FUNCTIONAL
DIFFERENTIAL
EQUATIONS

VOLUME 12
2005, NO 3-4
PP. 437–461

SHARP ESTIMATES OF SOLUTIONS TO NEUTRAL EQUATIONS IN SOBOLEV SPACES *

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Dedicated to Professor A.D.Myshkis on the occasion of his 85th-birthday

Abstract. In this paper, we obtain sharp estimates for the growth of strong solutions of difference differential equations of neutral type. Our work is based on existing results for the initial value problem for a related homogeneous neutral equation, by using the Riesz basis consisting of exponential solutions of the homogeneous equation.

Key Words. Functional differential equation, Sobolev space, Riesz Basis, sharp estimates, neutral type

1. Introduction. Despite its profound importance in the theory of control and the theory of dynamical systems and their applications, obtaining the sharpest estimates for solutions to functional differential equations remains to be a challenging task although some significant progress has been achieved, see, for example, [1]-[7], [22].

In this paper, we obtain some sharp estimates of the strong solutions for difference differential equations of neutral type. This is of course a well-known classical problem, and our result(estimate) is based on some previous work by one of the authors about the initial value problem for a corresponding homogeneous equation (see [8]-[14] for more details). As such, our work

* The research of V. Vlasov was accomplished with financial support of Russian Foundation for Basic Research (grants N 02-01-00790, N 04-01-00618). The work of J.Wu is supported by Natural Sciences and Engineering Research Council of Canada and by Canada Research Chair Program.

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depends heavily on the Riesz basisness of the system of exponential solutions (see [8]-[14]).

The remaining part of this paper consists of four sections. Section 2 provides the statement of the problem and the formulation of the results. The basic results are proved in Section 3. Section 4 gives examples to demonstrate that our estimates are sharp, and this section also contains some remarks, comments, and the comparison of our results with some earlier relevant works. Section 5, the Appendix, is devoted to the proof of the Riesz basis.

2. The Problems and Results. Denote by $W^p_{2,r}((a,b),\mathcal{C}^\tau)$, $(-\infty < a < b \leq +\infty)$, weighted Sobolov spaces of vector valued functions with values in $\mathcal{C}^\tau$, endowed with the norms:

$$\|V\|_{W^p_{2,r}(a,b)} = (\int_a^b exp(-2rt)(\sum_{j=0}^{p} \|V^{(j)}(t)\|^2)dt)^{1/2}, r \geq 0.$$

Here and in what follows, $W^p_{2,0} = W^p_2$, $V^{(j)}(t) = \frac{d^j}{dt^j}V(t), p, j = 1, 2, \cdots$; $\|.\|$ is the norm in the space $\mathcal{C}^\tau$.

We consider the following nonhomogeneous equation

$$(1) \qquad \begin{aligned} Du \;&\equiv\; \textstyle\sum_{j=0}^{n}(B_j(t-h_j) + D_j\frac{du}{dt}(t-h_j)) + \int_0^h B(s)u(t-s)ds \\ &+\; \textstyle\int_0^h D(s)u^{(1)}(t-s)ds = f(t), t > 0; \end{aligned}$$

subject to the usual Cauchy initial condition

$$(2) \qquad u(t) = g(t), \quad t \in [-h, 0].$$

Here $B_j, D_j (j = 0, 1, \cdots, n)$ are square $(\tau \times \tau)$ matrices with constant complex elements; the real numbers h_j are such that $0 = h_0 < h_1 < \cdots < h_n = h$; the elements $B_{ij}(s), D_{ij}(s) (i, j = 1, 2, \cdots, \tau)$ of matrices $B(s)$ and $D(s)$ belong to the space $L_2(0, h)$. Moreover, the non-homogeneous term f belongs to $L_2((0,T), \mathcal{C}^\tau)$ for an arbitrarily given $T > 0$, and the initial function g belongs to $W^1_2((-h, 0), \mathcal{C}^\tau)$.

DEFINITION 2.1. *A vector-valued function $u \in W^1_2((-h,T),\mathcal{C}^\tau)$ for arbitrary $T > 0$ is called a strong solution of the problem (1), (2), if $u(t)$ satisfies equation (1) almost everywhere on the semiaxis $\mathcal{R}_+ = (0, +\infty)$ and u satisfies the initial condition (2).*

In order to describe our main results we have to introduce certain notations.

We denote by $L(\lambda)$ the matrix-valued function

$$(3)\qquad \begin{aligned} L(\lambda) &= \sum_{j=0}^{n}(B_j+\lambda D_j)exp(-\lambda h_j) \\ &+\int_0^h B(s)exp(-\lambda s)ds+\lambda\int_0^h D(s)exp(-\lambda s)ds. \end{aligned}$$

Let $l(\lambda)=detL(\lambda)$ be the characteristic quasipolinomial (see [1] for more details) of the equation (1). We shall let λ_q denote a typical zero (referred as *characteristic number*) of the function $l(\lambda)$ with multiplicities ν_q, and we shall arrange these zeros, counting multiplicities, in increasing order of their modulus, and we will denote by Λ the set of all zeroes of the function $l(\lambda)$.

We denote the eigenvectors from the canonical system of eigen and associated (root) vectors, corresponding to the characteristic number λ_q, by $x_{q,j,0},(j=1,2,\cdots,\nu_q)$, their adjoint vectors of order s by $x_{q,j,s}$ $(s=1,2,\cdots,p_{qj})$. Here, the index j shows what is the number of the vector $x_{q,j,0}$ in a specially chosen basis of the subspace of the solutions of the equation $L(\lambda_q)x=0$. See [16], [17] for more details.

Therefore, we have the following set of exponential solutions of the equation (1) in homogeneous case $(f(t)\equiv 0)$

$$(4)\qquad y_{q,j,s}(t)=\exp(\lambda_q t)\left(\frac{t^s}{s!}x_{q,j,0}+\frac{t^{s-1}}{(s-1)!}x_{q,j,1}+\cdots+x_{q,j,s}\right).$$

Now we formulate two technical results which will be used in what follows.

LEMMA 2.2. *Suppose that $detD_0\neq 0, detD_n\neq 0$. Then each of the following values*

$$\aleph_+=\sup_{\lambda_q\in\Lambda} Re\lambda_q,\ \aleph_-=\inf_{\lambda_q\in\Lambda} Re\lambda_q,\ N=\max_{\lambda_q\in\Lambda}\nu_q$$

are finite.

We denote by $D(\lambda_q,\rho)$ the disk of radius $\rho>0$ with center in the point λ_q, define a domain $G(\Lambda,\rho)$ in the following way

$$G(\Lambda,\rho)\equiv C\setminus\left(\bigcup_{\lambda_q\in\Lambda} D(\lambda_q,\rho)\right).$$

LEMMA 2.3. *Suppose that $detD_0\neq 0, detD_n\neq 0$. Then there exists $\beta>0$ so that each of the following set of contours $\Gamma_n=\{\lambda\in C: Re\lambda=\aleph_++\beta,\gamma_n\leq Im\lambda\leq\gamma_{n+1}\}\cup\{\lambda\in C:\aleph_--\beta\leq Re\lambda\leq\aleph_++\beta, Im\lambda=\gamma_{n+1}\}\cup\{\lambda\in C: Re\lambda=\aleph_--\beta,\gamma_n\leq Im\lambda\leq\gamma_{n+1}\}\cup\{\lambda\in C:\aleph_--\beta\leq Re\lambda\leq\aleph_++\beta, Im\lambda=\gamma_n\},\beta>0$, belong to the domain $G(\Lambda,\rho)$ for sufficiently small $\rho>0$, and the following conditions are satisfied: there exist* *positive constants* δ, Δ *such that the sequence of real numbers* $\{\gamma_n\}$ *satisfy the inequalities:* $0 < \delta \leq \gamma_{n+1} - \gamma_n \leq \Delta < +\infty, n \in Z$, *and the number* $N(\Gamma_n)$ *of zeroes of the function* $l(\lambda)$ *(counting multiplicities), which lie in the domains* G_n *bounded by the contours* Γ_n, *are uniformly bounded with respect to n, i.e. there exists a constant* $M > 0$ *so that*

$$\max_n N(\Gamma_n) \leq M. \tag{5}$$

The above two lemmas can be reduced from the results of [15], and we refer to [9] for their proofs.

The next theorem is the main result of this article.

THEOREM 2.4. *Let* $detD_0 \neq 0, detD_n \neq 0$. *Suppose also that* $f \in L_2((0,T), C^r)$ *for an arbitrary given* $T > h$, *and* $g \in W_2^1((-h,0), C^r)$. *Then the problem (1), (2) has a unique solution and this solution* $u(t)$ *satisfies*

$$\begin{aligned} \|u\|_{W_2^1(t-h,t)} \leq\ & d_0(t+1)^{M-1} exp(\aleph_+ t)\|g\|_{W_2^1(-h,0)} \\ & + d_1\sqrt{t}\left(\int_0^t (t-s+1)^{2(M-1)} exp(2\aleph_+(t-s))\|f(s)\|^2 ds\right)^{1/2} \end{aligned} \tag{6}$$

for $t \in [h,T]$, *with constants* d_0 *and* d_1 *independent of functions g and f as well as the constant T.*

We add a few remarks to illustrate the significance of the above result.

REMARK 2.5. *If the set* Λ *of zeroes* λ_q *is separate, i.e.* $inf_{\lambda_p \neq \lambda_q}|\lambda_p - \lambda_q| > 0$, *then in the estimate (6) constant M (see (5)) may by replaced by* $N = max_{\lambda_q \in \Lambda} \nu_q$.

REMARK 2.6. *The estimate (6) is sharp in the following sense. It is impossible to change constant* $\aleph_+$ *by* $\aleph_+ - \epsilon$ *for every* $\epsilon > 0$. *Moreover,* $\sqrt{t}$ *can't be omitted. In Section 4, will have some examples to illustrate these.*

COROLLARY 2.7. *Suppose the conditions of the Theorem* 2.4 *are satisfied. Then we have the following*

$$\begin{aligned} \|u(t)\| \leq\ & d_2(t+1)^{M-1} exp(\aleph_+ t)\|g\|_{W_2^1(-h,0)} \\ & + d_3\sqrt{t}\left(\int_0^t (t-s+1)^{2(M-1)} exp(2\aleph_+(t-s))\|f(s)\|^2 ds\right)^{1/2} \end{aligned} \tag{7}$$

for $t \in [h,T]$, *with constants* d_2 *and* d_3 *independent of functions g and f.*

The above corollary immediately follows from Theorem 2.4 and trace theorem(see for example [21]).

We conclude this section with a result about the estimate of solutions of the problem (1)- (2) in the special case when the support of function f is compact.

PROPOSITION 2.8. *Let* $detD_0 \neq 0, detD_n \neq 0$, $g \in W_2^1((-h,0), C^\tau)$ *and assume that* f *has a compact support* Ω. *Then we have*

$$(8)\quad \begin{aligned} ||u||_{W_2^1(t-h,t)} \leq\ & d_0(t+1)^{M-1}exp(\aleph_+ t)||g||_{W_2^1(-h,0)} \\ & + d_2(\int_\Omega (t-s+1)^{2(M-1)}exp(2\aleph_+(t-s))||f(s)||^2 ds)^{1/2} \end{aligned}$$

for all $t \geq h$, *with constants* d_0 *and* d_2 *independent of the functions* f *and* g.

In the proofs to be provided next section, we shall use the solvability result of the problem (1), (2) in weighted Sobolev space $W^1_{2,r}((-h,+\infty), C^\tau)$ (see for example [8], [10]). In order to make the article self-contained, we state this solvability result below.

Let $L_{2,r}(R_+, C^\tau)$ be the Hilbert space of vector-valued functions equipped with the norm

$$||f||_{L_{2,r}} \equiv (\int_0^{+\infty} \exp(-2rt)||f(t)||^2 dt)^{1/2}, r \in R.$$

LEMMA 2.9. *Suppose* $detD_0 \neq 0$, $g \in W_2^1((-h,0), C^\tau)$, $f \in L_{2,r_0}(R_+, C^\tau)$ *for a certain* $r_0 \in R$. *Then there exists constant* $r_*(r_* > r_0)$ *such that for every* $r > r_*$, *problem (1)-(2) has a unique solution* $u \in W^1_{2,r}((-h,0), C^\tau)$ *satisfying the following inequality*

$$||u||_{W^1_{2,r}(-h,+\infty)} \leq c_r(||g||_{W_2^1(-h,0)} + ||f||_{L_{2,r}(R_+)})$$

with constant c_r *independent of the functions* f *and* g.

3. Proofs of the Main Results. We should emphasize that the main result Theorem 2.4 is based on the estimates of the solutions for homogeneous equation ($f(t) \equiv 0$) obtained earlier in [9], [10].

Also since equation (1) is linear, we shall consider the problem (1)-(2) in the case $g(t) \equiv 0$.

In the first step, we consider the problem (1)-(2) where the non-homogeneous term f has the support contained in $[0,h]$ and initial function $g = 0$.

Denote the solution of this problem by $u_h(t)$. Using the results in [9] and [10], we have

$$(9)\quad \int_0^{+\infty} exp(-2rt)(||u_h^{(1)}(t)||^2 + ||u_h(t)||^2)dt \leq c_r||f||^2_{L_2[0,h]}$$

for some constant r and constant c_r independent of the function f.

It follows immediately that

$$(10)\quad ||u_h||_{W_2^1[0,h]} \leq c_1||f||_{L_2[0,h]}$$

with the constant c_1 independent of the function f.

We then consider the following initial-value problem

$$(Dv)(t) = 0, t > h; \tag{11}$$

$$v(t) = u_h(t), t \in [0,h]. \tag{12}$$

Due to the uniqueness of the Cauchy initial value problem, we have $v(t) = u_h(t)$ for all $t > h$.

An estimate of the solution for the homogeneous equation (1) ($f(t) \equiv 0$) with initial data (2) was established earlier in [9], [10] and it has exactly the same form (6) with $f(t) \equiv 0$. Substituting g by u_h, u by v and segment $[-h, 0]$ by $[0, h]$, we obtain

$$\|v\|_{W_2^1[t-h,t]} \leq d_1(t+1)^{M-1} exp(\aleph_+ t)\|u_h\|_{W_2^1[0,h]}, t > h, \tag{13}$$

with constant d_1 independent of the function u_h.

From estimates (10)-(13) the following inequality follows

$$\|u_h\|_{W_2^1[t-h,t]} \leq d_2(t+1)^{M-1} exp(\aleph_+ t)\|f\|_{L_2[0,h]}, t > h, \tag{14}$$

with constant d_2 independent of the function f.

For the second step of the proof of Theorem 2.4, we need the following technical lemma whose proof is deferred to the end of this section.

LEMMA 3.1. *Suppose that $detD_0 \neq 0$, $detD_n \neq 0$, the support of f is contained in $[jh - h, jh]$ for some integer $j > 1$, the initial function $g = 0$. Then, for the solution of the problem (1)-(2) we have the following assertions:*

(i) $u(t) = 0$, for $t < jh - h, j = 1, 2, \cdots$;

(ii) For each $k > j$ we have

$$\|u\|_{W_2^1[kh-h,kh]} \leq d((k-j+1)h)^{M-1} exp(\aleph_+(k-j)h)\|f\|_{L_2[jh-h,jh]}, \tag{15}$$

with constant d independent of the function f.

For the proof of the estimate (6) on the whole interval, we need, in addition to Lemma 3.1, the following representation of the function f:

$$f(t) = \sum_{j=1}^{\infty} f_j(t), f_j(t) = \mathcal{X}(jh-h, jh)f(t), \tag{16}$$

where $\mathcal{X}(jh - h, jh)$ is a characteristic function of the interval $(jh - h, jh)$.

The essential part of the remaining part of the proof for the estimate (6) is the fact that the functions $f_j(t)$ with $j > k$ do not influence on the solution u on the segment $[0, kh]$. Also, as mentioned earlier, the estimates for solutions of the homogeneous equation are based on the fact that exponential solutions give a Riesz basis for the spaces involved. More precisely, we have the following

THEOREM 3.2. *Let $detD_0 \neq 0$ and $detD_n \neq 0$. Then the system of the subspace $W_n = Span_{\lambda_q \in G_n}\{y_{q,j,s}(t)\}$ forms the Riesz basis (unconditional basis) in the space $W_2^1((-h,0), C^\tau)$. If, in addition, we have $inf_{\lambda_p \neq \lambda_q}|\lambda_p - \lambda_q| > 0$, then the system of subspaces $V_{\lambda_q} = Span_{\lambda_q}\{y_{q,j,s}(t)\}$ forms the Riesz basis (unconditional basis) in the space $W_2^1((-h,0), C^\tau)$.*

The proofs of this results and estimates of the solutions of the homogeneous equation (1) can be found in [8], [9] (in the case $B(s) \equiv 0, D(s) \equiv 0$).

In our situation, the proofs are completely the same. Only certain technical details have to be added. We give the proof of Riesz basisness in the Appendix.

We should mention that more general results for the equations of arbitrary order m with matrix coefficients can be found in [13] and [14]. The most complete results about Riesz basisness of the system of exponential solutions in the scalar case ($\tau = 1$) for equations of neutral type and of arbitrary order were obtained in [12] and [26]. These articles also proved results about Riesz basisness of the system of divided differences constructed by the system of exponential solutions, these are rather useful in the situation when the set Λ is not separate. All results in [12] and [26] were obtained in the scale of Sobolev spaces of arbitrary orders ($s \geq m, s \neq l + 1/2, l \in \mathbb{N}$). We should also mention that Riesz basisness of exponential solutions $\{y_{q,j,s}(t)\}$ of homogeneous equation (1) in the space $M_2 \equiv \mathbb{C}^\tau \oplus L_2((-h,0), \mathbb{C}^\tau)$ was obtained in [27] under an additional condition (the set Λ is separate). In the particular case for the equation with one delay (in our notations: $detD_n \neq 0, D_j \neq 0, j = \overline{1, n-1}, D_0 = I, B_j \equiv 0, j = \overline{0, n}$), ne can find results about Riesz basisness of exponential solutions in [28].

Before getting into the details of the proofs of our main results, let us briefly indicate how this Riesz basis is used in obtaining the sharp estimates for the solutions of equation (1), in the simplest case of a scalar equation when all roots λ_q are simple. In this case, the system of the exponential solutions

$$z_q(t) = \frac{exp(\lambda_q t)}{|\lambda_q| + 1}, \lambda_q \in \Lambda,$$

forms the Riesz basis in the space $W_2^1((-h,0), C^\tau)$.

Denote by $\mathcal{V}$ the orthogonaizer, i.e. the bounded operator, acting in the space $W_2^1((-h,0),C^r)$ and turning the system $\{z_q(t)\}$ into orthonormal basis of the space $W_2^1((-h,0),C^r)$. Decompose the initial function g by the basis $\{z_q(t)\}$ as

$$g(s) = \sum_{\lambda_q \in \Lambda} c_q z_q(s), s \in [-h,0].$$

The desired estimate now follows from the following chain of the inequalities:

$$\begin{aligned} \|u(t+.)\|^2_{W_2^1(-h,0)} &= \|\mathcal{V}^{-1} \textstyle\sum_{\lambda_q\in\Lambda} c_q e^{\lambda_q t}(\mathcal{V}z_q)(.)\|^2_{W_2^1(-h,0)} \\ &\le \|\mathcal{V}^{-1}\|^2 exp(2\aleph_+ t) \textstyle\sum_{\lambda_q\in\Lambda} |c_q|^2 \\ &\le \|\mathcal{V}^{-1}\|^2 \|\mathcal{V}\|^2 exp(2\aleph_+ t)\|g\|^2_{W_2^1((-h,0)}. \end{aligned}$$

Note that we need the aforementioned Riesz basis in order to ensure that the bounded operator $\mathcal{V}$ has a bounded inverse $\mathcal{V}^{-1}$ (see [17] for more details).

Now we are ready to give

Proof of Lemma 3.1 The first assertion of Lemma 3.1 is valid due to the unique solvability of the problem (1)-(2) in the space $W_{2,r}^1((-h,+\infty),C^r)$ for some $r \in R$(see [8] for more details).

This means that if $u(t) = 0$ for $t \in [-h,0]$ and if the right-hand side $f(t) = 0$ for $t \in [0,T]$, then the solution $u(t) = 0$ for $t \in [0,T]$. In other words, the operator D is a causal operator(an operator of Volterra type).

In order to obtain the assertion (ii), we change the variables by $t = (jh-h)+\tau$ (note we are abusing τ here, and hope this will not cause any confusion) and set

$$\widetilde{u}(\tau) = u(jh-h+\tau), \widetilde{f}(\tau) = f(jh-h+\tau). \tag{17}$$

It is clear that $\widetilde{u}(\tau)$ is a solution of the following problem

$$(D\widetilde{u}(\tau)) = \widetilde{f}(\tau), \tau > 0; \tag{18}$$

and

$$\widetilde{u}(\tau) = 0, \tau \in [-h,0]. \tag{19}$$

Taking into account that the support of $\widetilde{f}$ is contained in $[0,h]$ due to the inequality (14), we have

$$\|\widetilde{u}\|_{W_2^1(t-h,t)} \le c_1(t+1)^{M-1} exp(\aleph_+ t)\|\widetilde{f}\|_{L_2(0,h)}, \; t > h, \tag{20}$$

with constant c_1 independent of the function f.

Therefore, from (20) we obtain

$$\|u\|_{W_2^1(jh-2h+t,jh-h+t)} \leq c_1(t+1)^{M-1}exp(\aleph_+ t)\|f\|_{L_2(jh-h,jh)}. \tag{21}$$

Using the change of variables $T = jh - h + t$ once more, we obtain from (21) the following inequality

$$\begin{aligned}&\|u\|_{W_2^1(T-h,T)}\\ &\leq c_1(1+T-(jh-h))^{M-1}exp(\aleph_+(T-(jh-h)))\|f\|_{L_2(jh-h,jh)}.\end{aligned} \tag{22}$$

Setting $T = kh$, we obtain from (22) the assertion (ii) of Lemma 2.2.

Proof of Theorem 2.4: First of all, we consider the case $T = kh, k \in N$. It is rather clear that functions $f_j(t)$ (see (16)) for $j > k$ do not have any influence to the solution $u(t)$ on the interval $[0, kh]$.

Denote by $u_j(t)$ the solution of the problem (1)-(2) for the right parts of $f = f_j$ and $g \equiv 0$. Then, for $t \leq kh$, we have the representation

$$u(t) = \sum_{j=1}^{k} u_j(t). \tag{23}$$

Using well-known inequality

$$(a_1 + a_2 + \cdots + a_k)^2 \leq k(a_1^2 + a_2^2 + \cdots + a_k^2), a_j \in R,$$

we obtain the estimate

$$\begin{aligned}\|u\|^2_{W_2^1(kh-h,kh)} &\leq (\textstyle\sum_{j=1}^k \|u_j\|_{W_2^1(kh-h,kh)})^2\\ &\leq k(\textstyle\sum_{j=1}^k \|u_j\|^2_{W_2^1(kh-h,kh)}).\end{aligned} \tag{24}$$

From the inequalities (15) and (24) for $t = kh$, we deduce the following estimate

$$\begin{aligned}&\|u\|^2_{W_2^1(kh-h,kh)}\\ &\leq c_1 k \textstyle\sum_{j=1}^k [\int_{jh-h}^{jh} (kh-\tau+1)^{2(M-1)} exp[2\aleph_+(kh-\tau)]\|f_j(\tau)\|^2 d\tau]\\ &= c_1 k \textstyle\int_0^{kh} (kh-\tau+1)^{2(M-1)} exp[2\aleph_+(kh-\tau)]\|f(\tau)\|^2 d\tau\\ &= c_1 \frac{t}{h} \textstyle\int_0^t (t-\tau+1)^{2(M-1)} exp[2\aleph_+(t-\tau)]\|f(\tau)\|^2 d\tau\end{aligned} \tag{25}$$

with constant c_1 independent of the function f.

Now, we consider the case of arbitrary real $t > h$. Let us choose such k that $kh \geq t$. It is evident that

$$\|u\|^2_{W_2^1(t-h,t)} \leq 2(\|u\|^2_{W_2^1(kh-2h,kh-h)} + \|u\|^2_{W_2^1(kh-h,kh)}). \tag{26}$$

Using the inequality (25), we have the following estimate

$$\|u\|^2_{W_2^1(t-h,t)}$$
$$\leq c_2(k-1)\int_0^{kh-h}(kh-h-\tau+1)^{2(M-1)}exp[2\aleph_+(kh-h-\tau)]\|f(\tau)\|^2d\tau$$
$$+c_2k\int_0^{kh}(kh-\tau+1)^{2(M-1)}exp[2\aleph_+(kh-\tau)]\|f(\tau)\|^2d\tau$$
$$\leq c_3k\int_0^{kh}(kh-\tau+1)^{2(M-1)}exp[2\aleph_+(kh-\tau)]\|f(\tau)\|^2d\tau \quad (27)$$

with constant c_2, c_3 independent of the function f.

The solution $u(\tau)$ on the segment $[0,t]$ does not depend on the function $f(\tau)$ for $\tau > t$, thus we can substitute the L_2-norm of the right-hand side of (27) on the interval $(0,kh)$ by the L_2-norm on the interval $(0,t)$.

Finally, we note that for $t \approx kh$ (for sufficiently large k), we have

$$exp(\aleph_+(kh-\tau)) \approx \exp(\aleph_+(t-\tau)),$$

$$(kh-\tau+1)^{M-1} \approx (t-\tau+1)^{M-1}.$$

From the inequality (27), we then obtain the assertion (6) of Theorem 2.4, completing the proof.

We conclude this section with a

Proof of Proposition 2.8. Due to the compactness of the support of the function f, there are only a finite number of nonzero terms in the representation (16). Thus there will be only a finite number functions $u_j(t)$ in the representation (23) of the solution, and hence the multiplier k in the estimate (24) may be substituted by a constant independent of k. As a result, the term $\sqrt{t}$ in the inequality (6) can be dropped.

4. Examples, Remarks and Comments. The following example shows that our estimates of solutions to the homogeneous equation ($f \equiv 0$) is sharp in the sense that it is impossible to replace the constant $\aleph_+$ by a constant $\aleph_+ - \epsilon$ for an arbitrary positive ϵ.

Example 1. We consider the following difference differential equation

$$\frac{du}{dt} - au(t) - \frac{du}{dt}(t-1) + au(t-1) = 0,\ t > 0, \quad (28)$$

It is known (see [20], and also [2], [3] (chapter 9)) that each root λ_q of the characteristic quasipolynomial

$$L_0(\lambda) = \lambda + a - e^{-\lambda}(\lambda - a)$$

of the equation (28) is on the imaginary axis ($Re\lambda_q = 0$) with multiplicity $\mu_q = 1$. Therefore, $\aleph_+ = \aleph_- = 0, N = \max_{\lambda_q \in \Lambda} \mu_q = 1$.

Under our definition of the solution for the initial value problem associated with the equation (28), the following estimate holds

$$||u||_{W_2^1(t-1,t)} \leq d||g||_{W_2^1(-1,0)}, t > 1, \tag{29}$$

with the constant d independent of the function g.

We should emphasize here that our conclusion does NOT contradict the result of [20], as we consider solutions from Sobolev space W_2^1, while in the article [20] the initial function g does not belong to the space $W_2(-1,0)$.

The following example shows the sharpness of the estimate for the non-homogeneous equation.

Example 2 Consider the following problem (in the scalar case with $m = 1$):

$$u^{(1)}(t) + u^{(1)}(t-1) = 1, t > 0, \tag{30}$$

with

$$u(t) = 0, t \in [-1, 0]. \tag{31}$$

The solutions can be constructed step by step as follows:

$$u(t) = \begin{cases} k, & t \in [2k-1, 2k], \\ t-k, & t \in [2k, 2k+1]. \end{cases}$$

Therefore, we have

$$\begin{gathered} ||u||_{W_2^1[n-1,n]} \approx n, \\ ||f||_{L_2[0,n]} \approx \sqrt{n}, n \in N. \end{gathered}$$

For the characteristic quasipolynomial

$$L_1(\lambda) = \lambda(1 + e^{-\lambda}),$$

the following assertions $\aleph_- = \aleph_+ = 0$, $M = N = \max_{\lambda_q \in \Lambda} \nu_q$ hold true.

For $t = n$, the right-hand side of the inequality (6) for our example likes

$$\sqrt{t}\Big(\int_0^t (t-\tau+1)^{2(M-1)} exp(2\aleph_+(t-\tau))|f(\tau)|^2\Big)^{1/2} \approx n$$

Thus it is impossible to substitute $\aleph_+$ by $\aleph_+ - \epsilon$ with some $\epsilon > 0$ and to omit $\sqrt{t}$.

We now make a few comments about existing results and the significance of our findings. First of all, the estimates similar to (6) for which the quantity $\aleph_+$ is replaced by $\aleph_+ + \epsilon$ with some $\epsilon > 0$ are well-known (see [1]-[7] for more details). In the so called critical and supercritical cases (the situations where the roots λ_q of the quasipolynomial $l(\lambda)$ approach or lie on the imaginary axis, more refined estimates are needed and can be can be found in [2], [3], [20]). A natural and important question is whether one can refine these estimates by setting $\epsilon = 0$. Theorem 2.4 gives, in a certain sense, a positive answer to this question.

We should emphasize the spectral character of our approach and we show how effective this approach is. Due to the fact that equation (1) has a convolution type, it is rather natural to use the method based on a Laplace transform. But then it is impossible to obtain the estimate (6), since any method based on the Laplace transform must involve the inverse of the Laplace transform that involves integration along lines parallel to the imaginary axis with a positive distance $\epsilon > 0$ to the spectra (the set of all zeroes Λ of the quasipolynomial $l(\lambda)$).

It is also relevant to remark that our main purpose here is to obtain the sharpest estimates for solutions of functional differential equations of neutral type. For the retarded type equations, the structure of the set of roots Λ is different and in particular, there is a dominating (with the most real part) zero λ_q of the characteristic quasipolynomial. In this case, Laplace transform method should be rather effective.

5. Appendix: Proof of Riesz Basisness of Exponential Solutions. The proof below of the Theorem 5.4 about Riesz basisness of exponential solutions is rather technical, so we start with a short discussion of the main steps involved in the proof.

To prove Theorem 5.4, we need to verify the conditions (Lemma 5.8 obtained in [16]) about Riesz basisness of root subspaces in terms of resolvent of an operator in an abstract Hilbert space. In order to do so, we must obtain the representation of the resolvent of the differential operator A subject to the nonlocal boundary conditions (32). It is thus important to note that this operator A is the generator of a C^0-semigroup of the shift operator along the trajectories of strong solutions of the homogeneous equation (1) ($f(t) \equiv 0$) (this construction is similar to the well-known one presented in [2] and[3] for the space of continuous functions.) Another important step in our long technical proof is to verify that the resolvent of the operator A satisfies the inequalities formulated in Lemma 5.8, and to obtain these estimates we will need the lower estimates of quasipolynomials (see [1] and [15] for more details).

We start by recalling some results characterizing the system of exponential solutions of equation (1).

PROPOSITION 5.1. *Let* $detD_0 \neq 0$. *Then exponential solutions (4) form a minimal system in the space* $W_2^1((-h,0),C^\tau)$.

LEMMA 5.2. *Let* $detD_0 \neq 0$, $detD_n \neq 0$. *Then there exist constants* α *and* β *such that the set* Λ *lies in the vertical strip* $\{\lambda : \alpha < Re\lambda < \beta\}$ *and the system of exponential solutions* $y_{q,j,s}(t)$ *is complete in the space* $W_2^1((-h,0),C^\tau)$.

In the next Lemma we give the estimates of matrix-valued function $L^{-1}(\lambda)$,

LEMMA 5.3. *Let* $detD_0 \neq 0$ *and* $detD_n \neq 0$. *Then there exists the system of contours* $\Gamma_n = \{\lambda \in C : Re\lambda = \alpha,\ \gamma_n \leq Im\lambda \leq \gamma_{n+1}\} \cup \{\lambda \in C : \alpha \leq Re\lambda \leq \beta,\ Im\lambda = \gamma_{n+1}\} \cup \{\lambda \in C : Re\lambda = \beta,\ \gamma_n \leq Im\lambda \leq \gamma_{n+1}\} \cup \{\lambda \in C : \alpha \leq Re\lambda \leq \beta,\ Im\lambda = \gamma_n\}$, *in the set* $G(\Lambda,\rho)$ *for sufficiently small* $\rho > 0$, *that satisfies the following conditions:*

(i) $0 < \delta \leq \gamma_{n+1} - \gamma_n \leq \Delta < +\infty$ *with some positive constants* δ *and* Δ;

(ii) there exists a constant K *such that:*

$$\sup_{\lambda \in \Gamma_n} |\lambda| ||L^{-1}(\lambda)|| \leq K, \quad n \in Z.$$

Recall that we denote by W_n subspaces of the space $W_2^1((-h,0),C^\tau)$ which are the span of all exponential solutions $y_{q,j,s}(t)$ corresponding to the numbers λ_q lying in the domains bounded by contours Γ_n; by V_{λ_q} subspaces of the space $W_2^1((-h,0),C^\tau)$ which are the span of all exponential solutions $y_{q,j,s}(t)$, corresponding to the number λ_q.

We recall the formulation of Theorem 3.2 about Riesz basisness. For the convenience of reference, we reformulate this as follows.

THEOREM 5.4. *Let* $detD_0 \neq 0$ *and* $detD_n \neq 0$. *Then the system of subspaces* $\{W_n\}_{n\in Z}$ *forms a Riesz basis of subspaces in the space* $W_2^1((-h,0),C^\tau)$.

The following theorem makes the previous one more precise in the case of a separate set Λ.

THEOREM 5.5. *Let* $detD_0 \neq 0$ *and* $detD_n \neq 0$. *Assume also the set* Λ *is separate, that is,* $\inf_{\lambda_p \neq \lambda_q} |\lambda_p - \lambda_q| > 0$. *Then the system of subspaces* $\{V_{\lambda_q}\}_{\lambda_q \in \Lambda}$ *forms a Riesz basis of subspaces in the space* $W_2^1((-h,0),C^\tau)$.

Let us consider operator $Ay = y^{(1)}$, acting in the space $W_2^1([-h,0],C^\tau)$

with the following domain:

$$DomA = \{y:\ y \in W_2^2([-h,0],C^\tau), \sum_{k=0}^{n}(B_k y(-h_k) + D_k y^{(1)}(-h_k)) + \int_0^h (B(s)y(-s) + D(s)y^{(1)}(-s))ds = 0\}. \tag{32}$$

We denote by $\{P_n\}_{n\in Z}$ a system of Riesz spectral projectors of the operator A, corresponding to contours Γ_n:

$$(P_n f) = -\frac{1}{2\pi i}\int_{\Gamma_n} R_A(\lambda) f d\lambda, \tag{33}$$

where $R_A(\lambda)$ is the resolvent of the operator A.

It can be easily shown that the system of exponential solutions (4) of equation (1) coincides with the system of eigen and associated vectors of the operator A defined in (32). Therefore, Theorem 3.2 (5.4) is a corollary of the following

THEOREM 5.6. *Let $detD_0 \neq 0$, $detD_n \neq 0$. Then the system of subspaces $W_n = P_n W_2^1([-h,0],C^\tau)$, corresponding to the system of projectors (33) with contours Γ_n, satisfying conditions of Lemma 5.3, form Riesz basis of subspaces in the space $W_2^1([-h,0],C^\tau)$.*

We shall prove Theorem 5.6 based on the following

PROPOSITION 5.7. *Suppose that $detD_0 \neq 0$, $detD_n \neq 0$. Then the matrix-valued function $L^{-1}(\lambda)$ satisfies the following estimates*

$$||L^{-1}(\lambda)|| \leq c(|\lambda|+1)^{-1}, \quad \lambda \in G(\Lambda,\rho)\cup\{Re\lambda > 0\}, \tag{34}$$

$$||L^{-1}(\lambda)|| \leq c_0(|\lambda|+1)^{-1}exp(Re\lambda h), \quad \lambda \in G(\Lambda,\rho)\cup\{Re\lambda < 0\}, \tag{35}$$

where c, c_0 are constants.

This proposition can be reduced from the results of chapter 12 of monograph [1] and the results of article [15]. The proof is based on the lower estimates of quasipolynomials. We give the proof of Proposition 5.7 at the end of the Appendix, for the sake of completeness.

We will also need the following result, and we refer to [16] for a proof.

LEMMA 5.8. *If for every elements $f, g \in W_2^1((-h,0),C^\tau) \equiv H$*

$$\sum_{n\in Z}\left|\int_{\Gamma_n}(R_A(\lambda)f,g)_H d\lambda\right| \leq const||f||_H||g||_H, \tag{36}$$

then the system of the subspaces W_n forms unconditional (Riesz) basis in the closure of it's span; and if additionally the system W_n is complete in H then it forms unconditional (Riesz) basis in the whole space $H \equiv W_2^1((-h,0),\mathbb{C}^r)$.

Let us now calculate the resolvent of the operator $A:\ R_A(\lambda)z = y$ or $y^{(1)} = \lambda y + z$. We have

$$y(t) = e^{\lambda t}(C + \int_0^t e^{-\lambda s} z(s)ds),$$

where constant vector C may be found from the conditions (32). Namely, using the equality

$$y^{(1)} = \lambda y + z,$$

we have from condition (32) the following

$$\begin{aligned}&\sum_{k=0}^n B_k e^{-\lambda h_k}[C + \int_0^{-h_k} e^{-\lambda s} z(s)ds]\\ &+D_k(\lambda e^{-\lambda h_k}[C + \int_0^{-h_k} e^{-\lambda s} z(s)ds] + z(-h_k))\\ &+\int_0^h (B(s)e^{-\lambda s}[C + \int_0^{-s} e^{-\lambda\tau} z(\tau)d\tau]\\ &+D(s)(\lambda e^{-\lambda s}[C + \int_0^{-s} e^{-\lambda\tau} z(\tau)d\tau] + z(-s))ds = 0.\end{aligned}$$

As $z \in W_2^1([-h,0], C^r)$, we obtain

$$\int_0^{-t} e^{-\lambda s} z(s)ds = \frac{1}{\lambda}(z(0) - z(-t))e^{\lambda t}) + \frac{1}{\lambda}\int_0^{-t} e^{-\lambda s} z^{(1)}(s)ds.$$

Hence condition (32) takes the form

$$\begin{aligned}&\sum_{k=0}^n (B_k e^{-\lambda h_k}[C + \int_0^{-h_k} e^{-\lambda s} z(s)ds]\\ &+D_k\lambda e^{-\lambda h_k}[C + \tfrac{z(0)}{\lambda} + \tfrac{1}{\lambda}\int_0^{-h_k} e^{-\lambda s} z^{(1)}(s)ds])\\ &+\int_0^h B(s)e^{-\lambda s}(C + \int_0^{-s} e^{-\lambda\tau} z(\tau)d\tau)\\ &+D(s)\lambda e^{-\lambda s}(C + \tfrac{z(0)}{\lambda} + \tfrac{1}{\lambda}\int_0^{-s} e^{-\lambda\tau} z^{(1)}(\tau)d\tau)ds = 0,\end{aligned}$$

or

$$-L(\lambda)C = \sum_{k=0}^{n} D_k e^{-\lambda h_k} z(0) + \sum_{k=0}^{n} (B_k e^{-\lambda h_k} \int_0^{-h_k} e^{-\lambda s} z(s) ds$$
$$+ D_k e^{-\lambda h_k} \int_0^{-h_k} e^{-\lambda s} z^{(1)}(s) ds) + \int_0^h D(s) e^{-\lambda s} z(0) ds$$
$$+ \int_0^h (B(s) e^{-\lambda s} \int_0^{-s} e^{-\lambda\tau} z(\tau) d\tau + D(s) e^{-\lambda s} \int_0^{-s} e^{-\lambda\tau} z^{(1)}(\tau) d\tau) ds.$$

Let us denote by $F(\lambda)$ the following vector-valued function:

$$F(\lambda) = L^{-1}(\lambda)[\sum_{k=0}^{n} B_k e^{-\lambda h_k} + \int_0^h B(s) e^{-\lambda s} ds] \frac{z(0)}{\lambda}$$
$$- L^{-1}(\lambda)[\sum_{k=0}^{n} (B_k e^{-\lambda h_k} \int_0^{-h_k} e^{-\lambda s} z(s) ds + D_k e^{-\lambda h_k} \int_0^{-h_k} e^{-\lambda s} z^{(1)}(s) ds)$$
$$+ \int_0^h (B(s) e^{-\lambda s} \int_0^{-s} e^{-\lambda\tau} z(\tau) d\tau + D(s) e^{-\lambda s} \int_0^{-s} e^{-\lambda\tau} z^{(1)}(\tau) d\tau) ds].$$

Then the resolvent of the operator A may be rewritten as

$$R_A(\lambda) f = -e^{\lambda t} F(\lambda) - e^{\lambda t} \frac{f(0)}{\lambda} + \int_0^t e^{\lambda(t-s)} f(s) ds. \tag{37}$$

We now rewrite the vector-valued function $F(\lambda)$ as

$$F(\lambda) = Q(\lambda) + L^{-1}(\lambda) P(\lambda), \tag{38}$$

where

$$Q(\lambda) = L^{-1}(\lambda)[\sum_{k=0}^{n} B_k e^{-\lambda h_k} + \int_0^h B(s) e^{-\lambda s} ds] \frac{f(0)}{\lambda},$$
$$P(\lambda) = \sum_{k=0}^{n} e^{-\lambda h_k} G_k(\lambda) + G_{n+1}(\lambda),$$
$$G_k(\lambda) = \int_0^{-h_k} e^{-\lambda s} (B_k f(s) + D_k f^{(1)}(s)) ds,$$
$$G_{n+1}(\lambda) = \int_0^h (B(s) e^{-\lambda s} \int_0^{-s} e^{-\lambda\tau} f(\tau) d\tau + D(s) e^{-\lambda s} \int_0^{-s} e^{-\lambda\tau} f^{(1)}(\tau) d\tau) ds.$$

Now let us give the estimate of the vector-valued function $F(\lambda)$. Note that the vector-valued functions $G_k(\lambda)$ are entire functions of exponential type (not more than h_k) and belong to Hardy space in every strip $\{\lambda : A \leq Re\lambda \leq B\}$. Moreover, the following inequalities hold:

$$\sup_{A\leq x\leq B} \int_{-\infty}^{+\infty} ||G_k(x+iy)||^2 dy \leq c_1 ||f||^2_{W_2^1(-h,0)} \tag{39}$$

with a constant c_1 independent of the function $f(t)$. Hence, we obtain

$$\sup_{A\leq x\leq B} \int_{-\infty}^{+\infty} ||P(x+iy)||^2 dy \leq c_2 ||f||^2_{W_2^1(-h,0)} \tag{40}$$

with a constant c_2 independent of the function $f(t)$.

By Proposition 5.7 and the trace theorem we derive the following estimates of the function $Q(\lambda)$ in the domain $\Pi_\rho(\alpha_1, \beta_1) = G(\Lambda, \rho) \cap \{\lambda : \alpha_1 < Re\lambda < \beta_1\}$

$$||Q(\lambda)|| \leq c_3(|\lambda|+1)^{-2} ||f||_{W_2^1(-h,0)}, \quad c_3 = const > 0. \tag{41}$$

Here α_1, β_1 are constants with $\alpha_1 \leq \alpha,\ \beta \leq \beta_1$.

Using representation (38), Proposition 5.7 and estimates (40) and (41) (for $Re\lambda = \alpha,\ Re\lambda = \beta$), we conclude that

$$\int_{-\infty}^{+\infty} (1+|\xi+i\mu|^2) ||F(\xi+i\mu)||^2 d\mu \leq c_4 ||f||^2_{W_2^1(-h,0)}, \quad \xi = \alpha, \beta; \tag{42}$$

with a constant c_4 independent of the function $f(t)$.

By Lemma 5.2, the system $\{W_n\}_{n\in Z}$ is complete in the space $W_2^1((-h,0), C^\tau)$. So we need only to verify inequality (36).

Due to Lemma 5.8 and (37), we need only to prove that

$$\sum_{n\in Z} \left| \int_{\Gamma_n} (e^{\lambda t} F(\lambda), g(t))_{W_2^1} d\lambda \right| \leq const ||f||_{W_2^1} ||g||_{W_2^1}. \tag{43}$$

This is due to Cauchy theorem - an integral of holomorphic functions along closed contours Γ_n is equal to zero. The second and third items in (37) are holomorphic functions (exept one simple pole). The third item is holomorphic everywhere in arbitrary bounded domain; the second item is holomorphic everywhere except simple pole $\lambda = 0$. So if we substitute resolvent (37) in the expression (36) the integrals of the second and third items will be equal to zero. Therefore, the inequality (36) will have form (43).

We have

$$(e^{\lambda t}F(\lambda), g(t))_{W_2^1((-h,0),C^r)} = (\lambda F(\lambda), g_1(\overline{\lambda}))_{C^r} + (F(\lambda), g_0(\overline{\lambda}))_{C^r},$$

where

$$g_1(\lambda) = \int_{-h}^{0} e^{\lambda t} g^{(1)}(t)dt, \quad \dot{g}_0(\lambda) = \int_{-h}^{0} e^{\lambda t} g(t)dt.$$

Thus (43), holds if

$$\sum_{n\in Z} |\int_{\Gamma_n} (\lambda^j F(\lambda), g_j(\overline{\lambda}))d\lambda| \leq const||f||_{W_2^1}||g||_{W_2^1}, \quad j = 0, 1.$$

We note that the vector-valued function g_1 and g_0 are entire functions of exponential type, belong to the Hardy space $H_2(A, B)$ in every strip

$$\{\lambda : \ A \leq Re\lambda \leq B\},$$

since Hardy theorem ensures that the Laplace transform of a function belonging to the space $L_2(0, +\infty)$ is an element of Hardy space $H_2(\mathbb{C}_+)$ in the right half plane $\{\lambda : Re\lambda > 0\}$.

In the integral representation of function $g_0(\lambda)$ and $g_1(\lambda)$ we change the variables from t to $(-t)$:

$$\hat{g}_1(\lambda) = \int_{0}^{h} e^{-\lambda t} g^{(1)}(-t)dt;$$

$$\hat{g}_0(\lambda) = \int_{0}^{h} e^{-\lambda t} g(-t)dt.$$

Functions $g^{(1)}(-t)$ and $g(-t)$ have compact support belonging to segment [0,h]. These functions are elements of the space $L_2((0, h), \mathbb{C}^r)$. Moreover, functions $exp(at)g^{(1)}(-t), \exp(at)g(-t)$ are also elements of space $L_2((0, h), \mathbb{C}^r)$ for arbitrary a. Due to this fact the Laplace transforms $g_1(\lambda)$ and $g(\lambda)$ belong to Hardy space $H_2(Re\lambda > -a)$ for arbitrary $a \in \mathbb{R}$. So these functions belong to Hardy space $H_2(\lambda : A \leq Re\lambda \leq B)$.

It is well-known that for Hardy space $H_2(\mathbb{C}_+)$ the following equality is valid

$$\|f\|_{L_2(\mathbb{R}_+)} = (\sup_{x>0} \int_{-\infty}^{+\infty} |\hat{f}(x+iy)|^2 dy)^{1/2},$$

here $\hat{f}(x+iy)$ is the Laplace transform of the function $f(t)$. From this equality we easily deduce the following estimate:

$$\sup_{A \le x \le B} \int_{-\infty}^{+\infty} \|g_j(x+iy)\|^2 dy \le k_j \|g^{(j)}\|^2_{L_2(-h,0)}, \quad j = 0, 1, \tag{44}$$

with constants k_0 and k_1 independent of the function g.

So, for $A = \alpha$, $B = \beta$ we obtain

$$\begin{aligned} &\sum_{n \in Z} | \int_{\xi + i\gamma_n}^{\xi + i\gamma_{n+1}} (\lambda^j F(\lambda), g_j(\overline{\lambda})) d\lambda | \\ &\le \int_{-\infty}^{+\infty} |((\xi + i\mu)^j F(\xi + i\mu), g_j(\xi - i\mu))| d\mu \\ &\le c_5 (\int_{-\infty}^{+\infty} (1 + |\xi + i\mu|^{2j}) \|F(\xi + i\mu)\|^2 d\mu)^{1/2} \|g^{(j)}\|_{L_2(-h,0)} \end{aligned} \tag{45}$$

with constant c_5 independent of the function $g(t)$, $(j = 0, 1,\ \xi = \alpha, \beta)$.

Therefore, from inequalities (42), (45) we have

$$\sum_{n \in Z} | \int_{\xi + i\gamma_n}^{\xi + i\gamma_{n+1}} (\lambda^j F(\lambda), g_j(\overline{\lambda})) d\lambda | \le c_7 \|f\|_{W_2^1(-h,0)} \|g^{(j)}\|_{L_2(-h,0)} \tag{46}$$

with constant c_6 independent of the functions f and g $(j - 0, 1, \xi = \alpha, \beta)$.

Thus, we obtain a part of the estimate (43) on the vertical sides of contours Γ_n. In order to prove the part of estimate (43) on the horizontal sides of contours Γ_n, we need the following proposition which is a significant modification of theorem 3.3.1 from [24].

Denote by $M_{\nu 2}(R)$ the set of all entire functions of exponential type ν, which belong to the space $L_2(R)$ as functions of real argument $t \in R$.

LEMMA 5.9. *Let $v(z) \in M_{\nu 2}(R)$, and let sequence of real numbers $\{t_n\}_{n \in Z}$ satisfy the condition: $0 < \delta \le t_{n+1} - t_n \le \Delta < +\infty$, with certain positive constants δ and Δ. Then the following inequality takes place:*

$$(\sum_{n \in Z} |v(t_n)|^2)^{1/2} \le \delta^{-1/2} (1 + \nu\Delta) (\int_{-\infty}^{+\infty} |v(t)|^2)^{1/2}.$$

We shall give the proof of Lemma 5.9 at the end of the Appendix.

According to representation (38), we have

$$(\lambda^j F(\lambda), g_j(\overline{\lambda})) = (\lambda^j Q(\lambda), g_j(\overline{\lambda})) + (\lambda^j L^{-1}(\lambda)P(\lambda), g_j(\overline{\lambda})).$$

Due to Lemma 5.3 and estimate (41), we have

$$||\lambda Q(\lambda)||\, |_{Im\lambda=\gamma_n} \leq c_8 \sup(|\lambda|+1)^{-1}|_{Im\lambda=\gamma_n} ||f||_{W_2^1(-h,0)}.$$

From the latter inequality we obtain the estimate

$$\int\limits_{\alpha+i\gamma_n}^{\beta+i\gamma_n} ||\lambda Q(\lambda)||^2 \, d\lambda \leq c_9(|n|+1)^{-2}||f||^2_{W_2^1(-h,0)} \tag{47}$$

with a constant c_9 independent of the function $f(t)$.

In turn, applying Lemma 5.9 to the vector-valued function $g_j(\lambda)$ and to $t_n = \gamma_n, t = y$, we have

$$\sum_{n=-\infty}^{n=+\infty} |(g_j(x+i\gamma_n), e_l)|^2 \leq c_{10} \int\limits_{-\infty}^{+\infty} |(g_j(x+iy), e_l)|^2 dy \leq c_{11} \int\limits_{-\infty}^{+\infty} ||g_j(x+iy)||^2 dy,$$

for $x \in [\alpha, \beta]$, $j = 0, 1$, where $\{e_l\}_{l=1}^r$ is an orthonormal basis of the space C^r. Then, due to (44), we have

$$\sum_{n=-\infty}^{n=+\infty} \int\limits_{\alpha}^{\beta} ||g_j(x+i\gamma_n)||^2 dx \leq c_{12} \sup_{\alpha\leq x\leq\beta} \int\limits_{-\infty}^{+\infty} ||g_j(x+iy)||^2 dy \leq c_{13}||g^{(j)}||_{L_2(-h,0)}, j = 0, 1, \tag{48}$$

with constants $c_{10}, c_{11}, c_{12}, c_{13}$ independent of the function $g(t)$.

Taking into account that function $(P(\lambda), e_l), l = 1, 2, ..., r, \lambda = iz$, also satisfy the conditions of Lemma 5.9, by analogy with estimate $g_j(\lambda)$, we obtain the inequality

$$\sum_{n=-\infty}^{n=+\infty} ||P(x+i\gamma_n)||^2_{C^r} \leq c_{14} \int\limits_{-\infty}^{+\infty} ||P(x+iy)||^2_{C^r} dy, \quad x \in [\alpha, \beta]. \tag{49}$$

Then, according to estimates (49) and (40), we have

$$\sum_{n=-\infty}^{n=+\infty} \int_{\alpha}^{\beta} ||P(x+i\gamma_n)||^2 dx \leq c_{15}||f||^2_{W_2^1(-h,0)}$$

with a constant c_{15}independent of the function $f(t)$.

From the latter inequality and estimate (Lemma 5.3)

$$\sup_{\lambda \in l_n} |\lambda| \, ||L^{-1}(\lambda)|| \leq K_0 = const, \quad n \in Z,$$

we obtain inequality

$$\sum_{n=-\infty}^{n=+\infty} \int_{l_n} ||\lambda L^{-1}(\lambda)P(\lambda)||^2 |d\lambda| \leq c_{16}||f||^2_{W_2^1(-h,0)}, \tag{50}$$

where $l_n = \{\lambda \in C : \; Im\lambda = \gamma_n, \; \alpha \leq Re\lambda \leq \beta\}$.

Taking into account representation (38) and estimates (47) and (50), we have

$$\sum_{n=-\infty}^{n=+\infty} \int_{l_n} ||\lambda F(\lambda)||^2 |d\lambda| \leq c_{17}||f||^2_{W_2^1(-h,0)} \tag{51}$$

with a constant c_{17} independent of the function $f(t)$.

Hence from estimates (48), (51) and inequality

$$\sum_{n=-\infty}^{n=+\infty} |\int_{l_n} (\lambda^j F(\lambda), g_j(\overline{\lambda}))d\lambda|$$
$$\leq (\sum_{n=-\infty}^{n=+\infty} \int_{l_n} ||\lambda^j F(\lambda)||^2 |d\lambda|)^{1/2} (\sum_{n=-\infty}^{n=+\infty} \int_{l_n} ||g_j(\overline{\lambda})||^2 |d\lambda|)^{1/2}, \quad j = 0, 1,$$

the following estimate follows:

$$\sum_{n=-\infty}^{n=+\infty} |\int_{l_n} (\lambda^j F(\lambda), g_j(\overline{\lambda}))d\lambda| \leq c_{18}||f||_{W_2^1(-h,0)}||g^{(j)}||_{W_2^1(-h,0)}, j = 0, 1, \tag{52}$$

with constant c_{18} independent of the functions $f(t)$ and $g(t)$.

So, according to Lemma 5.8, the sequence of subspaces $\{W_n\}_{n \in Z}$ forms an unconditional basis (Riesz basis) of the space $W_2^1((-h,0), C^r)$.

The estimate (6) of homogeneous equation (1) ($f(t) \equiv 0$) is a corollary of our results in this Appendix about Riesz basisness of the system $\{W_u\}$ and Theorem 1 in [23]. For another independent proof on the estimate of the homogeneous equation (in the scalar case $m = 1$), see [12], [26].

The proof of Theorem 5.5 is similar and thus is omitted.

Proof of Lemma 5.7. Let $v(z) \in M_\nu$ be an entire function of ν-type, such that $v(t) \in L_2(-\infty, +\infty)$. Then the following holds:

$$\int_{-\infty}^{+\infty} |v(t)|^2 dt = \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} |v(t)|^2 dt = \sum_{k\in\mathbb{Z}} |v(\xi_k)|^2 \Delta t_k,$$

where $\xi_k \in [t_k, t_{k+1}]$. Due to generalized Bernstein inequality (see [24])and Holder inequality, and using $\|a\| - \|b\| \le \|a-b\|$, we have

$$\begin{aligned}
&|(\textstyle\sum_{k\in\mathbb{Z}} |v(\xi_k)|^2 \Delta t_k)^{1/2} - (\sum_{k\in\mathbb{Z}} |v(t_k)|^2 \Delta t_k)^{1/2}| \\
&\le (\textstyle\sum_{k\in\mathbb{Z}} |v(\xi_k) - v(t_k)|^2 \Delta t_k)^{1/2} = (\sum_{k\in\mathbb{Z}} |\int_{t_k}^{\xi_k} v^{(1)}(t) dt|^2 \Delta t_k)^{1/2} \\
&\le (\textstyle\sum_{k\in\mathbb{Z}} (\int_{t_k}^{\xi_k} |v^{(1)}(t)|^2 dt)(\xi_k - t_k)\Delta t_k)^{1/2} \\
&\le (\textstyle\sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} |v^{(1)}(t)|^2 dt)^{1/2} \sup_{k\in\mathbb{Z}} (\Delta t_k) \\
&\le \Delta (\textstyle\int_{-\infty}^{+\infty} |v^{(1)}(t)|^2 dt)^{1/2} \\
&\le \Delta\nu \|v(t)\|_{L_2}.
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
&(\textstyle\sum_{k\in\mathbb{Z}} |v(t_k)|^2 \Delta t_k)^{1/2} \\
&= [(\textstyle\sum_{k\in\mathbb{Z}} |v(t_k)|^2 \Delta t_k)^{1/2} - (\sum_{k\in\mathbb{Z}} |v(\xi_k)|^2 \Delta t_k)^{1/2}] + (\sum_{k\in\mathbb{Z}} |v(\xi_k)|^2 \Delta t_k)^{1/2} \\
&\le (1 + \Delta\nu) \|v\|_{L_2}.
\end{aligned}$$

(53)

Hence, we have

$$\delta^{1/2} (\sum_{k\in\mathbb{Z}} |v(t_k)|^2)^{1/2} \le (\sum_{k\in\mathbb{Z}} |v(t_k)|^2 \Delta t_k)^{1/2}. \tag{54}$$

So from (53) and (54) we get the desired inequality.

We now turn to the proof of Proposition 5.7, that is a consequence of several results in [1] and [15]. To be more specific, we add a few comments here. The estimate of the matrix function $\mathcal{L}^{-1}(\lambda)$ outside the band

$\lambda :\leq Re\lambda \leq B, A < 0, B > 0$, can be established by a straightforward verification (see also [1], Chap.12). The estimates of the matrix function $L^{-1}(\lambda)$ on the set $G(\Lambda,\rho)\bigcap\{\lambda : A < Re\lambda < B\}$ can be derived using the lower estimate of the quasipolynomial $l(\lambda)$. Indeed, by virtue of the fact that $detD_0 \neq$ and $detD_n \neq 0$ the coefficients at λ^τ and $\lambda^\tau exp(-\lambda\tau h)$ are determined by the quantities $det(\lambda D_0 + B_0)$ and $det((\lambda D_n + B_n)exp(-\lambda h))$ (see [1],p.429, formula 12.2.12) that differ from zero if $|\lambda|$ is large. Then in correspondence with inequality (3.12) in [25], we obtain the estimate

$$|l(\lambda)| \geq c_{20}(|\lambda|^\tau + 1), \lambda \in G(\Lambda,\rho)\bigcap\{\lambda : A \leq Re\lambda \leq B\}. \tag{54}$$

Since the entries of the matrix function $L^{-1}(\lambda)$ are composed of cofactors of $L(\lambda)$ and quasipolynomial $l(\lambda)$, from estimate (55) we get inequalities (34) and(35) in the band $\{\lambda : A \leq Re\lambda \leq B\}, A < 0, B > 0$.

Therefore, the proof of the existence of the sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$, mentioned in Lemma 5.3, follows from Lemma 4 and 5 in [15] as follows: Consider τ different zeros $\lambda_{q_j} (j = 1, 2, \cdots, \tau)$ of the quasipolynomial $l(\lambda)$ and introduce the function

$$\eta(\lambda) = \exp(\frac{\lambda\tau h}{2})l(\lambda)/(\lambda - \lambda_{q_1})\cdots(\lambda - \lambda_{q_\tau}).$$

Using the fact that $detD_0 \neq 0$ and $detD_n \neq 0$, and repeating literally the corresponding arguments in [11], we get that the function $\eta_1(z) = \eta(-\frac{2\pi zi}{h\tau})$ satisfies the conditions of Lemma 5 in [15]. Also for sufficiently small $\rho > 0$, on every interval of unit length we can determine a number γ_n such that the straight line $Im\lambda = \gamma_n$ does not intersect the exceptional set $\bigcup_{\lambda_q\in\Lambda} D(\lambda_q, \rho)$. Hence we have the assertion of Lemma 5.3.

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