 + \tau]$, we have

$$\begin{aligned} u(t) &= [u(t_2 + \tau) + 1]\,\mathrm{e}^{t_2 + \tau - t} - 1 &&= \left(1 + \frac{1 - \mathrm{e}^{-\tau}}{1 + v}\right)\mathrm{e}^{t_2 + \tau - t} - 1, \\ v(t) &= [v(t_2 + \tau) - 1]\,\mathrm{e}^{t_2 + \tau - t} + 1 &&= \left(B - 1 + \mathrm{e}^{-\tau} - \frac{1 + B}{1 + v}\right)\mathrm{e}^{t_2 + \tau - t} + 1. \end{aligned} \tag{19}$$

This implies

$$\begin{aligned} u(t_3 + \tau) &= \frac{(1 + B)(1 + v) + B(1 - \mathrm{e}^{-\tau}) - (1 + B)\,\mathrm{e}^{\tau}}{(1 + B)\,\mathrm{e}^{\tau} - (1 + v)} > 0, \\ v(t_3 + \tau) &= \frac{(B - 1 + \mathrm{e}^{-\tau})(1 + v) - (1 + B)}{(1 + B)\,\mathrm{e}^{\tau} - (1 + v)} \cdot B + 1 > 0. \end{aligned}$$

From (18) and (19), it is easy to see that $(u_{t_3 + \tau}, v_{t_3 + \tau})^{\mathrm{T}} \in X^{+,+}$. Thus, by Theorem 3.1, we have $(u(t), v(t))^{\mathrm{T}} \to (0, B)^{\mathrm{T}}$ as $t \to \infty$.

Case 3. $m < v < M$. Using a similar argument as above, we have that $u(t_3 + \tau) < 0$ and that $v(t_3 + \tau) > 0$, which, together with the fact that $u(t_2 + \tau) = (1 - \mathrm{e}^{-\tau})/(1 + v) > 0$ and that $v(t_2 + \tau) = B + \mathrm{e}^{-\tau} - (1 + B)/(1 + v) > 0$, implies that there exists $t_4 \in (t_2 + \tau, t_3 + \tau)$ such that $u(t_4) = 0$. It follows from (19) that

$$t_4 = t_2 + \tau + \ln\left(1 + \frac{1 - \mathrm{e}^{-\tau}}{1 + v}\right) = \tau + \ln(v + 2 - \mathrm{e}^{-\tau}).$$

From (18) and (19), we see that $u(t - \tau) > 0$ and $v(t - \tau) > 0$ for $t \in (t_3 + \tau, t_4 + \tau)$. Therefore, $(u(t), v(t))^{\mathrm{T}}$ satisfies (6) for $t \in (t_3 + \tau, t_4 + \tau)$. Namely, for $t \in [t_3 + \tau, t_4 + \tau]$, we have

$$\begin{aligned} u(t) &= u(t_3 + \tau)\,\mathrm{e}^{t_3 + \tau - t} &&= \frac{(1 + B)(1 + v) + B(1 - \mathrm{e}^{-\tau}) - (1 + B)\,\mathrm{e}^{\tau}}{(1 + B)\,\mathrm{e}^{\tau} - (1 + v)}\,\mathrm{e}^{t_3 + \tau - t}, \\ v(t) &= [v(t_3 + \tau) - B]\,\mathrm{e}^{t_3 + \tau - t} + B &&= \left[1 - B + \frac{(\mathrm{e}^{-\tau} + B - 1)v + \mathrm{e}^{-\tau} - 2}{(1 + B)\,\mathrm{e}^{\tau} - (1 + v)} B\right]\mathrm{e}^{t_3 + \tau - t} + B. \end{aligned} \tag{20}$$

It follows that

$$u(t_4+\tau)=\frac{(1+B)(1+v)+B(1-e^{-\tau})-(1+B)e^{\tau}}{(2+v-e^{-\tau})Be^{\tau}}<0,$$
$$v(t_4+\tau)=\frac{(1-B)(1+B)e^{\tau}-(1-B)(1+v)}{(2+v-e^{-\tau})Be^{\tau}}+\frac{(e^{-\tau}+B-1)v+e^{-\tau}-2}{(2+v-e^{-\tau})e^{\tau}}+B>0.$$

Moreover, from (19) and (20), we see that $(u_{t_4+\tau}, v_{t_4+\tau})^{\mathrm{T}} \in X^{-,+}$ and $u(t_4+\tau)+v(t_4+\tau)>0$. Again by Lemma 2.2, the next zero of $u(t)\cdot v(t)$, i.e., its fifth zero in $[0,\infty)$, is $t_5=t_4+\tau+\ln[1-u(t_4+\tau)]$ and satisfies that $u(t_5)=0$ and

$$v(t_5)=\frac{u(t_4+\tau)+v(t_4+\tau)}{1-u(t_4+\tau)}=\frac{B(B+e^{-\tau})(e^{\tau}+1)v+B(e^{\tau}-1)(B-1)}{(Be^{\tau}-B-1)v+(1+3B)(e^{\tau}-1)+Be^{-\tau}}=f_1(v)>0,$$

where the function f_1 is defined as (13). If $f_1(v)\in(m,M)$, then we can repeat the same analysis and construction to get $f_1^2(v)=f_1(f_1(v))$ assuming that the initial condition is the value $f_1(v)$. If $f_1^2(v)\in(m,M)$, we continue to iterate f_1 to get a sequence:

$$v,\quad f_1(v),\quad f_1^2(v),\quad \ldots,\quad f_1^n(v),\quad \ldots,$$

where $f_1^n(v)=f_1(f_1^{n-1}(v))$.

Case 4. $0<v\le m$. Using a similar argument as above, we have $u(t_2+\tau)=(1-e^{-\tau})/(1+v)>0$ and $v(t_2+\tau)=B+e^{-\tau}-(1+B)/(1+v)\le 0$. From (17) and (18), we see that $(u_{t_2+\tau}, v_{t_2+\tau})^{\mathrm{T}}\in X^{+,-}$ and $u(t_2+\tau)+v(t_2+\tau)>0$. Let t_3 be the next zero of $u(t)\cdot v(t)$, i.e., the third zero in $[0,\infty)$, then $u(t-\tau)>0$ and $v(t-\tau)<0$ for $t\in(t_2+\tau,t_3+\tau)$. Thus, $(u(t),v(t))^{\mathrm{T}}$ satisfies (9) for $t\in(t_2+\tau,t_3+\tau)$. Namely, (19) holds for $t\in[t_2+\tau,t_3+\tau]$. This implies that

$$t_3=t_2+\tau+\ln\left(1-e^{-\tau}-B+\frac{1+B}{1+v}\right)=\tau+\ln[(1-e^{-\tau}-B)v+2-e^{-\tau}],$$
$$u(t_3+\tau)=\frac{(2e^{-\tau}+B-1)v+(1-e^{-\tau})(e^{-\tau}-2)}{(1-e^{-\tau}-B)v+2-e^{-\tau}}<0,\qquad v(t_3+\tau)=1-e^{-\tau}>0.$$

In view of $u(t_2+\tau)=((1-e^{-\tau})/(1+v))>0$, there exists $t_4\in(t_3,t_3+\tau)$ such that $u(t_4)=0$. This, together with (17), implies that

$$t_4=t_2+\tau+\ln\left(1+\frac{1-e^{-\tau}}{1+v}\right)=\tau+\ln(v+2-e^{-\tau}).$$

From (19), we see that $u(t-\tau)>0$ and $v(t-\tau)>0$ for $t\in(t_3+\tau,t_4+\tau)$. Therefore, for $t\in(t_3+\tau,t_4+\tau)$, $(u(t),v(t))^{\mathrm{T}}$ satisfies (6). Namely, for $t\in[t_3+\tau,t_4+\tau]$,

$$u(t)=u(t_3+\tau)e^{t_3+\tau-t}=\frac{(2e^{-\tau}+B-1)v+(1-e^{-\tau})(e^{-\tau}-2)}{(1-e^{-\tau}-B)v+2-e^{-\tau}}e^{t_3+\tau-t},$$
$$v(t)=[v(t_3+\tau)-B]e^{t_3+\tau-t}+B=(1-B-e^{-\tau})e^{t_3+\tau-t}+B. \tag{21}$$

It follows that

$$u(t_4+\tau)=\frac{(2\,\mathrm{e}^{-\tau}+B-1)v+(1-\mathrm{e}^{-\tau})(\mathrm{e}^{-\tau}-2)}{v+2-\mathrm{e}^{-\tau}}<0,$$
$$v(t_4+\tau)=(1-B-\mathrm{e}^{-\tau})\cdot\frac{(1-\mathrm{e}^{-\tau}-B)v+2-\mathrm{e}^{-\tau}}{v+2-\mathrm{e}^{-\tau}}+B>0.$$

By (19) and (21), we have $(u_{t_4+\tau}, v_{t_4+\tau})^{\mathrm{T}} \in X^{-,+}$ and $u(t_4+\tau)+v(t_4+\tau)>0$. This, together with Lemma 2.2, implies that the next zero of $u(t)\cdot v(t)$, i.e., its fifth zero in $[0,\infty)$, is $t_5=t_4+\tau+\ln[1-u(t_4+\tau)]$ and satisfies that $u(t_5)=0$ and

$$v(t_5)=\frac{u(t_4+\tau)+v(t_4+\tau)}{1-u(t_4+\tau)}=\frac{(B+\mathrm{e}^{-\tau})^2v}{(2-2\,\mathrm{e}^{-\tau}-B)v+(2-\mathrm{e}^{-\tau})^2}=f_2(v)>0,$$

where the function f_2 is defined as (14). If $f_1(v)\in(0,m]$, then we can repeat the same analysis and construction to get $f_2^2(v)=f_2(f_2(v))$ assuming that the initial condition is $f_2(v)$. If $f_2^2(v)\in(0,m]$, we continue to iterate f_2 to get a sequence:

$$v,\quad f_2(v),\quad f_2^2(v),\quad \ldots,\quad f_2^n(v),\quad \ldots,$$

where $f_2^n(v)=f_2(f_2^{n-1}(v))$.

As discussed above for Cases (1)–(4), using F and its iterates we can characterize the behavior of the solution $(u(t),v(t))^{\mathrm{T}}$ of system (5) with initial value $\Phi=(\varphi,\psi)^{\mathrm{T}}\in X^{-,+}$ and $\varphi(0)+\psi(0)\geq 0$. More precisely, if there exists some $v^*\in[0,M)$ such that $\lim_{n\to\infty}F^n(v)=v^*=F(v^*)$, then the solution $(u(t),v(t))^{\mathrm{T}}$ corresponding to v approaches the periodic solution $(u^*(t),v^*(t))^{\mathrm{T}}$ corresponding to v^* as $t\to\infty$. On the other hand, if there exists some integer $n^*\geq 1$ such that

$$F^{n^*-1}(v)<M\leq F^{n^*}(v),$$

then, according to the arguments in Cases (1) and (2), the solution $(u(t),v(t))^{\mathrm{T}}$ corresponding to v converges to $(0,B)^{\mathrm{T}}$ as $t\to\infty$.

Using the properties of $F(x)$ described in Lemma 2.8, we now describe the following main theorem.

Theorem 3.3. *Let $\eta=(\varphi(0)+\psi(0))/(1-\varphi(0))\geq 0$. The behaviors of the solution $(u(t),v(t))^{\mathrm{T}}$ of system (5) with initial value $\Phi=(\varphi,\psi)^{\mathrm{T}}\in X^{-,+}$ and $\varphi(0)+\psi(0)\geq 0$ are as follows:*

(i) *Suppose that $B=2(1-\mathrm{e}^{-\tau})$ and $\tau>\ln 2$. If $\eta\in[0,m]$, then $(u(t),v(t))^{\mathrm{T}}$ is eventually periodic and of the minimal period $\omega=2\ln(2\,\mathrm{e}^{\tau}-1)$; If $\eta\in(m,M)$, then $(u(t),v(t))^{\mathrm{T}}$ approaches the periodic solution corresponding to $\eta=m$ as $t\to\infty$; If $\eta\in[M,\infty)$, then $(u(t),v(t))^{\mathrm{T}}$ converges to $(0,B)^{\mathrm{T}}$ as $t\to\infty$.*

(ii) *Suppose that $B=2(1-\mathrm{e}^{-\tau})$ and $\tau<\ln 2$. If $\eta\in[0,m]$, then $(u(t),v(t))^{\mathrm{T}}$ is eventually periodic with the minimal period $\omega=2\ln(2\,\mathrm{e}^{\tau}-1)$; If $\eta\in(m,\infty)$, then $(u(t),v(t))^{\mathrm{T}}$ converges to $(0,B)^{\mathrm{T}}$ as $t\to\infty$.*

(iii) *Suppose that $0\leq B<2(1-\mathrm{e}^{-\tau})$ and $\tau\geq\ln 2$ or $0\leq B\leq B_1^*$ and $\tau<\ln 2$. Then $(u(t),v(t))^{\mathrm{T}}$ approaches the periodic solution $(q(t),-q(t))^{\mathrm{T}}$ as $t\to\infty$.*

(iv) *Suppose that $B>2(1-\mathrm{e}^{-\tau})$ and $\tau\leq\ln 2$ or $B\geq B_2^*$ and $\tau>\ln 2$. Then $(u(t),v(t))^{\mathrm{T}}$ converges to $(0,B)^{\mathrm{T}}$ as $t\to\infty$.*

(v) *Suppose that $B_1^*<B<2(1-\mathrm{e}^{-\tau})$ and $\tau<\ln 2$. Then there exists $T_1\geq 0$ and $\Phi_1=(\varphi_1,\psi_1)^{\mathrm{T}}\in X^{-,+}$ with $\varphi_1(0)+\psi_1(0)>0$ such that for $t\geq T_1$, the solution $(u^1(t),v^1(t))^{\mathrm{T}}$ of (5) with initial value Φ_1 is periodic.*

Moreover, as $t \to \infty$, *other solutions of system* (5) *with initial value* $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X^{-,+}$ *and* $\varphi(0) + \psi(0) > 0$ *either converge to* $(0, B)^{\mathrm{T}}$ *or approached the periodic solution* $(q(t), -q(t))^{\mathrm{T}}$.

(vi) *Suppose that* $2(1 - \mathrm{e}^{-\tau}) < B < B_2^*$ *and* $\tau > \ln 2$. *Then there exists* $T_2 \geq 0$ *and* $\Phi_2 = (\varphi_2, \psi_2)^{\mathrm{T}} \in X^{-,+}$ *with* $\varphi_2(0) + \psi_2(0) > 0$ *such that for* $t \geq T_2$, *the solution* $(u^2(t), v^2(t))^{\mathrm{T}}$ *of* (5) *with initial value* Φ_2 *is periodic with the minimal period* $\omega = 2\tau + \ln[(2 - 2\,\mathrm{e}^{-\tau} - B)x_2^* + (1 - \mathrm{e}^{-\tau})^2 + 3 - 2\,\mathrm{e}^{-\tau}]$, *where* x_2^* *is defined as in* Lemma 2.5. *Moreover, as* $t \to \infty$, *other solutions of system* (5) *with initial value* $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X^{-,+}$ *and* $\varphi(0) + \psi(0) > 0$ *either converge to* $(0, B)^{\mathrm{T}}$ *or approached the periodic solution* $(u^2(t), v^2(t))^{\mathrm{T}}$.

(vii) *Suppose that* $B = 1$ *and* $\tau = \ln 2$. *If* $\eta \in [0, M)$, *then* $(u(t), v(t))^{\mathrm{T}}$ *is eventually periodic*; *If* $\eta \in [M, \infty)$, *then* $(u(t), v(t))^{\mathrm{T}} \to (0, B)^{\mathrm{T}}$ *as* $t \to \infty$.

Theorem 3.3 means that B and τ can be regarded as bifurcation parameters. Moreover, by Lemma 2.1 and Theorem 3.3, we can easily determine the behaviors of the solution $(u(t), v(t))^{\mathrm{T}}$ of system (5) with initial value $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X^{+,-}$ and $\varphi(0) + \psi(0) \leq 0$.

4. Summary and outlook

The underlying purpose of this article is to help advance the idea that even the simplest two-neuron network also possesses a great deal of dynamics [1–3,8–11,16,17,22–24,29] and gain some useful insights into what we can achieve for a more general system of delay differential equations. In particular, a model equation with the McCulloch–Pitts nonlinearity describes a combination of analog and digital signal processing in a network of two neurons with delayed feedback. The piecewise-linear nature makes the network mathematically tractable without removing the possibilities for very interesting and varied dynamics. It was shown in [8–10,16,17,23] that the dynamics of Eq. (3) with can be understood in terms of the iteration of a one-dimensional map explicitly constructed from the delay and the synaptic weights. It might be argued that the *hard switching* of the model equation is not biologically reasonable, but similar models with very steep sigmoid switching appear to behave similarly and in fact, complexity of behavior seem to be lost as the gain is decreased (see, for example [5,18,27]). Rigorous treatment of the question of the high-gain limit has been done for fixed points and some limit cycles [25], but is not considered here.

Because the connection weights have a fundamental effect on the dynamics of the networks. Here, under the condition (H1), we rewrite (3) as (5) and only need to depict the dynamics of Eq. (5), i.e., the convergence of solutions, the existence, multiplicity and attractivity of periodic solutions. We have shown that if the initial data $\Phi = (\varphi, \psi)^{\mathrm{T}} \in X$ is given so that φ and ψ do not change sign on the initial time interval, then the dynamical behavior of the corresponding solution is completely determined by the sizes of B and τ and the relation between $\varphi(0)$ and $\psi(0)$. If $\varphi(0) + \psi(0) = 0$ and $(\varphi, \psi)^{\mathrm{T}} \in X^{+,-} \bigcup X^{-,+}$, then the solution is eventually periodic. If $(\varphi, \psi)^{\mathrm{T}} \in X^{+,+} \bigcup X^{-,-}$, then the solution is stabilized at the two stationary points $(0, B)^{\mathrm{T}}$ and $(0, -B)^{\mathrm{T}}$. If $(\varphi, \psi)^{\mathrm{T}} \in X^{+,-} \bigcup X^{+,-}$, then we can regard B and τ as bifurcation parameters to discuss the limiting behavior of the solution. For example, if $\tau < \ln 2$, then as the nonnegative number B is increased past the three critical values: 0, B_1^* and $2(1 - \mathrm{e}^{-\tau})$, the dynamics of model equation undergoes transition from one attracting periodic solution, to three periodic solutions (one is repulsive and the other two are attracting), then to infinitely many periodic solutions with a common period, and finally to only one repulsive periodic solution; If $\tau > \ln 2$, then as B is increased past the three critical values: 0, $2(1 - \mathrm{e}^{-\tau})$ and B_2^*, the dynamics of model equation undergoes transition from one attracting periodic solution, to three periodic solutions (one is attracting and the other two are repulsive), then to infinitely many periodic solutions with a common period, and finally to only one repulsive periodic solution; At last, if $\tau = \ln 2$, then as B is increased past $2(1 - \mathrm{e}^{-\tau})$, the dynamics of model equation undergoes transition from one attracting periodic solution, to infinite periodic solutions with a common period, and finally to only one repulsive periodic solution. These results are consistent with those of Edwards [4] where they applied a coherent method of analysis to an n-dimensional system and obtained surprising long and complex limit cycles.

By using the modern theory of monotone dynamical systems (Smith [26] and Hirsh [14]) and a discrete Liapunov functional introduced by Mallet-Paret and Sell [20] and the geometric approach of Krisztin et al. [19], Chen and Wu [1,2] considered model Eq. (3) with smooth sigmoid signal function. It is natural to ask whether the techniques developed in the aforementioned references which usually require a strictly monotone signal function can be generalized to the case where the signal function is monotone but not strictly monotone. The McCulloch–Pitts nonlinearity provides an important motivation from the viewpoint of artificial neural networks for understanding the dynamics of model Eq. (3) with a monotone but not strictly monotone nonlinearity, which could be a starting point towards the development of the aforementioned generalization. In this paper, the digital nature of the signal function allows us to relate Eq. (3) to four systems of simple linear nonhomogeneous ordinary differential equations and to depict the asymptotical behaviors of all solutions whose initial states do not oscillate. In future work, we desire to describe the dynamics of solutions of (3) and (4) with initial data in $X \setminus X_0$ (i.e., solutions whose initial states oscillate with high frequencies).

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