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Invariant manifolds of partial functional differential equations ☆

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Dedicated to the 60th anniversary of the birthday of Professor Toshiki Naito

Abstract

This paper is concerned with the existence, smoothness and attractivity of invariant manifolds for evolutionary processes on general Banach spaces when the nonlinear perturbation has a small global Lipschitz constant and locally C^k-smooth near the trivial solution. Such a nonlinear perturbation arises in many applications through the usual cut-off procedure, but the requirement in the existing literature that the nonlinear perturbation is globally C^k-smooth and has a globally small Lipschitz constant is hardly met in those systems for which the phase space does not allow a smooth cut-off function. Our general results are illustrated by and applied to partial functional differential equations for which the phase space $C([-r,0],\mathbb{X})$ (where $r>0$ and $\mathbb{X}$ being a Banach space) has no smooth inner product structure and for which the validity of variation-of-constants formula is still an interesting open problem.

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1. Introduction

Consider a partial functional differential equation in the abstract form

$$\dot{x}(t) = Ax + Fx_t + g(x_t), \tag{1.1}$$

where A is the generator of a C_0-semigroup of linear operators on a Banach space $\mathbb{X}$, $F \in L(\mathscr{C}; \mathbb{X})$ and $g \in C^k(\mathscr{C}, \mathbb{X})$, k is a positive integer, $g(0) = 0, Dg(0) = 0$, and $||g(\varphi) - g(\psi)|| \leqslant L||\varphi - \psi||$, $\forall \varphi, \psi \in \mathscr{C} := C([-r, 0], \mathbb{X})$ and L is a positive number. We will use the standard notations as in [34], some of which will be reviewed in Section 2. As is well known (see [31,34]), if A generates a compact semigroup, then the linear equation

$$\dot{x}(t) = Ax(t) + Fx_t \tag{1.2}$$

generates an eventually compact semigroup, so this semigroup has an exponential trichotomy. The existence and other properties of invariant manifolds for (1.1) with "sufficiently small" g have been considered in various papers (see [23,25,26,30] and the references therein), and it is expected that the existence, smoothness and attractivity of center-unstable, center and stable manifolds for Eq. (1.1) play important roles in the qualitative theory of (1.1) such as bifurcations (see e.g. [14,15,26,34,35]). However, all existing results on the existence of center-unstable, center and stable manifolds for Eq. (1.1) have been using the so-called Lyapunov–Perron method based on "variation-of-constants formula" in the phase space $\mathscr{C}$ of Memory [25,26], and as noted in our previous papers (see e.g. [19]), the validity of this formula in general is still open. The smoothness is an even more difficult issue (even for ordinary functional differential equations) as the phase space involved is infinite dimensional and does not allow smooth cut-off functions.

Much progress has been recently made for both theory and applications of invariant manifolds of general semiflows and evolutionary processes (see, for example, [2–7,10–12,14–17,23,30,32,34]). To our best knowledge, C^k-smoothness with $k \geqslant 1$ of center manifolds has usually been obtained under the assumption that the nonlinear perturbation is globally Lipschitz with a small Lipschitz constant AND is C^k-smooth. In many applications, one can use a cut-off function to the original nonlinearity so that the modified nonlinearity satisfies the above assumption. But if the underlying space does not allow a globally smooth cut-off function, as the case for functional differential equations, one cannot get a useful modified nonlinearity which meets both conditions: globally Lipschitz with a small Lipschitz constant AND globally C^k-smooth. One already faces this problem for ordinary functional differential equations, and this motivated the so-called method of contractions in a scale of Banach spaces by Vanderbauwhed and van Gils [32]. This method, together with the variation-of-constants formula in the light of suns and stars, allowed Dieckmann and van Gils [13] to provide a rigorous proof for the C^k-smoothness ($k \geqslant 1$) of center manifolds for ordinary functional differential equations.

The method of Dieckmann and van Gils [13] has then been extended by Kristin et al. [22] for the C^1-smoothness of the center-stable and center-unstable manifolds for maps defined in general Banach spaces. The C^1-smoothness result was later generalized by Faria et al. [16] to the general C^k-smoothness, and this generalization enables the authors to obtain a center manifold theory for partial functional differential equations. Unfortunately, this theory cannot be applied to obtain the local invariance of center manifolds as the center manifolds obtained in [16] depend on the time discretization. Moreover, the aforementioned work of Kristin et al. [22] and Faria et al. [16] is based on a variation-of-constants formula for iterations of maps and a natural way to extend these results to partial functional differential equations would require an analogous formula which, as pointed out above, is not available at this stage.

We also note that in [6], invariant manifolds and foliations for C^1 semigroups in Banach spaces were considered without using the variation-of-constants formula. This work treats directly C^1 semigroups rather than locally smooth equations, so its applications to Eq. (1.1) require a global Lipschitz condition on the nonlinear perturbation. The proofs of the main results on the C^1-smoothness there are based on a study of the C^1-smoothness of solutions to Lyapunov-Perron discrete equations (see [6, Section 2]). Moreover, the main idea in [6, Section 2] is to study the existence and C^1-smooth dependence on parameters of "coordinates" of the unique fixed point of a contraction with "bad" characters (in terminology of [6]), that is, the contraction may not depend on parameters C^1-smoothly. To overcome this the authors used the dominated convergence theorem in proving the C^1-smoothness of every "coordinate" of the fixed point. This procedure has no extension to the case of C^k-smoothness with arbitrary $k \geqslant 1$, so the method there does not work for C^k-smoothness case. As will be shown later in this paper, the C^k-smoothness of invariant manifolds can be proved, actually using the well-known assertion that contractions with "good" characters (i.e., they depend C^k-smoothly on parameters) have C^k-smooth fixed points (see e.g. [21,29]). Furthermore, our approach in this paper is not limited to autonomous equations, as will be shown later, because it arises from a popular method of studying the asymptotic behavior of nonautonomous evolution equations, called "evolution semigroups" (see e.g. [8] for a systematic presentation of this method for investigating exponential dichotomy of homogeneous linear evolution equations and [20] for almost periodicity of solutions of inhomogeneous linear evolution equations).

An important problem of dynamical systems is to investigate conditions for the existence of invariant foliations. In the finite-dimensional case well-known results in this direction can be found e.g. in [21]. Extensions to the infinite-dimensional case were made in [6,10]. In [10] a general situation, namely, evolutionary processes generated by a semilinear evolution equations (without delay), was considered. Meanwhile, in [6] a C^1-theory of invariant foliations was developed for general C^1 semigroups in Banach spaces. We will state a simple extension of a result in [6] on invariant foliations for C^1 semigroups to periodic evolutionary processes. The

C^k-theory of invariant foliations for general evolutionary processes is still an interesting question.

In Section 2, we give a proof of the existence and attractivity of center-unstable, center and stable manifolds for general evolutionary processes using the method of graph transforms as in [1]. Our general results apply to a large class of equations generating evolutionary processes that may not be strongly continuous. We then use some classical results about smoothness of invariant manifolds for maps (described in [21,28]) and the technique of "lifting" to obtain the smoothness of invariant manifolds. The smoothness result requires the nonlinear perturbation to be C^k-smooth, verification of which seems to be relatively simple, in particular, as will be shown in Section 3, for partial functional differential equations such verification can be obtained by some estimates based on the Gronwall inequality. In Section 4 we give several examples to illustrate the applications of the obtained results.

We conclude this introduction by listing some notations. $\mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the set of natural, real, complex numbers, respectively. $\mathbb{X}$ denotes a given (complex) Banach space with a fixed norm $||\cdot||$. For a given positive r, we denote by $\mathscr{C} := C([-r,0],\mathbb{X})$ the phase space for Eq. (1.1) which is the Banach space of all continuous maps from $[-r,0]$ into $\mathbb{X}$, equipped with sup-norm $||\varphi|| = \sup_{\theta\in[-r,0]}||\varphi(\theta)||$ for $\varphi\in\mathscr{C}$. If a continuous function $x : [\beta - r, \beta+\delta \to \mathbb{X}$ is given, then we obtain the mapping $[0,\delta)\ni t \mapsto x_t \in \mathscr{C}$, where $x_t(\theta) := x(t+\theta)$ $\forall\theta\in[-r,0], t\in[\beta,\beta+\delta$. Note that in the next section, we also use subscript t for a different purpose. This should be clear from the context.

The space of all bounded linear operators from a Banach space $\mathbb{X}$ to another Banach space $\mathbb{Y}$ is denoted by $L(\mathbb{X},\mathbb{Y})$. For a closed operator A acting on a Banach space $\mathbb{X}$, $D(A)$ and $R(A)$ denote its domain and range, respectively, and $\sigma_p(A)$ stands for the point spectrum of A. For a given mapping g from a Banach space $\mathbb{X}$ to another Banach space $\mathbb{Y}$ we set

$$\mathscr{L}ip(g) := \inf\{L \geqslant 0 : ||g(x) - g(y)|| \leqslant L||x-y||, \quad \forall x, y \in \mathbb{X}\}.$$

2. Integral manifolds of evolutionary processes

In this section, we consider the existence of stable, unstable, center-unstable and center manifolds for general evolutionary processes, in particular, for semigroups. We should emphasize that the process is not required to have the strong continuity in our discussions below and thus our results can be applied to a wide class of equations.

2.1. Definitions and preliminary results

In this section, we always fix a Banach space $\mathbb{X}$ and use the notation $\mathbb{X}_t$ to stand for a closed subspace of $\mathbb{X}$ parameterized by $t\in\mathbb{R}$. Obviously, each $\mathbb{X}_t$ is also a Banach space.

Definition 2.1. Let $\{\mathbb{X}_t, t\in\mathbb{R}\}$ be a family of Banach spaces which are *uniformly isomorphic* to each other (i.e. there exists a constant $a>0$ so that for each pair $t,s\in\mathbb{R}$ with $0\leqslant t-s\leqslant 1$ there is a linear invertible operator $S:\mathbb{X}_t\to\mathbb{X}_s$ such that $\max\{\|S\|,\|S^{-1}\|\}<a$). A family of (possibly nonlinear) operators $X(t,s):\mathbb{X}_s\to\mathbb{X}_t$, $(t,s)\in\Delta:=\{(t,s)\in\mathbb{R}\times\mathbb{R}: t\geqslant s\}$, is said to be *an evolutionary process in* $\mathbb{X}$ if the following conditions hold:

(i) $X(t,t)=I_t, \forall t\in\mathbb{R}$, where I_t is the identity on $\mathbb{X}_t$;
(ii) $X(t,s)X(s,r)=X(t,r),\ \forall (t,r),(r,s)\in\Delta$;
(iii) $\|X(t,s)x-X(t,s)y\|\leqslant Ke^{\omega(t-s)}\|x-y\|,\ \forall x,y\in\mathbb{X}_s$, where K,ω are positive constants.

An evolutionary process $(X(t,s))_{t\geqslant s}$ is said to be *linear* if $X(t,s)\in L(\mathbb{X}_s,\mathbb{X}_t)$ for $(t,s)\in\Delta$. An evolutionary process $(X(t,s))_{t\geqslant s}$ is said to be *strongly continuous* if for every fixed $x\in\mathbb{X}$ the function $\Delta\ni(t,s)\mapsto X(t,s)(x)$ is continuous. This strong continuity will not be required in the remaining part of this paper. An evolutionary process $(X(t,s))_{t\geqslant s}$ is said to be *periodic with period* $T>0$ if

$$X(t+T,s+T)=X(t,s),\quad \forall (t,s)\in\Delta.$$

In what follows, for convenience, we will make the *standing assumption* that all evolutionary processes under consideration have the property

$$X(t,s)(0)=0,\quad \forall (t,s)\in\Delta. \tag{2.1}$$

For linear evolutionary processes, we have the following notion of exponential trichotomy.

Definition 2.2. A given linear evolutionary process $((U(t,s))_{t\geqslant s}$ is said to have an *exponential trichotomy* if there are three families of projections $(P_j(t))_{t\in\mathbb{R}}, j=1,2,3$, on $\mathbb{X}_t, t\in\mathbb{R}$, positive constants N,α,β with $\alpha<\beta$ such that the following conditions are satisfied:

(i) $\sup_{t\in\mathbb{R}}\|P_j(t)\|<\infty, j=1,2,3$;
(ii) $P_1(t)+P_2(t)+P_3(t)=I_t, \forall t\in\mathbb{R}, P_j(t)P_i(t)=0,\ \forall j\neq i$;
(iii) $P_j(t)U(t,s)=U(t,s)P_j(s)$, for all $t\geqslant s, j=1,2,3$;
(iv) $U(t,s)|_{ImP_2}, U(t,s)|_{ImP_3(s)}$ are homeomorphisms from $ImP_2(s)$ and $ImP_3(s)$ onto $ImP_2(t)$ and $ImP_3(t)$ for all $t\geqslant s$, respectively;
(v) The following estimates hold:

$$\|U(t,s)P_1(s)x\|\leqslant Ne^{-\beta(t-s)}\|P_1(s)x\|,\quad (\forall (t,s)\in\Delta, x\in\mathbb{X}_s),$$

$$\|U(s,t)P_2(t)x\|\leqslant Ne^{-\beta(t-s)}\|P_2(t)x\|,\quad (\forall (t,s)\in\Delta, x\in\mathbb{X}_t),$$

$$\|U(t,s)P_3(t)x\|\leqslant Ne^{\alpha|t-s|}\|P_3(s)x\|,\quad (\forall (t,s)\in\Delta, x\in\mathbb{X}_s).$$

Note that in the above definition, we define $y := U(s,t)P_2(t)x$ with $t \geqslant s$ and $x \in \mathbb{X}_t$ as the inverse of $U(t,s)y = P_2(t)x$ in $P_2(s)\mathbb{X}$. The process $(U(t,s))_{t\geqslant s}$ is said to have an *exponential dichotomy* if the family of projections $P_3(t)$ is trivial, i.e., $P_3(t) = 0, \ \forall t \in \mathbb{R}$.

Remark 2.3. Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup of linear operators on a Banach space $\mathbb{X}$ such that there is a $t_0 > 0$ for which $T(t)$ is compact for all $t \geqslant t_0$. As will be shown, this eventual compactness of the semigroup is satisfied by Eq. (1.1) with $g \equiv 0$, when A is the usual elliptic operator. We define a process $(U(t,s))_{t\geqslant s}$ by $U(t,s) := T(t-s)$ for all $(t,s) \in \Delta$. It is easy to see that $(U(t,s))_{t\geqslant s}$ is a linear evolutionary process. We now claim that the process has an exponential trichotomy with an appropriate choice of projections. In fact, since the operator $T(t_0)$ is compact, its spectrum $\sigma(T(t_0))$ consists of at most countably many points with at most one limit point $0 \in \mathbb{C}$. This property yields that $\sigma(T(t_0))$ consists of three disjoint compact sets $\sigma_1, \sigma_2, \sigma_3$, where σ_1 is contained in $\{||z|| < 1\}$, σ_2 is contained in $\{|z| > 1\}$ and σ_3 is on the unit circle $\{||z|| = 1\}$. Obviously, σ_2 and σ_3 consist of finitely many points. Hence, one can choose a simple contour γ inside the unit disc of $\mathbb{C}$ which encloses the origin and σ_1. Next, using the Riesz projection

$$P_1 := \frac{1}{2\pi i} \int_\gamma (\lambda I - T(t_0))^{-1} \, \mathrm{d}\lambda,$$

we can show easily that $P_1 T(t) = T(t) P_1, \ \forall t \geqslant 0$. Obviously, there are positive constants M, δ such that $||P_1 T(t) P_1|| \leqslant M e^{-\delta t}, \ \forall t \geqslant 0$. Furthermore, if $Q := I - P_1$, then *Im* Q is of finite-dimension and $QT(t) = T(t)Q$ for $t \in \mathbb{R}$ with $t \geqslant 0$. Consider now the strongly continuous semigroup $(T_Q(t))_{t\geqslant 0}$ on the finite-dimensional space *Im* Q, where $T_Q(t) := QT(t)Q$. Since $\sigma_2 \cup \sigma_3 = \sigma(T_Q(t_0))$, $T_Q(t)$ can be extended to a group on *Im* Q. As is well known in the theory of ordinary differential equations, in *Im* Q there are projections P_2, P_3 and positive constants K, α, β such that α can be chosen as small as required, for instance $\alpha < \delta$, and the following estimates hold:

$$P_2 + P_3 = Q, \quad P_2 P_3 = 0,$$

$$||P_2 T_Q(-t) P_2|| \leqslant K e^{-\beta t}, \quad \forall t > 0,$$

$$||P_3 T_Q P_3|| \leqslant K e^{\alpha |t|}, \quad \forall t \in \mathbb{R}.$$

Summing up the above discussions, we conclude that the evolutionary process $(U(t,s))_{t\geqslant s}$ defined by $U(t,s) = T(t-s)$ has an exponential trichotomy with projections $P_j, j = 1,2,3$, and positive constants N, α, β', where

$$\beta' := \min\{\log \sup_{\lambda \in \sigma_1} |\lambda|, \beta\},$$
$$N = \max\{K, M\}.$$

We now give the definition of integral manifolds for evolutionary processes.

Definition 2.4. For an evolutionary process $(X(t,s))_{t\geqslant s}$ in $\mathbb{X}$, a set $M \subset \cup_{t\in\mathbb{R}} \{\{t\} \times \mathbb{X}_t\}$ is said to be an *integral manifold* if for every $t\in\mathbb{R}$ the phase space $\mathbb{X}_t$ is split into a direct sum $\mathbb{X}_t = \mathbb{X}_t^1 \oplus \mathbb{X}_t^2$ with projections $P_1(t)$ and $P_2(t)$ such that

$$\sup_{t\in\mathbb{R}} ||P_j(t)|| < \infty, \quad j = 1,2 \tag{2.2}$$

and there exists a family of Lipschitz continuous mappings $g_t : \mathbb{X}_t^1 \to \mathbb{X}_t^2,\ t\in\mathbb{R}$, with Lipschitz coefficients independent of t so that

$$M = \{(t, x, g_t(x)) \in \mathbb{R} \times \mathbb{X}_t^1 \times \mathbb{X}_t^2\}$$

and

$$X(t,s)(gr(g_s)) = gr(g_t), \quad (t,s)\in\Delta.$$

Here and in what follows, $gr(f)$ denotes the graph of a mapping f, and we will abuse the notation and identify $\mathbb{X}_1^1 \oplus \mathbb{X}_t^2$ with $\mathbb{X}_t^1 \times \mathbb{X}_t^2$, namely, we write $(x,y) = x + y,\ \forall x\in\mathbb{X}_t^1, y\in\mathbb{X}_t^2$. We will also write $M_t = \{(x, g_t(x)) \in \mathbb{X}_t^1 \times \mathbb{X}_t^2\}$ for $t\in\mathbb{R}$.

In the case of nonlinear semigroups, we have the following notion of invariant manifolds with a slightly restricted meaning.

Definition 2.5. Let $(V(t))_{t\geqslant 0}$ be a semigroup of (possibly nonlinear) operators on the Banach space $\mathbb{X}$. A set $N \subset \mathbb{X}$ is said to be *an invariant manifold* for $(V(t))_{t\geqslant 0}$ if the phase space $\mathbb{X}$ is split into a direct sum $\mathbb{X} = \mathbb{X}^1 \oplus \mathbb{X}^2$ and there exists a Lipschitz continuous mapping $g : \mathbb{X}^1 \to \mathbb{X}^2$ so that $N = gr(g)$ and $V(t)N = N$ for $t\in\mathbb{R}$ with $t\geqslant 0$.

Obviously, if N is an invariant manifold of a semigroup $(V(t))_{t\geqslant 0}$, then $\mathbb{R} \times N$ is an integral manifold of the evolutionary process $(X(t,s))_{t\geqslant s} := (V(t-s))_{t\geqslant s}$.

An integral manifold M (invariant manifold N, respectively) is said to be of class C^k if the mappings g_t (the mapping g, respectively) are of class C^k. In this case, we say that M (N, respectively) is a integral C^k-manifold (invariant C^k-manifold, respectively).

Definition 2.6. Let $(U(t,s))_{t\geqslant s}$ with $U(t,s) : \mathbb{X}_s \to \mathbb{X}_t$ for $(t,s)\in\Delta$ be a linear evolutionary process and let ε be a positive constant. A nonlinear evolutionary process $(X(t,s))_{t\geqslant s}$ with $X(t,s) : \mathbb{X}_s \to \mathbb{X}_t$ for $(t,s)\in\Delta$ is said to be ε-*close* to $(U(t,s))_{t\geqslant s}$ (with exponent μ) if there are positive constants μ, η such that $\eta e^{\mu} < \varepsilon$ and

$$||\phi(t,s)x - \phi(t,s)y|| \leqslant \eta e^{\mu(t-s)} ||x-y||, \quad \forall (t,s)\in\Delta; x,y\in\mathbb{X}_s, \tag{2.3}$$

where

$$\phi(t,s)x := X(t,s)x - U(t,s)x, \quad \forall (t,s) \in \Delta, x \in \mathbb{X}_s.$$

In the case where $(U(t,s))_{t \geqslant s}$ and $(X(t,s))_{t \geqslant s}$ are determined by semigroups of operators $(U(t))_{t \geqslant 0}$ and $(X(t))_{t \geqslant 0}$, respectively, we say that the semigroup $(X(t))_{t \geqslant 0}$ is ε-close to the semigroup $(U(t))_{t \geqslant 0}$ if the process $(X(t,s))_{t \geqslant s}$ is ε-close to $(U(t,s))_{t \geqslant s}$ in the above sense.

In the sequel we will need the Implicit Function Theorem for Lipschitz continuous mappings (see [24,28]) which we state in the following lemma.

Lemma 2.7. *Assume that $\mathbb{X}$ is a Banach space and $L : \mathbb{X} \to \mathbb{X}$ is an invertible bounded linear operator. Let $\phi : \mathbb{X} \to \mathbb{X}$ be a Lipschitz continuous mapping with*

$$\mathscr{L}ip(\phi) < ||L^{-1}||^{-1}.$$

Then

(i) $(L + \phi)$ *is invertible with a Lipschitz continuous inverse, and*

$$\mathscr{L}ip[(L+\phi)^{-1}] \leqslant \frac{1}{||L^{-1}||^{-1} - \mathscr{L}ip(\phi)};$$

(ii) *if* $(L+\phi)^{-1} = L^{-1} + \psi$, *then*

$$\psi(x) = -L^{-1}\phi(L^{-1}x + \psi(x)) = -L^{-1}\phi((L+\phi)^{-1}x), \ \forall x \in \mathbb{X}$$

and

$$||\psi(x) - \psi(y)|| \leqslant \frac{||L^{-1}||\mathscr{L}ip(\phi)}{||L^{-1}||^{-1} - \mathscr{L}ip(\phi)} ||x-y||, \ \forall x, y \in \mathbb{X}. \tag{2.4}$$

We also need a stable and unstable manifold theorem for a map defined in a Banach space in our "lifting" procedure. Let A be a bounded linear operator acting on a Banach space $\mathbb{X}$ and let F be a Lipschitz continuous (nonlinear) operator acting on $\mathbb{X}$ such that $F(0) = 0$.

Definition 2.8. For a given a positive real ρ, a bounded linear operator A acting on a Banach space $\mathbb{X}$ is said to be *ρ-pseudo-hyperbolic* if $\sigma(A) \cap \{z \in \mathbb{C} : |z| = \rho\} = \emptyset$. In particular, the operator A is said to be *hyperbolic* if it is 1-pseudo-hyperbolic.

For a given ρ-pseudo-hyperbolic operator A on a Banach space $\mathbb{X}$ we consider the Riesz projection P corresponding to the spectral set $\sigma(A) \cap \{|z| < \rho\}$. Let $\mathbb{X} = Im\,P \oplus Ker\,P$ be the canonical splitting of $\mathbb{X}$ with respect to the projection P. Then we define $A_1 := A|_{Im\ P}$ and $A_2 := A|_{Ker\ P}$.

We have

Lemma 2.9. *Let A be a ρ-pseudo-hyperbolic operator acting on $\mathbb{X}$ and let F be a Lipschitz continuous mapping such that $F(0) = 0$. Then, under the above notations, the following assertions hold:*

(i) Existence of Lipschitz manifolds: *For every positive constant δ one can find a positive ε_0, depending on $||A_1||$, $||A_2^{-1}||$ and δ such that if*

$$\mathscr{L}ip(F - A) < \varepsilon, \quad 0 < \varepsilon < \varepsilon_0,$$

then, there exist exactly two Lipshitz continuous mappings $g : Im\ P \to Ker\ P$ and $h : Ker\ P \to Im\ P$ with $\mathscr{L}ip(g) \leqslant \delta$, $\mathscr{L}ip(h) \leqslant \delta$ such that their graphs $W^{s,\rho} := gr(g)$, $W^{u,\rho} := gr(h)$ *have the following properties*:

(a) $FW^{u,\rho} = W^{u,\rho}$;

(b) $F^{-1}W^{s,\rho} = W^{s,\rho}$.

(ii) Dynamical characterizations: *The following holds*:

$$W^{s,\rho} = \{z \in \mathbb{X} |\ \lim_{n \to +\infty} \rho^{-n} f^n(z) = 0\} \quad \textit{and}$$

$$W^{u,\rho} = \{z \in \mathbb{X} |\ \forall n \in \mathbb{N}\ \exists z_{-n} \in \mathbb{X} : f^n(z_{-n}) = z, \ \lim_{n \to +\infty} \rho^n z_{-n} = 0\}.$$

(iii) C^k-smoothness: *If F is of class C^k in $\mathbb{X}$ (in a neighborhood of $0 \in \mathbb{X}$, respectively), then,*

(a) *g and h are of class C^1 (in a neighborhood of 0, respectively)*;

(b) *If $||A_2^{-1}||||A_1||^j < 1$ for all $1 \leqslant j \leqslant k$, then $W^{s,\rho}$ is of class C^k, and if $||A_2^{-1}||^j||A_1|| < 1$ for all $1 \leqslant j \leqslant k$, then $W^{u,\rho}$ is of class C^k.*

Proof. For the proof of the lemma, we refer the reader to [27 Section 5; 37, p. 171]. □

2.2. The case of exponential dichotomy

This subsection is a preparatory step for proving the existence and smoothness of invariant manifolds in a more general case of exponential trichotomy. Our later general results will be based on the ones here.

2.2.1. Unstable manifolds

We start with the following result:

Theorem 2.10. *Let $(U(t,s))_{t\geqslant s}$ be a given linear process which has an exponential dichotomy. Then, there exist positive constants ε_0, δ such that for every given nonlinear process $(X(t,s))_{t\geqslant s}$ which is ε-close to $(U(t,s)_{t\geqslant s}$ with $0<\varepsilon<\varepsilon_0$, there exists a unique integral manifold $M\subset \mathbb{R}\times\mathbb{X}$ for the process $(X(t,s)_{t\geqslant s}$ determined by the graphs of a family of Lipschitz continuous mappings $(g_t)_{t\in\mathbb{R}}, g_t:\mathbb{X}_t^2\to\mathbb{X}_t^1$ with $\mathscr{L}ip(g_t)\leqslant\delta,\ \forall t\in\mathbb{R}$; here $\mathbb{X}_t^1,\mathbb{X}_t^2, t\in\mathbb{R}$ are determined from the exponential dichotomy of the process $(U(t,s))_{t\geqslant s}$. Moreover, this integral manifold has the following properties*:

(i) $X(t,s)M_s = M_t,\ \forall (t,s)\in\Delta$;

(ii) *It attracts exponentially all orbits of the process $(X(t,s))_{t\geqslant s}$ in the following sense: there are positive constants $\tilde{K},\tilde{\eta}$ such that for every $x\in\mathbb{X}$*

$$d(X(t,s)x, M_t)\leqslant \tilde{K}e^{-\tilde{\eta}(t-s)}\, d(x, M_s), \quad \forall (t,s)\in\Delta, \tag{2.5}$$

(iii) *For any $\tilde{\delta}>0$ there exists $\tilde{\varepsilon}>0$ so that if $0<\varepsilon<\tilde{\varepsilon}$, then*

$$\sup_{t\in\mathbb{R}} \mathscr{L}ip(g_t)\leqslant\tilde{\delta}. \tag{2.6}$$

Proof. This result was obtained in [1, Section 3]. For the sake of later reference, we sketch here the proof, based on several lemmas.

Let $\mathbb{X}_t^j := P_j(t)\mathbb{X}_t$ for $j=1,2$, where projections $P_j(t), j=1,2$ are as in Definition 2.2. We define the space O_δ as follows:

$$O_\delta := \left\{ g=(g_t)_{t\in\mathbb{R}} \mid g_t:\mathbb{X}_t^2\to\mathbb{X}_t^1, g_t(0)=0, \mathscr{L}ip(g_t)\leqslant\delta \right\} \tag{2.7}$$

with the metric

$$d(g,h) := \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{t\in\mathbb{R}, ||x||\leqslant k} ||g_t(x)-h_t(x)||, \quad g,h\in O_\delta. \tag{2.8}$$

It is easy to see that (O_δ, d) is a complete metric space.

First of all, we note that using Lemma 2.7 one can easily prove the following:

Lemma 2.11. *Let $(U(t,s))_{t\geqslant s}$ have an exponential dichotomy with positive constants N,β and projections $P_1(t), P_2(t), t\in\mathbb{R}$ as in Definition 2.2. Under the above notations, for every positive constant h_0, if*

$$\delta<\frac{1}{2N}, \quad \varepsilon<\frac{e^{-\mu h_0}}{2N}, \tag{2.9}$$

then, for every $g \in O_\delta$ *and* $(t,s) \in \Delta$ *such that* $0 \leqslant t - s \leqslant h_0$ *the mappings*

$$P_2(t)U(t,s)(g_s(\cdot) + \cdot) : \mathbb{X}_s^2 \to \mathbb{X}_t^2,$$

$$P_2(t)X(t,s)(g_s(\cdot) + \cdot) : \mathbb{X}_s^2 \to \mathbb{X}_t^2$$

are homeomorphisms.

The next lemma allows us to define graph transforms.

Lemma 2.12. *Let* ε *and* δ *satisfy* (2.9). *Then, the mapping* Γ^h *with* $0 \leqslant h < h_0$ *given by the formula*

$$O_\delta \ni g \mapsto \Gamma^h g \in O_{\delta'}, \tag{2.10}$$

$$gr((\Gamma^h g)_t) = X(t, t-h)(gr(g_{t-h}), \quad \forall t \in \mathbb{R} \tag{2.11}$$

is well defined, where

$$\delta'(\varepsilon, \delta, h) := \frac{\delta N e^{-\beta h} + 2\varepsilon e^{\mu h}}{(1/N)e^{\beta h} - 2\varepsilon e^{\mu h}}. \tag{2.12}$$

The next lemma ensures that the graph transforms defined above have fixed points.

Lemma 2.13. *Let* $h_0 = k$ *be a fixed natural number such that*

$$N e^{-\beta k} = q < \tfrac{1}{2}, \tag{2.13}$$

and let ε, δ *satisfy*

$$\begin{aligned} &0 < \delta < \frac{1}{2N}, \\ &0 < \varepsilon < \min\left\{\frac{e^{-2\mu k}}{2N}, \frac{\delta(q^{-1} - q)}{2(1+\delta)} e^{-2\mu k}\right\}, \\ &0 < \varepsilon < \left(\frac{1}{2} - \delta\right) \sup_{t \in \mathbb{R}} \max\{||P_1(t)||, ||P_2(t)||\}. \end{aligned}$$

Then $\Gamma^k : O_\delta \to O_\delta$ *is a (strict) contraction.*

The key step leading to the proof of the contractiveness of Γ^k is the estimate

$$\begin{aligned} &||P_1(t)X(t, t-k)x - (\Gamma^k g)_t (P_2(t)X(t, t-k)x)|| \\ &\quad \leqslant q' ||P_1(t-k)x - g_{t-k}(P_2(t-k)x)||, \ \forall g \in O_\delta, \end{aligned} \tag{2.14}$$

where q' is a constant such that $0 < q' < 1$. Next, for sufficiently small ε and δ we can apply the above lemmas to prove that the unique fixed point g of Γ^k in Lemma 2.13 is also a fixed point of Γ^h provided $0 \leqslant h \leqslant k$. In fact, for $\delta'(\varepsilon, \delta, h)$ defined by (2.12), there are positive constants ε_0, δ_0 such that

$$\delta_1 := \sup_{(\varepsilon,\delta,h) \in [0,\varepsilon_0]\times[0,\delta_0]\times[0,2k]} \delta'(\varepsilon, \delta, h) < \frac{1}{4N}. \tag{2.15}$$

Now letting

$$\begin{aligned} &0 < \delta < \min\{\delta_0, \delta_1\}, \\ &0 < \varepsilon < \min\left\{\varepsilon_0, \frac{e^{-2\mu k}}{2N}, \frac{\delta(q^{-1} - q)}{2(1+\delta)} e^{-2\mu k}\right\}, \\ &0 < \varepsilon < \left(\frac{1}{2} - \delta_1\right) \sup_{t \in \mathbb{R}} \max\{||P_1(t)||, ||P_2(t)||\}, \end{aligned}$$

by Lemmas 2.11–2.13, we have that

(i) $O_\delta \subset O_{\delta_1}$;
(ii) $\Gamma^\xi : O_\delta \to O_{\delta_1}$, for all $0 \leqslant \xi \leqslant 2k$;
(iii) $\Gamma^k : O_{\delta_1} \to O_{\delta_1}$ and $\Gamma^k(O_\delta) \subset O_\delta$;
(iv) In O_{δ_1} the operator Γ^k has a unique fixed point $g \in O_\delta$.

Thus, for $h \in [0, k]$, by the definition of the operator Γ^{k+h} (see (2.11)), we have $\Gamma^{h+k} = \Gamma^h \Gamma^k : O_\delta \to O_{\delta_1}$ and $\Gamma^{h+k} = \Gamma^k \Gamma^h : O_\delta \to O_{\delta_1}$. Next, for $h \in [0, k]$,

$$O_{\delta_1} \ni \Gamma^h g = \Gamma^h(\Gamma^k g) = \Gamma^{h+k} g = \Gamma^k(\Gamma^h g) \in O_{\delta_1}.$$

By the uniqueness of the fixed point g of Γ^k in O_{δ_1}, we have $\Gamma^h g = g$ for all $h \in [0, k]$.

The above result yields immediately

$$gr(g_t) = X(t, s)(gr(g_s)), \quad \forall (t, s) \in \Delta.$$

This proves the existence of a unstable manifold M and (i). We now prove (2.5). Let $g = (g_t)_{t \in \mathbb{R}}$ be the fixed point of Γ^k. By (2.14) and the bounded growth

$$\mathscr{L}ip(X(t, s)) \leqslant K e^{\omega(t-s)}, \ \forall (t, s) \in \Delta,$$

we can easily show that there are positive constants $\tilde{K}$ and $\tilde{\eta}$ independent of $(t, s) \in \Delta$ and $x \in \mathbb{X}$ such that

$$||P_1(t)X(t, s)(x) - g_t(P_2(t)X(t, s)(x))|| \leqslant \tilde{K} e^{-\tilde{\eta}(t-s)} ||P_1(s)x - g_s(P_2(s)x||. \tag{2.16}$$

To see how (2.5) follows from (2.16), we need the following

Lemma 2.14. *Let $Y = U \oplus V$ be a Banach space which is the direct sum of two Banach subspaces U, V with projections $P : Y \to U$, $Q : Y \to V$, respectively. Assume further that $g : U \to V$ is a Lipschitz continuous mapping with $\mathscr{L}ip(g) < 1$. Then, for any $y \in Y$,*

$$d(y, gr(g)) := \inf_{z \in U} \|y - (z + g(z))\| \geqslant \frac{1}{\|P\| + \|Q\|} \|Qy - g(Py)\|. \tag{2.17}$$

Proof. For any $y \in Y$ we have

$$\|y\| = \|Py + Qy\| \leqslant \|Py\| + \|Qy\| \leqslant (\|P\| + \|Q\|)\|y\|,$$

i.e., the norm $\|y\|_* := \|Py\| + \|Qy\|$ is equivalent to the original norm $\|y\|$. We have

$$\begin{aligned} d(y, gr(g)) &= \inf_{u \in U} \|y - (u + g(u))\| \\ &\geqslant \frac{1}{\|P\| + \|Q\|} \inf_{u \in U} \{\|Py - u\| + \|Qy - g(u))\|\} \\ &\geqslant \frac{1}{\|P\| + \|Q\|} \inf_{u \in U} \{\|Qy - g(Py)\| - \|g(Py) - g(u)\| + \|Py - u\|\} \\ &\geqslant \frac{1}{\|P\| + \|Q\|} \inf_{u \in U} \{\|Qy - g(Py)\| + (1 - \mathscr{L}ip(g))\|Py - u\|\} \\ &\geqslant \frac{1}{\|P\| + \|Q\|} \inf_{u \in U} \|Qy - g(Py)\|. \qquad \square \end{aligned} \tag{2.18}$$

Now we can apply (2.17) to (2.16) to get (2.5).

By the above discussions, for every $\delta_0 > 0$ there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the unique fixed point $g = (g_t)_{t \in \mathbb{R}}$ of Γ^k satisfies $\mathscr{L}ip(g_t) \leqslant \delta \leqslant \delta_0, \ \forall t \in \mathbb{R}$. Hence, (2.6) holds. $\square$

Proposition 2.15. Let all the conditions of Theorem 2.10 be satisfied. Moreover, assume that $(X(t,s))_{t \geqslant s}$ is T-periodic (generated by a semiflow, respectively). Then, the family of Lipschitz continuous mappings $g = (g_t)_{t \in \mathbb{R}}$ has the property that $g_t = g_{t+T}, \ \forall t \in \mathbb{R}$ (g_t is independent of $t \in \mathbb{R}$, respectively).

Proof. Consider the translation S^τ on O_δ given by $(S^\tau g)_t = g_{t+\tau}$, $\forall g, \in O_\delta, \ \tau \in \mathbb{R}, t \in \mathbb{R}$. By the T-periodicity of the process $(X(t,s))_{t \geqslant s}$ (the autonomousness of $(X(t,s))_{t \geqslant s}$, respectively) we can show that if g is a fixed point of Γ^k, then so is $S^T g$ (so is $S^\tau g, \ \forall \tau \in \mathbb{R}$, respectively). By the uniqueness of the fixed point in O_δ, we have $S^T g = g$ ($S^\tau g = g, \ \forall \tau \in \mathbb{R}$, respectively), completing the proof. $\square$

By the above proposition, if $(X(t,s))_{t \geqslant s}$ is generated by a semiflow, then the unstable integral manifold obtained in Theorem 2.10 is invariant.

2.2.2. Stable manifolds

If the process $(X(t,s))_{t\geqslant s}$ is invertible, the existence of a stable integral manifold can be easily obtained by considering the unstable manifold of its inverse process. However, in the infinite-dimensional case we frequently encounter non-invertible evolutionary processes. For this reason we will have to deal with stable integral manifolds directly. Our method below is based on a similar approach, developed in [21, Section 5] for mappings.

Theorem 2.16. *Let $(X(t,s))_{t\geqslant s}$ be an evolutionary process and let $(U(t,s))_{t\geqslant s}$ be a linear evolutionary process having an exponential dichotomy. Then, there exists a positive constant ε_0 such that if $(X(t,s))_{t\geqslant s}$ is ε-close to $(U(t,s))_{t\geqslant s}$ with $0<\varepsilon<\varepsilon_0$, then, the set*

$$M := \{(s,x)\in\mathbb{R}\times\mathbb{X} : \lim_{t\to+\infty} X(t,s)x = 0\} \tag{2.19}$$

is an integral manifold, called the stable integral manifold of $(X(t,s))_{t\geqslant s}$, represented by the graphs of a family of Lipschitz continuous mappings $g=(g_t)_{t\in\mathbb{R}}$, where $g_t:\mathbb{X}_1^t\to\mathbb{X}_2^t$, $\forall t\in\mathbb{R}$. Moreover, for every $\tilde{\delta}>0$ there exists $\tilde{\varepsilon}>0$ so that, if $0<\varepsilon<\tilde{\varepsilon}$,

$$\sup_{t\in\mathbb{R}} \mathscr{L}ip(g_t)\leqslant\tilde{\delta}. \tag{2.20}$$

Proof. First, for a fixed $0<\theta<1$, we choose $k\in\mathbb{N}$ such that for all $t\in\mathbb{R}$

$$||P_1(t)U(t,t-k)P_1(t-k)||\leqslant\theta, \tag{2.21}$$

$$||P_2(t-k)U(t-k,t)P_2(t)||\leqslant\theta. \tag{2.22}$$

Let Σ be the set of all families $g=(g_t)_{t\in\mathbb{R}}$, $g_t:\mathbb{X}_1^t\to\mathbb{X}_2^t$ such that $g_t(0)=0$, $\forall t\in\mathbb{R}$, and

$$||g||_* := \sup_{t\in\mathbb{R}}\ \sup_{y\neq 0}\frac{||g_t(y)||}{||y||}<+\infty.$$

For a positive constant γ let

$$\Sigma(\gamma) := \{g\in\Sigma : \mathscr{L}ip(g) := \sup_{t\in\mathbb{R}}\mathscr{L}ip(g_t)\leqslant\gamma\}.$$

It is not hard to prove that Σ is a Banach space with the norm $||\cdot||_*$ defined as above. Consider the graph transform G defined on $\Sigma(\gamma)$ by the formula

$$gr((Gg)_{t-k}) := [X(t,t-k)]^{-1}\{gr(g_t)\},\ \ \forall t\in\mathbb{R}, g\in\Sigma(\gamma), \tag{2.23}$$

where k is defined by (2.21) and (2.22). Note that $[X(t,t-k)]^{-1}$ is, in general, set valued. The next result justifies the use of notations of (2.23) and shows that G is well defined.

Lemma 2.17. *If $\varepsilon_0 > 0$ is sufficiently small, then for every $g \in \Sigma(\gamma)$ there is a unique $h \in \Sigma(\gamma)$ such that*

$$gr(h_{t-k}) = [X(t,t-k)]^{-1}\{gr(g_t)\}, \ \forall t \in \mathbb{R}.$$

Proof. The assertion of the lemma is equivalent to the following: for every $x \in \mathbb{X}_1^{t-k}$ there is a unique $y \in \mathbb{X}_2^{t-k}$ such that $(x,y) \in [X(t,t-k)]^{-1}\{gr(g_t)\}$ and the mapping $h_{t-k} : x \mapsto y$ is Lipschitz continuous with $\mathscr{L}ip(h_{t-k}) \leqslant \gamma$. Recall that, by abusing notations, we will identify (x,y) with $x+y$ for $x \in \mathbb{X}_1^t$, $y \in \mathbb{X}_2^t$ if this does not cause any confusion. Now $(x,y) \in [X(t,t-k)]^{-1}\{gr(g_t)\}$ if and only if

$$g_t(P_1(t)X(t,t-k)(x+y)) - P_2(t)X(t,t-k)((x+y)) = 0.$$

In the remaining part of this subsection, for the sake of simplicity of notations we will denote

$$\begin{aligned} &P := P_1(t), \ Q := P_2(t), \ X := X(t,t-k), \\ &U := U(t,t-k), \ U_2^{-1} := P_2(t-k)U(t-k,t)P_2(t). \end{aligned}$$

Hence, we get the equation for y as follows

$$y = U_2^{-1}[g_t(PX(x+y)) - Q(X(x+y) - U(x+y))]. \tag{2.24}$$

Write the right-hand side of (2.24) by $F(x+y)$, and note that

$$\mathscr{L}ip(X(t,s) - U(t,s)) < \eta e^{\mu(t-s)}, \ \forall (t,s) \in \Delta, \tag{2.25}$$

with $\eta e^{\mu} < \varepsilon$. Then, by definition, for every $x \in \mathbb{X}_t^{t-k}, y \in \mathbb{X}_2^{t-k}$, $F(x,y) \in \mathbb{X}_2^{t-k}$. We now show that if $Y_x := \{(u,v) \in \mathbb{X}_1^{t-k} \times \mathbb{X}_2^{t-k} : ||u|| \leqslant \gamma||x||\}$, then $||F(x,\cdot)|| \leqslant \gamma||x||$, i.e., $F(x,\cdot)$ leaves Y_x invariant. In fact,

$$||F(x,y)|| \leqslant \theta[\gamma||PX(x+y)|| + p\eta e^{\mu k}||x+y||],$$

where

$$p := \sup_{t \in \mathbb{R}} \max\{||P_1(t)||, ||P_2(t)||\}. \tag{2.26}$$

For $||y||\leqslant\gamma||x||$ we have

$$\begin{aligned}||PX(x+y)||&\leqslant ||P(X(x+y)-U(x+y))||+||PU(x+y)||\\&\leqslant p\eta e^{\mu k}(1+\gamma)||x||+\theta||x||\\&=[\theta+(1+\gamma)p\eta e^{\mu k}]||x||.\end{aligned}\tag{2.27}$$

Therefore,

$$\begin{aligned}||F(x,y)||&\leqslant \theta[\gamma(\theta+(1+\gamma)p\eta e^{\mu k})||x||+p\eta e^{\mu k}||x+y||]\\&=\eta\theta[\gamma(\theta+(1+\gamma)pe^{\mu k})+pe^{\mu k}(1+\gamma)]||x||.\end{aligned}\tag{2.28}$$

Hence, for small η, $F(x,\cdot)$ leaves Y_x invariant.

Next, we will show that *under the above assumptions and notations, $F(x,\cdot)$ is a contraction in Y_x*. In fact, we have

$$\begin{aligned}||F(x,y)-F(x,y')||\leqslant \theta[&||g_t(PX(x+y))-g_t(PX(x+y'))||\\&+p\eta e^{\mu k}||y-y'||].\end{aligned}\tag{2.29}$$

On the other hand,

$$\begin{aligned}||g_t(PX(x+y))-g_t(PX(x+y'))||&\leqslant \gamma||PX(x+y)-PX(x+y')||\\&\leqslant \gamma[||(PX(x+y)-PU(x+y))\\&\quad-(PX(x+y')-PU(x+y'))||\\&\quad+||PU(y-y')||].\end{aligned}$$

Using the assumption on the commutativeness of P with $U(t,s)$ we have

$$PU(y-y')=P_1(t)UP_1(t-k)(y-y')=0.$$

Hence,

$$||g_t(PX(x+y))-g_t(PX(x+y'))||\leqslant\eta\gamma pe^{\mu k}||y-y'||.$$

Consequently,

$$||F(x,y)-F(x,y')||\leqslant\theta\eta pe^{\mu k}(1+\gamma)||y-y'||.\tag{2.30}$$

Therefore, for small η, $F(x,\cdot)$ is a contraction in Y_x. By the above claim there exists a mapping $h_{t-k}:\mathbb{X}_1^{t-k}\ni x\mapsto h_{t-k}(x)\in\mathbb{X}_2^{t-k}$, where $h_{t-k}(x)$ is the fixed point of $F(x,\cdot)$ in Y_x.

We now show that this mapping is Lipschitz continuous with Lipschitz coefficient $\mathscr{L}ip(h_{t-k}) \leqslant \gamma$. In fact, letting (x, y) and $(x', y') \in X(t-k, t)(gr(g_t))$, we have $F(x, y) - F(x', y') = y - y'$. Therefore,

$$||F(x,y) - F(x',y')|| \leqslant \theta\{||g_t(PX(x+y)) - g_t(PX(x'+y'))|| \\ + p\eta e^{\mu k}||(x+y) - (x'+y')||\}. \tag{2.31}$$

On the other hand,

$$\begin{aligned} ||g_t(PX(x+y)) - g_t(PX(x'+y'))|| &\leqslant \gamma\{P(X(x+y) - U(x+y)) \\ &\quad - P(X(x'+y') - U(x'+y'))|| \\ &\quad + ||PU(x+y-x'-y')||\} \\ &\leqslant \gamma\{\theta||x-x'|| + p\eta e^{\mu k}\}[||x-x'|| + ||y-y'||] \\ &= \gamma(\theta + p\eta e^{\mu k})||x-x'|| + \gamma p\eta e^{\mu k}||y-y'||. \end{aligned} \tag{2.32}$$

Therefore,

$$\begin{aligned} ||y-y'|| &= ||F(x,y) - F(x'+y')|| \\ &\leqslant \theta\gamma p\eta e^{\mu k}||y-y'|| + \theta p\eta e^{\mu k}||y-y'|| \\ &\quad + \theta\gamma(\theta + p\eta e^{\mu k})||x-x'|| + \theta p\eta e^{\mu k}||x-x'||. \end{aligned}$$

Finally, we arrive at

$$||y-y'|| \leqslant \frac{\theta\gamma(\theta + p\eta e^{\mu k}) + \theta p\eta e^{\mu k}}{1 - \theta\gamma p\eta e^{\mu k}) - \theta p\eta e^{\mu k}}||x-x'||. \tag{2.33}$$

Thus, for sufficiently small $\eta > 0$ we have $||y-y'|| \leqslant \gamma||x-x'||$, i.e., $\mathscr{L}ip(h_{t-k}) \leqslant \gamma$. $\square$

Hence, by the above lemma, we have shown that if $\varepsilon > 0$ is small, then the graph transform G is well defined as a mapping acting on $\Sigma(\gamma)$. Moreover, we have

Lemma 2.18. *Under the above assumptions and notations, for small ε, the graph transform G is a contraction in $\Sigma(\gamma)$.*

Proof. Let $g, h \in \Sigma(\gamma)$ and let $y := (Gg)_{t-k}(x), y' := (Gh)_{t-k}(x)$. Then, we have

$$\begin{aligned}\frac{||y-y'||}{||x||} &\leqslant \frac{\theta}{||x||}||\{||g_t(PX(x+y)) - Q(X(x+y) - U(x+y))\} \\ &\quad - \{h_t(PX(x+y') + Q(X(x+y') - U(x+y'))\}|| \\ &\leqslant \frac{\theta}{||x||}\{||g_t(PX(x+y)) - h_t(PX(x+y')|| + p\eta e^{\mu k}||y-y'||\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}||g_t(PX(x+y) - h_t(PX(x+y'))|| &\leqslant ||g_t(PX(x+y) - h_t(PX(x+y)|| \\ &\quad + ||h_t(PX(x+y)) - h_t(PX(x+y'))|| \\ &\leqslant ||PX(x+y)||||g-h||_* \\ &\quad + \gamma||PX(x+y) - PX(x+y')||.\end{aligned}$$

We have, using $||y|| \leqslant \gamma||x||$, that

$$\begin{aligned}||PX(x+y)|| &\leqslant ||P(X(x+y) - U(x+y))|| + ||PU(x+y)|| \\ &\leqslant \{p\eta e^{\mu k}(1+\gamma) + \theta\}||x||.\end{aligned}$$

Thus,

$$\begin{aligned}&||g_t(PX(x+y) - h_t(P_1X(x+y'))|| \\ &\quad \leqslant \{p\eta e^{\mu k}(1+\gamma) + \theta\}||x||\,||g-h||_* + \gamma p\eta e^{\mu k}||y-y'||.\end{aligned}$$

Therefore,

$$\frac{||y-y'||}{||x||} \leqslant \theta\{(p\eta e^{\mu k}(1+\gamma) + \theta\}||g-h||_* + (1+\gamma)p\eta e^{\mu k}\frac{||y-y'||}{||x||}. \tag{2.34}$$

Finally,

$$||Gg - Gh||_* \leqslant \frac{\theta\{\theta + \eta p(1+\gamma)e^{\mu k}\}}{1 - \eta\theta p(1+\gamma)e^{\mu k}}||g-h||_*. \tag{2.35}$$

Since $0 < \theta < 1$, this yields that for small $\eta > 0$, the graph transform G is a contraction in $\Sigma(\gamma)$. $\square$

By the above lemma, for small $\eta > 0$ the graph transform G has a unique fixed point, say $g \in \Sigma(\gamma)$.

Consider the space $\mathscr{B} := \{v : \mathbb{R} \to \mathbb{X} : \sup_{t\in\mathbb{R}} ||v(t)|| < \infty\}$ and $\mathscr{B}_j := \{v \in \mathscr{B} : v(t) \in Im\ P_j(t),\ \forall t \in \mathbb{R}\}$ for $j = 1, 2$. Let the operators f, A acting on $\mathscr{B}$ be defined by the formulas

$$[fv](t) := X(t, t-k)v(t-k), \ \forall t \in \mathbb{R}, \ v \in \mathscr{B},$$

$$[Av](t) := U(t, t-k)v(t-k), \ \forall t \in \mathbb{R}, \ v \in \mathscr{B}.$$

Therefore, for $\varepsilon := \eta e^{\mu k}$, A is hyperbolic and $\mathscr{L}ip(f - A) \leqslant \varepsilon$. We define a mapping $\chi : \mathscr{B}_1 \to \mathscr{B}_2$ by the formula

$$[\chi v_1](t) := g_t(v_1(t)), \ \forall t \in \mathbb{R}, \ v_1 \in \mathscr{B}_1. \tag{2.36}$$

Obviously, $\mathscr{L}ip(\chi) \leqslant \sup_{t \in \mathbb{R}} \mathscr{L}ip(g_t)$. We want to show that $gr(\chi)$ *is the stable invariant manifold of* f. We first show that

$$f^{-1}(gr(\chi)) = gr(\chi). \tag{2.37}$$

We claim that

$$f^{-1}(gr(\chi)) \supset gr(\chi). \tag{2.38}$$

Let $(u, \chi(u)) \in gr(\chi)$ for some $u \in \mathscr{B}_1$. We have to find $v \in \mathscr{B}_1$ such that

$$f(u, \chi(u)) = (v, \chi(v)).$$

By definition, letting $(u, \chi(u)) := x$ we have

$$\begin{aligned}[f(x)](t) &= X(t, t-k)(x(t-k)) \\ &= X(t, t-k)(u(t-k), g_{t-k}(u(t-k)), \quad \forall t \in \mathbb{R}.\end{aligned}$$

By Lemma 2.17, since g is the unique fixed point of G, $X(t, t-k)(u(t-k), g_{t-k}(u(t-k)) \in gr(g_t)$, i.e., for all $\in \mathbb{R}$,

$$P_1(t)X(t, t-k)(u(t-k), g_{t-k}(u(t-k)) \in Im\ P_1(t)$$

and

$$P_2(t)X(t, t-k)(u(t-k), g_{t-k}(u(t-k)) = g_t(P_1(t)X(t, t-k)(u(t-k), g_{t-k}(u(t-k)).$$

Hence, if we set

$$v(t) := P_1(t)X(t, t-k)(u(t-k), \ \forall t \in \mathbb{R},$$

then, by definition, $v \in \mathscr{B}_1$ and $f(x) = (v, \chi(v)) \in gr(\chi)$.

Now we prove

$$f^{-1}(gr(\chi) \subset gr(\chi). \tag{2.39}$$

For every $y \in f^{-1}(gr(\chi))$, we have $f(y) \in gr(\chi)$, and hence, there is $u \in \mathscr{B}_1$ such that $f(y) = (u, \chi(u))$. By definition, for every $t \in \mathbb{R}$,

$$X(t, t-k)(y(t-k)) = (u(t), g_t(u(t))).$$

Hence, by Lemma 2.17, $y(t-k) \in gr(g_{t-k})$ for all $t \in \mathbb{R}$, i.e.,

$$P_2(t-k)y(t-k) = g_{t-k}(P_1(t-k)y(t-k)), \forall t \in \mathbb{R}.$$

Therefore, $y \in gr(\chi)$. Finally, (2.38) and (2.39) prove (2.37).

By Lemma 2.9, for sufficiently small $\varepsilon > 0$, there is a unique Lipschitz mapping $\mathscr{B}_1 \to \mathscr{B}_2$ with Lipschitz coefficient less than γ whose graph is the unique stable invariant manifold of the mapping f with $\mathscr{L}ip(f - A) < \varepsilon$. By the above discussion and since $\chi : \mathscr{B}_1 \to \mathscr{B}_2$ is Lipschitz continuous with $\mathscr{L}ip(\chi) \leqslant \gamma$ we conclude that $gr(\chi)$ is the stable invariant manifold of f.

Now, for $(x, g_s(x)) \in gr(g_s)$, we define

$$v_x(t) = \begin{cases} (x, g_s(x)), & t = s, \\ 0, & \forall t \neq s. \end{cases}$$

Observe that $g_t(0) = 0,\ \forall t \in \mathbb{R}$. Therefore, $v_x \in gr(\chi)$. Using the characterization of the stable invariant manifold of f, we have

$$0 = \lim_{n \to +\infty} ||f^n v_x|| = \lim_{n \to +\infty} ||X(s + nk, s)(x)||.$$

This, combined with the bounded growth of $(X(t,s))_{t \geqslant s}$, i.e., $||X(t,s)(x)|| \leqslant Ke^{\omega(t-s)}||x||$, implies that

$$\begin{aligned} 0 \leqslant & \lim_{t \to +\infty} ||X(t,s)x|| \\ = & \lim_{t \to +\infty} ||X\left(t, s + \left[\frac{t-s}{k}\right]k\right) X\left(s + \left[\frac{t-s}{k}\right]k, s\right)(x)|| \\ \leqslant & Ke^{\omega k} \lim_{n \to +\infty} ||X(s + nk, s)(x)|| \\ = & 0. \end{aligned} \tag{2.40}$$

On the other hand, if $x \notin gr(g_s)$, then $v_x \notin gr(\chi)$. By the characterization of the stable manifold of f,

$$\limsup_{t \to +\infty} ||X(t,s)(x)|| \geqslant \limsup_{n \to +\infty} ||f^n v_x|| = \infty. \tag{2.41}$$

Hence, $M_s := gr(g_s)$ coincides with $\{x \in \mathbb{X}_s | \lim_{t \to +\infty} X(t,s)(x) = 0\}$. That is,

$$\begin{aligned} M &:= \{(s,x) \in \mathbb{R} \times \mathbb{X} | x = \in gr(g_s)\} \\ &= \{(s,x) \in \mathbb{R} \times \mathbb{X} | \lim_{t \to +\infty} X(t,s)(x) = 0\}. \end{aligned}$$

In particular, $X(t,s)M_s \subset M_t$, $\forall (t,s) \in \Delta$. Finally, we note that $\sup_{t\in\mathbb{R}} \mathscr{L}ip(g_t) \leqslant \gamma$, which can be made as small as possible if ε is small. The proof of the theorem is then complete. $\square$

2.3. The case of exponential trichotomy

2.3.1. Lipschitz continuity, invariance and attractivity

We now apply Theorem 2.10 to prove the existence of center-unstable and center manifolds for a nonlinear process $(X(t,s))_{t\geqslant s}$ with exponential trichotomy.

Theorem 2.19. *Let $(U(t,s))_{t\geqslant s}$ be a linear evolutionary process having an exponential trichotomy in a Banach space $\mathbb{X}$ with positive constants K, α, β and projections $P_j(t), j = 1,2,3$, respectively, given in Definition* 2.2. *Then, for every sufficiently small $\delta > 0$, there exists a positive constant ε_0 such that every non-linear evolutionary process $(X(t,s))_{t\geqslant s}$ in $\mathbb{X}$, which is ε-close to $(U(t,s))_{t\geqslant s}$ with $0 < \varepsilon < \varepsilon_0$, possesses a unique integral manifold $M = \{(t, M_t),\ t \in \mathbb{R}\}$, called a center-unstable manifold, that is represented by the graphs of a family of Lipschitz continuous mappings $g = (g_t)_{t\in\mathbb{R}}$, $g_t : Im(P_2(t) + P_3(t)) \to Im\ P_1(t)$, with $\mathscr{L}ip(g_t) \leqslant \delta$, such that $M_t = gr(g_t)$, $\forall t \in \mathbb{R}$, have the following properties*:

(i) $X(t,s)gr(g_s) = gr(g_t)$, $\forall (t,s) \in \Delta$.
(ii) *There are positive constants $\hat{K}, \tilde{\eta}$ such that, for every $x \in \mathbb{X}$,*

$$d(X(t,s)(x), M_t) \leqslant \hat{K} e^{-\tilde{\eta}(t-s)} d(x, M_s), \ \forall (t,s) \in \Delta. \tag{2.42}$$

Proof. Set $P(t) := P_1(t)$ and $Q(t) := P_2(t) + P_3(t)$. Consider the following "change of variables"

$$U^*(t,s)x := e^{\gamma(t-s)} U(t,s)x, \quad \forall (t,s) \in \Delta, x \in \mathbb{X}, \tag{2.43}$$

$$X^*(t,s)x := e^{\gamma t} X(t,s)(e^{-\gamma s}x), \quad \forall (t,s) \in \Delta, x \in \mathbb{X}, \tag{2.44}$$

where α, β are given in Definition 2.2, and $\gamma := (\alpha + \beta)/2$.

We claim that $U^*(t,s)$ has an exponential dichotomy with the projections $P(t)$ and $Q(t), t \in \mathbb{R}$. In fact, it suffices to check the estimates as in Definition 2.2. We have

$$\begin{aligned} ||U^*(t,s)P(s)x|| &\leqslant N e^{\gamma(t-s)} e^{-\beta(t-s)} ||P(s)x|| \\ &\leqslant N e^{\frac{\alpha-\beta}{2}(t-s)} ||P(s)x||, \quad \forall (t,s) \in \Delta, x \in \mathbb{X}. \end{aligned}$$

On the other hand, if $(s,t)\in\Delta, x\in\mathbb{X}$, then

$$\begin{aligned}||U^*(t,s)x(I-P(s))x|| &\leqslant ||U^*(t,s)P_2(s)x|| + ||U^*(t,s)P_3(s)x|| \\ &\leqslant Ne^{\gamma(t-s)}e^{-\beta(t-s)}||P_2(s)x|| \\ &\quad + Ne^{\gamma(t-s)}e^{\alpha(s-t)}||P_3(s)x|| \\ &= Ne^{-\frac{\beta-\alpha}{2}(s-t)}(||P_2(s)x|| + ||P_3(s)x||).\end{aligned}$$

Taking into account assumption (i) in Definition 2.2 we finally get the estimate

$$||U^*(t,s)Q(s)x|| \leqslant 2pNe^{\frac{\alpha-\beta}{2}(s-t)}||Q(s)x||, \quad \forall (s,t)\in\Delta, x\in\mathbb{X}, \tag{2.45}$$

where

$$p := \sup_{t\in\mathbb{R}} \{||P_1(t)||, ||P_2(t)||, ||P_3(t)||\} < \infty. \tag{2.46}$$

This justifies the claim.

Set $\phi^*(t,s)x := X^*(t,s) - U^*(t,s)x$, and assume that $(X(t,s))_{t\geqslant s}$ is ε-close to $(U(t,s))_{t\geqslant s}$ (with exponent μ), i.e., there are positive η,μ such that $\eta e^{\mu} < \varepsilon$ and

$$\mathscr{L}ip(X(t,s) - U(t,s)) \leqslant \eta e^{\mu(t-s)}, \ \forall (t,s)\in\Delta. \tag{2.47}$$

Then, $\mathscr{L}ip(\phi^*) \leqslant \eta e^{(\gamma+\mu)(t-s)}$, i.e.,

$$||\phi^*(t,s)x - \phi^*(t,s)y|| \leqslant \eta e^{(\gamma+\mu)(t-s)}||x-y||, \quad \forall x,y\in\mathbb{X}, (t,s)\in\Delta. \tag{2.48}$$

Therefore, for any $\tilde{\varepsilon} > 0$ there exists $\varepsilon_0 = \varepsilon_0(\tilde{\varepsilon}) > 0$ so that if $(X(t,s))_{t\geqslant s}$ is ε-close to $(U(t,s))_{t\geqslant s}$ (with exponent μ), then $(X^*(t,s))_{t\geqslant s}$ is $\tilde{\varepsilon}$-close to $(U^*(t,s))_{t\geqslant s}$ (with exponent $\gamma+\mu$). Hence, by Theorem 2.10 for sufficiently small $\delta > 0$ there exists a number $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then there exists a unstable integral manifold

$$N \subset \mathbb{R}\times\mathbb{X} \text{ with } N_t = gr(d_t) \text{ for } t\in\mathbb{R} \tag{2.49}$$

for the process $(X^*(t,s))_{t\geqslant s}$, where $d_t : Im\ Q(t) \to Im\ P(t)$ and $\mathscr{L}ip(d_t) \leqslant \delta$. Let us define

$$g_t(x) := e^{-\gamma t}d_t(e^{\gamma t}x), \ \forall t\in\mathbb{R}, \ x\in Im\ Q(t). \tag{2.50}$$

Then, for all $(t,s)\in \Delta$, by using $gr(d_\xi) = e^{\gamma\xi}gr(g_\xi),\ \forall \xi\in\mathbb{R}$, we have

$$\begin{aligned} gr(d_t) &= X^*(t,s)(gr(d_s)),\\ e^{\gamma t}gr(g_t) &= e^{\gamma t}X(t,s)(e^{-\gamma s}e^{\gamma s}gr(g_s)),\\ gr(g_t) &= X(t,s)(gr(g_s)). \end{aligned}$$

Therefore, $M := \{(t, gr(g_t)),\ t\in\mathbb{R}\}$ is an integral manifold of $(X(t,s))_{t\geqslant s}$. Now, for every $x\in\mathbb{X}$ we define $y = e^{-\gamma s}x$. By Theorem 2.10, there are positive constants $\tilde{K}$ and $\tilde{\eta}$ independent of t, s, x such that

$$\begin{aligned} &d(X^*(t,s)y, N_t)\leqslant \tilde{K}e^{-\tilde{\eta}(t-s)}d(y, N_s),\\ &d(e^{\gamma t}X(t,s)(e^{-\gamma s}y), e^{\gamma t}M_t)\leqslant \tilde{K}e^{-\tilde{\eta}(t-s)}d(y, e^{\gamma s}M_s). \end{aligned}$$

Therefore,

$$\begin{aligned} d(X(t,s)(e^{-\gamma s}y), M_t) &\leqslant \tilde{K}e^{-\gamma t}e^{-\tilde{\eta}(t-s)}d(y, e^{\gamma s}M_s),\\ d(X(t,s)(x), M_t) &\leqslant \tilde{K}e^{-\gamma t}e^{-\tilde{\eta}(t-s)}e^{\gamma s}d(e^{-\gamma s}y, M_s)\\ &\leqslant \tilde{K}e^{-\gamma t}e^{-\tilde{\eta}(t-s)}e^{\gamma s}d(x, M_s),\\ &\leqslant \tilde{K}e^{-(\gamma+\tilde{\eta})(t-s)}d(x, M_s). \end{aligned}$$

This shows the attractivity of the center-unstable manifold M. $\square$

Remark 2.20.

(i) In Theorem 2.19 if $P_2(t), t\in\mathbb{R}$, are trivial projections, then the obtained center-unstable manifold is called *a center manifold*. Obviously, this center manifold attracts exponentially every point of the space $\mathbb{X}$.

(ii) By the uniqueness of the (global) center-unstable manifold obtained in Theorem 2.19 (uniqueness as a fixed point of a contractive map, it is easy to see that, in case $(X(t,s))_{t\geqslant s}$ is T-periodic (autonomous, i.e., it is generated by a semiflow, respectively), the family of mappings $g = (g_t)_{t\in\mathbb{R}}$, whose graphs represent the center-unstable manifold M of the process $(X(t,s))_{t\geqslant s}$ in Theorem 2.19 possesses property that $g_{t+T} = g_t,\ \forall t\in\mathbb{R}$ ($g_{t+\tau} = g_t,\ \forall t\in\mathbb{R}$, respectively).

Definition 2.21. Let $(X(t,s))_{t\geqslant s}$ be an evolutionary process in $\mathbb{X}$. A function $v:\mathbb{R}\to\mathbb{X}$ is said to be a trajectory of $(X(t,s))_{t\geqslant s}$ if $v(t) = X(t,s)(v(s)),\ \forall (t,s)\in\Delta$.

Proposition 2.22. *Let* $(X(t,s))_{t\geqslant s}$ *and* $(U(t,s))_{t\geqslant s}$ *satisfy all conditions of Theorem* 2.19 *and let* v *be a trajectory of* $(X(t,s))_{t\geqslant s}$ *such that*

$$\lim_{s\to-\infty} e^{\gamma s}v(s) = 0, \tag{2.51}$$

where $\gamma = (\alpha+\beta)/2$, *with* α, β *being defined in Definition* 2.2. *Then,* $v(t)\in M_t$, $\forall t\in\mathbb{R}$, *where* $M = \{(t, M_t),\ t\in\mathbb{R}\}\subset\mathbb{R}\times\mathbb{X}$ *is the center-unstable manifold of* $(X(t,s))_{t\geqslant s}$.

Proof. Consider the change of variables (2.43),(2.44). Let f, T be the lifting operators of the processes $(X^*(t,s))_{t\geqslant s}, (U^*(t,s))_{t\geqslant s}$ in $\mathscr{B}$, i.e., the operators defined by the formula

$$fu(t) = X^*(t,t-k)(u(t-k)),\ \ Tu(t) = U^*(t,t-k)u(t-k),\ \ \forall t\in\mathbb{R}, u\in\mathscr{B}, \tag{2.52}$$

where $k\in\mathbb{N}$. As is shown, f and $(X^*(t,s))_{t\geqslant s}$ have unstable manifolds W^u and $N = \{(t, N_t)\}$, respectively, and $W^u = \{v\in\mathscr{B} : v(t)\in N_t,\ \ \forall t\in\mathbb{R}\}$. For every fixed $s\in\mathbb{R}$ we define

$$w_s(t) = \begin{cases} e^{\gamma s}v(s), & t = s, \\ 0, & \forall t\neq s. \end{cases} \tag{2.53}$$

We have

$$\begin{aligned}
[fw_s](t) &= X^*(t,t-k)(w_s(t-k)) \\
&= e^{\gamma t}X(t,t-k)(e^{-\gamma(t-k)}w_s(t-k)) \\
&= \begin{cases} e^{\gamma(s+k)}v(s+k), & t = s \\ 0, & \forall t\neq s \end{cases} \\
&= w_{s+k}(t).
\end{aligned}$$

Therefore,

$$w_s\in f^{-1}(w_{s+k}),$$

and so,

$$w_{s-nk}\in f^{-n}(w_s),\ \ \forall n\in\mathbb{N}, \tag{2.54}$$

On the other hand, $||w_{s-nk}|| = ||e^{\gamma(s-nk)}v(s-nk)||$ which tends to 0 as $n\to+\infty$. By Lemma 2.9, $w_s\in W^u$. This yields that $w_s(s) = e^{\gamma s}v(s)\in N_s$. Hence, as in the proof of Theorem 2.19, since $M_s = e^{-\gamma s}N_s$, we have $v(s)\in M_s$. $\square$

Theorem 2.23. *Let* $(U(t,s))_{t\geqslant s}$ *be a linear evolutionary process having an exponential trichotomy in a Banach space* $\mathbb{X}$. *Then there exists a positive constant* ε_0 *such that for every nonlinear evolutionary process* $(X(t,s))_{t\geqslant s}$ *in* $\mathbb{X}$ *which is* ε*-close to* $(U(t,s))_{t\geqslant s}$, *there exists an integral manifold* $C = \{(t, C_t), t \in \mathbb{R}\}$, *called a center manifold, for* $(X(t,s))_{t \geqslant s}$, *that is represented by a family of Lipschitz continuous mappings* $(k_t)_{t \in \mathbb{R}}$, *and is invariant under* $(X(t,s))_{t \geqslant s}$, *i.e.,* $X(t,s)C_s = C_t, \ \forall (t,s) \in \Delta$. *Moreover, if* v *is a trajectory of* $(X(t,s))_{t \geqslant s}$ *such that* $\lim_{t \to \infty} e^{-\gamma |t|} v(t) = 0$, *then* v *is contained in* C, *i.e.,* $v(t) \in C_t, \ \forall t \in \mathbb{R}$.

Proof. Let us make the change of variables as in the proof of Theorem 2.19. As a result, we obtain the center-unstable manifold $M = \{(t, M_t), t \in \mathbb{R}\}$ for $(X(t,s))_{t \geqslant s}$ that is represented by the graphs of a family of Lipschitz continuous mappings $(g_t)_{t \in \mathbb{R}}$. We then consider the processes $(Y(t,s))_{t \geqslant s}$ and $(V(t,s))_{t \geqslant s}$, defined by

$$Y(t,s)y := Q(t)X(t,s)(g_s(y) + y), \quad \forall (t,s) \in \Delta, y \in Im \ Q(s), \tag{2.55}$$

$$V(t,s)y := Q(t)U(t,s)y, \quad \forall (t,s) \in \Delta, y \in Im \ Q(s). \tag{2.56}$$

By the commutativeness of $Q(t)$ with $(U(t,s))_{t \geqslant s}$, we can easily show that $(V(t,s))_{t \geqslant s}$ is a linear evolutionary process. As for $(Y(t,s))_{t \geqslant s}$, note that by the invariance of the integral manifold M, if $z = g_s(y) + y \in M_s$, then $X(t,s)(z) \in M_t$. This means that $X(t,s)(z) = g_t(Q(t)X(t,s)(z)) + Q(t)X(t,s)(z)$. Hence, for any $r \leqslant s \leqslant t$, $x \in Im \ Q(r)$, we have

$$\begin{aligned}
Y(t,s)Y(s,r)(x) &= Q(t)X(t,s)(g_s(Y(s,r)(x) + Y(s,r)(x)) \\
&= Q(t)X(t,s)(g_s(Q(s)X(s,r)(g_r(x) + x) + Q(s)X(s,r)(g_r(x) + x)) \\
&= Q(t)X(t,s)(X(s,r)(g_r(x) + x) \\
&= Q(t)X(t,r)(g_r(x) + x) \\
&= Y(t,r)(x).
\end{aligned}$$

Next, since $\mathscr{L}ip(g_t) \leqslant \delta$ for some $\delta > 0$, we have

$$\begin{aligned}
||Y(t,s)(x) - Y(t,s)(y)|| &\leqslant p\mathscr{L}ip(X(t,s))||(g_r(x) + x) - (g_r(y) + y)|| \\
&\leqslant p(1+\delta)Ke^{\omega(t-s)}||x - y||, \ \forall x, y \in Im \ Q(s).
\end{aligned} \tag{2.57}$$

This shows that $(Y(t,s))_{t \geqslant s}$ is an evolutionary process.

Set $\psi(t,s)y = Y(t,s)y - V(t,s)y, \quad \forall t \geqslant s, y \in Im \ Q(s)$. It is easy to see that $(Y(t,s))_{t \geqslant s}$ and $(V(t,s))_{t \geqslant s}$ are evolutionary processes. Moreover, since $\lim_{\varepsilon \to 0}$

$\sup_{t\in\mathbb{R}}\mathscr{L}ip(g_t)=0$ and since $(X(t,s))_{\geqslant s}$ is ε-close to $(U(t,s))_{t\geqslant s}$, we have

$$\begin{aligned}
||\psi(t,s)z-\psi(t,s)w|| &= ||[Y(t,s)z-V(t,s)z)]-[Y(t,s)w-V(t,s)w]|| \\
&= ||Q(t)X(t,s)(g_s(z)+z)-Q(t)U(t,s)z \\
&\quad -Q(t)X(t,s)(g_s(w)+w)+Q(t)U(t,s)(w)|| \\
&\leqslant ||Q(t)X(t,s)(g_s(z)+z)-Q(t)U(t,s)(g_s(z)+z) \\
&\quad -Q(t)X(t,s)(g_s(w)+w)+Q(t)U(t,s)(g_s(w)+w)|| \\
&\quad +||Q(t)U(t,s)(g_s(z))-Q(t)U(t,s)(g_s(w))|| \\
&\leqslant \eta e^{\mu(t-s)}\sup_{t\in\mathbb{R}}||Q(t)||[1+\sup_{t\in\mathbb{R}}\mathscr{L}ip(g_t)]||z-w|| \\
&\quad +\sup_{t\in\mathbb{R}}||Q(t)||\sup_{t\in\mathbb{R}}\mathscr{L}ip(g_t)||z-w|| \\
&\leqslant \delta_1(\varepsilon)e^{\mu(t-s)}||z-w||, \qquad (2.58)
\end{aligned}$$

where $\lim_{\varepsilon\downarrow 0}\delta_1(\varepsilon)=0$. This means that if $(X(t,s))_{t\geqslant s}$ is sufficiently close to $(U(t,s))_{t\geqslant s}$ then so is $(Y(t,s))_{t\geqslant s}$ to $(V(t,s))_{t\geqslant s}$.

We now return to the process $(Y(t,s))_{t\geqslant s}$. By the definition of $(V(t,s))_{t\geqslant s}$ and Definition 2.2, we have

$$||V(s,t)z||\leqslant Npe^{\alpha(t-s)}||z||, \quad \forall (t,s)\in\Delta, z\in Im\ Q(t), \qquad (2.59)$$

where p is defined by (2.46). Now we check the conditions of Lemma 2.7 for the linear operator $L:=V(t,s)$, and the Lipschitz mapping $\psi:=Y(t,s)-V(t,s)$ with $0\leqslant t-s\leqslant 1$. By (2.58), there is a constant $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$, then

$$\begin{aligned}
\delta_1(\varepsilon)e^{\omega} &< a^{-1}||V^{-1}(t,s)||^{-1} \\
&= a^{-1}||V(s,t)||^{-1} \\
&\leqslant a^{-1}N^{-1}p^{-1}e^{-\alpha} \\
&\leqslant a^{-1}N^{-1}p^{-1}e^{-\alpha(t-s)}, \ \forall 0\leqslant t-s\leqslant 1, \qquad (2.60)
\end{aligned}$$

where $a>0$ is a positive constant defined by $a=\sup_{t,s\in\mathbb{R}}||S(s,t)||$, $S(s,t)$ is the isomorphism from $Im\ Q(s)$ to $Im\ Q(t)$ for all $t,s\in\mathbb{R}$ as in Definition 2.1. We now can apply Lemma 2.7 to the Lipschitz mappings $L:=S(s,t)V(t,s)$ and $\phi:=S(s,t)Y(t,s)-L$ for all $0\leqslant t-s\leqslant 1$. As a result, we obtain that there is a positive $\varepsilon_0>0$ such that if $(X(t,s))_{t\geqslant s}$ is ε-close to $(U(t,s))_{t\geqslant s}$ with $0<\varepsilon<\varepsilon_0$, then $(Y(t,s))_{t\geqslant s}$ has an inverse for all $0\leqslant t-s\leqslant 1$. Moreover, by Lemma 2.7, if $Z(t,s)$ is the inverse of $Y(t,s)$ for $0 \leqslant t-s \leqslant 1$, then, for $\varphi(t,s) := Z(t,s) - V(s,t)$, we have

$$\begin{aligned} \mathscr{L}ip(\varphi(t,s)) &\leqslant \frac{||V(s,t)||\mathscr{L}ip(\psi)}{||V(s,t)||^{-1} - \mathscr{L}ip(\psi)} \\ &\leqslant \frac{Npe^{\alpha(t-s)}\delta_1(\varepsilon)e^{\mu(t-s)}}{N^{-1}p^{-1}e^{-\alpha(t-s)} - \delta_1(\varepsilon)e^{\omega(t-s)}}. \end{aligned} \tag{2.61}$$

For arbitrary $(t,s) \in \varDelta$, the invertibility of $Y(t,s)$ follows from that of $Y(t,[t]), Y([t],[t]-1), \cdots, Y([s]+1,s)$, where $[\xi]$ denotes the largest integer n such that $n \leqslant \xi$. Let $(Z(t,s))_{t\geqslant s}$ be the inverse process of $(Y(t,s))_{t\geqslant s}$. Now using (2.59), (2.58) and (2.61) we obtain that

(i) $(Z(t,s))_{t\geqslant s}$ is an evolutionary process;
(ii) For every $\eta > 0$ there is a positive constant $\varepsilon_1 > 0$ such that if $(X(t,s))_{t\geqslant s}$ is ε-close to $(U(t,s))_{t\geqslant s}$ with $0 < \varepsilon < \varepsilon_1$, then $(Z(t,s))_{t\geqslant s}$ is η-close to $(V(s,t))_{t\geqslant s}$.

Thus, for sufficiently small ε_1, by Theorem 2.10 and a change of variables as in the proof of Theorem 2.19, we can prove that there exists an integral manifold I for $(Z(t,s))_{t\geqslant s}$, that is represented by a family of Lipschitz continuous mappings $(h_t)_{t\in\mathbb{R}}$, $h_t : Im\ P_3(t) \to Im\ P_2(t)$.

Summing up the above discussions, we obtain the existence of the so-called "center" integral manifold C for the process $(X(t,s))_{t\geqslant s}$, defined by

$$C = \{(t,x) \in \mathbb{R} \times \mathbb{X} | x = g_t(h_t(z)+z) + h_t(z) + z, z \in Im\ P_3(t)\}. \tag{2.62}$$

In fact, $C = \{(t, C_t), t \in \mathbb{R}\}$, where C_t is represented by the Lipschitz continuous mapping

$$k_t :, Im\ P_3(t) \to Im\ P_1(t) \oplus Im\ P_2(t),$$

$$Im\ P_3(t) \ni z \mapsto k_t(z) := g_t(h_t(z)+z) + h_t(z) + z.$$

We now claim that C is invariant under $(X(t,s))_{t\geqslant s}$, i.e., $X(t,s)C_s = C_t,\ \forall (t,s) \in \varDelta$. Set $x := g_s(h_s(z)+z) + h_s(z) + z \in C_s$. Then, since $C_s \subset M_s$, there is $y \in Im\ P_2(t) \oplus Im\ P_3(t)$ such that

$$X(t,s)x = g_t(y) + y.$$

On the other hand, since I is an integral manifold of $(Y(t,s))_{\geqslant s}$, there is $w \in Im\ P_3(t)$ such that $Q(t)X(t,s)(h_s(z)+z) = h_t(w) + w$. Thus, $y = Q(t)X(t,s)Q(s)x = h_t(w) + w$. This shows that $X(t,s)x = g_t(h_t(w)+w) + h_t(w) + w \in C_t$, i.e., $X(t,s)C_s \subset C_t$. Conversely, suppose that $x = g_t(h_t(w)+w) + h_t(w) + w \in C_t$, then there is $y \in Im\ Q(s)$ such that $X(t,s)(g_s(y)+y) = x$, and there exists $z \in Im\ P_3(s)$ such that $Q(t)X(t,s)(h_s(z)+z) = h_t(w) + w$. From the uniqueness of the decomposition, we get that $h_s(z) + z = y$. So $x = X(t,s)(g_s(h_s(z)+z) + h_s(z) + z)$. This shows that $C_t \subset X(t,s)C_s,\ \forall (t,s) \in \varDelta$. Finally, $C_t = X(t,s)C_s,\ \forall (t,s) \in \varDelta$.

Applying repeatedly Proposition 2.22 to $(X(t,s))_{t\geq s}$ and $(Y(t,s))_{t\geq s}$ respectively, we obtain that the center manifold C obtained above does contain all trajectory v of $(X(t,s))_{t\geq s}$ such that $\lim_{t\to\infty} e^{-\gamma|t|}v(t) = 0$. $\square$

By a similar argument as above, we obtain the following result of stable manifolds.

Theorem 2.24. *Let $(U(t,s))_{t\geq s}$ be a linear evolutionary process having an exponential trichotomy in a Banach space $\mathbb{X}$. Then there exists a positive constant ε_0 such that for every nonlinear evolutionary process $(X(t,s))_{t\geq s}$ in $\mathbb{X}$ which is ε-close to $(U(t,s))_{t\geq s}$, there exists an integral manifold $N = \{(t,N_t), t\in\mathbb{R}\}$, called a stable manifold, for $(X(t,s))_{t\geq s}$, that is represented by a family of Lipschitz continuous mappings $(h_t)_{t\in\mathbb{R}}$, and is invariant under $(X(t,s))_{t\geq s}$, i.e., $X(t,s)N_s \subset N_t$, $\forall (t,s)\in\Delta$. Moreover, for every $s\in\mathbb{R}$, the following characterization holds*:

$$N_s = \{x\in\mathbb{X} : \lim_{t\to+\infty} e^{\gamma t}X(t,s)(x) = 0\}. \tag{2.63}$$

We now turn our attention to the case of semiflows. By abusing terminology, we will say that a semiflow has some properties if the induced evolutionary process has the same properties. With this convention, as in Remark 2.20 we have the following:

Theorem 2.25. *Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup of linear operators having an exponential trichotomy. Then there exists a positive constant ε_0 such that for every semiflow $(T(t))_{t\geq 0}$ in $\mathbb{X}$ which is ε-close to $(S(t))_{t\geq 0}$ and $0<\varepsilon<\varepsilon_0$, there exists a (center) invariant manifold C for $(T(t))_{t\geq 0}$. This invariant manifold contains all trajectories v satisfying $\lim_{t\to\infty} e^{-\delta|t|}v(t) = 0$ with sufficiently small $\delta>0$.*

In particular, the center manifold C contains all bounded periodic trajectories.

2.3.2. The smoothness of integral manifolds

We now consider the smoothness of the integral manifolds of evolutionary processes.

Definition 2.26. Let k be a natural number and $(X(t,s))_{t\geq s}$ be an evolutionary process. Then,

(i) $(X(t,s))_{t\geq s}$ is said to be *C^k-regular* if for every $(t,s)\in\Delta$ the mapping $X(t,s) : \mathbb{X}_s \to \mathbb{X}_t$ is of class C^k;
(ii) $(X(t,s))_{t\geq s}$ is said to be *locally C^k-regular* if there is a positive real ρ such that for every $t\geq s\in\mathbb{R}$ the mapping $X(t,s)|_{\{\|x<\|\rho\}}$ is of class C^k.

In what follows for any $r>0$, let $B_r(\mathbb{X}) = \{x\in\mathbb{X} | \|x\|<r\}$.

Definition 2.27. An integral manifold M, represented by the graph of $(g_t)_{t\in\mathbb{R}}$ is said to be *locally of class* C^k if there is a positive number r such that for each $t\in\mathbb{R}$ the mapping $g_t|_{\{\|x\|<r\}}$ is of class C^k.

With this notion we have:

Theorem 2.28. *Let* $(U(t,s))_{t\geqslant s}$ *be a linear* T*-periodic evolutionary process having exponential trichotomy in the Banach space* $\mathbb{X}$ *with the exponents* α *and* β *such that* $k\alpha<\beta$ *for some positive integer* k*. Then there exist* $\varepsilon_0>0$ *such that if a* T*-periodic evolutionary process* $(X(t,s))_{t\geqslant s}$ *is* ε*-close to* $(U(t,s))_{t\geqslant s}$ *with* $\varepsilon<\varepsilon_0$*, and if* $(X(t,s))_{t\geqslant s}$ *is locally* C^k*-regular, then the center-unstable, center and stable integral manifolds of* $(X(t,s))_{t\geqslant s}$ *obtained in Theorems* 2.19, 2.23, 2.24 *are locally of class* C^k.

Proof. We consider first the case of stable and center-unstable manifolds. By Remark 2.20, for sufficiently small δ and ε a unique stable manifold W^s of $(X(t,s))_{t\geqslant s}$ exists, and is represented by the graphs of a family of Lipschitz mappings $g=(g_t)_{t\in\mathbb{R}},\ g_t: Im\ P_1(t)\to Im\ P_2(t)\oplus Im\ P_3(t)$ such that $g_t=g_{t+T}$. This yields in particular that for every fixed $t\in\mathbb{R}$, $X(t+T,t)(gr(g_t))=gr(g_t)$. On the other hand, by applying Lemma 2.9 for the mappings $A:=U(t+nT,t)$ and $F:=X(t+nT,t)$ with a fixed sufficiently large natural number n and for $\rho=e^{(\alpha-\beta)/2}<1$, we obtain that there are positive constants ε_0 and δ independent of $t\in\mathbb{R}$ such that for every $t\in\mathbb{R}$ the graph transform $\Gamma_{X(t+nT,t)}$ of the mapping $X(t+nT,t)$ has g_t as a unique fixed point. Therefore, for every $t\in\mathbb{R}$, the mapping g_t is of class C^k by Lemma 2.9.

To obtain the C^k-smoothness of center manifold obtained in (2.62) we first note that in the proof of Theorem 2.23 the process $(Y(t,s))_{t\geqslant s}$ is C^k-regular (using (2.55) and C^k-smoothness of $g_t,\ t\in\mathbb{R}$) and invertible. This yields that its inverse process $(Z(t,s))_{t\geqslant s}$ is C^k-regular. Consequently, the family of mappings $(h_t)_{t\in\mathbb{R}}$ is C^k-smooth. By using the above conclusion of C^k-smoothness of stable and center-unstable manifolds this shows that the family of mappings representing the center manifold C in (2.62) is C^k-smooth. $\square$

2.4. *Invariant foliations*

Let $(X(t,s))_{t\geqslant s}$ be a T-periodic evolutionary process on $\mathbb{X}$. If $(X(t,s))_{t\geqslant s}$ is a C^1 semiflow sufficiently close to a linear semigroup having an exponential trichotomy on $\mathbb{X}$, then the C^1-theory of invariant foliations in [6] applies. This result can be easily extended to periodic evolutionary processes as follows. Let $(U(t,s))_{t\geqslant s}$ be a T-periodic linear evolutionary process having an exponential trichotomy with positive constants as well as projections as in Definition 2.2. In this subsection we will denote $X_1(t)=Im\ P_2(t),\ X_2(t)=Ker\ P_2(t)$. As shown in Theorem 2.19, for $(X(t,s))_{t\geqslant s}$ sufficiently close to $(U(t,s))_{t\geqslant s}$, there exists the center unstable integral manifold $M=\{(t,M_t),t\in\mathbb{R}\}$ to the process $(X(t,s))_{t\geqslant s}$.

The following result is a simple extension of [6, Theorem 1.1 (iii)] to periodic processes.

Theorem 2.29. *Let $(U(t,s))_{t\geq s}$ be a linear T-periodic evolutionary process having exponential trichotomy in the Banach space $\mathbb{X}$ with the exponents α and β such that $k\alpha<\beta$ for some positive integer k. Then there exist $\varepsilon_0>0$ such that if a T-periodic evolutionary process $(X(t,s))_{t\geq s}$ is ε-close to $(U(t,s))_{t\geq s}$ with $\varepsilon<\varepsilon_0$, then there exists a unique center- unstable integral manifold $M=\{(t,M_t),t\in\mathbb{R}\}$ to $(X(t,s))_{t\geq s}$. Moreover, for every $s\in\mathbb{R}$ there is a continuous map $h_s:\mathbb{X}\times X_2(s)\to X_1(s)$ such that $h_{s+T}=h_s$, $\forall s\in\mathbb{R}$ and for each $\xi\in M_s$, $h_s(\xi,Q_2(s)\xi)=P_2(s)\xi$ (here $Q_2(s):=I-P_2(s)$), the manifold $M_{s,\xi}:=\{h_s(\xi,x_2)+x_2:x_2\in X_2\}$ passing through ξ has the following properties*:

(i)
$$X(t,s)M_{s,\xi}\subset M_{t,X(t,s)\xi},\ \forall t\geq s;$$

$$M_{s,\xi}=\{y\in\mathbb{X}:\limsup_{\mathbb{R}\ni t\to+\infty}\frac{1}{t}\ln\|X(t,t_0)y-X(t,t_0)x\|\leq\frac{1}{2}(\alpha+\beta)\};$$

(ii) *For every fixed $s\in\mathbb{R}$, the map $h_s(\xi,x_2)$ is Lipschitz continuous in $x_2\in X_2$, uniformly in ξ;*

(iii) *For every fixed $s\in\mathbb{R}$, $x\in\mathbb{X}$, $M_{s,x}\cap M_s$ consists of exactly a single point. In particular,*

$$M_{s,\xi}\cap M_{s,\eta}=\varnothing,\quad (\xi,\eta\in M_s,\xi\neq\eta)$$

$$\bigcup_{\xi\in M_s}M_{s,\xi}=\mathbb{X};$$

(iv) *If the operators $X(t+T,t)$, $t\in\mathbb{R}$ are C^1-smooth, then the maps $h_s(\xi,x_2)$ is C^1-smooth in $x_2\in X_2$.*

Proof. Applying [6, Theorem 3.1] to $X(s,s-T)$ for every $s\in\mathbb{R}$ we get the invariant foliation in $\mathbb{X}$ with respect these maps. The characterization of the foliations in term of Lyapunov exponents and the ε-closeness (i.e. estimate of the form (2.3)) allow us to show that these foliations are actually for the process. Details are left to the reader. □

3. Integral manifolds for partial functional differential equations

This section will be devoted to applications of the results obtained in the previous section to partial functional differential equations (PFDE). We emphasize that the results so far on the existence of invariant manifolds of (PFDE) are mainly based on a "variation-of-constants formula" in the phase space $\mathscr{C}$ of Memory [25,26], and as noted in our previous papers (see e.g. [19]), the validity of this formula in general is still open. In this section we will give a proof of the existence and smoothness of invariant manifolds of PFDE in the general case. The case where a compactness assumption is imposed has been studied in [27] using a new "variation-of-constants formula" in the phase space $\mathscr{C}$. It may be noted that this method has no extension to the general case.

3.1. Evolutionary processes associated with partial functional differential equations

In this subsection, we consider the evolutionary processes generated by partial functional differential equations of the form

$$\dot{x}(t) = Ax(t) + F(t)x_t + g(t, x_t), \tag{3.1}$$

where A generates a C_0-semigroup, $F(t) \in L(\mathscr{C}, \mathbb{X})$ is strongly continuous, i.e., for each $\phi \in \mathscr{C}$ the function $\mathbb{R} \ni t \to F(t)\phi \in \mathbb{X}$ is continuous, $\sup_{t \in \mathbb{R}} ||F(t)|| < \infty$, $g(t, \phi)$ is continuous in $(t, \phi) \in \mathbb{R} \times \mathscr{C}$, $g(t, 0) = 0$, $\forall t \in \mathbb{R}$ and there is a positive constant L such that $||g(t, \phi) - g(t, \psi)|| \leqslant L||\phi - \psi||$, $\forall \psi, \psi \in \mathscr{C}$, $\forall t \in \mathbb{R}$.

In the sequel, we will need some technical lemmas. Consider the Cauchy problem

$$\begin{cases} x(t) = T(t)\phi(0) + \int_s^t T(t - \xi)F(\xi)x_\xi \, d\xi, & \forall t \geqslant s, \\ x_s = \phi \in \mathscr{C}. \end{cases} \tag{3.2}$$

Let $U(t, s)\phi := x_t$, where $x(t)$ is the solution to the above Cauchy problem. Using a standard argument (see, for example, [34]), we obtain

Lemma 3.1. *Under the above assumptions, the linear equation*

$$\dot{x}(t) = Ax(t) + F(t)x_t \tag{3.3}$$

generates a strongly continuous linear evolutionary process $(U(t, s))_{t \geqslant s}$ on $\mathscr{C}$.

We can also use a standard method to prove the existence, uniqueness and continuous dependence on initial data for mild solutions to the Cauchy problem

$$\begin{cases} x(t) = T(t)\phi(0) + \int_s^t T(t - \xi)[F(\xi)x_\xi + g(\xi, x_\xi)] \, d\xi, & \forall t \geqslant s, \\ x_s = \phi \in \mathscr{C}. \end{cases} \tag{3.4}$$

Now if we set $X(t, s)(\phi) := x_t$, where $x(\cdot)$ is the mild solution to the Cauchy problem Eq. (3.4), then we have

(i) $X(t, s)(0) = 0$, for all $t \geqslant s$ with $t, s \in \mathbb{R}$;
(ii) $X(t, t) = I$, for all $t \in \mathbb{R}$;
(iii) $X(t, r)X(r, s) = X(t, s)$, for all $t \geqslant r \geqslant s$ and $t, r, s \in \mathbb{R}$;
(iv) For every $\phi \in \mathscr{C}$, the mapping $X(t, s)(\phi)$ is continuous in (t, s) with $t \geqslant s$.

Moreover, we can prove

Lemma 3.2. *Under the above assumptions*, Eq. (3.1) *generates an evolutionary process in* $\mathscr{C}$

Proof. It remains to show that there are positive constants K, ω such that

$$||X(t,s)(\phi) - X(t,s)(\psi)|| \leqslant Ke^{\omega(t-s)}||\phi - \psi||, \quad \forall \phi, \psi \in \mathscr{C}. \tag{3.5}$$

By definition, $X(t,s)(\phi)(\theta) = x(t+\theta, \phi), \theta \in [-r, 0]$, where $x(t,\phi)$ is a solution to the following integral equation

$$\begin{cases} x(t) = T(t)\phi(0) + \int_s^t T(t-\xi)[F(\xi)x_\xi + g(\xi, x_\xi)]d\xi, \quad \forall t \geqslant s, \\ x_s = \phi \in \mathscr{C}. \end{cases} \tag{3.6}$$

Let us define $x(t) := x(t,\phi), y(t) := x(t,\psi)$. Then

$$\begin{aligned} ||X(t,s)(\phi) - X(t,s)(\psi)|| &= \sup_{-r\leqslant\theta\leqslant 0} ||X(t,s)(\phi)(\theta) - X(t,s)(\psi)(\theta)|| \\ &= \sup_{-r\leqslant\theta\leqslant 0} ||x(t+\theta) - y(t+\theta)|| \\ &\leqslant \sup_{-r\leqslant\theta\leqslant 0} \sup_{t+\theta\geqslant 0} [||T(t+\theta)||||\phi - \psi||_{\mathscr{C}} \\ &\quad + \int_s^{t+\theta} ||T(t+\theta-\xi)||(\sup_{t\in\mathbb{R}} ||F(t)|| + 2L)||x_\xi - y_\xi||d\xi], \end{aligned}$$

where $L := \sup_{t\in\mathbb{R}} \mathscr{L}ip(g(t,\cdot))$. Set $u(\xi) := ||x_\xi - y_\xi||$ for $s \leqslant \xi \leqslant t$. Let $\tilde{N}$ and $\tilde{\omega}$ be given so that $||T(t)|| \leqslant \tilde{N}e^{\tilde{\omega}t}$ for all $t \geqslant 0$. Then

$$u(t) \leqslant \tilde{N}e^{\tilde{\omega}t}u(s) + \int_s^t \tilde{N}e^{\tilde{\omega}(t-\xi)}[\sup_{t\in\mathbb{R}} ||F(t)|| + 2\mathscr{L}ip(g)]u(\xi)d\xi.$$

Setting $v(t) := e^{-\omega t}u(t)$ and noting that $v(\xi) \geqslant 0$, we have by the Gronwall inequality that

$$v(t) \leqslant v(s)\tilde{N}e^{\tilde{N}(\sup_{t\in\mathbb{R}} ||F||+2L)(t-s)}, \ \forall t \geqslant s.$$

Therefore,

$$u(t) \leqslant u(s)\tilde{N}e^{\tilde{N}(\sup_{t\in\mathbb{R}} ||F||+2L+\tilde{\omega})(t-s)}, \ \forall t \geqslant s. \tag{3.7}$$

Hence, $(X(t,s))_{t\geqslant s}$ is an evolutionary process. □

Lemma 3.3. *Under the assumptions of Lemma* 3.2, *for every* $\delta > 0$ *there exists* $\varepsilon_0 > 0$ *such that if* $\sup_{t\in\mathbb{R}} \mathscr{L}ip(g(t,\cdot) < \varepsilon_0$, *then* $(X(t,s))_{t\geqslant s}$ *is* δ*-close* $(U(t,s))_{t\geqslant s}$.

Proof. Set $V(t,s)(\phi) = X(t,s)(\phi) - U(t,s)\phi, \quad \forall t \geqslant s, \phi \in \mathscr{C}$. Below we will denote $\varepsilon := 2\sup_{t\in\mathbb{R}} \mathscr{L}ip(g_t)$ which, without loss of generality, is assumed to be positive. Obviously,

$$\mathscr{L}ip(V(t,s)) \leqslant \mathscr{L}ip(X(t,s)) + \mathscr{L}ip(U(t,s)) < \infty.$$

Let us denote by u, v, a, b the solutions to the following Cauchy problems, respectively,

$$\begin{cases} u(t) = T(t)\phi(s) + \int_s^t T(t-\xi)[F(\xi)u_\xi + g(\xi, u_\xi)]d\xi, & \forall t \geqslant s, \\ u_s = \phi \in \mathscr{C}, \end{cases}$$

$$\begin{cases} v(t) = T(t)\phi(s) + \int_s^t T(t-\xi)F(\xi)v_\xi d\xi, & \forall t \geqslant s, \\ v_s = \phi \in \mathscr{C}, \end{cases}$$

$$\begin{cases} a(t) = T(t)\psi(s) + \int_s^t T(t-\xi)[F(\xi)a_\xi + g(\xi, a_\xi)]d\xi, & \forall t \geqslant s, \\ a_s = \psi \in \mathscr{C}, \end{cases}$$

$$\begin{cases} b(t) = T(t)\psi(s) + \int_s^t T(t-\xi)F(\xi)b_\xi d\xi, & \forall t \geqslant s, \\ b_s = \phi \in \mathscr{C}. \end{cases}$$

We have

$$u(t) - v(t) = \int_s^t T(t-\xi)[F(\xi)(u_\xi - v_\xi) + g(\xi, u_\xi)]d\xi, \tag{3.8}$$

$$a(t) - b(t) = \int_s^t T(t-\xi)[F(\xi)(a_\xi - b_\xi) + g(\xi, a_\xi)]d\xi. \tag{3.9}$$

Using (3.7) we can show that there are positive constants $K, \Omega > \omega$ independent of ϕ, ψ such that

$$||u_\xi - a_\xi|| \leqslant K e^{\Omega\xi} ||\phi - \psi||, \quad \forall \xi \geqslant s. \tag{3.10}$$

Hence,

$$\begin{aligned} ||[u(t) - v(t)] - [a(t) - b(t)]|| \leqslant & \int_s^t N e^{\omega(t-\xi)} \sup_{t\in\mathbb{R}} ||F|| ||(u_\xi - v_\xi) - (a_\xi - b_\xi)|| d\xi \\ & + \int_s^t N e^{\omega(t-\xi)} \varepsilon K e^{\Omega\xi} ||\phi - \psi|| d\xi. \end{aligned}$$

Set $w(\xi) := e^{-\omega\xi} ||(u_\xi - v_\xi) - (a_\xi - b_\xi)||, \quad \forall s \leqslant \xi \leqslant t$. Then, by the Gronwall inequality and the inequality $e^x - 1 \leqslant x e^x \ \forall x \geqslant 0$, we get

$$w(\xi) \leqslant \varepsilon K N \int_s^t e^{(\Omega-\omega)\eta} d\eta ||\phi - \psi|| e^{Nm\xi} \tag{3.11}$$

$$\leqslant \varepsilon KN(t-s)e^{(\Omega-\omega)(t-s)}\, e^{Nm\xi}||\phi-\psi||, \tag{3.12}$$

where $m := \sup_{t\in\mathbb{R}} ||F(t)||$. Thus

$$\begin{aligned} ||[u(t)-v(t)]-[a(t)-b(t)]|| &\leqslant \varepsilon KN(t-s)e^{(\Omega-\omega)(t-s)}e^{(Nm-\omega)(t-s)}||\phi-\psi|| \\ &\leqslant \varepsilon KNe^{(\Omega+Nm)(t-s)}||\phi-\psi||. \end{aligned}$$

By definition, letting $s\leqslant t\leqslant s+1$ we have

$$\begin{aligned} ||V(t,s)(\phi)-V(t,s)(\psi)|| &= \sup_{\theta\in[-r,0]} ||[u(t+\theta)-v(t+\theta)]-[a(t+\theta)-b(t+\theta)]|| \\ &\leqslant \varepsilon KNe^{(\Omega+Nm)(t-s)}\, ||\phi-\psi|| \\ &= N(\varepsilon)e^{\mu}||\phi-\psi||, \end{aligned} \tag{3.13}$$

where $\lim_{\varepsilon\downarrow 0} N(\varepsilon)=0$ and $N(\varepsilon)$ is independent of μ. Now Lemma 3.3 follows from (3.13). $\square$

As an immediate consequence of the previous lemmas and Theorems 2.19, 2.23, 2.24 we have:

Theorem 3.4. *Assume that*

(i) *A generates a C_0-semigroup of linear operators*;
(ii) *$F(t)\in L(\mathscr{C},\mathbb{X})$ is strongly continuous such that* $\sup_{t\in\mathbb{R}}||F(t)||<\infty$;
(iii) *the solution evolutionary process $(U(t,s))_{t\geqslant s}$ in $\mathscr{C}$ associated with the equation*

$$\dot{x}(t) = Ax(t)+F(t)x_t, \quad t\geqslant 0,$$

has an exponential trichotomy.

Then, for sufficiently small $\sup_{t\in\mathbb{R}} \mathscr{L}ip(g(t,\cdot)$ *the evolutionary process $(X(t,s))_{t\geqslant s}$ in $\mathscr{C}$ associated with the perturbed equation*

$$\dot{x}(t) = Ax(t)+F(t)x_t+g(t,x_t), \quad t\geqslant 0, \tag{3.14}$$

has center-unstable, center and unstable integral manifolds in $\mathscr{C}$. If (3.14) *is time independent, then these manifolds are invariant under the corresponding semiflows.*

We now consider the smoothness of the above integral manifolds. We start with the study of the smoothness of global integral manifolds. To this end, we consider the following equation

$$\dot{x}(t) = Ax(t)+f(t,x_t), \tag{3.15}$$

where A is the generator of a C_0-semigroup, $f(t,\phi)$ is continuous in $(t,\phi)\in[a,b]\times\mathscr{C}$ and is Lipschitz continuous in $\phi\in\mathscr{C}$ uniformly in $t\in[a,b]$, i.e., there is a positive constant K such that

$$||f(t,\phi)-f(t,\psi)||\leqslant K||\phi-\psi||,\quad \forall t\in[a,b],\phi,\psi\in\mathscr{C}.$$

Next, we recall the well-known procedure of proving the existence and uniqueness of mild solutions of the Cauchy problem corresponding to Eq. (3.15)

$$\begin{cases} u_a=\phi\in\mathscr{C}, \\ u(t)=T(t-a)\phi(0)+\int_a^t T(t-\xi)f(\xi,u_\xi)d\xi, \quad \forall t\in[a,b]. \end{cases} \tag{3.16}$$

For every $\phi\in\mathscr{C}, u\in C([a-r,b],\mathbb{X})$, let us consider the operator

$$[\mathscr{F}(\phi,u)](t)=\begin{cases} \phi(t-a), & \forall t\in[a-r,a], \\ T(t-a)\phi(0)+\int_a^t T(t-\xi)f(\xi,u_\xi)d\xi, & \forall t\in[a,b]. \end{cases} \tag{3.17}$$

It is easy to see that $\mathscr{F}:\mathscr{C}\times C([a-r,b],\mathbb{X})\ni(\phi,u)\mapsto\mathscr{F}(\phi,u)\in C([a-r,b],\mathbb{X})$. Moreover, for sufficiently small $b-a$ (independent of $\phi\in\mathscr{C}$), $\mathscr{F}(\phi,\cdot)$ is a strict contraction (see e.g. [31, 45, p. 38]). Obviously, the unique solution to the Cauchy problem (3.16) is the unique fixed point of $\mathscr{F}(\phi,\cdot)$. For a given positive ρ we define $B(\rho):=\{\phi\in\mathscr{C}:||\phi||<\rho\}$ and $C_\rho:=\{u\in C([-r,b],\mathbb{X}):||u(t)||<\rho,\ \forall t\in[-r,b]\}$. Now assume that $f(t,\phi)$ is differentiable with respect to ϕ up to order $k\in\mathbb{N}$ and $D^j_\phi f(t,\phi)$ is continuous in $(t,\phi)\in[a,b]\times B(\rho)$ for $j=1,\dots,k$.

Lemma 3.5. *With the above notation, the mapping $\mathscr{C}\times C_\rho\ni(\phi,u)\mapsto\mathscr{F}(\phi,u)\in C([a-r,b],\mathbb{X})$ is differentiable up to order k.*

Proof. By the definition of $\mathscr{F}$, the derivative of $\mathscr{F}(\phi,u)$ with respect to ϕ is the following bounded linear operator $D_\phi\mathscr{F}(\phi,u):C([-r,0],\mathbb{X})\ni\psi\mapsto D\mathscr{F}(\phi,u)\psi\in C([a-r,b],\mathbb{X})$

$$[D_\phi\mathscr{F}(\phi,u)\psi](t)=\begin{cases} \psi(t), & t\in[a-r,a], \\ T(t)\psi(a), & t\in[a,b]. \end{cases}$$

On the other hand, by Henry [18, Lemma 3.4.3, p. 64] the derivative of the mapping $C_\rho\ni u\mapsto\mathscr{F}(\phi,u)\in C([-r,0],\mathbb{X})$ is the following operator:

$$[D_u\mathscr{F}(\phi,u)\psi](t)=\begin{cases} 0, & t\in[a-r,a], \\ \int_a^t & T(t-\xi)D_uf(\xi,u_\xi)\psi_\xi d\xi. \end{cases} \tag{3.18}$$

Obviously, $D_\phi\mathscr{F}$ is independent of (ϕ,u), so it is of class C^∞. On the other hand, by the assumptions and (3.18), $D_u\mathscr{F}$ is of class C^{k-1}. Note that all other nonzero partial derivatives of $\mathscr{F}$ with respect to ϕ and u are $D^j_u\mathscr{F}, j=2,\dots k$. This yields that $\mathscr{F}$ is of class C^k. $\square$

We need the following result on the smooth dependence of mild solutions of Eq. (3.15) on the initial data.

Lemma 3.6. *Let A be the generator of a C_0-semigroup and let $f(t,\phi)$ be Lipschitz continuous in $\phi \in \mathscr{C}$ uniformly in $t \in [a,b]$, differentiable up to order k in $\phi \in B(\rho)$. Moreover, assume that $f(t,0)=0$ for $t \in [a,b]$, $D_u^j f(t,\phi)$ is continuous in $(t,\phi) \in [a,b] \times B(\rho)$ for all $j = 1,\dots,k$, and*

$$\sup_{(t,\phi)\in[a,b]\times B(\rho)} ||D_\phi^j f(t,\phi)|| < \infty \quad j = 1,\dots,k.$$

Then, the solution $u(t,\phi)$ of the Cauchy problem (3.16) *depends C^k-smoothly on $\phi \in B(\rho)$ uniformly in $t \in [a-r,b]$, i.e., the mapping $B(\rho) \ni \phi \mapsto u(\cdot,\phi) \in C([a-r,b],\mathbb{X})$ is of class C^k.*

Proof. Set $G(\phi,u) := \mathscr{F}(\phi,u) - u$, for $(\phi,u) \in \mathscr{C} \times B(\rho)$. Obviously, if u^a is the solution of the Cauchy problem (3.16) with $\phi = \phi_a$, then $G(\phi_a, u^a) = 0$. Moreover, G is differentiable with respect to $(\phi,u) \in \mathscr{C} \times B(\rho)$ up to order k. We have $D_u G(\phi,u) = D_u \mathscr{F}(\phi,u) - I$. Note that the assertion of the theorem can be proved for $b := b'$ with sufficiently small $b' - a$ because of the continuation principle of mild solutions. For instance, we can choose

$$(b'-a)Ke^{\omega(b'-a)} \sup_{(t,\psi)\in[a,b]\times B(\rho)} ||D_\psi f(t,\psi)|| < 1, \tag{3.19}$$

where K, ω are positive constants such that $||T(t)|| \leqslant Ke^{\omega t}$, $\forall t \geqslant 0$. With this assumption, $D_u G(\phi,u)$ is invertible. In view of Lemma 3.5 we are in a position to apply the Implicit Function Theorem (see e.g. [9, p. 25] or [18, Section 1.2.6,pp. 12–13])) to conclude that the mapping $B(\rho) \ni \phi \mapsto u(\phi) \in C([a-r,b'],\mathbb{X})$ is of class C^k, i.e, the solution $u(\cdot,\phi)$ to the Cauchy problem (3.16), depends C^k-smoothly on ϕ uniformly in $t \in [a-r,b']$, so by the continuation principle, the conclusion holds true for $t \in [a-r,b]$. □

As a consequence of the above lemma we have the following.

Corollary 3.7. Let A generate a C_0-semigroup and let $f(\cdot,\cdot) : \mathbb{R} \times \mathscr{C} \to \mathbb{X}$ be continuous and satisfy the following conditions:

(i) $f(t,\phi)$ is continuously differentiable in $\phi \in B(\rho)$ up to order $k \in \mathbb{N}$ for a given positive real ρ;
(ii) For every $j = 1,\dots,k$ the following holds

$$\sup_{(t,\phi)\in\mathbb{R}\times B(\rho)} ||D_\phi^j f(t,\phi)|| < \infty.$$

Then, Eq. (3.15) generates an evolutionary process $(X(t,s))_{t\geqslant s}$ in $\mathscr{C}$ which is C^k-regular in $B(\rho)$.

Proof. In view of Lemma 3.6, for a fixed positive real T, letting $a := \tau; b = \tau + T$ we have

$$\mathscr{U} : B(\rho) \ni \phi \mapsto u(\phi) \in C([\tau - r, \tau + T], \mathbb{X})$$

is of class C^k for any $\tau \in \mathbb{R}$. So is the mapping

$$B(\rho) \ni \phi \mapsto u(\cdot, \phi)|_{[\tau+T-r,\tau+T]} \in C([-r, b], \mathbb{X}).$$

Hence, by definition,

$$X(T + \tau, \tau) : \mathscr{C} \ni \phi \mapsto u(\cdot, \phi)|_{[T+\tau-r,T+\tau]}$$

is of class C^k with respect to $\phi \in B(\rho)$. $\square$

As an immediate consequence of Theorem 2.28 and the above corollary we have:

Theorem 3.8. *Assume that*

(i) *A generates a C_0-semigroup of linear operators*;
(ii) $F(t) \in L(\mathscr{C}, \mathbb{X})$ *is strongly continuous such that* $\sup_{t\in\mathbb{R}} ||F(t)|| < \infty$, $F(t + T) = f(t)$, $\forall t \in \mathbb{R}$ *with certain positive* T;
(iii) *the solution evolutionary process* $(U(t,s))_{t \geqslant s}$ *in* $\mathscr{C}$ *associated with the equation*

$$\dot{x}(t) = Ax(t) + F(t)x_t, \quad t \in \mathbb{R},$$

has an exponential trichotomy with the exponents α *and* β *such that* $k\alpha < \beta$ *for a positive integer* k;
(iv) $g(t, x)$ *satisfies* $g(t, 0) = 0$, $g(t + T, x) = g(t, x)$, $\forall x \in \mathbb{X}, t \in \mathbb{R}$, $D^j_u g(t, \phi)$ *is continuous in* $(t, \phi) \in \mathbb{R} \times \mathscr{C}$ *and for every* $\rho > 0$ *and* $j = 1, \ldots, k$,

$$\sup_{(t,\phi)\in\mathbb{R}\times B(\rho)} ||D^j_\phi g(t, \phi)|| < \infty \quad j = 1, \ldots, k.$$

Then, for sufficiently small $\sup_{t\in\mathbb{R}} \mathscr{L}ip(g(t, \cdot))$ *the evolutionary process* $(U(t,s))_{t\geqslant s}$ *in* $\mathscr{C}$ *associated with the perturbed equation* (3.14) *has center-unstable*, *center*, *stable integral* C^k*-manifolds in* $\mathscr{C}$.

3.2. Local integral manifolds and smoothness

The local version of the above results can be derived by using the cut-off technique. In fact, we will prove the following:

Theorem 3.9. *Assume that*

(i) *A generates a strongly continuous semigroup*, $F \in L(\mathscr{C}, \mathbb{X})$;

(ii) *The solution semigroup associated with the equation* $\dot{x}(t) = Ax(t) + Fx_t$ *has an exponential dichotomy with the exponents* α *and* β *such that* $k\alpha < \beta$ *for a positive integer* k;

(iii) $g \in C^k(B(\rho_1), \mathbb{X})$ *for positive constant* ρ_1 *and integer* k, *with* $g(0) = 0, Dg(0) = 0$.

Then there exists a positive constant $\rho < \rho_1$ *such that the equation*

$$\dot{x}(t) = Ax(t) + Fx_t + g(x_t) \tag{3.20}$$

has local center-unstable, center and stable invariant C^k*-manifolds contained in* $B(\rho)$.

Proof. For a fixed $0 < \rho < \rho_1$ we define the cut-off mapping

$$G_\rho(\phi) = \begin{cases} g(\phi), & \forall \phi \in \mathscr{C} \text{ with } ||\phi|| \leqslant \rho/2, \\ g\left(\dfrac{\rho}{||\phi||}\phi\right), & \forall \phi \in \mathscr{C} \text{ with } ||\phi > \rho. \end{cases}$$

Obviously, in $B(\rho)$ we have $\mathscr{L}ip(g|_{B(\rho)}) \leqslant \sup_{\phi \in B(\rho)} ||Dg(\phi)||$. As is shown in [33, Proposition 3.10, p.95], G_ρ is globally Lipschitz continuous with

$$\mathscr{L}ip(G_\rho) \leqslant 2\mathscr{L}ip(g|_{B(\rho)}) \leqslant 2 \sup_{\phi \in B(\rho)} ||Dg(\phi)||.$$

Because of the continuous differentiability of g in $B(\rho_1)$, if we choose ρ sufficiently small, then so becomes $\mathscr{L}ip(G_\rho)$. If the solution semigroup associated with Eq. (3.14) has an exponential trichotomy, then there exist center-unstable, center and stable invariant manifolds $M, C, N \subset \mathscr{C}$ for the equation

$$\dot{x}(t) = Ax(t) + Fx_t + G_\rho(x_t). \tag{3.21}$$

Moreover, by our results in the previous section, this center manifold is C^k-smooth in $B(\rho)$. Suppose that Eq. (3.21) generates a nonlinear semigroup $(V(t))_{t \geqslant 0}$ in $\mathscr{C}$. By the definition of G_ρ it may be seen that if $\phi \in B(\rho)$ and $T > 0$ such that $V(t)\phi \in B(\rho)$ for all $t \in [0, T]$, then $V(t)\phi$ is a mild solution of the equation

$$\dot{x}(t) = Ax(t) + Fx_t + g(x_t). \tag{3.22}$$

Hence, $M_\rho := M \cap B(\rho)$, $C_\rho := C \cap B(\rho)$, $N_\rho := N \cap B(\rho)$, are invariant C^k-manifolds which we call *a local center-unstable, center and stable invariant manifolds* of Eq. (3.22), respectively. $\square$

Remark 3.10.

(i) As in the case of ordinary differential equations, local center-unstable and center invariant manifolds of Eq. (3.22) may not be unique. They depend on the cut off functions. However, using the characterization of stable manifolds one can show that in a neighborhood of the origin $B(\rho')$, $N_\rho \cap B(\rho')$ is independent of the choice of $\rho > \rho'$, i.e., it is unique.

(ii) Although there may be more than one local center manifolds, by Theorems 2.23 and 3.9, any local center manifolds obtained in Theorem 3.9 should contain small mild solutions $x(\cdot)$ of Eq. (3.22) with $\sup_{t \in \mathbb{R}} ||x(t)|| < \rho$.

(iii) The local center unstable manifold C_ρ is locally positively invariant in the sense that if $\phi \in \mathscr{C}$ and the solution x_t^ϕ of (3.22) belongs to $B(\rho)$ for all $t \in [0, T]$ with a constant $T > 0$, then $x_t^\phi \in C_\rho$ for all $t \in [0, T]$. This is, of course, obvious since C is positively invariant and hence $V(t)\phi \in \mathscr{C}$ for all $t \geqslant 0$ from which $x_t^\phi = V(t)\phi \in \mathscr{C} \cap B(\rho) = C_\rho$ for all $t \in [0, T]$.

4. An example

In this section, as an example we consider the Hutchinson equation with diffusion

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} - au(t-1,x)[1+u(t,x)], \quad t>0, x \in (0,\pi),$$
$$\frac{\partial u(t,x)}{\partial t} = 0, \quad x = 0, \pi,$$

where $d > 0, a > 0$. This equation can be re-written in the following abstract form in the phase space $\mathscr{C} := C([-1,0], X)$:

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(a)(u_t) + f(u_t, a), \tag{4.1}$$

where $X = \{v \in W^{2,2}(0,\pi) : v' = 0 \text{ at } x = 0, \pi\}$, $d\Delta v = d(\partial^2/\partial x^2)$ on X, $L(a)(v) = -av(-1)$, $f(v,a) = -av(0)v(-1)$. For further information on this equation and its applications we refer the reader to [14,25,26,34,36].

It is well-known (see e.g. [30]) that $d\Delta$ generates a compact semigroup in X. By the well-known facts from the theory of partial functional differential equations (see e.g. [31,34]) the linear equation

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(a)(u_t) \tag{4.2}$$

generates in $\mathscr{C}$ a solution semigroup of linear operators $(T(t))_{t \geqslant 0}$ with $T(t)$ compact for every $t > 1$. Obviously, $u = 0$ is an equilibrium of (4.1). By Remark 2.3, the solution semigroup $(T(t))_{t \geqslant 0}$ of (4.2) has an exponential trichotomy. Since $f(\cdot, a)$ is C^k-smooth for any $k \geqslant 1$, we can apply our above results to claim that Eq. (4.1) has C^k-smooth local invariant manifolds around $u = 0$. Moreover, the dimensions of the center and unstable manifolds are finite. We refer the reader to [14] for more information on applications of the center manifold of the above equation to the Hopf bifurcation as a passes through $\pi/2$.

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