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MULTIPLE PERIODIC PATTERNS VIA DISCRETE NEURAL NETS WITH DELAYED FEEDBACK LOOPS*

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We show that a discrete dynamical system with delayed arguments that describe the computational performance of a neural network with a loop structure can exhibit the coexistence of huge number of periodic solutions, and we describe the domains of attraction for the stable periodic orbits. This demonstrates a great potential of discrete time delayed neural nets with a loop structure for the purpose of storage and retrieval of periodic patterns.

Keywords: Neural nets; discrete; delay; periodic orbit; multistability; loop.

1. Introduction

One of the many important tasks that artificial neural networks can fulfil is associative memory storage. In this context, a network is required to possess as large capacity as possible for retrievable memories. Translating into the language of dynamical systems, this demands the coexistence of as many as possible either stable equilibria or stable periodic orbits (depending on the pattern to be stored) for the system describing the computational performance of the network. This is exactly the so-called multistability problem in dynamical systems theory.

It has been shown that time delay provides an efficient mechanism for a network to store and retrieve periodic patterns, and biologically these delays arise due to axonal conduction time, distances of inter-neurons and the finite switching speeds of amplifiers. We refer to the monographs by Milton [1996] and Wu [2001], and Wu and Zhang [2002] and the references therein for extensive literatures in the area. It turns out, however, that the coexistence of multiple periodic solutions must be accompanied by a huge number of discrete delays or a complicated distributed delay. In particular, for a network of two coupled neurons with delayed monotone feedbacks

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and with a single delay, although the series of papers by Chen and Wu [1999, 2000, 2001a–2001c], and Chen *et al.* [2000] established the coexistence of multiple periodic orbits and gave a detailed description of their domains of attraction and the structure of the global attractors, all these periodic orbits, except one, are unfortunately unstable.

In the case when the state of a network is updated discretely, the situation seems to be pleasantly different. Zhou and Wu [2002a, 2002b] considered the following network of two identical neurons with excitatory interactions

$$\begin{cases} x(n) = \beta x(n-1) + \alpha f(y(n-k)) \,, \\ y(n) = \beta y(n-1) + \alpha f(x(n-k)) \,, \end{cases} \tag{1}$$

where $n \in \mathbf{N}$ (the set of all nonnegative integers), $\alpha > 0$ and $k \geq 1$ is a fixed integer, $f : R \to R$ satisfies the following conditions

$$(\mathbf{H1}) \quad \begin{cases} |f(x) - 1| \leq \varepsilon & \text{if } x \in (r,\, R], \\ |f(x) + 1| \leq \varepsilon & \text{if } x \in [-R,\, -r), \end{cases} \qquad \text{for some constants } \varepsilon > 0 \text{ and } R > r > 0 \,.$$

$$(\mathbf{H2}) \quad |f(x) - f(y)| \leq L|x-y| \quad \text{if} \quad x,\ y \in [-R,\, -r) \quad \text{or} \quad x,\ y \in (r,\, R], \quad \text{where } L > 0 \text{ is a constant} \,.$$

They obtained some results on the existence of k-periodic and $2k$-periodic orbits. In a more recent work, Wu and Zhang [2002] explored the existence of periodic orbits with all possible periods and found that (1) indeed allows coexistence of a very large number of stable periodic orbits. More precisely, under (H1)–(H2) and some other conditions on the parameters involved, Wu and Zhang [2002] developed an elementary way to compute the total number of stable p-periodic orbits of (1) for each $p|2k$, and they were also able to describe the domains of attraction for these p-periodic solutions.

This seems to be surprising in the sense that if f is monotonically increasing then the continuous analogue of (1) in terms of a delay differential system does not have any stable periodic orbits. The potential for applications to associative memory storage of the aforementioned surprising results is, however, very limited. This is because model (1) involves a very small amount of neurons and small number of parameters, and hence it is difficult, if not impossible, to train such a network via any learning scheme to store a large number of independent periodic patterns, despite the fact that such a small network does have a large number of periodic orbits. It is therefore very natural to ask whether the surprising results of Wu and Zhang [2002] can be extended to large networks with complicated connection topology. For a large network with arbitrary connections, this seems to be an extremely challenging question. We are thus forced to work on networks with special and biologically motivated connection topology. As the work by Foss *et al.* [1996, 1997, 2000] showed that time delayed recurrent loops have potentially huge capacity for encoding information in the form of temporally patterned spike trains, we shall consider a network of m neurons arrayed in a ring with possiblly bidirectional nearest neighbor connections.

The rest of the paper is organized as follows. Section 2 is dedicated to the basic setting up and preparation, and Sec. 3 contains the main results and proofs.

2. Preliminaries

Consider a discrete-time network of m neurons arranged in a ring and connected by excitatory feedback with delay. Let $\beta_i \in (0,\, 1)$ denote the internal decay rate, $(a_i,\, b_i)$ be the excitatory synaptic weights of the ith neuron to its nearest neurons in the ring and let $f : \mathbf{R} \to \mathbf{R}$ be the signal transmission function. Then we have

$$\begin{aligned} x_i(n) = \beta_i x_i(n-1) &+ a_i f(x_{i+1}(n-k_{i+1})) \\ &+ b_i f(x_{i-1}(n-k_{i-1})) \,, \end{aligned} \tag{2}$$

where $i \equiv 1, \ldots, m (\mathrm{mod}\ m)$, $a_i,\, b_i \geq 0$, $n \in \mathbf{N}^+ = \{n;\ n \text{ is a positive integer}\}$. We shall assume the following

$$(\mathbf{A1}) \quad \begin{cases} \text{either} & \displaystyle\min_{1\leq i,j,\leq m} \left\{ \frac{a_i - b_i}{\beta_i} - \frac{a_j + b_j}{1-\beta_j} \right\} > 0, \\ \text{or} & \displaystyle\min_{1\leq i,j,\leq m} \left\{ \frac{b_i - a_i}{\beta_i} - \frac{a_j + b_j}{1-\beta_j} \right\} > 0. \end{cases}$$

This is true if $a_i > b_i$ (or $b_i > a_i$) for all $i (\mathrm{mod}\ m)$ corresponding to the forward (or backward) dominant connection, and if β_i, $1 \leq i \leq m$, are all sufficiently small. We also assume that the signal transmission function $f : \mathbf{R} \to \mathbf{R}$ satisfies the following conditions:

$$(\mathbf{A2}) \quad \begin{cases} |f(x)-1| \le \varepsilon & \text{if } x \in (r,\, R], \\ |f(x)+1| \le \varepsilon & \text{if } x \in [-R,\, -r), \end{cases}$$

where, $0 < \varepsilon < \overline{\varepsilon}$, $0 \le r < a^*$, $R \ge b^*$, and

$$\overline{\varepsilon} = \frac{\min_{1\le i,j,\le m}\left\{\dfrac{|a_i-b_i|}{\beta_i} - \dfrac{a_j+b_j}{1-\beta_j}\right\}}{\max_{1\le i\le m}\left\{\dfrac{a_i+b_i}{\beta_i(1-\beta_i)}\right\}}$$

$$b^* = \max_{1\le i\le m}\left(\frac{a_i+b_i}{1-\beta_i}\right)(1+\varepsilon)\,,$$

$$a^* = \min_{1\le i\le m}\{[|a_i-b_i| - (a_i+b_i)\varepsilon] - \beta_i b^*\}\,.$$

In addition, we need the following Lipschitz continuity of f:

(A3) There exists $L \in (0,\, \max_{1\le i\le m}\{(1-\beta_i/a_i + b_i)\})$ such that

$$|f(x)-f(y)| \le L|x-y| \quad \text{if } x,\ y \in [-R,\, -r) \text{ or } x,\ y \in (r,\, R]\,.$$

Note that if $\varepsilon = 0$, $r = 0$, and $R = \infty$, then f becomes the widely used McCulloch–Pitts nonlinearity given by

$$f(x) = \begin{cases} 1, & x \ge 0, \\ -1, & x < 0, \end{cases} \tag{3}$$

and in this case, $L = 0$. We also note that, the frequently used sigmoid functions with high gains also satisfy (A2) and (A3).

By a solution of (2), we mean a sequence $\{(x_1(n), \dots, x_m(n))\}$ of points in $\mathbf{R}^m$ that is defined for every integer $n \ge \min\{-k_j,\, j = 1, \dots, m\}$ and satisfies (2) for $n \in \mathbf{N} = \mathbf{N}^+ \cup \{0\}$. It is evident that system (2) can be written as a difference system without delay but in a higher dimensional space. Indeed, there are many ways to do this, and in what follows, we will choose one that is convenient to our problem. To this end, we proceed as below.

Let $k'_0 = 0$, $k'_l = \sum_{j=1}^{l} k_j$, for $l = 1, \dots, m-1$ and $k := k'_m = \sum_{j=1}^{m} k_j$. For $i = k'_l + j$, with $l = 0, \dots, m-1$, and $j = 1, \dots, k_{l+1}$, let $w_i(n) = x_{l+1}(n+j-k-1)$. Then, in terms of the new variables $w_i(n)$, $i = 1, \dots, k$, (2) can be written as

$$w(n+1) = F(w(n))\,, \tag{4}$$

where $F : \mathbf{R}^k \to \mathbf{R}^k$ is given, for $w = w(w_1, \dots, w_k) \in \mathbf{R}^k$, by the following

$$F_j(w) = \begin{cases} \beta_j w_{k'_l} + a_j f(w_{k'_l+1}) + b_j f(w_{k'_{l-1}+1}) & \text{for } j = k'_l,\ l = 1, \dots, m-1; \\ \beta_m w_k + a_m f(w_1) + b_m f(w_{k'_{m-1}+1}) & \text{for } j = k = k'_m; \\ w_{j+1} & \text{for other } j. \end{cases} \tag{5}$$

We denote by $\{w(n,\, w^0)\}_{n\in\mathbf{N}}$ the solution of (5) with initial value $w(0) = w^0 \in \mathbf{R}^k$. For $w = \{w_1, \dots, w_k\} \in \mathbf{R}^k$, let $|w| = \max\{|w_j|,\ j = 1, \dots, k\}$.

Let

$$\Sigma = \{\sigma = (\sigma_1, \dots, \sigma_k) \in \mathbf{R}^k :\ \sigma_j \in \{-1,\, 1\},\ j = 1, \dots, k\}\,. \tag{6}$$

For any $n \in \mathbf{N}$, let $[n] \in \{1, \dots, k\}$ be uniquely given so that $n \equiv [n] \pmod k$. Define a mapping $\pi :\ \Sigma \to \Sigma$ by

$$(\pi\sigma)_j = \sigma_{[j+1]} \quad \text{for} \quad j \in \{j = 1, \dots, k\} \tag{7}$$

for any $\sigma \in \Sigma$. As usual, the mapping $\pi^p :\ \Sigma \to \Sigma$ with $p \ge 2$ is given by $\pi^p\sigma = \pi(\pi^{p-1}\sigma)$ for $\sigma \in \Sigma$ inductively. Clearly,

$$\pi^k\sigma = \sigma \quad \text{for every} \quad \sigma \in \Sigma\,. \tag{8}$$

Moreover, for any $p \in \{1,\, 2, \dots, k-1\}$ and for any $\sigma \in \Sigma$, we have

$$(\pi^p\sigma)_j = \sigma_{[j+p]} \quad \text{for} \quad j \in \{j = 1, \dots, k\}\,. \tag{9}$$

Denote the set of all fixed points of π by

$$\Sigma_1 = \{\sigma \in \Sigma :\ \pi\sigma = \sigma\}\,.$$

Similarly, for any $p \in \{2, \dots, k\}$, denote the set of all p-periodic points of π by

$$\Sigma_p = \{\sigma \in \Sigma :\ \pi^p\sigma = \sigma \text{ and } \pi^q\sigma \ne \sigma \text{ for any } q \in \{1, \dots, p-1\}\}\,. \tag{10}$$

We will need the following technical lemma.

Lemma 2.1. *The following statements hold*:

(i) *Let* $p \in \{1, \dots, k\}$. *If* $p|k$, *then* $\Sigma_p \ne \emptyset$, *and* *the number of elements in* Σ_p *is given by*

$$N(p)=\begin{cases}2 & \text{if } p=1,\\ 2^p-2 & \text{if } p \text{ is a prime integer,}\\ 2^p-\displaystyle\sum_{q|p,q<p} N(q) & \text{otherwise.}\end{cases} \tag{11}$$

(ii) *If* $\Sigma_p \neq \emptyset$, *then* $p|k$.
(iii) $\Sigma = \cup_{p|k}\Sigma_p$.

Proof. (iii) is a direct consequence of (i) and (ii) as well as the fact that $\Sigma = \cup_{p=1}^{k}\Sigma_p$ The result in (i) can be obtained by simple counting. Indeed, let $S(p)$ be the number of points fixed by σ^p in Σ. Clearly, $S(p) = 2^p$. On the other hand,

$$S(p) = \sum_{q|p} N(q) = N(p) + \sum_{q|p,q<p} N(q)\,, \tag{12}$$

which gives (11). We now prove (ii). Assume $\Sigma_p \neq \emptyset$. Obviously, we only need to consider the case when $p \in \{2, \ldots, k-1\}$ and $\Sigma_p \neq \emptyset$. Fix a $\sigma \in \Sigma_p$, and define a sequence $\{s_n\}_{n\in\mathbf{N}}$ by

$$s_n = \sigma_{[n]}\,. \tag{13}$$

Then, $s_{n+k} = s_n$ for $n \in \mathbf{N}$, implying that k is a period of $\{s_n\}_{n\in\mathbf{N}}$. By (9) and $\pi^p\sigma = \sigma$, we have

$$\begin{cases}\sigma_j = \sigma_{j+p} & \text{for } j = 1, \ldots, k-p,\\ \sigma_j = \sigma_{p-s} & \text{for } j = k-s \text{ with } s = 0, \ldots, p-1.\end{cases} \tag{14}$$

Note that for $j = k-s$ with $s \in \{0, \ldots, p-1\}$, we have $j+p = k+(p-s)$ with $p-s \in \{1, \ldots, p\}$. So, $[j+p] = p-s$. This, together with (13) and (14), yields

$$s_n = s_{n+p} \quad \text{for} \quad n = 1, \ldots, k-p\,,$$

and

$$s_n = \sigma_{[n+p]} = s_{n+p} \quad \text{for} \quad n = k-(p-1), \ldots, k\,.$$

For $n \geq k+1$, we have $n = lk+q$ with some $l \geq 1$ and $q \in \{1, \ldots, k-1\}$. Therefore

$$s_{n+p} = s_{lk+q+p} = s_{q+p} = s_q = s_n\,.$$

Thus, p is the minimum period of $\{s_n\}_{n\in\mathbf{N}}$. As $p < k$ and as k is a period of $\{s_n\}_{n\in\mathbf{N}}$, we conclude that $p|k$. This proves (ii) and completes the proof of the lemma. ■

Remark 2.2. The formula (11) for $N(p)$ is given in a recursive way. As we shall see $N(p)$ is precisely the number of stable p-periodic solutions of (4). We should remark that the well-known Möbius inversion formula gives an explicit formula for $N(p)$. To this end, let us write $p = \prod_1^l p_i^{m_i}$, where p_i, $i = 1, 2, \ldots, l$, are primes. For every subset I of $\{1, \ldots, l\}$, let $p_I = \prod_{i\in I} p_i$. Then by (12) and the Möbius inversion theorem (see [Hardy & Wright, 1979, pp. 234–236]), we obtain

$$N(p) = \sum_{I\subset\{1,\ldots,l\}} (-1)^{|I|}2^{p/p_I}\,. \tag{15}$$

3. Multistability of Periodic Orbits

In what follows, we only consider the case $a_i > b_i$ for $i \pmod m$ in (A1). The backword dominant case can be similarly dealt with. In the sequel, when we talk about a p-periodic solution of (4), we always means that p is the minimum period of the solution. We first establish the following existence result.

Theorem 3.1. *Assume that (A1) and (A2) hold. Then, for any positive integer* p *with* $p|k$ *and for every* $\sigma \in \Sigma_p$, *(4) has a* p*-periodic solution.*

In order to prove this theorem, we need some preparation. First of all, by (A1), it is easy to see that for any fixed $\varepsilon \in (0, \overline{\varepsilon})$, we have

$$b^* < \min_{1\leq i\leq m}\left(\frac{a_i-b_i}{\beta_i}\right) - \max_{1\leq i\leq m}\left(\frac{a_i+b_i}{\beta_i}\right)\varepsilon$$

Therefore,

$$b^* < \min_{1\leq i\leq m}\left\{\frac{(a_i-b_i)-(a_i+b_i)\varepsilon}{\beta_i}\right\}$$

Thus, $a^* > 0$. Let $c^* = \min\{a^*-r,\, R-b^*\}$. Fix $c \in [0, c^*)$, define $a_c = a^*-c$ and $b_c = b^*+c$. Denote

$$\Omega(\sigma, c) = \{w \in \mathbf{R}^k :\ a_c \leq |w_j| \leq b_c,\ \operatorname{sign}\{w_j\} = \sigma_j,\ j = 1, \ldots, k\} \tag{16}$$

and

$$\Omega_3(c) = \{(u_1, u_2, u_3) \in \mathbf{R}^3 :\ a_c \leq |u_j| \leq b_c,\ j = 1, 2, 3\}\,. \tag{17}$$

Let $g_i : \mathbf{R}^3 \to \mathbf{R}$, $i = 1, \ldots, m$, be given by $g_i(u_1, u_2, u_3) = \beta_i u_1 + a_i f(u_2) + b_i f(u_3)$. Then, we can have the following

Lemma 3.2. *Assume that (A1) and (A2) hold. Then, for every* $c \in (0, c^*)$ *and* $(u_1, u_2, u_3) \in \Omega_3(c)$

and $i \in \{1, \dots, m\}$, *we have*

$$\operatorname{sign} g_i(u_1, u_2, u_3) = \operatorname{sign} u_2$$

and
$$a_c \le |g_i(u_1, u_2, u_3)| \le b_c\,. \tag{18}$$

Proof. It is clear that

$$g_i(-|u_1|, u_2, -|u_3|) \le g_i(u_1, u_2, u_3) \le g_i(|u_1|, u_2, |u_3|)\,. \tag{19}$$

In the case where $u_2 > 0$, we note from (A1) and (A2) that

$$\begin{aligned} g_i(|u_1|, u_2, |u_3|) &\le \beta_i b^* + a_i(1+\varepsilon) + b_i(1+\varepsilon) + \beta_i c \\ &\le b^* + c = b_c \end{aligned} \tag{20}$$

and

$$\begin{aligned} &g_i(-|u_1|, u_2, -|u_3|) \\ &\quad\ge -\beta_i b^* + a_i(1-\varepsilon) - b_i(1+\varepsilon) - \beta_i c \\ &\quad\ge a^* - c = a_c\,. \end{aligned} \tag{21}$$

This, together with (20) and (21), gives (18).

In the case where $u_2 < 0$, we note from (A1), (A2) and (19) that

$$\begin{aligned} g_i(|u_1|, u_2, |u_3|) &\le \beta_i b^* - a_i(1-\varepsilon) + b_i(1+\varepsilon) + \beta_i c \\ &\le -[(a_i - b_i) - (a_i + b_i)\varepsilon - \beta_i b^*] + c \\ &\le -a^* + c = -a_c \end{aligned} \tag{22}$$

and

$$\begin{aligned} &g_i(-|u_1|, u_2, -|u_3|) \\ &\quad\ge -\beta_i b^* - a_i(1+\varepsilon) - b_i(1+\varepsilon) - \beta_i c \\ &\quad\ge -(b^* + c) + [(1-\beta_i)b^* - (a_i + b_i)(1+\varepsilon)] \\ &\quad\ge -(b^* + c) = -b_c\,. \end{aligned} \tag{23}$$

This together with (22) and (23), gives (18). This completes the proof. ■

Proof of Theorem 3.1. We first prove that for any $\sigma \in \Sigma$ and $c \in [0, c^*)$, $F(\Omega(\sigma, c)) \subset \Omega(\pi(\sigma), c)$. Let $\sigma = (\sigma_1, \dots, \sigma_k) \in \Sigma$ and $w = (w_1, \dots, w_k) \in \Omega(\sigma, c)$. Then, $\pi\sigma = (\sigma_2, \dots, \sigma_k, \sigma_1)$, $a_c \le |w_j| \le b_c$ and $\operatorname{sign}(w_j) = \sigma_j$ for $j = 1, 2, \dots, k$. If $j \ne k'_l$, $l = 1, \dots, m$, by (5), we know that $F_j(w) = w_{j+1}$ for $j \ne k'_l$, $l = 1, \dots, m$, and hence,

$$a_c \le |F_j(w)| =\le b_c \quad \text{and} \quad \operatorname{sign}(F_j(w)) = \sigma_{j+1}\,. \tag{24}$$

For $j = k'_l$, $l = 1, \dots, m$, by (5) and Lemma 3.2, we see that (24) is also true. This shows that $F(w) \in \Omega(\pi\sigma, c)$. Now, if $p|k$ and $\sigma \in \Sigma_p$ ($\Sigma_p \ne \emptyset$ by Lemma 2.1), then we have $F^p(\Omega(\sigma, c)) \subset \Omega(\pi^p(\sigma), c) = \Omega(\sigma, c)$. Note that $\Omega(\sigma, c)$ is bounded, closed, convex and nonempty. By Schauder's fixed point theorem, the continuous mapping F^p has a fixed point which gives a p-periodic solution of (4), in the form $\{w(n, w^\sigma)\}_{n\in\mathbf{N}}$ with $w_\sigma \in \Omega(\sigma, c)$. This completes the proof of Theorem 3.1. ■

Next, we address the stability of the multiplicity periodic solutions obtained above.

Theorem 3.3. *Assume (A1), (A2) and (A3) hold.*

(i) *For any integer p with $p|k$ and each $\sigma \in \Sigma_p$, (4) has a unique p-periodic $w(n, w^\sigma)$ (denote by $w^\sigma(n)$) with $w^\sigma \in \Omega(\sigma, 0)$, and this solution is uniformly asymptotically stable. More precisely, let*

$$r(\sigma) = \min\{|w_j^\sigma| - (a^* - c^*),\ (b^* + c^*) - |w_j^\sigma|;\ j = 1, \dots, k\}\,.$$

Then for any w^0 with $|w^0 - w^\sigma| < r(\sigma)$, we have

$$|w(n, w^0) - w(n, w^\sigma)| \le C\xi^n |w^0 - w^\sigma| \quad \text{for } n = 0, 1, \dots\,,$$

where $\xi = \max_{1\le i\le m}[\beta_i + (a_i + b_i)L]^{1/k}$ and $C = \xi^{1-k}$.

(ii) *Let*

$$\Omega = \{w \in \mathbf{R}^k :\ r < |w_j| < R,\ j = 1, \dots, k\}\,. \tag{25}$$

If $\{w(n)\}_{n\in\mathbf{N}}$ is a p-periodic solution of (4) with $w(n) \in \Omega$ for $n = 1, \dots, p$, then $p|k$ and there exists a unique $\sigma \in \Sigma_p$ such that $w(n) = w^\sigma(n)$ for $n = 1, 2, \dots$.

(iii) *For any solution $\{w(n, w^0)\}_{n\in\mathbf{N}}$ of (4) with $a^* - c^* < |w_j^0| < b^* + c^*$ for $1 \le j \le k$, there exists a unique integer $p \in \mathbf{N}$ with $p|k$ and a unique $\sigma \in \Sigma_p$ such that*

$$|w(n, w^0) - w^\sigma(n)| \le C\xi^n |w^0 - w^\sigma(0)| \quad \text{for } n = 0, 1, \dots\,.$$

(iv) *For each integer $p \in \mathbf{N}$ with $p|k$, (4) has $N(p)$ p-periodic solutions in Ω, and these solutions are uniformly asymptotically stable. For any integer $p \in \mathbf{N}$ with $p \not| k$, (4) has no p-periodic solution in Ω.*

To prove this theorem, we need some lemmas.

Lemma 3.4. *If $c \in [0, c^*)$, $\sigma \in \Sigma_p$ and $p|k$, then*

$$|F^k(w') - F^k(w'')| \le \xi^k |w' - w''| \quad \text{for} \quad w'w'' \in \Omega(\sigma, c)\,. \tag{26}$$

Proof. By the definition of $\Omega(\sigma, c)$ and using Lemma 3.2, we have

$$w'_j w''_j > 0\,, \quad F_j^k(w')F_j^k(w'') > 0 \quad \text{for } j = 1, \ldots, k\,.$$

Therefore, (00) follows from the definition of $|\cdot|$ in $\mathbf{R}^k$, Eq. (5) and (A3). ■

Lemma 3.5. *If $\{w(n, w^0)\}_{n\in\mathbf{N}}$ is a p-periodic solution of (4) with $w(j, w^0) \in \Omega$ for $j = 1, \ldots, p$. Then $|w(j, w^0)| \le b^*$ for $j = 1, \ldots, p$.*

Proof. By way of contradiction, assume that the conclusion is not true. Then, we can obtain a p-periodic solution $x(n) = \{x_1(n), \ldots, x_m(n)\}$ of (2) from $\{w(n, w^0)\}_{n\in\mathbf{N}}$, with $|x_i(n_0)| > b^*$ for some $i \in \{1, \ldots, m\}$. We first consider the case $x_i(n_0) > b^*$. Then $b^* < x_i(n_0) < R$, and we can write $x(n_0) = b^* + \delta_0$, with some $\delta_0 > 0$. It follows from (2) and (A1)–(A2) that

$$\begin{aligned} x_1(n_0 - 1) &= \frac{1}{\beta_1}[x_1(n_0) - a_1 f(x_2(n_0 - k_2)) \\ &\quad - b_1 f(x_m(n - k_m))] \\ &\ge \frac{1}{\beta_1}[\delta_0 + b^* - (a_1 + b_1)(1 + \varepsilon)] \\ &= \frac{1}{\beta_1}\delta_0 + b^*\,. \end{aligned}$$

Let $\beta = \min_{1\le i\le m}\{\beta_i\}$, repeating the above argument, we get

$$x_1(n_0 - m) \ge \frac{1}{\beta}[x_1(n_0 - m + 1) - b^*] + b^*\,, \quad m = 1, 2, \ldots, k\,.$$

In particular,

$$x_1(n_0 - p) \ge \frac{1}{\beta^p}\delta_0 + b^* > b^* + \delta_0 = x_1(n_0)\,,$$

a contradiction to the p-periodicity. Similarly, we can exclude $x(n_0) < -b^*$. This completes the proof. ■

Lemma 3.6. *Let*

$$\Omega^* = \{w \in \mathbf{R}^k : \ r < |w_j| \le b^* \text{ and } \operatorname{sign}(w_j) = \sigma_j,\ j = 1, \ldots, k\}\,.$$

Then,

$$|F^k(w') - F^k(w'')| \le \xi^k |w' - w''| \quad \text{for} \quad w', w'' \in \Omega^*(\sigma)\,. \tag{27}$$

Proof. Using the same argument as that in the proof for Lemma 3.2, we see that for any fixed $i \in \{1, \ldots, m\}$,

$$\operatorname{sign} g_i(u_1, u_2, u_3) = \operatorname{sign} u_2$$

and

$$r < |g_i(u_1, u_2, u_3)| < b^*$$

for $(u_1, u_2, u_3) \in \Omega_3^*$, where

$$\Omega_3^* := \{(u_1, u_2, u_3) \in \mathbf{R}^3 : \ r < |u_j| \le b^*,\ j = 1, 2, 3\}\,.$$

Therefore, $F^k(\Omega^*(\sigma)) \subset \Omega^*(\pi^k\sigma)$. Furthermore,

$$F_k^l(w')F_k^l(w'') > 0 \quad \text{for } j = 1, 2, \ldots, k \quad \text{and} \quad w', w'' \in \Omega^*(\sigma)\,.$$

Combining this with (5) and (A1)–(A2), we obtain (27). ■

Lemma 3.7. *If $\{w(n, w^0)\}_{n\in\mathbf{N}}$ is a p-periodic solution of (4) with $w(j, w^0) \in \Omega$ for $j = 1, \ldots, p$, then $p|k$ and $w(n, w^0) = w(n, w^\sigma)$ for some $\sigma \in \Sigma_p$ and $\{w(n, w^0)\}$ is one of the p-periodic solutions obtained in Theorem 3.1.*

Proof. By Lemma 2.1, we have $\Omega = \cup_{p|k} \cup_{\sigma\in\Sigma_p} \Omega(\sigma)$. Therefore, there exist $q|k$ and $\sigma \in \Sigma_q$ such that $w^0 \in \Omega(\sigma)$. Moreover, Lemma 3.5 implies $w(j, w^0) \in \Omega^*(\sigma)$ for $j = 1, \ldots, p$. On the other hand, by Theorem 3.1, there exists $w^\sigma \in \Omega(\sigma, c)$ such that $\{w(n, w^\sigma)\}_{n\in\mathbf{N}}$ is a p-periodic solution of (4). Now by Lemma 3.4, we also know that $w(j, w^\sigma) \in \Omega^*(\sigma)$ for $j = 1, 2, \ldots$. Thus, for any $j \in \mathbf{N}$, we have

$$\begin{aligned} &|w(j, w^0) - w(j, w^\sigma)| \\ &\quad = |w(pqk + j, w^0) - w(pqk + j, w^\sigma)| \\ &\quad = |(F^k)^{pq}(w(j, w^0)) - (F^k)^{pq}(w(j, w^\sigma))| \\ &\quad \le \xi^{pqk}|w(j, w^0) - w(j, w^\sigma)|\,. \end{aligned}$$

Therefore, we must have $w(j, w^0) = w(j, w^\sigma)$ for $j \in \mathbf{N}$ and $p = q$. ■

Proof of Theorem 3.3. (i) We can obtain the existence and uniqueness of p-periodic solution $\{w(n, w^\sigma)\}_{n\in\mathbf{N}}$ with $w^\sigma \in \Omega(\sigma, 0)$ by using Theorem 3.1 and Lemma 3.7. For any w^0 with $|w^0 - w^\sigma| < r(\sigma)$, by the definition of $r(\sigma)$, we can find $c \in [0, c^*)$ such that $w^0 \in \Omega(\sigma, c)$. Note from (5) that for any $l \in \{1, \ldots, k-1\}$, we have

$$|F^l(w') - F^l(w'')| \le |w' - w''| \quad \text{for} \quad w',\, w'' \in \Omega(\sigma, c)\,.$$

Let $n = sk + q$ with some $q \in \{1, \ldots, k-1\}$, then

$$\begin{aligned}|w(n, w^0) - w(n, w^\sigma)| &= |F^{(sk+q)}(w^0) - F^{(sk+q)}(w^\sigma)| \\ &\le |F^{sk}(w^0) - F^{sk}(w^\sigma)|\,.\end{aligned}$$

By Lemma 3.6, we have

$$\begin{aligned}|F^{sk}(w^0) - F^{sk}(w^\sigma)| &\le \xi^{ks}|w^0 - w^\sigma| \\ &= C\xi^{sk+k-1}|w^0 - w^\sigma| \\ &\le C\xi^n|w^0 - w^\sigma|\,,\end{aligned}$$

completing the proof of (i).

(ii) follows easily from Lemma 3.7. For (iii), we can find $c \in [0, c^*)$ such that $a^* - c \le |w_j^0| \le b^* + c$. Let $\sigma \in \Sigma$ so that $\sigma_j = \mathrm{sign}(w_j^0)$, then $w^0 \in \Omega(\sigma, c)$. Now, (i) and the result in Lemma 3.4 give the rest of proof. For (iv), we first notice from Lemma 3.7 that the period p of any given periodic solution of (4) must divide k, i.e. $p|k$. This, together with (i) and the definition of $N(p)$, proves (iv). The proof is complete. ■

The above theorem gives a very clear description about the number of periodic solutions of (4), as well as some estimates about the domains of the attraction of these periodic solutions. Note that different periodic solutions may give the same orbit, and therefore, it would also be desirable to consider the number of the periodic *orbits* of (4).

Definition 3.8. Two periodic solutions $w(n, w')$ and $w(n, w'')$ of (4) are said to be equivalent to each other, if there exists $q \in \mathbf{N}$ such that

$$w(n, w') = w(n+q, w'') \quad \text{for } n = 0, 1, \ldots\,. \tag{28}$$

Clearly, two equivalent periodic solutions $w(\cdot, w')$ and $w(\cdot, w'')$ of (3) give the same orbit

$$\begin{aligned}O(w') &:= \{w(n, w');\, n = 0, 1, \ldots\} \\ &= \{w(n, w'');\, n = 0, 1, \ldots\} \\ &=: O(w'')\,.\end{aligned}$$

Lemma 3.9. *For any fixed $p \in \mathbf{N}$ with $p|k$, and any given σ, $\overline{\sigma} \in \Sigma_p$ with $\sigma \ne \overline{\sigma}$, w^σ and $w^{\overline{\sigma}}$ are equivalent to each other if and only if there exists $q \in \{1, \ldots, p-1\}$ such that $\overline{\sigma} = \pi^q\sigma$.*

Proof. Let $w^0 = w(0, \sigma)$ and $\overline{w}^0 = w(0, \overline{\sigma})$. Define $\{\sigma^{(n)}\}_{n\ge0}$ and $\{\overline{\sigma}^{(n)}\}_{n\ge0}$ by $\sigma^{(n)} = (\sigma_1^{(n)}, \ldots, \sigma_k^{(n)})$ and $\overline{\sigma}^{(n)} = (\overline{\sigma}_1^{(n)}, \ldots, \overline{\sigma^{(n)}}_k)$, where

$$\begin{aligned}\sigma^{(0)} = \sigma\,,\quad \sigma_j^{(n)} &= \mathrm{sign}\, F_j^n(w^0) \quad \text{for } n \ge 1 \\ &\text{and} \quad j = 1, \ldots, k\,, \\ \overline{\sigma}^{(0)} = \overline{\sigma}\,,\quad \overline{\sigma}_j^{(n)} &= \mathrm{sign}\, F_j^n(\overline{w}^0) \quad \text{for } n \ge 1 \\ &\text{and} \quad j = 1, \ldots, k\,.\end{aligned}$$

From the result in the proof for Theorem 3.1, it follows that for any $p \in \mathbf{N}$ with $p|k$ and for any $\sigma \in \Sigma_p$, we have

$$\sigma^{(n+l)} = \pi^l\sigma^{(n)} \quad \text{and} \quad \sigma^{(n+p)} = \sigma^{(n)} \quad \text{for } n = 0, 1, \ldots, l = 1, \ldots, p-1\,.$$

Since w^σ and $w^{\overline{\sigma}}$ are equivalent to each other, we can find $q \in \{1, \ldots, p-1\}$ such that $w^{\overline{\sigma}}(n) = w^\sigma(n+q)$ for $n \ge 0$. Note that $w^\sigma(n) = F^n(w^0)$ and $w^{\overline{\sigma}}(n) = F^n(\overline{w}^0)$. Therefore, for any $j \in \{1, \ldots, k\}$ and $n \ge 0$, we have

$$\begin{aligned}\overline{\sigma}_j^{(n)} &= \mathrm{sign}\, w_j(n, \overline{w}^0) \\ &= \mathrm{sign}\, w_j(n+q, \sigma) \\ &= \sigma_j(n+q) \\ &= \pi^q\sigma_j^{(n)}\,.\end{aligned}$$

So, we get $\overline{\sigma} = \overline{\sigma}^{(0)} = \pi^q\sigma^{(0)} = \pi^q\sigma$.

Conversely, assume σ, $\overline{\sigma} \in \Sigma_p$ are given so that

$$\overline{\sigma} = \pi^q\sigma \quad \text{for some } q \in \{1, \ldots, p-1\}\,. \tag{29}$$

Let w^σ and $w^{\overline{\sigma}}$ with $w^\sigma(0) = w^0$ and $w^{\overline{\sigma}}(0) = \overline{w}^0$ be p-periodic solutions of (4). Since $F^q : \Omega(\sigma, 0) \to \Omega(\pi^q\sigma, 0)$, we have that $w(n+q, w^0) = w(n, F^q(w^0))$ is a p-periodic solution of (3) with $F^q(w^0) \in \Omega(\pi^q\sigma, 0)$. Note that (29) gives

$$\mathrm{sign}\, \overline{w}_j^0 = \overline{\sigma}_j = (\pi^q\sigma)_j = \mathrm{sign}\, F_j^q(w^0)\,.$$

This, together with $a^* \le |\overline{w}_j^0| \le b^*$ for $j = 1, \ldots, k$, yields $\overline{w}^0 \in \Omega(\pi^q\sigma, 0)$. From the uniqueness of a p-periodic solution of (3), we then have

$$\begin{aligned}w^\sigma(n+q) &= w(n+q, w^0) \\ &= w(n, F^q(w^0)) \\ &= w^{\pi^q\sigma}(n) \\ &= w^{\overline{\sigma}}(n) \quad \text{for } n = 0, 1, \ldots\,.\end{aligned}$$

Therefore, w^σ and $w^{\overline{\sigma}}$ are equivalent to each other. ■

Theorem 3.10. *Let $N^*(p)$ be the number of p-periodic solutions of (2) which are not equivalent to each other as p-periodic solutions of (4). Then for each $p \in \mathbf{N}$ with $p|k$, we have $N^*(p) = N(p)/p$.*

Proof. We have shown that $N(p)$ is exactly the number of elements of Σ_p. For each $p \in \Sigma_p$, the p-periodic solution w^σ is equivalent to each of the following p-periodic solutions $w^{\pi\sigma}, \ldots, w^{\pi^{p-1}\sigma}$. As $\pi^q\sigma \neq \sigma$ for any $q \in \{1, \ldots, p-1\}$, we conclude that $w^{\pi^i\sigma}$ and $w^{\pi^j\sigma}$ are not equivalent to each other when $i \neq j$ and $i, j \in \{1, \ldots, p-1\}$. Therefore, for each w^σ, there are exactly $(p-1)$ equivalent p-periodic solutions. This completes the proof. ■

Remark 3.1. In (2), if $b_i = 0$, $i = 1, \ldots, m$, then (2) becomes a system of m neural networks with *uni-directed connections*.

4. Conclusions and Remarks

We conclude that the network (2) allows the coexistence of $N^*(p)$ stable p-periodic orbits for every $p|k = \sum_{j=1}^m k_j$. This is in sharp contrast to the corresponding continuous model. For instance, it was shown in [Wu, 1998] that for all excitatory continuous networks with ring structure, the dominant dynamic is the convergence to equilibria, and in the case of delayed excitatory ring connections which corresponds to the continuous version of (2), synchronous/phase locked oscillations may exist but cannot be stable. This seems to suggest that using discrete networks are more efficient than continuous ones, as far as the storage of periodic patterns is concerned. Also formula (11), Theorem 3.10 and the fact that $k = \sum_{j=1}^m k_j$ give useful information on how the number of neurons and the delays would contribute to the capacity of stable periodic orbits in a discrete network with loop delayed feedback. For more general connection structures, we expect that similar results will hold. We note that the case where $m = 2$ was addressed in [Wu & Zhang, 2002]. By allowing the arbitrary number of neurons and allowing different synaptic coefficients in the bidirectional loop structure, we have $4m$-parameters (p_i, a_i, b_i, k_i for $1 \leq i \leq m$). This is extremely important in potential applications of neural networks with delayed feedback to storing and retrieval of large amount of prescribed periodic patterns. How to train such a network with periodic patterns remains an open and important work. We complete this section with a list of numbers $N^*(p)$ for p between 1 and 20 to illustrate how large a network's capacity can be in terms of stable p-periodic orbits: 2, 1, 2, 3, 6, 9, 18, 30, 56, 99, 186, 335, 630, 1161, 2182, 4080, 7710, 14532, 27594, 52377. .

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