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Periodic Solutions of Delay Differential Equations with a Small Parameter: Existence, Stability and Asymptotic Expansion*

Chunhua Ou[1] **and Jianhong Wu**[1,2]

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We consider a scalar delay differential equation with a small parameter, and employ Walther's method to obtain a result on the existence and stability of a slowly oscillatory periodic solution that represents a refinement of the estimate for the Lipschitz constant of a returning map. We also develop a matching method and obtain asymptotic expansions of the slowly oscillatory periodic solution and its minimal period.

KEY WORDS: Periodic solutions; stability; Walther's method; matching method; asymptotic expansion.

AMS subject classifications: 34K15; 34K20; 34C25.

1. INTRODUCTION

We are here concerned with the delay differential equation

$$x' = -ax(t) - bf_\varepsilon(x(t-1)), \tag{1.1}$$

where a is real, $b > 0$ and ε is a small positive parmeter, $f_\varepsilon(x)$ is a nonlinear function which approximates the *sign* function (in a sense to be made precise later), and satisfies a sign condition

$$f_\varepsilon(x)x > 0 \quad \text{for } x \neq 0.$$

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*Dedicated to Professor Shui-Nee Chow on the occasion of his 60th birthday.

[1] Laboratory for Industrial and Applied Mathematics, York University, Toronto, Canada M3J 1P3. E-mail: wujh@mathstat.yorku.ca

[2] To whom correspondence should be addressed.

We shall concentrate on the case where $a > 0$, for the case where $a \leqslant 0$ the idea is entirely similar. Such an equation arises naturally from describing synchronized activities of a network of neurons with delayed negative feedback (see [1, 8]).

Our focus here is on the existence, stability and asymptotic expansion of periodic solutions when ε is small. A new constructive method was recently developed by Walther in [3, 4] to obtain not only existence but also stability of a slowly oscillatory periodic solution, based on a careful construction of a closed cone and a contractive returning map whose unique fixed point gives the periodic solution. This method, referred to as Walther's method in the sequel, was also employed to deal with equations with state-dependent delay in [5, 6] and systems of delay differential equations arising from phase-locking in neural networks [9].

Our first purpose is to refine some of the estimates in [3, 4] for the Lipschitz constant of the returning map, enabling us to obtain a sharper sufficient condition for Eq. (1.1) to have a stable slowly oscillatory periodic solution. Our second purpose is to develop the matching method in order to derive explicitly asymptotic expansions of the slowly oscillatory periodic solution.

2. RETURNING MAP AND A SHARP ESTIMATE FOR ITS LIPSCHITZ CONSTANT

In this section, we first recall the notion of a returning map introduced in [3, 4], and we then derive a sharper estimate of the Lipschitz constant for this map. We note that even in the case where $f_\varepsilon(x)$ is monotone and $f'_\varepsilon(x)$ is increasing, our estimate represents an improvement over the work of Walther [3, 4]. In particular, for the nonlinearity

$$f_\varepsilon(x) = \int_0^{x/\varepsilon} \frac{1}{1+t^\gamma} dt,$$

we conclude that Eq. (1.1) has a stable periodic solution if $\gamma > \sqrt{2}$ and ε is sufficiently small.

We first specify that by a solution of Eq. (1.1), we mean either a continuous real function on R which is differentiable and satisfies Eq. (1.1) almost everywhere, or a continuous real function on some interval $[t_0 - 1, \infty)$, $t_0 \in R$, which is differentiable and satisfies Eq (1.1) almost everywhere on (t_0, ∞). Denote by C the space of all continuous real functions on the interval $[-1, 0]$, with the norm given by $||\phi|| = \max_{-1 \leqslant t \leqslant 0} |\phi(t)|$ for $\phi \in C$. For each $\phi \in C$, one can solve Eq. (1.1) step by step on consecutive intervals [0, 1], [1, 2],..., and obtain a solution $x(t)$ on $[-1,\infty)$ with $x(t)=\phi(t)$ for $-1\leqslant t\leqslant 0$.

We start with solving Eq. (1.1) when $f_\varepsilon(x)$ is exactly the sign function. The following is straightforward, and was obtained in [4].

Lemma 1. *Suppose that y is the solution of the following equation*

$$y'(t)=-ay(t)-b\ \mathrm{sign}(y(t-1)) \tag{2.1}$$

with the initial function $\phi(t)>0$ for $t\in[-1,0)$ and $\phi(0)=0$. Then $y(t)$ is eventually periodic in the sense that $y(t+T)=y(t)$ for $t\geqslant 0$, where

$$T=2+\frac{2}{a}\ln(2-e^{-a}),$$

and in the interval $[0,T]$, $y(t)$ has the following expression:

$$y(t)=\begin{cases}\frac{b}{a}(e^{-at}-1), & 0\leqslant t\leqslant 1,\\ \frac{b}{a}((1-2e^{a})e^{-at}+1), & 1\leqslant t\leqslant t_1,\\ \frac{b}{a}((2-e^{-a})e^{-a(t-t_1)}-1), & t_1\leqslant t\leqslant T\end{cases} \tag{2.2}$$

with

$$t_1=\frac{T}{2}+1=1+\frac{1}{a}\ln(2e^{a}-1).$$

We should mention that this periodic solution $y(t)$ is an S-solution, i.e.,

$$y(t)=-y\left(t-\frac{T}{2}\right)$$

and also satisfies Eq. (2.1) for t in the interval $(-\infty,\infty)$. The period T as a function of a is decreasing in a, and $T\to 4$ as $a\to 0$, and $T\to 2$ as $a\to\infty$.

Our next step is to study periodic solutions of Eq. (1.1) when the nonlinearity $f_\varepsilon(x)$ is continuous, bounded and close to the sign function $\mathrm{sign}(x)$ when ε is small. More precisely, we define

$$N_\varepsilon=\{\ f: f \text{ is continuous and odd, } |f(x)|\leqslant M \text{ for } x\in R,\\ |f(x)-1|\leqslant\delta(\varepsilon) \text{ if } x\geqslant\beta(\varepsilon)\}, \tag{2.3}$$

where M is a positive constant, $\beta(\varepsilon)$ and $\delta(\varepsilon)$ tend to zero when ε tends to zero. Like in [3, 4], the symmetry restriction for $f_\varepsilon(x)$ can be removed, but it is used here to shorten our presentation.

We shall fix a f_ε in N_ε, and restrict the initial data ϕ to the following closed convex set

$$A_\varepsilon = \{\phi \in C : \beta(\varepsilon) \leqslant \phi(t) \text{ for} -1 \leqslant t \leqslant 0, \phi(0) = \beta(\varepsilon)\}.$$

Note that for given $\phi \in A_\varepsilon$, Eq. (1.1) has a unique solution $x = x^{\phi, f_\varepsilon}$ which exists in the interval $[-1, \infty)$. The relation

$$F_{f_\varepsilon}(t, \phi) = x_t, \qquad x = x^{\phi, f_\varepsilon}, \qquad x_t(s) = x(t+s), \quad -1 \leqslant s \leqslant 0$$

defines a continuous semiflow $F = F_{f_\varepsilon}$ on C.

Now, we introduce some notations frequently used in the field of asymptotic analysis (see [2, 7]).

Let $h(\varepsilon)$ and $g(\varepsilon)$ be two real functions defined in R. By

$$h(\varepsilon) = O(g(\varepsilon)) \quad \text{as } \varepsilon \to 0,$$

we mean that there is a constant $\mathbf{K} > 0$ and a neighborhood U of the origin such that $|f(\varepsilon)| \leqslant K|g(\varepsilon)|$ for all ε in U. By

$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as } \varepsilon \to 0,$$

we mean that for every small $\eta > 0$, there exists a neighborhood U_η of the origin such that $|f(\varepsilon)| \leqslant \eta |g(\varepsilon)|$ for all ε in U_η. If $f(\varepsilon)/g(\varepsilon)$ tends to unity, then we write

$$f(\varepsilon) \sim g(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Finally, let $\{\varphi_n(\varepsilon, t)\}$ be an asymptotic sequence as $\varepsilon \to 0$, i.e., $\varphi_{n+1}(\varepsilon, t) = o(\varphi_n(\varepsilon, t))$ as $\varepsilon \to 0$ for every $n \geqslant 0$, where t belongs to some interval $[a, b]$, and suppose that $f(\varepsilon, t)$ has the asymptotic expansion

$$f(\varepsilon, t) = \sum_{n=0}^{N-1} f_n(\varepsilon, t) + O(\varphi_N(\varepsilon, t)) \quad \text{as } \varepsilon \to 0.$$

If the neighborhood U in the definition of the O-symbol is independent of $t \in [a, b]$, then we say the above expansion is a uniform asymtotic expansion with respect to t in $[a, b]$.

With the above notations, we can now state the following result concerning the solution $x = x^{\phi, f_\varepsilon}$.

Lemma 2. *Let x^{ϕ, f_ε} be the solution of Eq. (1.1) with the initial function $\phi \in A_\varepsilon$. Then there exist two points w and s, $0 < w < s$,*

$$w = \frac{2\beta(\varepsilon)}{b} + o(\beta(\varepsilon)), \qquad s = 1 + \frac{1}{a}\ln(2 - e^{-a}) + O(\beta(\varepsilon), \delta(\varepsilon)), \quad \varepsilon \to 0, \quad (2.4)$$

so that $x^{\phi,f_\varepsilon}(w) = x^{\phi,f_\varepsilon}(s) = -\beta(\varepsilon)$. *Moreover*

$$x^{\phi,f_\varepsilon}(t) = y(t) + O(\beta(\varepsilon), \delta(\varepsilon)) \quad \text{for } 0 \leqslant t \leqslant s \tag{2.5}$$

and

$$x^{\phi,f_\varepsilon}(t) \leqslant -\beta(\varepsilon), \quad t \in [s-1, s],$$

where y *is the function defined in (2.2), and*

$$O(\beta(\varepsilon), \delta(\varepsilon)) = O(\max\{\beta(\varepsilon), \delta(\varepsilon)\}).$$

Remark 1. It follows from (1.1) and (2.5) that there exist two positive constants $K_1(a,b)$ and $K_2(a,b)$ which are independent of ε such that

$$|x^{\phi,f_\varepsilon}(t)| \leqslant K_1, \tag{2.6}$$

$$\left|\frac{dx^{\phi,f_\varepsilon}(t)}{dt}\right| \leqslant K_2 \tag{2.7}$$

for $t \geqslant 0$ and

$$1 < s = 1 + \frac{1}{a}\ln(2 - e^{-a}) + O(\beta(\varepsilon), \delta(\varepsilon)) < 2,$$

when ε is small.

The proof of this lemma involves the use of the variation-of-constants formula given by

$$x(t) = x(0)e^{-at} - b\int_0^t e^{-a(t-s)} f_\varepsilon(x(s-1))ds. \tag{2.8}$$

Note that when $x(s-1)$ lies in the interval $(-\beta(\varepsilon), \beta(\varepsilon))$, the value of $f_\varepsilon(x(s-1))$ may change abruptly, hence we call the corresponding interval of s the *inner layer*. Likewise, an interval outside this region is called an *outer layer*. In order to obtain the result in Lemma 2, we need to split the integral interval in (2.8) into inner and outer layers.

Proof of Lemma 2. Since $\phi \in A_\varepsilon$, we have $\phi(t) \geqslant \beta(\varepsilon)$ and

$$1 - \delta(\varepsilon) \leqslant f_\varepsilon(\phi(t)) \leqslant 1 + \delta(\varepsilon)$$

for $t \in [-1, 0]$. Inserting this into Eq. (1.1) gives

$$-ax - b(1 + \delta(\varepsilon)) \leqslant x'(t) \leqslant -ax - b(1 - \delta(\varepsilon)), \quad t \in [0, 1].$$

Therefore, we have

$$g_-(t) \leqslant x(t) \leqslant g_+(t), \quad t \in [0, 1], \tag{2.9}$$

where

$$g_+(t) = -\frac{b(1-\delta)}{a} + \left(\beta(\varepsilon) + \frac{b(1-\delta)}{a}\right)e^{-at}$$

and

$$g_-(t) = -\frac{b(1+\delta(\varepsilon))}{a} + \left(\beta(\varepsilon) + \frac{b(1+\delta)}{a}\right)e^{-at}.$$

Define three points w_-, w, w_+ by

$$g_-(w_-) = -\beta(\varepsilon), \qquad x(w) = -\beta(\varepsilon), \qquad g_+(w_+) = -\beta(\varepsilon).$$

We can readily deduce that

$$\frac{1}{a}\ln\frac{b(1+\delta)+a\beta(\varepsilon)}{b(1+\delta)-a\beta(\varepsilon)} = w_- \leqslant w \leqslant w_+ = \frac{1}{a}\ln\frac{b(1-\delta)+a\beta(\varepsilon)}{b(1-\delta)-a\beta(\varepsilon)},$$

and we obtain that w_+ and w_- have the following approximations:

$$w_+ = \frac{2\beta(\varepsilon)}{b} + o(\beta(\varepsilon)), \qquad w_- = \frac{2\beta(\varepsilon)}{b} + o(\beta(\varepsilon)). \tag{2.10}$$

It follows from (2.9) that

$$x(t) = y(t) + O(\beta(\varepsilon), \delta(\varepsilon)) \tag{2.11}$$

for $t \in [0, 1]$, where y is the function defined in (2.2). In particular, at $t = 1$, (2.11) becomes

$$x(1) = y(1) + O(\beta(\varepsilon), \delta(\varepsilon)) = \frac{b}{a}(e^{-a} - 1) + O(\beta(\varepsilon), \delta(\varepsilon)). \tag{2.12}$$

Next we shall give the estimate for $x(t)$ when $t > 1$, by using (2.11) and (2.12). For $t \in [1, 1 + w_+]$, using $|f_\varepsilon| \leqslant M$ and Eq. (1.1) we have

$$\begin{aligned} x(t) &= x(1)e^{-a(t-1)} - b\int_1^t e^{-a(t-s)} f_\varepsilon(x(s-1))ds \\ &= x(1) + O(\beta(\varepsilon), \delta(\varepsilon)). \end{aligned} \tag{2.13}$$

Also by the explicit function $y(t)$, we can easily show that

$$y(t) = y(1) + O(\beta(\varepsilon), \delta(\varepsilon)), \quad t \in [1, 1 + w_+]. \tag{2.14}$$

Combining (2.12)–(2.14) gives

$$x(t) = y(t) + O(\beta(\varepsilon), \delta(\varepsilon)), \quad t \in [1, 1 + w_+]. \tag{2.15}$$

In particular, at $t=1+w_+$, we have

$$x(1+w_+)=y(1+w_+)+O(\beta(\varepsilon),\delta(\varepsilon)) \tag{2.16}$$

as $\varepsilon\to 0$.

Finally when $t\geqslant 1+w_+$, using the information that $x(t)<-\beta(\varepsilon)$ for $t\in[w_+,1+w_+]$ (due to (2.9) and (2.15)), we obtain from Eq. (1.1) that

$$-ax-b(-1+\delta)\leqslant x'(t)\leqslant -ax-b(-1-\delta), \tag{2.17}$$

or equivalently

$$G_-(t)\leqslant x(t)\leqslant G_+(t), \tag{2.18}$$

where

$$G_+(t)=x(1+w_+)e^{-a(t-1-w_+)}+\frac{b(1+\delta)}{a}(1-e^{-a(t-1-w_+)})$$

and

$$G_-(t)=x(1+w_+)e^{-a(t-1-w_+)}+\frac{b(1-\delta)}{a}(1-e^{-a(t-1-w_+)}).$$

As before, define three points s_-,s,s_+ by

$$G_-(s-)=-\beta(\varepsilon),\qquad x(s)=-\beta(\varepsilon),\qquad G_+(s_+)=-\beta(\varepsilon).$$

Using (2.2), (2.10), (2.16) and (2.18), we can easily deduce

$$x(t)=y(t)+O(\beta(\varepsilon),\delta(\varepsilon)),\quad t\in[1+w_+,s]$$

and

$$s_+\leqslant s\leqslant s_-,$$

where

$$s_+=1+\frac{1}{a}\ln(2-e^{-a})+O(\beta(\varepsilon),\delta(\varepsilon)),$$

$$s_-=1+\frac{1}{a}\ln(2-e^{-a})+O(\beta(\varepsilon),\delta(\varepsilon)).$$

Thus we have proved (2.4) and (2.5). It remains to show $x^{\phi,f}(t)\leqslant-\beta(\varepsilon)$ for $t\in[s-1,s]$. Indeed, for $t\in[s-1,1+w_+]$, by (2.5) we have $x^{\phi,f}(t)\leqslant-\beta(\varepsilon)$. When $t\in[1+w_+,s]$, by (2.17) we know that $x^{\phi,f}(t)$ is increasing and attains the value $-\beta(\varepsilon)$ at the point $t=s$. This completes the proof. □

Lemma 2 enables us to construct a returning map

$$R_{f_\varepsilon} : A_\varepsilon \ni \phi \mapsto -F_{f_\varepsilon}(q,\phi) \in A_\varepsilon, \tag{2.19}$$

where $q = s > 1$ satisfies

$$x^{\phi, f_\varepsilon}(q) = -\beta(\varepsilon).$$

In addition, for any initial function $\phi \in A_\varepsilon$, there exists a constant $k > 0$ which is independent of ε and ϕ so that for $q-1 \leqslant t \leqslant q$, we have

$$-x^{\phi, f_\varepsilon}(t) \geqslant k(q-t) + \beta(\varepsilon). \tag{2.20}$$

To see this, we recall from Lemma 2 that

$$q = s = 1 + \frac{1}{a}\ \ln(2 - e^{-a}) + O(\beta(\varepsilon), \delta(\varepsilon)).$$

Choose a point $p = 1 + (1/2a)\ln(2 - e^{-a})$. It follows from Lemma 2 that for $t \in [q-1, p]$, the solution $x^{\phi, f_\varepsilon}(t)$ has a negative upper bound which only depends on the coefficients a and b. For $t \in [p, s]$, using the fact $x(t) \leqslant -\beta(\varepsilon)$ and $x(t-1) \leqslant -\beta(\varepsilon)$, we have from (1.1) that

$$x'(t) = -ax - bf_\varepsilon(x(t-1)) \geqslant b(1 - \delta(\varepsilon)).$$

Therefore, in the whole interval $[q-1, q]$, we can choose a constant $k = k(a, b)$, dependent only on a, b, so that (2.20) holds.

Due to this fact, we can further restrict the initial data to the following subset of A_ε:

$$A_\varepsilon^k = \{\phi \in A_\varepsilon; \phi(t) \geqslant -kt + \beta(\varepsilon) \text{ for } t \in [-1, 0]\}.$$

It is shown in [3, 4] that for every fixed point ϕ of R_{f_ε}, the solution $x = x^{\phi, f_\varepsilon}(t)$ extends to a periodic S-solution with the minimal period $T_\varepsilon = 2q$. Here by an S-solution, we mean $x(t)$ satisfies $x(t+q) = -x(t)$ for $t \in R$.

Next we shall give an estimation of a Lipschitz constant for R_{f_ε} in (2.19), restricted to A_ε^k. A Lipschitz constant for a given map $T : D_T \to Y, D_T \subset X$, X and Y being normed linear spaces, is given by

$$L(T) = \sup_{\xi \in D_T, \eta \in D_T, \xi \neq \eta} \frac{\|T(\xi) - T(\eta)\|}{\|\xi - \eta\|}.$$

In the case where $D_T = X = R, \beta \in R$, and $f_\varepsilon = T$, we set

$$L_\beta = L_\beta(f_\varepsilon) = L(f_\varepsilon|_{[\beta,\infty)}).$$

Theorem 1. *R_{f_ε} restricted to A_ε^k is Lipschitz continuous, with an upper bound of the Lipschitz constant given by*

$$\left(bL_{\beta(\varepsilon)}+b^2L_{\beta(\varepsilon)}^2+9\beta^2L(f_\varepsilon)L_{k/2}\right)\left(1+\frac{K_2}{b(1-\delta(\varepsilon))}\right).$$

Proof. Step 1. For $\phi,\bar{\phi}$ in A_ε^k, it follows from Lemma 2 that there exist $w<s, \bar{w}<\bar{s}, s,\bar{s}>1$ so that

$$x^{\phi,f_\varepsilon}(w)=x^{\phi,f_\varepsilon}(s)=-\beta(\varepsilon)$$

and

$$x^{\bar{\phi},f_\varepsilon}(\bar{w})=x^{\bar{\phi},f_\varepsilon}(\bar{s})=-\beta(\varepsilon).$$

By (2.4) in Lemma 2 again, we have

$$w\sim\frac{2\beta(\varepsilon)}{b},\qquad \bar{w}\sim\frac{2\beta(\varepsilon)}{b}\quad \text{as } \varepsilon\to 0.$$

Denote

$$\eta=\max(w,\bar{w}). \tag{2.21}$$

Then we have from (1.1)

$$\begin{aligned}|x^{\phi,f_\varepsilon}(t)-x^{\bar{\phi},f_\varepsilon}(t)| &= b\int_0^t e^{-a(t-s)}|f_\varepsilon(\phi(s-1))-f_\varepsilon(\bar{\phi}(s-1))|ds\\ &\leqslant \eta bL_{k/2}\|\phi-\bar{\phi}\| \end{aligned}\tag{2.22}$$

for any $t\in[0,\eta]$, and

$$|x^{\phi,f_\varepsilon}(t)-x^{\bar{\phi},f_\varepsilon}(t)|\leqslant bL_{\beta(\varepsilon)}\|\phi-\bar{\phi}\| \tag{2.23}$$

for any $t\in[0,1]$. Here to obtain (2.22), we have made use of the fact that

$$\phi(t-1)>\frac{k}{2},\bar{\phi}(t-1)>\frac{k}{2}$$

for t in the small region $[0,\eta]$.

Step 2. For $t\in[1,1+\eta]$, we have by (2.22) and (2.23) that

$$\begin{aligned}&|x^{\phi,f_\varepsilon}(t)-x^{\bar{\phi},f_\varepsilon}(t)|\\ &\quad\leqslant bL_{\beta(\varepsilon)}\|\phi-\bar{\phi}\|+b\int_1^t e^{-a(t-s)}|f_\varepsilon(x(s-1))-f_\varepsilon(\bar{x}(s-1))|ds\\ &\quad\leqslant bL_{\beta(\varepsilon)}\|\phi-\bar{\phi}\|+bL(f_\varepsilon)\int_1^{1+\eta}|x(s-1)-\bar{x}(s-1)|ds\\ &\quad\leqslant \{bL_{\beta(\varepsilon)}+b^2L(f_\varepsilon)L_{k/2}\eta^2\}\|\phi-\bar{\phi}\|. \end{aligned}\tag{2.24}$$

Step 3. We may assume, without loss of generality, that $s \leqslant \bar{s} < 2$. The Case where $\bar{s} < s < 2$ can be dealt with similarly. Note that when $t \in [1+\eta, s]$, we get from (2.23) that

$$|x^{\phi, f_\varepsilon}(t-1) - x^{\bar{\phi}, f_\varepsilon}(t-1)| \leqslant bL_{\beta(\varepsilon)} \|\phi - \bar{\phi}\|,$$

hence by (1.1) and (2.24), we obtain

$$\begin{aligned} |x^{\phi, f_\varepsilon}(t) - x^{\bar{\phi}, f_\varepsilon}(t)| &\leqslant |x^{\phi, f_\varepsilon}(1+\eta) - x^{\bar{\phi}, f_\varepsilon}(1+\eta)| \\ &\quad + b\int_{1+\eta}^{t} e^{-a(t-s)} |f_\varepsilon(x(s-1)) - f_\varepsilon(\bar{x}(s-1))| ds \\ &\leqslant \{bL_{\beta(\varepsilon)} + b^2 L(f_\varepsilon) L_{k/2}\eta^2\} \|\phi - \bar{\phi}\| + b^2 L^2_{\beta(\varepsilon)} \|\phi - \bar{\phi}\| \\ &= L_{R_s} \|\phi - \bar{\phi}\|, \end{aligned} \tag{2.25}$$

where

$$L_{R_s} = bL_{\beta(\varepsilon)} + b^2 L^2_{\beta(\varepsilon)} + b^2 L(f_\varepsilon) L_{k/2}\eta^2.$$

In particular, at the point $t = s$, we have

$$|-\beta(\varepsilon) - x^{\bar{\phi}, f_\varepsilon}(s)| = |x^{\phi, f_\varepsilon}(s) - x^{\bar{\phi}, f_\varepsilon}(s)| \leqslant L_{R_s} \|\phi - \bar{\phi}\|. \tag{2.26}$$

Step 4. For the solution $x^{\bar{\phi}, f_\varepsilon}(s)$, $t \in [s, \bar{s}]$, by Lemma 2 we can readily derive that

$$x'(t) = -ax - bf_\varepsilon(x(t-1)) \geqslant -bf_\varepsilon(x(t-1)) \geqslant b(1-\delta).$$

Then

$$\begin{aligned} |x^{\bar{\phi}, f_\varepsilon}(s) - (-\beta(\varepsilon))| &= |x^{\bar{\phi}, f_\varepsilon}(s) - x^{\bar{\phi}, f_\varepsilon}(\bar{s})| \\ &= \Big| \int_s^{\bar{s}} \dot{x}(s) \mathrm{d}s \Big| \\ &\geqslant b(1-\delta(\varepsilon)) |s - \bar{s}|, \end{aligned}$$

which implies, by (2.26), that

$$|s - \bar{s}| \leqslant \frac{L_{R_s} \|\phi - \bar{\phi}\|}{b(1-\delta(\varepsilon))}. \tag{2.27}$$

Step 5. Note that

$$\begin{aligned} |R_{f_\varepsilon}(\bar{\phi}) - R_{f_\varepsilon}(\phi)| &= |F_{f_\varepsilon}(\bar{s}, \bar{\phi}) - |F_{f_\varepsilon}(s, \phi)| \\ &\leqslant |F_{f_\varepsilon}(s, \bar{\phi}) - F_{f_\varepsilon}(s, \phi)| + |F_{f_\varepsilon}(\bar{s}, \bar{\phi}) - F_{f_\varepsilon}(s, \bar{\phi})|. \end{aligned}$$

By (2.7), the value of $|F_{f_\varepsilon}(\bar{s},\bar{\phi}) - F_{f_\varepsilon}(s,\bar{\phi})|$ is bounded by

$$\begin{aligned} |x_{\bar{s}}^{\bar{\phi},f}(\theta) - x_s^{\bar{\phi},f}(\theta)| &= \Big|\int_{s+\theta}^{\bar{s}+\theta} \dot{x}(u)du\Big| \\ &\leqslant K_2|\bar{s}-s| \\ &\leqslant \frac{K_2 L_{R_s}\|\phi-\bar{\phi}\|}{b(1-\delta(\varepsilon))}, \quad -1 \leqslant \theta \leqslant 0, \end{aligned}$$

and the value of $|F_{f_\varepsilon}(s,\bar{\phi}) - F_{f_\varepsilon}(s,\phi)|$ is bounded by the right-hand side of (2.25).

Therefore, the Lipschitz constant for R_{f_ε} is bounded by

$$L_{R_s}\left(1+\frac{K_2}{b(1-\delta(\varepsilon))}\right). \tag{2.28}$$

Finally, by (2.21) and Lemma 2, we have $0<\eta \leqslant (3/b)\beta(\varepsilon)$ as ε is small. Inserting this into (2.28), we obtain the estimate of the Lipschitz constant of R_{f_ε} in our theorem. This completes the proof. □

Remark 2. If

$$f_\varepsilon(x) = \alpha(r)\int_0^{x/\varepsilon} \frac{1}{1+t^r}dt$$

with $\alpha(r)$ properly chosen so that

$$\alpha(r)\int_0^{\infty} \frac{1}{1+x^r}dx = 1,$$

we can choose $\beta(\varepsilon) = \varepsilon^s$ with $0 < s < 1$. Then $L(f_\varepsilon) = O(1/\varepsilon), L_{k/2} = O(\varepsilon^{r-1}), L_{\beta(\varepsilon)} = O(\varepsilon^{r-1-rs})$
and

$$L(f_\varepsilon)L_{k/2}\beta^2 = O(\varepsilon^{r-2+2s}).$$

If $r>\sqrt{2}$, we can take $s = 1-(\sqrt{2}/2)$ so that

$$r-2+2s>0, \qquad r-1-rs>0.$$

Hence,

$$L_{\beta(\varepsilon)} \to 0, \qquad L(f_\varepsilon)L_{k/2}\beta^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

Then we can conclude from Theorem 1 that the Lipschitz constant for R_{f_ε} approaches zero as ε tends to zero provided that $r>\sqrt{2}$. Our result removes the assumptions of the monotonicity of the functions f_ε and f'_ε in [3], and extends the parameter r from $(3/2,\infty)$ to $(\sqrt{2},\infty)$.

3. ASYMPTOTIC ANALYSIS OF PERIODIC SOLUTIONS

In Section 2, we obtained an estimate for the Lipschitz constant R_{f_ε}. Using the argument of [3, 4], we know that if the function f_ε is chosen so that $R_{f_\varepsilon} < 1$, then there exists a stable periodic solution for problem (1.1).

In this section, we give an approximation formula for this stable periodic solution. We know that the minimal period of the periodic solution tends to a constant as ε tends to zero, this information is however not sufficient for the approximation formula of the periodic solution. We need to derive the exact functional relation between the minimal period and ε. We shall obtain this by developing the approach of asymptotic analysis, a powerful tool to solve ODEs and PDEs with a small parameter over large intervals. In particular, we shall apply the matching method to derive the approximation formula for the minimal period, and an asymptotic expansion of the periodic solution.

Since the function $f_\varepsilon(x)$ is odd, we would like to seek an S-solution for the original equation (1.1). For ease of computation, we also assume that $f_\varepsilon(x)$ is a smooth function. Suppose the periodic solution is extended to the entire interval $(-\infty, \infty)$ and has the series expansion

$$x(t) = \sum_{i=0}^{\infty} \phi_i(t, \varepsilon), \quad t \in (-T_\varepsilon, 0), \tag{3.1}$$

where $\phi_i(t, \varepsilon)$ vanishes at $t = 0$ for all $i \geqslant 0$, and T_ε is the half period of the solution to be determined later. By the fact that it is an S-solution, we deduce that

$$x(t) = -\sum_{i=0}^{\infty} \phi_i(t - T_\varepsilon, \varepsilon) \tag{3.2}$$

for $t \in [0, T_\varepsilon]$. Using Eqs. (1.1) and (3.1), we can compute $x(t)$ for $t \in [0, 1]$ as follows:

$$\begin{aligned} x(t) &= \int_0^t -be^{-a(t-s)} f_\varepsilon\left(\sum_{i=0}^{\infty} \phi_i(s-1, \varepsilon)\right) ds \\ &= \int_0^t -be^{-a(t-s)} ds - \int_0^t -be^{-a(t-s)}(-1 + f_\varepsilon(\phi_0(s-1, \varepsilon)))ds \\ &\quad - \int_0^t be^{-a(t-s)} \sum_{n=1}^{\infty} \left(f_\varepsilon\left(\sum_{i=0}^{n} \phi_i(s-1, \varepsilon)\right) - f_\varepsilon\left(\sum_{i=0}^{n-1} \phi_i(s-1, \varepsilon)\right)\right) ds \\ &= \sum_{n=0}^{\infty} \varphi_n(t, \varepsilon), \end{aligned}$$

where

$$\varphi_0(t,\varepsilon)=\int_0^t -be^{-a(t-s)}ds=\frac{b}{a}(e^{-at}-1), \tag{3.3}$$

$$\varphi_1(t,\varepsilon)=\int_0^t -be^{-a(t-s)}(-1+f_\varepsilon(\phi_0(s-1,\varepsilon)))ds, \tag{3.4}$$

and

$$\varphi_n(t,\varepsilon)=\int_0^t -be^{-a(t-s)}\left(f_\varepsilon\Big(\sum_{i=0}^{n-1}\phi_i(s-1,\varepsilon)\Big)-f_\varepsilon\Big(\sum_{i=0}^{n-2}\phi_i(s-1,\varepsilon)\Big)\right), \quad n\geqslant 2. \tag{3.5}$$

Note that $x(T_\varepsilon)=0$. Moreover, by Remark 1, $T_\varepsilon<2$ when ε is small. Integrating Eq. (1.1) gives, for $t\in(1,T_\varepsilon]$, that

$$\begin{aligned} x(t) &= \int_{T\varepsilon}^t -be^{-a(t-s)}f_\varepsilon\Big(\sum_{n=0}^{\infty}\varphi_n(s-1,\varepsilon)\Big)\mathrm{d}s \\ &= \sum_{n=0}^{\infty} w_n(t,\varepsilon), \end{aligned}$$

where

$$w_0(t,\varepsilon)=\int_{T_\varepsilon}^t be^{-a(t-s)}ds=\frac{b}{a}\Big(1-e^{a(T_\varepsilon-t)}\Big), \tag{3.6}$$

$$w_1(t,\varepsilon)=\int_{T_\varepsilon}^t -be^{-a(t-s)}(1+f_\varepsilon(\varphi_0(s-1,\varepsilon)+\varphi_1(s-1,\varepsilon)))ds \tag{3.7}$$

and

$$\begin{aligned} w_n(t,\varepsilon) = \int_{T_\varepsilon}^t -be^{-a(t-s)}\Big(f_\varepsilon\Big(\sum_{i=0}^{n}\varphi_i(s-1,\varepsilon)\Big) \\ -f_\varepsilon\Big(\sum_{i=0}^{n-1}\varphi_i(s-1,\varepsilon)\Big)\Big)ds, \quad n\geqslant 2, \end{aligned} \tag{3.8}$$

for $t\in(1,T_\varepsilon)$.

Now from (3.2), we have

$$-\sum_{i=0}^{\infty}\phi_i(t-T_\varepsilon,\varepsilon)=\sum_{n=0}^{\infty}w_n(t,\varepsilon) \tag{3.9}$$

for $t \in (1, T_\varepsilon)$ and

$$-\sum_{i=0}^{\infty} \phi_i(t - T_\varepsilon, \varepsilon) = \sum_{n=0}^{\infty} \varphi_n(t, \varepsilon) \tag{3.10}$$

for $t \in [0, 1]$. Equating the corresponding terms of (3.9) and (3.10), respectively, yields

$$\phi_0(t, \varepsilon) = \begin{cases} -\frac{b}{a}(1 - e^{-at}), & t \in (1 - T_\varepsilon, 0), \\ -\frac{b}{a}(e^{-a(t+T_\varepsilon)} - 1), & t \in [-T_\varepsilon, 1 - T_\varepsilon], \end{cases} \tag{3.11}$$

and

$$\phi_n(t, \varepsilon) = \begin{cases} -w_n(t + T_\varepsilon, \varepsilon), & t \in (1 - T_\varepsilon, 0], \\ -\varphi_n(t + T_\varepsilon, \varepsilon), & t \in [-T_\varepsilon, 1 - T_\varepsilon] \end{cases} \tag{3.12}$$

for $n \geqslant 1$.

Next we shall deduce the formula for T_ε. Using the continuity of the solution at $t = 1$, we can match $\sum_{n=0}^{\infty} \varphi_n(t, \varepsilon)$ and $\sum_{n=0}^{\infty} w_n(t, \varepsilon)$ to obtain

$$\sum_{n=0}^{\infty} w_n(1, \varepsilon) - \sum_{n=0}^{\infty} \varphi_n(1, \varepsilon) = 0, \tag{3.13}$$

or

$$T_\varepsilon = 1 + \frac{\ln(2 - e^{-a} + \frac{a}{b} \sum_{n=1}^{\infty} (w_n(1, \varepsilon) - \varphi_n(1, \varepsilon)))}{a}. \tag{3.14}$$

Remark 3. From the formulae of φ_0, w_0 and ϕ_0 in (3.3), (3.6) and (3.11), we find that there exist three positive constants $k_1(a, b), k_2(a, b)$ and $k_3(a, b)$, independent of ε, such that

$$\varphi_0(t, \varepsilon) \leqslant -k_1(a, b)t, \qquad t \in [0, 1],$$

$$w_1(t, \varepsilon) \leqslant k_2(a, b)(t - T_\varepsilon), \qquad t \in (1, T_\varepsilon]$$

and

$$\phi_0(t, \varepsilon) \geqslant -k_3(a, b)t, \quad t \in [-1, 0].$$

We now discuss the convergence of the series $\sum_{n=0}^{\infty} \varphi_n(t, \varepsilon)$ and $\sum_{n=0}^{\infty} w_n(t, \varepsilon)$ in (3.9), (3.10) and (3.14). Under minor technical conditions, similar to those required in Theorem 1, on the nonlinear function $f_\varepsilon(x)$, we not only establish uniform convergence of these series, but also prove that these series can be viewed as uniform asymptotic expansions in terms of the variable ε with respect the parameter t. Specifically, we have the following result:

Theorem 2. *Assume that $f_\varepsilon(x)$ in N_ε is a Lipschitz-continous function in R such that*

$$L_{c_1}(f)\to 0,\quad L_{c_2\beta(\varepsilon)}(f)\to 0,\quad L_{c_1}L(f)\beta^2(\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0 \tag{3.15}$$

for any fixed positive constants c_1 and c_2 independent of ε, where $L_{\beta(x)}$, L_{c_i}, $L(f)$ and $\beta(\varepsilon)$ are defined in Section 2. Then the series $\sum_{n=0}^{\infty}\varphi_n(t,\varepsilon)$ is uniformly convergent for $t\in[0,1]$, and so is the series $\sum_{n=0}^{\infty}w_n(t,\varepsilon)$ for $t\in(1,T_\varepsilon]$. Moreover, for the series $\sum_{n=0}^{\infty}\phi_n(t,\varepsilon)$, we have for $t\in[-T_\varepsilon,0]$

$$|\phi_{n+1}(t,\varepsilon)|=o\left(\sup_{t\in[-T_\varepsilon,0]}|\phi_n(t,\varepsilon)|\right),\quad n\geqslant 2,\quad \text{as } \varepsilon\to 0. \tag{3.16}$$

Proof. For any function $\bar\phi$ defined in the interval $[-T_\varepsilon,0]$, when $t\geqslant 0$ we define the returning map $R_{\bar\phi}(t)$ with $t\in[0,T_\varepsilon]$ as follows:

$$R_{\bar\phi}(t)=\begin{cases} u(t)=\int_0^t -be^{-a(t-s)}f_\varepsilon(\bar\phi(s-1,\varepsilon))ds, & t\in[0,1],\\ \int_{T_\varepsilon}^t -be^{-a(t-s)}f_\varepsilon(u(s-1,\varepsilon))ds, & t\in(1,T_\varepsilon]. \end{cases} \tag{3.17}$$

Then in terms of the returning map $R_{\bar\phi}(t)$, we can establish the following relations:

$$R_{\phi_0}(t)=\begin{cases} \varphi_0(t,\varepsilon)+\varphi_1(t,\varepsilon), & 0\leqslant t\leqslant 1,\\ w_0(t,\varepsilon)+w_1(t,\varepsilon), & 1<t\leqslant T_\varepsilon, \end{cases}$$

$$R_{\left(\sum_{i=0}^{n-1}\phi_i\right)}=\begin{cases} \sum_{i=0}^{n}\varphi_i, & 0\leqslant t\leqslant 1,\\ \sum_{i=0}^{n}w_i, & 1<t\leqslant T_\varepsilon, \end{cases} \tag{3.18}$$

and

$$R_{\left(\sum_{i=0}^{n-1}\phi_i\right)}-R_{\left(\sum_{i=0}^{n-2}\phi_i\right)}=\begin{cases} \varphi_n, & 0\leqslant t\leqslant 1,\\ w_n, & 1<t\leqslant T_\varepsilon. \end{cases}$$

For the function ϕ, the notation

$$\|\phi_i\|=\sup_{t\in[-T_\varepsilon,0]}|\phi_i(t)| \tag{3.19}$$

is well defined, although ϕ_i is not continuous at the point $t^*=1-T_\varepsilon$. This is because both $\lim_{t\to t^*-0}\phi_i$ and $\lim_{t\to t^*+0}\phi_i$ exist.

Next, we show that under condition (3.15), the returning map $R_\phi(t)$ satisfies

$$|R_{(\sum_{i=0}^{n}\phi_i)} - R_{\left(\sum_{i=0}^{n-1}\phi_i\right)}| = o(\|\phi_n\|), \quad \varepsilon \to 0, \;\; n \geqslant 2,$$

or, by (3.12) and (3.18), equivalently

$$|\phi_{n+1}(t)| = o(\|\phi_n\|), \quad \varepsilon \to 0, \;\; n \geqslant 2. \tag{3.20}$$

We shall only prove (3.20) in the case when $n=2$. When $n>2$, (3.20) can be obtained by induction and the proof is omitted.

We can easily show that for the functions ϕ_0, φ_0 and w_0, there exist three points η_1^0, η_2^0 and η_3^0 such that

$$\eta_1^0 \in (1-T_\varepsilon, 0], \qquad \eta_2^0 \in [0, 1], \qquad \eta_3^0 \in (1, T_\varepsilon]$$

and

$$\phi_0(\eta_1^0, \varepsilon) = \beta(\varepsilon), \quad \varphi_0(\eta_2^0, \varepsilon) = -\beta(\varepsilon), \quad w_0(\eta_3^0, \varepsilon) = -\beta(\varepsilon). \tag{3.21}$$

After some straightforward computations, we obtain

$$\eta_1^0 \sim -\frac{\beta(\varepsilon)}{b}, \qquad \eta_2^0 \sim \frac{\beta(\varepsilon)}{b}, \qquad \eta_3^0 - T_\varepsilon = \eta_1^0 \sim \frac{-\beta(\varepsilon)}{b} \tag{3.22}$$

as $\varepsilon \to 0$. Note that $\phi_0 \geqslant \beta(\varepsilon)$ for $t \in [-1, \eta_1^0]$, $\varphi_0(t,\varepsilon) \leqslant -\beta(\varepsilon)$ for $t \in [\eta_2^0, 1]$ and $w_0(t,\varepsilon) \leqslant -\beta(\varepsilon)$ for $t \in (1, \eta_3^0]$.

Using the above information, we can now estimate $\phi_1(t,\varepsilon), \varphi_1(t,\varepsilon)$ and $w_1(t,\varepsilon)$.

For $t \in \left[0, \eta_2^0\right]$, $|1 - f_\varepsilon(\phi_0(t-1))| \leqslant \delta(\varepsilon)$. By (3.4), we have

$$\begin{aligned} |\varphi_1(t,\varepsilon)| &\leqslant \int_0^t be^{-a(t-s)}|-1 + f_\varepsilon\;\,(\phi_0(s-1,\varepsilon))|ds \\ &\leqslant \int_0^t b\delta(\varepsilon)ds = b\delta(\varepsilon)t \leqslant b\delta(\varepsilon)\eta_2^0 \\ &= O(\delta(\varepsilon)\beta(\varepsilon)) \end{aligned} \tag{3.23}$$

for $t \in \left[0, \eta_2^0\right]$. For $t \in [\eta_2^0, 1]$, we have that

$$\begin{aligned} |\varphi_1(t,\varepsilon)| &\leqslant \int_0^t be^{-a(t-s)}|-1 + f_\varepsilon\;\,(\phi(s-1,\varepsilon))|ds \\ &\leqslant bt\delta(\varepsilon), \quad t \in \left[\eta_2^0, 1+\eta_1^0\right], \end{aligned} \tag{3.24}$$

and, by using $|f_\varepsilon| \leqslant M$, that

$$|\varphi_1(t,\varepsilon)| = O(\delta(\varepsilon), \beta(\varepsilon)), \quad t \in \left[1+\eta_1^0, 1\right]. \tag{3.25}$$

Therefore, in the whole interval [0, 1], we have

$$\max_{t\in[0,1]} |\varphi_1(t,\varepsilon)| = O(\delta(\varepsilon), \beta(\varepsilon)). \tag{3.26}$$

A combination of (3.23)–(3.26) gives

$$|\varphi_1(t,\varepsilon)| \leqslant k_{\varphi_1} t \tag{3.27}$$

for $t \in [0, 1]$, where

$$k_{\varphi_1} = b\delta(\varepsilon) + 2\max_{t\in[0,1]} |\varphi_1(t,\varepsilon)|.$$

Indeed, (3.27) is obviously true for $t\in\left[0, 1+\eta_1^0\right]$. For $t\in\left[1+\eta_1^0, 1\right]$, we can show (3.27) holds by the fact that $2t \geqslant 2\left(1+\eta_1^0\right) \geqslant 1$ as ε is sufficiently small. In a manner similar to that in the argument of (3.27), we can readily deduce

$$|w_1(t,\varepsilon)| \leqslant k_{w_1}(T_\varepsilon - t) \tag{3.28}$$

for all $t \in (1, T_\varepsilon]$, where $k_{w_1} = O(\beta(\varepsilon), \delta(\varepsilon))$. Returning to ϕ_1, we obtain that

$$|\phi_1(t,\varepsilon)| \leqslant -k_{\phi_1} t \tag{3.29}$$

for $t\in[-1, 0]$, where $k_{\phi_1} > 0$ and satisfies

$$k_{\phi_1} = O(\beta(\varepsilon), \delta(\varepsilon)).$$

Next, we shall establish the estimates for $\phi_2(t,\varepsilon), \varphi_2(t,\varepsilon)$ and $w_2(t,\varepsilon)$. For $t\in\left[0, \eta_2^0\right]$, we can show by Remark 3 and (3.29) that

$$\phi_0(t-1,\varepsilon) > \frac{k_3(a,b)}{2}, \qquad \phi_0(t-1,\varepsilon) + \phi_1(t-1,\varepsilon) > \frac{k_3(a,b)}{2}, \tag{3.30}$$

which leads to the following estimate of $\varphi_2(t,\varepsilon)$ on $\left[0, \eta_2^0\right]$:

$$\begin{aligned} |\varphi_2(t,\varepsilon)| &\leqslant \int_0^t be^{-a(t-s)} |f_\varepsilon(\phi_0(s-1,\varepsilon)+\phi_1(s-1,\varepsilon)) - f_\varepsilon(\phi_0(s-1,\varepsilon))| ds \\ &\leqslant bL_{k_3/2} |\eta_2^0| \|\phi_1\|. \end{aligned} \tag{3.31}$$

For $t\in\left[0, 1+\eta_1^0\right]$, it follows again from Remark 3 and (3.29) that

$$\phi_0(t-1,\varepsilon) \geqslant -k_3(t-1) \geqslant k_3\left|\eta_1^0\right| \tag{3.32}$$

and

$$\begin{aligned}\phi_0(t-1,\varepsilon)+\phi_1(t,\varepsilon) &\geqslant -k_3(t-1)+k_{\phi_1}(t-1)\\ &\geqslant (k_3-k_{\phi_1})\left|\eta_1^0\right| \\ &\geqslant \frac{k_3}{2}|\eta_1^0|.\end{aligned} \tag{3.33}$$

Recall that $\eta_1^0 \sim -\beta(\varepsilon)/b$ as $\varepsilon \to 0$. Then $\left|\eta_1^0\right| \geqslant \beta(\varepsilon)/2b$ as long as ε is sufficiently small. Inserting this into (3.32) and (3.33) yields

$$\phi_0(t-1,\varepsilon) \geqslant \frac{k_3\beta(\varepsilon)}{4b}, \qquad \phi_0(t-1,\varepsilon)+\phi_1(t,\varepsilon) \geqslant \frac{k_3\beta(\varepsilon)}{4b}, \quad t\in\left[0,1+\eta_1^0\right]. \tag{3.34}$$

Using the above estimate and (3.5), we have

$$\begin{aligned}|\varphi_2(t,\varepsilon)| &\leqslant \int_0^t be^{-a(t-s)}|f_\varepsilon(\phi_0+\phi_1)-f_\varepsilon(\phi_0(s-1,\varepsilon))|ds \\ &\leqslant bL_{k_3\beta/4b}\|\phi_1\|, \quad t\in\left[0,1+\eta_1^0\right].\end{aligned} \tag{3.35}$$

Finally, for $t\in\left[1+\eta_1^0,1\right]$, by (3.35) and the fact that $|f_\varepsilon|\leqslant M$, we have

$$\begin{aligned}|\varphi_2(t,\varepsilon)| &\leqslant \int_0^t be^{-a(t-s)}|f_\varepsilon(\phi_0+\phi_1)-f_\varepsilon(\phi_0(s-1,\varepsilon))|ds \\ &\leqslant \int_0^{1+\eta_1^0} be^{-a(t-s)}|f_\varepsilon(\phi_0+\phi_1)-f_\varepsilon(\phi_0(s-1,\varepsilon))|ds \\ &\quad +\int_{1+\eta_1^0}^t be^{-a(t-s)}|f_\varepsilon(\phi_0+\phi_1)-f_\varepsilon(\phi_0(s-1,\varepsilon))|ds \\ &\leqslant bL_{k_3\beta/4b}\|\phi_1\|+2bM|\eta_1^0|.\end{aligned} \tag{3.36}$$

Also inserting $|\phi_1(t-1,\varepsilon)|\leqslant k_{\phi_1}\left|\eta_1^0\right|$ into the second term of (3.36), we obtain

$$|\varphi_2(t,\varepsilon)|\leqslant bL_{k_3\beta/4b}\|\phi_1\|+bL(f)k_{\phi_1}\left|\eta_1^0\right|^2.$$

Coupling these two facts gives

$$|\varphi_2(t,\varepsilon)|\leqslant bL_{k_3\beta/4b}\|\phi_1\|+\min\{2bM\left|\eta_1^0\right|, bL(f)k_{\phi_1}\left|\eta_1^0\right|^2\}. \tag{3.37}$$

Then as before, we can show by (3.29) and (3.37) that there exists a positive constant $k_{\varphi_2}=O(\beta(\varepsilon),\delta(\varepsilon))$ such that

$$|\varphi_2(t,\varepsilon)|\leqslant k_{\varphi_2}t, \quad t\in[0,1]. \tag{3.38}$$

Similar argument yields the following estimate of $w_2(t,\varepsilon)$:

$$|w_2(t,\varepsilon)| \leqslant k_{w_2}(T_\varepsilon - t), \quad t \in (1, T_\varepsilon], \tag{3.39}$$

where $k_{w_2} = O(\beta(\varepsilon), \delta(\varepsilon))$.

By Lemma 2, we have $1 < T_\varepsilon = 1 + (1/a)\ \mathrm{In}(2 - e^{-a}) + O(\beta(\varepsilon), \delta(\varepsilon)) < 2$. It then follows from (3.27), (3.38) and Remark 3 that there exists a constant $c > 0$, independent of ε, such that for $t \in \left[\eta_1^0 + T_\varepsilon - 1, T_\varepsilon - 1\right]$, we have

$$\varphi_0(t,\varepsilon) + \varphi_1(t,\varepsilon) < -c, \;\; \varphi_0(t,\varepsilon) + \varphi_1(t,\varepsilon) + \varphi_2(t,\varepsilon) < -c. \tag{3.40}$$

In view of (3.8), (3.40) and the fact $T_\varepsilon < 2$, we can obtain, for $t \in \left[\eta_1^0 + T_\varepsilon, T_\varepsilon\right]$, that

$$\begin{aligned} |w_2(t,\varepsilon)| &\leqslant \left| \int_{T_\varepsilon}^t -be^{-a(t-s)} \left(f_\varepsilon\Big(\sum_{i=0}^{2} \varphi(s-1)\Big) - f_\varepsilon\Big(\sum_{i=0}^{1} \varphi(s-1)\Big)\right) ds \right| \\ &\leqslant be^{2a} \left|\eta_1^0\right| L_c(f_\varepsilon) \max_{t\in[0,1]} \left|\varphi_2(t,\varepsilon)\right|. \end{aligned}$$

Returning to ϕ_2 and using (3.12), we have

$$|\phi_2(t,\varepsilon)| \leqslant be^{2a}\left|\eta_1^0\right| L_c(f_\varepsilon)\|\phi_2\|, \quad t \in \left[\eta_1^0, 0\right]. \tag{3.41}$$

By virtue of (3.38) and (3.39), we can use an argument similar to that used for (3.30) and (3.34) to obtain

$$\phi_0(s,\varepsilon) + \phi_1(s,\varepsilon) + \phi_2(s,\varepsilon) > \frac{k_3(a,b)}{2}, \quad t \in \left(-1, -1 + \eta_2^0\right] \tag{3.42}$$

and

$$\phi_0(s,\varepsilon) + \phi_1(s,\varepsilon) + \phi_2(s,\varepsilon) \geqslant \frac{k_3(a,b)\beta(\varepsilon)}{4b}, \quad t \in \left(-1, \eta_1^0\right]. \tag{3.43}$$

Now we are ready to prove (3.20). By the definition of $R_{\bar\phi}(t)$ in (3.17), we can obtain from (3.30) and (3.42) that

$$|R_{(\phi_0+\phi_1+\phi_2)}(t) - R_{(\phi_0+\phi_1)}(t)| \leqslant bL_{\frac{k_3}{2}} \left|\eta_2^0\right| \cdot \|\phi_2\|, \quad t \in \left[0, \eta_2^0\right]. \tag{3.44}$$

Making use of (3.34) and (3.43), we have

$$|R_{(\phi_0+\phi_1+\phi_2)}(t) - R_{(\phi_0+\phi_1)}(t)| \leqslant bL_{k_3\beta(\varepsilon)/4b}\|\phi_2\|, \quad t \in \left[0, 1 + \eta_1^0\right]. \tag{3.45}$$

For $t \in [1+\eta_1^0, 1]$, we derive by (3.40), (3.41) and (3.45) that

$$\begin{aligned}
&|R_{(\phi_0+\phi_1+\phi_2)}(t) - R_{(\phi_0+\phi_1)}(t)| \\
&\leqslant \int_0^{1+\eta_1^0} be^{-a(t-s)}|f_\varepsilon(\phi_0+\phi_1+\phi_2) - f_\varepsilon(\phi_0+\phi_1(s-1))|ds \\
&\quad + \int_{1+\eta_1^0}^{t} be^{-a(t-s)}|f_\varepsilon(\phi_0+\phi_1+\phi_2) - f_\varepsilon(\phi_0+\phi_1(s-1))|ds \\
&\leqslant bL_{k_3\beta(\varepsilon)/4b}||\phi_2|| + b^2e^{2a}\left|\eta_1^0\right|^2 L(f_\varepsilon)L_c(f_\epsilon)\|\phi_2\|.
\end{aligned} \tag{3.46}$$

Coupling (3.44)–(3.46) yields for $t \in [0, 1]$, that

$$|\varphi_3| = |R_{(\phi_0+\phi_1+\phi_2)}(t) - R_{(\phi_0+\phi_1)}(t)| \leqslant L_1\|\phi_2\|, \tag{3.47}$$

where

$$L_1 = \max\left\{bL_{k_3/2}\left|\eta_2^0\right|, bL_{k_3\beta(\varepsilon)/4b} + b^2e^{2a}L(f_\varepsilon)L_c(f_\varepsilon)\left|\eta_1^0\right|^2\right\}.$$

Using (3.15) and (3.22), we have

$$L_1 \to 0 \quad \text{as } \varepsilon \to 0.$$

Using a similar argument, and using (3.44)–(3.46), we can deduce that for $t \in (1, T_\varepsilon]$ there is a constant $L_2(\varepsilon)$ so that

$$L_2 \to 0 \quad \text{as } \varepsilon \to 0$$

and

$$\begin{aligned}
|w_3(t,\varepsilon)| &= |R_{(\phi_0+\phi_1+\phi_2)}(t) - R_{(\phi_0+\phi_1)}(t)| \\
&\leqslant \int_{T_\varepsilon}^{t} be^{-a(t-s)}|f_\varepsilon(\varphi_0+\varphi_1+\varphi_2+\varphi_3) - f_\varepsilon(\varphi_0+\varphi_1+\varphi_2(s-1)|ds \\
&\leqslant L_2(\varepsilon)\|\phi_2\|.
\end{aligned}$$

Returning to ϕ_3 and using (3.12) again, we conclude by the above formula and (3.47) that for $t \in [-T_\varepsilon, 0]$

$$|\phi_3(t)| = o(\|\phi_2\|) \quad \text{as } \varepsilon \to 0.$$

Moreover, by induction, we can derive that for any $n \geqslant 2$

$$|\phi_{n+1}| = o(\|\phi_n\|) \quad \text{as } \varepsilon \to 0,$$

and hence that the series $\sum^\infty \|\phi\|_n$ is convergent. By the M-test of the uniform convergence, we know that $\sum_{n=0}^\infty \phi_n(t,\varepsilon)$ is uniformly convergent for t in $[-T_\varepsilon, 1-T_\varepsilon]$ or in $(1-T_\varepsilon, 0]$, which means by (3.12) that

$\sum_{n=0}^{\infty} w_n(t,\varepsilon)$ is uniformly convergent for $t\in(1,T_\varepsilon]$ and so is $\sum_{n=0}^{\infty}\varphi_n(t,\varepsilon)$ for $t\in[0,1]$. This completes the proof of Theorem 2. $\square$

For illustration, we now give an example to show to deduce the term $\phi_n(t,\varepsilon)$ and T_ε in (3.13) and (3.14). We take $f_\varepsilon(x)=\tanh(x/\varepsilon)$ and $\beta(\varepsilon)=N\varepsilon\log(1/\varepsilon)$, where $N\geqslant 10$ so that when $x\geqslant\beta(\varepsilon)$

$$|f_\varepsilon(x)-1|\leqslant e^{-x/\varepsilon}\leqslant\varepsilon^N\leqslant\varepsilon^{10}. \tag{3.48}$$

Then we can see that all of the conditions in Theorem 2 are satisfied.

By (3.4), we get

$$\varphi_1=\int_0^t -be^{-a(t-s)}\left(-1+\tanh\left(\frac{\phi_0(s-1)}{\varepsilon}\right)\right)ds. \tag{3.49}$$

To compute the integral for $\varphi_1(t,\varepsilon)$ more explicitly, we need to split the interval [0, 1] into outer and inner regions. Recall that

$$\eta_1^0\sim-\frac{\beta(\varepsilon)}{b},\qquad \eta_2^0\sim\frac{\beta(\varepsilon)}{b}.$$

We have from (3.49) and (3.48) that

$$\varphi_1(t,\varepsilon)=O(\varepsilon^N),\quad t\in\left[0,1+\eta_1^0\right], \tag{3.50}$$

and

$$\begin{aligned}\varphi_1(1,\varepsilon)&=\int_0^1 -be^{-a(1-s)}\left(-1+\tanh\left(\frac{\phi_0(s-1)}{\varepsilon}\right)\right)ds\\&=\int_0^{1+\eta_1^0} -be^{-a(1-s)}\left(-1+\tanh\left(\frac{\phi_0(s-1)}{\varepsilon}\right)\right)ds\\&\quad+\int_{1+\eta_1^0}^1 -be^{-a(1-s)}\left(-1+\tanh\left(\frac{\phi_0(s-1)}{\varepsilon}\right)\right)ds\\&=O(\varepsilon^N)+I,\end{aligned}$$

Where

$$\begin{aligned}I&=\int_{1+\eta_1^0}^1 -be^{-a(1-s)}\left(-1+\tanh\left(\frac{\phi_0(s-1)}{\varepsilon}\right)\right)ds\\&=\int_{1+\eta_1^0}^1 2be^{-a(1-s)}\frac{e^{-\phi_0(s-1)/\varepsilon}}{e^{\phi_0(s-1)/\varepsilon}+e^{-\phi_0(s-1)/\varepsilon}}ds\\&=\int_{\eta_1^0}^0 2be^{at}\frac{e^{-\phi_0(t)/\varepsilon}}{e^{\phi_0(t)/\varepsilon}+e^{-\phi_0(t)/\varepsilon}}dt.\end{aligned}$$

Using the facts that $\phi_0'(t) = -be^{-at}$, $e^{2at} \sim 1 + 2at$ as $t \to 0$,

$$\int_{\eta_1^0}^{0} -2\phi_0'(t)\frac{e^{-\phi_0(t)/\varepsilon}}{e^{\phi_0(t)/\varepsilon} + e^{-\phi_0(t)/\varepsilon}}dt = \varepsilon \ln 2 + O(\varepsilon^N)$$

and

$$\int_{\eta_1^0}^{0} -2t\phi_0'(t)\frac{e^{-\phi_0(t)/\varepsilon}}{e^{\phi_0(t)/\varepsilon} + e^{-\phi_0(t)/\varepsilon}}\mathrm{d}t \leqslant \int_{\eta_1^0}^{0} -t\phi_0'(t)e^{-\phi_0(t)/\varepsilon}dt = O(\varepsilon^2),$$

we can evaluate the integral I as

$$I = \varepsilon \ln 2 + O(\varepsilon^2),$$

and hence

$$\varphi_1(1, \varepsilon) = \varepsilon \ln 2 + O(\varepsilon^2).$$

Similarly, we can estimate $w_1(t, \varepsilon)$ as

$$w_1(t, \varepsilon) = \varepsilon \ln 2 + O(\varepsilon^2).$$

Therefore,

$$w_1(1, \varepsilon) - \varphi_1(1, \varepsilon) = O(\varepsilon^2). \tag{3.51}$$

We can continue the above analysis for φ_2 and w_2 to obtain

$$\begin{aligned}
\varphi_2(1, \varepsilon) &= \int_0^1 -be^{-a(1-s)}(f_\varepsilon(\phi_0 + \phi_1) - f_\varepsilon(\phi_0))ds \\
&= \int_0^1 -be^{-a(1-s)}(-1 + \tanh\left(\frac{\phi_0(s-1) + \phi_1(s-1)}{\varepsilon}\right)ds \\
&\quad - \int_0^1 -be^{-a(1-s)}(-1 + \tanh\left(\frac{\phi_0(s-1)}{\varepsilon}\right)ds \\
&= \varepsilon \ln 2 + O(\varepsilon^2) - [\varepsilon \ln 2 + O(\varepsilon^2)] \\
&= O(\varepsilon^2)
\end{aligned} \tag{3.52}$$

and

$$w_2 = O(\varepsilon^2). \tag{3.53}$$

Since $f_\varepsilon(x) = \tanh(x/\varepsilon)$ satisfies all of the conditions in Theorem 2, we can use (3.51)–(3.53) and Theorem 2 to obtain

$$\sum_{n=1}^{\infty}(w_n(1, \varepsilon) - \varphi_n(1, \varepsilon)) = O(\varepsilon^2),$$

hence

$$T_\varepsilon = 1 + \frac{\ln\left(2 - e^{-a} + \frac{a}{b}\sum_{n=1}^{\infty}(w_n(1,\varepsilon) - \varphi_n(1,\varepsilon))\right)}{a}$$
$$= 1 + \frac{1}{a}\ \ln(2 - e^{-a}) + O(\varepsilon^2).$$

Therefore, we conclude that for $f_\varepsilon(x) = \tanh(x/\varepsilon)$, the minimal period $2T_\varepsilon$ of the periodic solution is approximated by

$$2T_0 + O(\varepsilon^2)$$

and the periodic solution $x(t)$ has the following uniform asymptotic expansion

$$x(t) = \sum_{n=0}^{N-1} \phi_n(t,\varepsilon) + O(\|\phi_N\|) \quad \text{as } \varepsilon \to 0$$

with respect to $t \in [-T_\varepsilon, 0]$.

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